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Convergence of block iterative methods for linear systems arising in the numerical solution of Euler equations

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Summary. We discuss block matrices of the form $A = [A_{ij}]$, where A_{ij} is a $k \times k$ symmetric matrix, A_{ii} is positive definite and A_{ij} is negative semidefinite. These matrices are natural block-generalizations of Z-matrices and M-matrices. Matrices of this type arise in the numerical solution of Euler equations in fluid flow computations. We discuss properties of these matrices, in particular we prove convergence of block iterative methods for linear systems with such system matrices.

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1. Introduction

The theory of M- and Z-matrices was developed in the last 50 years, starting with the paper of Ostrowski [O1] in 1937, followed by the work of Varga [V1] and Young [Y1] in the 50's and 60's, the papers of Fiedler and Ptak [F2], the book of Berman and Plemmons [B2] and the work of many others.

It has been stressed in [V1] and [Y1] that at least some interest in this topic comes from its important applications in the studies of the convergence of iterative schemes for linear systems arising in the numerical solution of partial differential equations. This led to many generalizations and modifications of this theory.

Here we present a further generalization. Recall that a real $n \times n$ matrix $A = [a_{ij}]$ is a Z-matrix if $a_{ij} \leq 0$ for $i \neq j$. If in addition A^{-1} exists and is elementwise nonnegative, it is called an M-matrix.

In this paper we study block matrices $A = [A_{ij}] \in \mathbb{C}^{km,km}$, where the blocks $A_{ij} \in \mathbb{C}^{k,k}$ are Hermitian matrices and the off diagonal blocks A_{ij}, $i \neq j$ are

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negative semidefinite. As for blocksize $k = 1$ these matrices are Z-matrices, we denote this class by Z_m^k.

Matrices of this type arise for example in the numerical solution of 2-D or 3-D Euler equations in fluid dynamics [H1], [D1]. In Sect. 5 we discuss these matrices. In our study of the class Z_m^k we always have these examples in the background. In particular we study convergence results for iterative methods for the abovementioned linear systems.

In the case $k = 1$, i.e., the classical case, there are many equivalent conditions, which are necessary and sufficient for a Z-matrix to be an M-matrix, e.g. [B2]. In our generalization it cannot be expected that these are equivalent, and hence it is not at all clear, which subclass of Z_m^k is the right one to replace the class of M-matrices. We opted for a diagonal dominance criterion. So we call a matrix $A = [A_{ij}] \in Z_m^k$ a generalized M-matrix, if there exists a positive vector $u^{\mathrm{T}} = [u_1, \ldots, u_m]$ such that $R_i(u) := \sum_{j=1}^m u_j A_{ij}$ is positive definite for $i = 1, \ldots, m$. The class of these matrices is denoted by M_m^k.

After presenting the notation and some preliminaries in Sect. 2, we study some general properties of the classes Z_m^k and M_m^k in Sect. 3. In particular we give conditions when matrices in Z_m^k are in M_m^k. We show that Hermitian matrices in M_m^k are positive definite and exhibit a subclass of M_m^k which is invariant under Gaussian elimination (Theorem 3.24).

In Sect. 4 we study the convergence of the Jacobi iterative method and show that some of the other M-matrix properties do not generalize to M_m^k.

2. Notation and preliminaries

In this paper we use the following notation:

Let n be a natural number. Then we denote by

$\langle n \rangle$ – the set $\{1, \ldots, n\}$;
$\mathbb{C}^{n,n}$ – the set of complex $n \times n$ matrices, $\mathbb{C}^{n,1} =: \mathbb{C}^n$;
$\mathbb{R}^{n,n}$ – the set of real $n \times n$ matrices, $\mathbb{R}^{n,1} =: \mathbb{R}^n$;
$\mathbb{R}_+^n$ – the set of positive vectors in $\mathbb{R}^n$;
I_n – the $n \times n$ identity matrix, the n may be omitted;
e_i – the i-th unit vector.

Let $A \in \mathbb{C}^{n,n}$. Then, we denote by

A^* – the conjugate transpose of A;
A^{T} – the transpose of A;
$\sigma(A)$ – the spectrum of A;
$\varrho(A)$ – the spectral radius of A, i.e. $\varrho(A) = \max\{|\lambda| \,|\lambda \in \sigma(A)\}$;
$W(A)$ – the field of values of A, i.e. $W(A) = \{x^*Ax | x \in \mathbb{C}^n,\ x^*x = 1\}$;
$\mathcal{N}(A)$ – the right null space of A, i.e. $\mathcal{N}(A) = \{x \in \mathbb{C}^n | Ax = 0\}$.

Let $A \in \mathbb{C}^{n,n}$ be Hermitian and let $A = Q^*DQ = Q^*(D_1 - D_2)Q$ be its spectral decomposition with Q unitary, where D is written as the difference of two diagonal matrices D_1, D_2 with nonnegative diagonal elements and $D_1 D_2 = 0$. Then we set $A^+ := Q^* D_1 Q$, $A^- := Q^* D_2 Q$ and $|A| := A^+ + A^-$.

2.1 Definitions. i) Let $A, B \in \mathbb{C}^{n,n}$ be Hermitian. A is *positive definite* if $x^*Ax > 0$ for all $x \in \mathbb{C}^n \backslash \{0\}$, and A is *positive semidefinite* if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$. We denote this by $A > 0$ and $A \geq 0$, respectively. Analogously we write $A < 0$ if $-A > 0$ and $A \leq 0$ if $-A \geq 0$.

ii) Let $A \in \mathbb{C}^{n,n}$. We call A *positive definite* if $A + A^* > 0$ and *positive semidefinite* if $A + A^* \geq 0$.

iii) For $A, B \in \mathbb{C}^{n,n}$, we write $A > B$, $A \geq B$, $A < B$, $A \leq B$ if $A - B > 0$, $A - B \geq 0$, $A - B < 0$, $A - B \leq 0$, respectively.

If $A = Q^*DQ \in \mathbb{C}^{n,n}$ is Hermitian positive definite, then we denote by $A^{1/2}$ the Hermitian positive definite matrix Q^*D_1Q, where for $D = \operatorname{diag}(d_1, \ldots, d_n)$, $D_1 = \operatorname{diag}(d_1^{1/2}, \ldots, d_n^{1/2})$.

2.2 Definition. Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$. Then,
- A is a *Z-matrix* if $a_{ij} \leq 0$ for $i \neq j$, $i, j = 1, \ldots, n$;
- A is an *M-matrix* if A is a Z-matrix, $B = [b_{ij}] = A^{-1}$ exists and $b_{ij} \geq 0$ for all $i, j = 1, \ldots, n$.

2.3 Notation. By Z_m^k we denote the set of matrices $\{A \in \mathbb{C}^{m \cdot k, m \cdot k} | A = [A_{ij}],$ $A_{ij} \in \mathbb{C}^{k,k}$ Hermitian for $i, j = 1, \ldots, m$ and $A_{ij} \leq 0$ for $i, j = 1, \ldots m$, $i \neq j\}$ and we set $\hat{Z}_m^k := \{A = [A_{ij}] \in Z_m^k | A_{ii} > 0,\ i = 1, \ldots, m\}$.

We furthermore define $M_m^k = \{A \in \hat{Z}_m^k |$ there exists $u \in \mathbb{R}_+^n$ such that $\sum_{j=1}^m u_j A_{ij} > 0$ for all $i = 1, \ldots, m\}$.

2.4 Definition. Let $A = [a_{ij}] \in \mathbb{C}^{m,m}$, $B = [b_{ij}] \in \mathbb{C}^{k,k}$. Then the (right) Kronecker product of A and B, denoted by $A \otimes B$, is defined to be the matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & \ldots & a_{1m}B \\ a_{21}B & \ldots & \vdots \\ \vdots & & \vdots \\ a_{m1}B & \ldots & a_{mm}B \end{bmatrix} \in \mathbb{C}^{km \cdot km}.$$

2.5 Definition. Let $A = [A_{ij}] \in \mathbb{C}^{mk,mk}$ with $A_{ij} \in \mathbb{C}^{k,k}$. Then, we define the block graph G_A of A as the nondirected graph of vertices $1, \ldots, m$ and edges $\{i, j\}$, $i \neq j$, where $\{i, j\}$ is an edge of G_A if $A_{ij} \neq 0$ or $A_{ji} \neq 0$. By $E(G_A)$ we denote the edge set of G_A. A is called *block acyclic* if G_A is a *forest*, i.e. G_A is either a tree or a collection of trees. A vertex of G_A that has less than two neighbors is called a *leaf*.

For properties of acyclic matrices see [B1].

Besides this we consider also directed graphs D_A, which are obtained from G_A by introducing various different orientations on the edges of G_A.

The set of directed edges or *arcs* of D_A is again denoted by $E(D_A)$. Note that then an arc is an ordered pair (i, j).

For the discussion of block iterative methods we need the following definitions, e.g. [B2].

2.6 Definition. Let $A = [A_{ij}] \in \mathbb{C}^{mk,mk}$ with block $A_{ij} \in \mathbb{C}^{k,k}$ and nonsingular diagonal blocks. Then, the *block Jacobi matrix* corresponding to A is the matrix $J = D^{-1}(L+U)$, where $-L$, $-U$ are the block strictly lower and upper triangular part of A. The Jacobi matrix J is called *weakly cyclic* of *index* $p \geq 2$ if there exists a permutation matrix P such that PAP^{T} has the block form

$$(2.7)\qquad \begin{bmatrix} 0 & \dots & 0 & B_{1p} \\ B_{21} & 0 & & 0 \\ & \ddots & \ddots & \vdots \\ & 0 & B_{p,p-1} & 0 \end{bmatrix}$$

where all the null diagonal blocks are square. A is called *p-cyclic* if J is weakly cyclic of index p. A is called *consistently ordered p-cyclic* if A *p-cyclic* and if all the eigenvalues of the matrix $J(\alpha) = \alpha L + \alpha^{1-p} U$ are independent of α for all $\alpha \neq 0$.

Note that this is a special case of the usual definition of weakly cyclic matrices, since here we have blocks A_{ij} which are all of size $k \times k$. We conclude this section with some simple Lemmas.

2.8 Lemma. *Let $A \in \mathbb{C}^{n,n}$ be Hermitian, $A \geq 0$. Then $A^2 \leq A$ if and only if $A \leq I$.*

Proof. The proof is obvious, e.g. [H2, p. 470]. □

2.9 Lemma. *Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \hat{Z}_2^k$ be Hermitian. If $A_{22} + A_{21} > 0$, $A_{11} + A_{12} \geq 0$, then $A > 0$.*

Proof. It is well known (cf. [H2, p. 472]) that $A > 0$ if and only if $A_{11} > 0$ and $A_{22} - A_{21} A_{11}^{-1} A_{12} > 0$. The first condition holds trivially, since $A \in \hat{Z}_2^k$. For the second condition we get

$$A_{22} - A_{21} A_{11}^{-1} A_{12} = (A_{22} + A_{21}) + (-A_{12} - A_{21} A_{11}^{-1} A_{12}) .$$

The first term is positive definite and the second is positive semidefinite by Lemma 2.8, since $A_{11}^{-1/2}(-A_{12}) A_{11}^{-1/2} \leq I$. □

3. Positive definiteness and invariance under Gaussian elimination

In this section we list several results for Z_m^k, which generalize some results for Z-matrices or M-matrices. We begin with a generalized diagonal dominance result.

3.1 Proposition. *i) Let $A = [A_{ij}] \in Z_m^k$ and suppose that*

$$(3.2)\qquad A_{ii} + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^{m} (A_{ij} + A_{ji}) \geq 0 \quad i = 1, \dots, m .$$

Then A is positive semidefinite.

ii) Let $A = [A_{ij}] \in Z_m^k$ and suppose that

$$(3.3)\qquad A_{ii} + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^{m} (A_{ij} + A_{ji}) > 0 \quad i = 1, \dots, m .$$

Then A is positive definite.

Proof. Let $B = [B_{ij}] = A + A^*$.

Then

$$B = \left(\sum_{\substack{i,j=1 \\ i>j}}^{n} \begin{bmatrix} 0 & & & & \\ & -B_{ij} & & B_{ij} & \\ & & 0 & & \\ & B_{ij} & & -B_{ij} & \\ & & & & 0 \end{bmatrix} \right)$$

$$+ \begin{bmatrix} B_{11} + \sum_{j \neq 1} B_{ij} & & \\ & \ddots & \\ & & B_{mm} + \sum_{j \neq m} B_{mj} \end{bmatrix} .$$

Each summand in the first term is of the form

$$-(e_i - e_j)(e_i - e_j)^{\mathrm{T}} \otimes B_{ij}$$

and hence is positive semidefinite since $-B_{ij} \geq 0$.

(3.2) or (3.3) imply that

$$\begin{bmatrix} B_{11} + \sum_{j \neq 1} B_{ij} & & \\ & \ddots & \\ & & B_{mm} + \sum_{j \neq m} B_{mj} \end{bmatrix}$$

is positive definite or positive semidefinite, respectively and together we get i) and ii), respectively. □

It is well known in the case $k = 1$ that (3.3) is not a necessary condition and that (3.2) is not a sufficient condition for A to be positive definite.

In the following we discuss other conditions for $A \in Z_m^k$ to be positive definite, which generalize conditions for M-matrices to matrices in M_m^k.

3.4 Theorem. *Let $A \in \mathbb{C}^{mk,mk}$ and $A + A^* \in M_m^k$. Let $A = D - N$ with $D = \operatorname{diag}(A_{11}, \ldots, A_{m,m})$. Define for $t \in \mathbb{R}$*

$$\hat{A}_t = D + D^* - (\mathrm{e}^{\mathrm{i}t} N + \mathrm{e}^{-\mathrm{i}t} N^*) . \tag{3.5}$$

Then, $\hat{A}_t > 0$.

Proof. There exists $u \in \mathbb{R}_+^m$ such that

$$\sum_{i=1}^{m} (A_{ij} + A_{ji}) u_j > 0 , \quad i = 1, \ldots, m .$$

Replacing A by $\operatorname{diag}(u_1 I_k, \ldots, u_m I_k) A \operatorname{diag}(u_1 I_k, \ldots, u_m I_k)$, we see that we may assume $u_i = 1$, $i = 1, \ldots, m$. But then

$$\hat{A}_t = - \sum_{\substack{\ell=1 \\ \ell<j}}^{m} \left[\begin{bmatrix} 1 & \mathrm{e}^{\mathrm{i}t} \\ \mathrm{e}^{-\mathrm{i}t} & 1 \end{bmatrix} \otimes A_{\ell j} + \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}t} \\ \mathrm{e}^{\mathrm{i}t} & 1 \end{bmatrix} \otimes A_{j\ell} \right]$$

$$+ \begin{bmatrix} \sum_{j=1}^{m} (A_{1j} + A_{j1}) & & \\ & \ddots & \\ & & \sum_{j=1}^{m} (A_{mj} + A_{jm}) \end{bmatrix} > 0 . \quad \Box \tag{3.6}$$

3.7 Corollary. *Let $A \in M_m^k$ be Hermitian. Then, $A > 0$ and for $D = \mathrm{diag}(A_{11}, \ldots, A_{mm})$ we have $2D - A > 0$.*

Proof. Apply Theorem 3.4 with $t = 0$ and $t = \pi$. □

We now discuss conditions which are sufficient for a matrix to be in M_m^k. As a preparation we prove:

3.8 Lemma. *Let $A \in Z_m^k$ and assume that the following condition holds:*

(3.9) *For each $J \subset \langle m \rangle$ there exists $i \in J$ such that* $\sum_{j \in J} A_{ij} > 0$.

Then, there exists a permutation π such that for $\hat{A}_{ij} = A_{\pi(i),\pi(j)}$

$$\sum_{j \geq i} \hat{A}_{ij} > 0 , \quad i = 1, \ldots, m .$$

Proof. Choose $\pi(1)$ such that $\sum_{j=1}^{m} A_{\pi(1),j} > 0$ and $\mathscr{I}_1 = \{\pi(1)\}$. Then choose successively $\pi(2), \pi(3), \ldots, \pi(m)$ such that

$$\sum_{j \notin \mathscr{I}_{s-1}} A_{\pi(s),j} > 0 , \ \pi(s) \notin \mathscr{I}_{s-1} ,$$

where $\mathscr{I}_{s-1} = \{\pi(1), \ldots, \pi(s-1)\}$. This construction is always possible by (3.9). □

Using Lemma 3.8 we can prove a characterization of M_m^k.

3.11 Theorem. *Let $A \in Z_m^k$, let $u \in \mathbb{R}_+^m$ and let*

$$R_i(u) := \sum_{j=1}^{m} A_{ij} u_j \geq 0 \quad \textit{for all} \quad i = 1, \ldots, m . \tag{3.12}$$

Assume that there exists a permutation π of $\langle m \rangle$ such that

$$\sum_{j=i}^{m} A_{\pi(i),\pi(j)} u_{\pi(j)} > 0 \quad \textit{for all} \quad i = 1, \ldots, m . \tag{3.13}$$

Then, $A \in M_m^k$.

Proof. W.l.o.g. we may assume that $u = [1, \ldots, 1]^{\mathrm{T}}$ and $\pi(i) = i$, $i = 1, \ldots, m$. This we can always achieve by replacing A with $\Delta P^{\mathrm{T}} A P \Delta$, where P is the block permutation matrix defined by π and $\Delta = \mathrm{diag}(u_1 I_k, \ldots, u_m I_k)$.

We now construct $v \in \mathbb{R}_+^m$ such that $R_i(v) > 0$, $i = 1, \ldots, m$. Choose $v^{(1)} = u$. Then $R_1(v^{(1)}) > 0$ and $R_i(v^{(1)}) \geq 0$, $i = 2, \ldots, m$ by (3.13). If we have constructed $v^{(s-1)}$ such that $R_i(v^{(s-1)}) \geq 0$, $i = 1, \ldots, m$ and $R_i(v^{(s-1)}) > 0$, for $i \leq s-1$, then we set

$$v_i^{(s)} = \begin{cases} (1 - \varepsilon_s) v_i^{(s-1)} & \text{for } i < s \\ v_i^{(s-1)} & \text{for } i \geq s \end{cases} \tag{3.14}$$

with suitable $\varepsilon_s \in (0,1)$. By continuity it follows that $R_i(v^{(s)}) > 0$ for $i \le s-1$. Then

$$R_s(v^{(s)}) = \varepsilon_s \sum_{j\ge s} A_{sj} v_j^{(s-1)} + (1-\varepsilon_s) \sum_{j=1}^{m} A_{sj} v_j^{(s-1)}$$
$$= \varepsilon_s \sum_{j\ge s} A_{sj} v_j^{(s-1)} + (1-\varepsilon_s) R_s(v_j^{(s-1)}) > 0$$

by (3.13). For $i > s$ we have

$$R_i(v^{(s)}) = R_i(v^{(s-1)}) - \varepsilon_s \sum_{j<s} A_{ij} v_j^{(s-1)} \ge R_i(v^{(s-1)}) \ge 0\,,$$

since $A \in Z_m^k$. Setting $v := v^{(m)}$ we obtain $R_i(v) > 0$ for $i = 1, \ldots, m$, and thus $A \in M_m^k$. □

In general the assumptions of Theorem 3.11 are difficult to check. But as in the M-matrix case, there are graph theoretical conditions that imply the assumptions of Theorem 3.11. We discuss such conditions now.

3.15 Definition. Let $A \in Z_m^k$ with block graph G_A. An orientation on the edges of G_A, yielding a directed graph D_A is called *admissable* if the following conditions hold:

a) the vertex set of D_A is $\langle m \rangle$;

b) if $\{i,j\} \in E(G_A)$ then $(i,j) \in E(D_A)$ or $(j,i) \in E(D_A)$, and if $(i,j) \in E(D_A)$ then $\{i,j\} \in E(G_A)$ and $A_{ij} \neq 0$;

c) for $i = 1, \ldots, m$

$$A_{ii} + \sum_{\substack{j\neq i \\ (i,j)\in E(D_A)}} A_{ij} > 0\,,$$

d) D_A has no cycles.

Note that condition d) implies also that D_A has no 2-cycles, i.e. if $(i,j) \in E(D_A)$ then $(j,i) \notin E(D_A)$.

The assumptions of Theorem 3.11 with $u = [1, \ldots, 1]^{\mathrm{T}}$ guarantee the existence of an admissable directed graph D_A as we show now:

3.16 Theorem. *Let $A \in Z_m^k$ satisfy the assumptions of Theorem 3.11 with $u = [1, \ldots, 1]^{\mathrm{T}}$. Then, there exists an admissable graph D_A.*

Proof. If $\{i,j\} \in E(G_A)$, choose (i,j) in D_A if and only if $\pi^{-1}(j) > \pi^{-1}(i)$, i.e. $(\pi(i), \pi(j)) \in E(D_A)$ if and only if $i < j$. It follows that D_A has no cycles. But then for all $i \in \langle m \rangle$

$$A_{\pi(i),\pi(i)} + \sum_{(\pi(i),\pi(j))\in E(D_A)} A_{\pi(i),\pi(j)} = A_{\pi(i),\pi(i)} + \sum_{j>i} A_{\pi(i),\pi(j)} > 0\,.$$

by (3.13). Hence, D_A is admissable for A. □

For the next result we need the following Lemma, which is probably well-known.

3.17 Lemma. *Let D be a directed graph with vertex set $\langle m\rangle$ that does not contain cycles. Then, the vertices of D can be ordered in such a way that if $(i,j) \in E(D)$, then $i < j$.*

Proof. We proceed by induction. For $m = 1$ the assertion holds trivially. Let now $v \in E(D)$ and let

$$(3.18)\qquad \begin{aligned} P(v) &= \{w \in \langle m\rangle \backslash \{v\} \mid \text{there exists a path } D \text{ from } w \text{ to } v\}\,, \\ S(v) &= \langle m\rangle \backslash P(v)\,. \end{aligned}$$

By inductive assumption we can order the induced subgraph of D given by the vertices in $P(v)$ (if not empty) in the required way, with vertex numbers $1,\dots,k$. We label v by $k+1$ and then by inductive assumption we can order the vertices in $S(v)$ as $k+2,\dots,m$. (Note that $P(v)$ or $S(v)$ may be empty in which case the ordering is trivial.) Since we have assumed that D does not contain cycles, it follows that there exists no edge from a vertex in $S(v)$ to a vertex in $P(v)$. Hence, we have the required ordering. □

With this we can prove:

3.19 Theorem. *Let $A \in Z_m^k$ and let $R_i(e) \geq 0$ for $i = 1,\dots,m$, where R_i is as in (3.12) and $e = [1\dots1]^T$. Assume that there exists an admissable directed graph D_A for A. Then, we can order rows and columns of A such that*

$$(3.20)\qquad \sum_{j\geq i} A_{ij} > 0\,, \quad i = 1,\dots,m\,.$$

In particular $A \in M_m^k$.

Proof. By Lemma 3.17 we can order the vertices of D_A such that if $(i,j) \in E(D_A)$ then $i < j$. Let $i < j$ and $A_{ij} \neq 0$, then $(i,j) \in D_A$, since otherwise $(j,i) \in D_A$, which is impossible, since it would imply $j < i$. Hence,

$$A_{ii} + \sum_{j>i} A_{ij} = A_{ii} + \sum_{\substack{j\in\langle m\rangle \\ (i,j)\in E(D_A)}} A_{ij} > 0$$

by Definition 3.15 c). Hence, (3.20) holds. $A \in M_m^k$ then follows by Theorem 3.11. □

It follows that in order to test whether a matrix is in M_m^k, we have to test all possible graphs D_A for admissability. In general this is an expensive test, but for some special cases it is quite useful. Here we discuss the case that A is block tridiagonal.

3.21 Corollary. *Let $A \in \hat{Z}_m^k$ be block tridiagonal,*

$$A = \begin{bmatrix} A_1 & -B_1 & & \\ -C_1 & \ddots & \ddots & \\ & \ddots & \ddots & -B_{m-1} \\ & & -C_{m-1} & A_m \end{bmatrix}$$

with $A_i > 0$, $B_i, C_i \geq 0$, $i = 1,\dots,m$, $B_m = C_m = 0$. Suppose that

(3.22) $$A_i \geq (B_i + C_i), (B_{i-1} + C_{i+1}) \quad i = 1, \ldots, m-1\,.$$

If there exists an index $j \in \{1, \ldots, m-1\}$ such that

$$A_i > (B_i + C_i) \quad \textit{for all } i = 1, \ldots j \textit{ and } A_{i+1} > (B_i + C_i) \quad \textit{for all } i = j, \ldots, m-1\,, \tag{3.23}$$

then $A + A^* \in M_m^k$ and hence, A is positive definite.

Proof. W.l.o.g. we may assume that G_{A+A^*} is strongly connected, since we may otherwise consider subproblems. Then, we have that the directed graph

$$1 \to 2 \to \ldots \to (j-1) \to j \leftarrow (j+1) \leftarrow \ldots \leftarrow m$$

is admissable for $A + A^*$. Hence, by Theorem 3.19 $A + A^* \in M_m^k$ and by Corollary 3.7 we obtain $A + A^* > 0$. □

For other acyclic graphs we can obtain similar corollaries.

Since the class M_m^k generalizes the class of M-matrices and since it is known that the class of M-matrices is invariant under Gaussian elimination, e.g. Fan [F1], we may ask whether M_m^k is invariant under block Gaussian elimination. In general this is not true but we have

3.24 Theorem. *Let $M_{\mathrm{H},a} \subset M_m^k$ be the subclass of matrices in M_m^k that are Hermitian and block acyclic. Let $A \in M_{\mathrm{H},a}$ and let ℓ be a leaf of G_A and let $\{s, \ell\} \in E(G_A)$. Let $L = [L_{ij}] \in \mathbb{C}^{mk,mk}$ with*

$$L_{ij} = \begin{cases} I & i = j \\ -A_{\ell s} A_{ss}^{-1} & i = l\,,\ j = s \\ 0 & \textit{otherwise.} \end{cases} \tag{3.25}$$

Then,

$$\tilde{A} := [\tilde{A}_{ij}] := LAL^* \in M_{\mathrm{H},a}\,.$$

Proof. Multiplication with L from the left changes only elements in row ℓ and multiplication with L^* from the right changes only elements in column ℓ. Thus, we obtain

$$\tilde{A}_{ij} = \begin{cases} A_{ij} & i, j \neq \ell \\ A_{ij} - A_{is} A_{ss}^{-1} A_{sj} & i = \ell, j \neq \ell \text{ or } i \neq \ell, j = \ell\,. \end{cases} \tag{3.27}$$

Now suppose that for $j \neq \ell, s$, $\{\ell, j\}$ is an edge of G_A. Then $\{j, s\}$ is not an edge of G_A, since otherwise G_A would contain the cycle $\{\ell, j\}$, $\{s, j\}$, $\{\ell, s\}$. Thus, the only blocks in $\tilde{A}$ that are different from the corresponding blocks in A are

$$\tilde{A}_{\ell s} = \tilde{A}_{s\ell} = 0\,, \quad \tilde{A}_{\ell\ell} = A_{\ell\ell} - A_{\ell s} A_{ss}^{-1} A_{s\ell}\,. \tag{3.28}$$

It follows immediately that $\tilde{A}$ is block acyclic and Hermitian. It remains to show that there exists a vector $v \in \mathbb{R}_+^m$ such that

$$\sum_{j=1}^{m} \tilde{A}_{ij} v_j > 0\,, \quad \text{for all } i \in \langle m \rangle\,.$$

Now $A \in M_m^k$ implies that there exists $u \in \mathbb{R}_+^m$ such that

$$\sum_{j=1}^{m} A_{ij}u_j > 0\,, \quad \text{for all } i \in \langle m \rangle\,.$$

Setting $v = u$, we have

$$\sum_{j=1}^{m} \tilde{A}_{ij}v_j = \sum_{\substack{j=1\\ j\neq s}}^{m} A_{ij}v_j + A_{is}v_s\,, \quad \text{for all } i \in \langle m \rangle\,. \tag{3.29}$$

For $i \neq \ell, s$ we obtain

$$\sum_{j=1}^{m} \tilde{A}_{ij}v_j = \sum_{j=1}^{m} A_{ij}v_j > 0\,,$$

since $A_{is} = 0$. For $i = s$ we trivially have

$$\sum_{j=1}^{m} \tilde{A}_{sj}v_j = A_{ss}v_s > 0\,.$$

For $i = \ell$ we obtain

$$\begin{aligned}
\sum_{j=1}^{m} \tilde{A}_{\ell j}v_j &= \sum_{\substack{j=1\\ j\neq s,\ell}}^{m} A_{\ell j}v_j + \tilde{A}_{\ell\ell}v_\ell + \tilde{A}_{\ell s}v_s \\
&= \sum_{\substack{j=1\\ j\neq \ell}}^{m} A_{\ell j}u_j + A_{\ell s}u_s + (A_{\ell\ell} - A_{\ell s}A_{ss}^{-1}A_{s\ell})u_\ell \\
&= \sum_{j=1}^{m} A_{\ell j}u_j - (A_{\ell s}u_s + A_{\ell s}A_{ss}^{-1}A_{s\ell}u_\ell)\,.
\end{aligned}$$

Now the first term is positive definite, since $A \in M_m^k$, which also implies that

$$\sum_{j=1}^{m} A_{sj}u_j = A_{ss}u_s + A_{s\ell}u_\ell > 0\,, \tag{3.30}$$

since s has only one neighbor in G_A. But (3.30) implies that

$$I_k > -\frac{u_\ell}{u_s}\,A_{ss}^{-1/2}A_{s\ell}A_{ss}^{-1/2}$$

and from Lemma 2.10 we obtain

$$\begin{aligned}
-\frac{u_\ell}{u_s}A_{ss}^{-1/2}A_{s\ell}A_{ss}^{-1/2} &> \left(\frac{u_\ell}{u_s}\right)^2 \left(A_{ss}^{-1/2}A_{s\ell}A_{ss}^{-1/2}\right)^2 \\
&= \left(\frac{u_\ell}{u_s}\right)^2 A_{ss}^{-1/2}A_{s\ell}A_{ss}^{-1}A_{s\ell}A_{ss}^{-1/2}\,.
\end{aligned}$$

Therefore,

$$-A_{\ell s}u_s - A_{\ell s}A_{ss}^{-1}A_{s\ell}u_\ell > 0\,,$$

since $A_{\ell s} = A_{s\ell}^* = A_{s\ell}$. □

Thus, the subclass of Hermitian, block acyclic matrices in M_m^k is invariant under block Gaussian elimination. In general block Gaussian elimination destroys already the symmetry of the off-diagonal blocks.

Another interesting property of Hermitian block acyclic matrices is the following.

3.31 Lemma. *Let $M = [M_{ij}] \in \mathbb{C}^{mk,mk}$ be Hermitian with blocks $M_{ij} \in \mathbb{C}^{k,k}$. If M is block acyclic, then there exists a unitary block diagonal matrix $U = \mathrm{diag}(U_1, \ldots, U_m)$, with $U_i \in \mathbb{C}^{k,k}$, $i = 1, \ldots, m$, such that $A := [A_{ij}] = UMU^* \in Z_m^k$.*

Proof. Since M is block acyclic, it is obvious that G_M can have at most $m-1$ edges $\{i_1, j_1\}, \ldots, \{i_{m-1}, j_{m-1}\}$, e.g. [B1]. If G_M has less than $m-1$ edges then M is the direct sum of smaller matrices, which can be treated separately. Thus, we may assume w.l.o.g. that G_M has exactly $m-1$ edges.

Let j_1 be a vertex of G_M. Choose $U_{j1} = I$, and for all edges $\{j_1, j_\ell\}$ of G_M let $M_{j_1,j_\ell} = A_{j_1,j_\ell}U_{j\ell}$ be the *polar decomposition* of M_{j_1,j_ℓ} with $U_{j\ell}$ unitary and $A_{j_1j_\ell}$ Hermitian negative semidefinite and $\mathrm{rank}(A_{j_1,j_\ell}) = \mathrm{rank}(M_{j_1j_\ell})$, e.g. [H2, p. 156]. (Note that usually the Hermitian factor is chosen positive semidefinite, but we may just choose the negative of the unitary factor to obtain the required form.) It follows that for all edges $\{j_1, j_\ell\}$ we have $A_{j_1,j_\ell} = A_{j_1,j_\ell}^* = U_{j_1}M_{j_1,j_\ell}U_{j_\ell}^*$ as required.

For all the vertices $j_\ell \neq j_1$ we can now consider the edges $\{j_\ell, j_s\}$, with $j_s \neq j_\ell, j_1$ and perform the polar decompositions $U_{j_\ell}M_{j_\ell,j_s} = A_{j_\ell,j_s}U_{j_s}$ with U_{j_s} unitary and A_{j_ℓ,j_s} Hermitian negative semidefinite.

Proceeding like this with all the edges $\{j_s, j_t\}$ that were not considered before, we can exhaust the whole graph. Since M was acyclic, no previously considered vertex occurs again and this finite procedure completely determines U, A. □

From Lemma 3.31 we can conclude that some of the previous results also hold for Hermitian matrices which have nonhermitian blocks.

4. Convergence of Jacobi's method and general results

Another important characteristic of M-matrices in comparison to Z-matrices is the convergence of the Jacobi iterative method for a linear system $Ax = b$ (e.g. [B2]).

A natural generalization of this method for matrices $A = [A_{ij}] \in \hat{Z}_m^k$ is the block Jacobi iteration for $Ax = b$ defined by $D = \mathrm{diag}(A_{11}, \ldots, A_{mm})$, $N = D - A$ and

$$x_{i+1} = D^{-1}Nx_i + D^{-1}b\,, \quad i = 1, 2, 3, \ldots \tag{4.1}$$

It is wellknown [B2] that (4.1) converges for all initial x_0 if and only if $\varrho(D^{-1}N) < 1$. For the proof of convergence of (4.1) we can employ the following Lemma:

4.2 Lemma. *Let $A = D - N \in \mathbb{C}^{n,n}$ be such that*

$$D + D^* > 0 \tag{4.3}$$

$$A_t = D + D^* - (e^{it}N + e^{-it}N^*) > 0 \quad \textit{for all } t \in \mathbb{R}\,. \tag{4.4}$$

Then $\varrho(D^{-1}N) < 1$. If $A_t \geq 0$ for all $t \in \mathbb{R}$, then $\varrho(D^{-1}N) \leq 1$.

Proof. There exists $\xi \neq 0$ such that $\lambda D\xi = N\xi$, with $|\lambda| = \lambda e^{i\tau} = \varrho(D^{-1}N)$. Then obviously $|\lambda| D\xi = e^{i\tau}N\xi$ and hence

$$|\lambda|\xi^* D\xi = e^{i\tau}\xi^* N\xi\,. \tag{4.5}$$

From (4.3), (4.4) for $t = \tau$ and (4.5) we get $\xi^*(D + D^*)\xi > 0$ and

$$\begin{aligned} \xi^*(D + D^*)\xi &> e^{i\tau}\xi^* N\xi + e^{-i\tau}\xi^* N^*\xi \\ &= |\lambda|\xi^* D\xi + |\lambda|\xi^* D^*\xi = |\lambda|\xi^*(D + D^*)\xi\,, \end{aligned}$$

which implies $|\lambda| < 1$. The second part follows analogously. □

We immediately have an analogue to Proposition 3.1.

4.6 Proposition. *Let $A = [A_{ij}] \in \hat{Z}^k_m$ and let D, N be as in (4.1). If A satisfies (3.2), then $\varrho(D^{-1}N) \leq 1$, and if A satisfies (3.3), then $\varrho(D^{-1}N) < 1$.*

Proof. Let $\hat{A}_t = D - \frac{e^{it}N + e^{-it}N^*}{2} =: [\tilde{A}_{ij}]$. Then $\tilde{A}_{ii} = A_{ii}$, $\tilde{A}_{ij} = -\frac{1}{2}(e^{it}A_{ij} + e^{-it}A_{ji})$ for $i \neq j$ and

$$\begin{aligned} \hat{A}_t = &-\frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{m} \left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & A_{ij} & 0 & e^{it}A_{ij} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-it}A_{ij} & 0 & A_{ij} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & A_{ji} & 0 & e^{-it}A_{ji} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & e^{it}A_{ji} & 0 & A_{ji} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \\ &+ \begin{bmatrix} A_{11} + \frac{1}{2}\sum_{j\neq 1}(A_{1j} + A_{j1}) & & \\ & \ddots & \\ & & A_{mm} + \frac{1}{2}\sum_{j\neq m}(A_{mj} + A_{jm}) \end{bmatrix}. \end{aligned}$$

By (3.2), (3.3) we obtain $\hat{A}_t \geq 0$ or $\hat{A}_t > 0$ respectively; hence, the proof follows by Lemma 4.3. □

The analogous result to Theorem 3.4 is then

4.7 Theorem. *Let $A = D - N$, D, N as in equation (4.1) and $A + A^* \in M^k_m$, then $\varrho(D^{-1}N) < 1$.*

Proof. Apply Theorem 3.4 and Lemma 4.2. □

As a corollary we then obtain convergence of (4.1) for all block tridiagonal matrices as in Corollary 3.21. We omit the statement of the Corollary here.

The analysis of convergence results for other iterative methods like the Gauss–Seidel method or the SOR is currently under investigation in the project of a PhD thesis and partial results have been obtained.

Theorems 3.4 and 4.7 generalize results for M-matrices, cf. [B2]. It is natural to ask:

(4.8) Which of the equivalent conditions for M-matrices in [B2] still hold in M_m^k?

(4.9) Which of the equivalent condidions for M-matrices have a nontrivial block analogue?

One of the conditions in the case $k = 1$, which is not satisfied for $k > 1$, even if $A \in Z_m^k$ is positive definite, is N_{38} in [B2] stating

$$A^{-1} \text{ exists and } A^{-1} \text{ is elementwise nonnegative.} \tag{4.10}$$

Consider the following example:

4.11 Example. Let

$$A = \begin{bmatrix} 660 & 160 & -1 & 0 \\ 160 & 40 & 0 & -1 \\ -1 & 0 & 340 & -80 \\ 0 & -1 & -80 & 20 \end{bmatrix} .$$

$A \in Z_2^2$ is positive definite, but a MATLAB experiment yields

$$A^{-1} = \begin{bmatrix} 0.1301 & -0.5309 & -0.0997 & -0.4252 \\ -0.5309 & 2.1925 & 0.4120 & 1.7575 \\ -0.0997 & 0.4120 & 0.1274 & 0.5302 \\ -0.4252 & 1.7575 & 0.5302 & 2.2588 \end{bmatrix} .$$

Neither is A^{-1} elementwise nonnegative nor are all the 2×2 blocks of A^{-1} positive semidefinite.

Also not satisfied is condition M_{36} in [B2], stating:

$A = [a_{ij}]$ has all positive diagonal elements and there exists a diagonal matrix D with positive diagonal elements such that $AD = [a_{ij}d_j]$ is strictly diagonally dominant, i.e.

$$a_{ii}d_i > \sum_{j \neq i} |a_{ij}| d_j \quad i = 1, \ldots, n . \tag{4.12}$$

Consider the following example:

4.13 Example.

$$A = \begin{bmatrix} 1.5 & 1 & -1 & -1 \\ 1 & 1.5 & -1 & -1 \\ -1 & -1 & 1.5 & 1 \\ -1 & -1 & 1 & 1.5 \end{bmatrix} \in Z_2^2$$

is positive definite. Suppose $d_1, d_2, d_3, d_4 > 0$ such that (4.12) holds, then $1.5d_1 > d_2 + d_3 + d_4$, $1.5d_2 > d_1 + d_3 + d_4$, $1.5d_3 > d_1 + d_2 + d_4$, $1.5d_4 > d_1 + d_2 + d_3$. This implies $d_2 > 2d_3 + 2d_4$, $d_3 > 2d_2 + 2d_4$, $d_4 > 2d_2 + 2d_3$, from which we get $-3d_3 > 6d_4$, $-3d_4 > 6d_3$ which is not possible if $d_3, d_4 > 0$. Thus, (4.12) does not hold and thus, $A \notin M_2^2$.

One obviously has to generalize the diagonal dominance in the block fashion described in Sect. 3.

Example 4.11 also serves to show that matrices in $\hat{Z}_m^k$ that are positive definite are not necessarily H-matrices. (A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix if $\mathscr{M}(A)$ is an M-matrix, where $\mathscr{M}(A) = [b_{ij}]$ with $b_{ij} := \begin{cases} |a_{ij}| & j = i \\ -|a_{ij}| & j \neq i \end{cases}$ $i, j = 1, \ldots, n$).

In Theorem 3.24 we have shown that the class of Hermitian, block acyclic matrices in M_m^k is invariant under block Gaussian elimination, if it is applied to leafs. For general and even positive definite matrices in Z_m^k this is, however, not the case, since the symmetry of the off diagonal blocks is destroyed. This is another property of M-matrices [F1], which does not carry over to the block case.

It is known that any principal submatrix of a Z-matrix has at least one real eigenvalue [E1], [M1]. This is generally not true for Z_m^k, since $A \in Z_m^k$ can have all eigenvalues complex.

4.14 Example. We have that

$$A = \begin{bmatrix} 1.096 & 0.016 & -1.000 & 0.000 \\ 0.016 & 1.034 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.064 & -0.008 \\ 0.000 & -0.100 & -0.008 & 1.032 \end{bmatrix} \in Z_2^2$$

is positive definite, but the eigenvalues rounded to 4 digits are $1.0097 \pm \mathrm{i}\,0.0388, 1.1043 + \mathrm{i}\,0.0363$.

Observe that all the negative examples (4.11), (4.13), (4.14) have an acyclic block graph and hence these properties do not even hold in the acyclic case.

5. Application to special case from fluid flow computations

In this section we now discuss matrices arising in special cases in the numerical solution of Euler equations [H1]. These matrices have the form

$$M := \begin{bmatrix} T_1 & S_1 & & \\ S_2 & T_1 & \ddots & \\ & \ddots & \ddots & S_1 \\ & & S_2 & T_1 \end{bmatrix} \in \mathbb{C}^{p\cdot r\cdot k, p\cdot r\cdot k}, \tag{5.1}$$

where $T_1, S_1, S_2 \in \mathbb{C}^{r\cdot k, r\cdot k}$ are defined by

$$T_1 := \begin{bmatrix} C & -A^- & & \\ -A^+ & C & \ddots & \\ & \ddots & \ddots & -A^- \\ & & -A^+ & C \end{bmatrix}, \quad S_2 := \begin{bmatrix} -B^+ & & \\ & \ddots & \\ & & -B^+ \end{bmatrix}, \tag{5.2}$$

$$S_1 := \begin{bmatrix} -B^- & & \\ & \ddots & \\ & & -B^- \end{bmatrix}.$$

Here $A = A^+ - A^- \in \mathbb{C}^{k,k}$, $B = B^+ - B^- \in \mathbb{C}^{k,k}$ are decompositions of symmetric (indefinite) matrices A, B in positive semidefinite parts A^+, B^+ and negative semidefinite parts $-A^-, -B^-$, and $C = |A| + |B| = A^+ + A^- + B^+ + B^-$.

Other finite element approaches yield matrices with similar block structure but different matrices in each row. Also sometimes the matrices A, B are not Hermitian but only have real eigenvalues [D1]. Here we only discuss the case that the blocks are Hermitian.

Consider first the following simple Lemma.

5.3 Lemma. *Let $C_1, C_2, D_1, D_2 \in \mathbb{C}^{n,n}$ be Hermitian positive definite and let $A, B \in \mathbb{C}^{m,m}$ be Hermitian with $A, B \geq 0$, $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$. Let $A = A^+ - A^-$, $B = B^+ - B^-$ be the decompositions of A, B in its positive and negative semidefinite parts. Then $T := C_1 \otimes A^+ + C_2 \otimes A^- + D_1 \otimes B^+ + D_2 \otimes B^+ > 0$.*

Proof. Obviously $T \geq 0$ and $A + B > 0$. If $x \in \mathbb{C}^{mn,mn}$ satisfies $Tx = 0$, then

$$(C_1 \otimes A^+)x = (C_2 \otimes A^-)x = (D_1 \otimes B^+)x = (D_2 \otimes B^-)x = 0\,.$$

Using $C_1 \otimes A^+ = (C_1 \otimes I)(I \otimes A^+)$ and the nonsingularity of $C_1 \times I$, we get $(I \otimes A^+)x = 0$ and analogously

$$(I \otimes A^-)x = (I \otimes B^+)x = (I \otimes B^-)x = 0\,.$$

Hence, $[I \otimes (A^+ + A^- + B^+ + B^-)]x = 0$. But, since $A + B$ nonsingular implies $|A| + |B|$ nonsingular, we obtain $x = 0$, and hence $T > 0$. □

Using this Lemma we obtain:

5.4 Proposition. *Let M be as in (5.1), (5.2).*
If $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$ then

$$T_1 \in \hat{Z}_r^k\,, \quad \textit{and} \tag{5.5}$$

$$M \in \hat{Z}_{rp}^k\,. \tag{5.6}$$

Furthermore T_1 and M are positive definite.

Proof. $\mathcal{N}(A) = \mathcal{N}(|A|)$ and $\mathcal{N}(B) = \mathcal{N}(|B|)$, therefore $C > 0$. Thus, (5.5), (5.6) follow trivially.

To show that T_1 is positive definite, observe that

$$\begin{aligned} T_1 + T_1^* &= \begin{bmatrix} 2|A| & -|A| & & \\ -|A| & \ddots & \ddots & \\ & \ddots & \ddots & -|A| \\ & & -|A| & 2|A| \end{bmatrix} + 2\begin{bmatrix} |B| & & \\ & \ddots & \\ & & |B| \end{bmatrix} \\ &= H \otimes |A| + 2I_r \otimes |B|\,, \end{aligned}$$

where

$$H := \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}. \tag{5.7}$$

Applying Lemma 5.3 we obtain that T_1 is positive definite.

To prove that M is positive definite, observe that

$$M + M^* = \underbrace{\mathrm{diag}(H,\dots,H)}_{>0} \otimes |A| + \underbrace{\begin{bmatrix} 2I & -I & & & \\ -I & 2I & -I & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & 2I \end{bmatrix}}_{>0} \otimes |B| .$$

M being positive definite then follows by Lemma 5.3. □

Observe that the matrices T_1, M are not necessarily in M_r^k, M_{rp}^k respectively, as the following example shows:

5.8 Example. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad B = 0, m = 3 .$$

Then

$$T_1 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \in \hat{Z}_3^2$$

is positive definite by Proposition 5.4 but there exists no $u \in \mathbb{R}_+^3$ such that $\sum_{j=1}^3 A_{ij} u_j > 0$.

We now prove that the obvious block Jacobi method converges for M.

5.9 Proposition. *Let M be as in (5.1), (5.2) and $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$ and let*

$$D = \begin{bmatrix} C & & \\ & \ddots & \\ & & C \end{bmatrix} \in \mathbb{C}^{p \cdot r \cdot k} , \quad N = D - M .$$

Then, $\varrho(D^{-1}N) < 1$.

Proof. In order to apply Lemma 4.2, we show that $M_t = D - \frac{e^{it}N + e^{-it}N^*}{2} > 0$ for all $t \in \mathbb{R}$.

$$D - \frac{e^{it}N + e^{-it}N^*}{2} = \frac{1}{2}(H_1 \otimes A^+ + H_1^T \otimes A^- + H_2 \otimes B^+ + H_2^T \otimes B^-) ,$$

with

$$H_1 = \begin{bmatrix} 2 & e^{it} & & & & & & & \\ e^{-it} & 2 & \ddots & & & & & & \\ & \ddots & \ddots & e^{it} & & & & & \\ & & e^{-it} & 2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 2 & e^{it} & & \\ & & & & & e^{-it} & 2 & \ddots & \\ & & & & & & \ddots & \ddots & e^{it} \\ & & & & & & & e^{-it} & 2 \end{bmatrix}$$

and

$$H_2 = \begin{bmatrix} 2I & e^{it}I & & \\ e^{-it}I & 2I & \ddots & \\ & \ddots & \ddots & e^{it}I \\ & & e^{-it}I & 2I \end{bmatrix} .$$

The matrices H_1, H_2 are trivially positive definite and then Lemma 5.3 implies $M_t > 0$. Applying Lemma 4.2 finishes the proof. □

Thus, having shown convergence of one of the natural block-Jacobi methods, we can now discuss the convergence of the corresponding block SOR method:

Let $M = [A_{ij}] \in \mathbb{C}^{km,km}$, with $A_{ij} \in \mathbb{C}^{k,k}$. Then, the block SOR iteration for $Mx = b$ is defined by

$$x_{i+1} = H_\omega x_i + (D - \omega L)^{-1} b\,, \quad i = 1, 2, 3, \dots\,, \tag{5.10}$$

where $H_\omega = (D - \omega L)^{-1}((1-\omega)D + \omega U)$, $M = D - L - U$, $D = \mathrm{diag}(C, \dots, C)$ and L, U are the block lower and upper triangular parts of M.

The relationship between the spectral radius of H_ω and $D^{-1}(L+U)$ is given by the following well known result of Varga, e.g. [V1], [V2], [Y1] or [B2].

5.11 Theorem. *Let M, H_ω be as in (5.10) and let M be consistently ordered p-cyclic. If $\omega \neq 0$ and $\lambda \neq 0$ is an eigenvalue of the block SOR matrix H_ω and if δ satisfies*

$$(\lambda + \omega - 1)^p = \lambda^{p-1} \omega^p \delta^p \tag{5.12}$$

then $\delta \in \sigma(D^{-1}(L+U))$. Conversely if $\delta \in \sigma(D^{-1}(L+U))$ and λ satisfies (5.12) then $\lambda \in \sigma(H_\omega)$.

Another immediate consequence of Theorem 5.11 is the convergence result for the block Gauss-Seidel method, which is (5.10) with $\omega = 1$ and a block version of the Stein–Rosenberg Theorem, e.g. [V2, p. 70].

5.13 Corollary. *Let M, D, N, H_ω be as in (5.10) and suppose M is consistently ordered 2-cyclic. Then one and only one of the following mutually exclusive relations is valid:*

i) $\varrho(D^{-1}N) = \varrho(H_1) = 0\,,$
ii) $0 < \varrho(H_1) < \varrho(D^{-1}N) < 1\,,$
iii) $1 = \varrho(D^{-1}N) = \varrho(H_1)\,,$
iv) $1 < \varrho(D^{-1}N) < \varrho(H_1)\,,$

Proof. The proof follows directly from Theorem 5.11 by inserting $\omega = 1$ in 5.12. □

Using this result we can now determine real parameters ω such that the block SOR method converges if the block Jacobi method converges. This result is probably well known, but we did not find a reference.

5.14 Theorem. *Let M, D, N, H_ω be as in (5.10) and suppose M is consistently ordered and 2-cyclic. Let $\varrho = \varrho(D^{-1}N) < 1$. If $0 < \omega < \frac{2}{1+\varrho}$ then $\varrho(H_\omega) < 1$.*

Proof. a) Let $0 < \omega \leq 1$ and $\lambda \in \sigma(H_\omega)$, $|\lambda| = z$. Then

$$(z - (1-\omega))^2 \leq |\lambda - (1-\omega)|^2 = |\lambda + \omega - 1|^2 = |\lambda|\omega^2|\delta|^2 \leq |\lambda|\omega^2\varrho^2 ,$$

where $\delta \in \sigma(D^{-1}N)$. Hence

$$(z - (1-\omega))^2 \leq z\omega^2\varrho^2$$

and by elementary considerations we infer $z < 1$. This shows $\varrho(H_\omega) < 1$.

b) Let $\omega > 1$. Then if as above $\lambda \in \sigma(H_\omega)$, $|\lambda| = z$, we have

$$(z - \omega + 1)^2 \leq |\lambda - \omega + 1|^2 = |\lambda|\omega^2|\delta|^2 \leq |\lambda|\omega^2\varrho^2 ,$$

i.e. z lies between the two real roots of $q(z) = (z - \omega + 1)^2 - z\omega^2\varrho^2$.

As $q(0) > 0$, and $q'(0) \leq 0$ it is obvious that we can infer $z < 1$ if and only if $q(1) > 0$, i.e. if and only if

$$(2-\omega)^2 - \omega^2\varrho^2 > 0 , \quad \text{i.e. } \varrho < \frac{2-\omega}{\omega} \text{ or } \omega < \frac{2}{1+\varrho} . \quad \square$$

We now apply these results to the matrix M in (5.1), (5.2).

5.15 Theorem. *The matrix M in (5.1), (5.2) is a consistently ordered 2-cyclic matrix.*

Proof. The block graph G_M of M is the same as the standard graph of the finite difference approximation of the Laplace operator, hence M is weakly cyclic of index 2, e.g. [V2]. Let L, U be the lower and upper triangular part of $D - M$ and $D = \operatorname{diag}(C, \ldots, C)$, then $J(\alpha) = \alpha L + \alpha^{-1}U = \Delta^{-1}J(1)\Delta$ by a diagonal similarity with $\Delta = \operatorname{diag}(\Delta_1, \alpha\Delta_1, \ldots, \alpha^{p-1}\Delta_1)$, where

$$\Delta_1 = \begin{bmatrix} \alpha I_k & & & \\ & \alpha^2 I_k & & \\ & & \ddots & \\ & & & \alpha^r I_k \end{bmatrix} \in \mathbb{C}^{rk,rk} .$$

Thus, M is consistently ordered. □

Thus we can summarize the results as follows:

5.16 Corollary. *Let M be as in (5.1), (5.2) then we have*

i) the block Jacobi method converges;
ii) the block SOR method converges if $0 < \omega < \frac{2}{1+\varrho(D^{-1}N)}$.

6. Conclusion

We have generalized several results for Z-matrices to block matrices in Z_m^k. Positive definiteness, invariance under Gaussian elimination, diagonal dominance and convergence of the block Jacobi methods are generalized to the block case.

For the special case arising in the numerical solution of Euler equations, we also have given convergence results for block Jacobi block Gauss–Seidel and block SOR methods. There are many open problems for matrices in Z_m^k and also for the applications in numerical solutions of partial differential equations, it would be important to generalize the described results to matrices which have off diagonal blocks with real nonpositive eigenvalues, which are not necessarily Hermitian.

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