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Invariant cones in solvable Lie algebras

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A subset C of a finite dimensional real vector space V is called a cone if it is closed topologically, additively and under multiplication with nonnegative scalars. In this note we shall consider cones C in (solvable) Lie algebras $\mathfrak{g}$ which are invariant under all inner automorphisms of $\mathfrak{g}$. Such cones are simply called invariant. A whole chapter in the monumental book [1] is devoted to the subject of invariant cones. So, I am not going to repeat the whole story, but rather restrict myself to recall some basic facts needed in the sequel or at least useful for a better understanding. In particular, I will say nothing on the classification of invariant cones in simple Lie algebras. To get an overview over the possible invariant cones it is justified to restrict first the attention to pointed generating cones. Pointed means that C contains no lines. Generating means that $C-C=\mathfrak{g}$ or, equivalently, that C has a non-empty interior. The justification is that for any invariant cone C in $\mathfrak{g}$ the generated vector space $C-C$ and the largest vector subspace $C\cap(-C)$ of C are ideals in $\mathfrak{g}$. And C defines an invariant pointed generating cone in the subquotient $C-C/C\cap(-C)$. By the results described in [1] the structure of the solvable Lie algebras accommodating an invariant pointed generating cone is known as well as a construction principle for those cones. The results are as follows.

Theorem A (see [1, III.2.14/15] or [4]) *If the Lie algebra $\mathfrak{g}$ permits an invariant pointed generating cone then $\mathfrak{g}$ contains a "compactly embedded" Cartan subalgebra $\mathfrak{h}$, i.e., $\mathfrak{h}$ is a Cartan subalgebra such that* $\operatorname{Exp}\operatorname{ad}\mathfrak{h}$ *is a relatively compact subgroup of* $\operatorname{Aut}(\mathfrak{g})$. *Such an $\mathfrak{h}$ is necessarily abelian. If C is an invariant pointed generating cone in $\mathfrak{g}$ then $C\cap\mathfrak{h}$ is a pointed generating cone in $\mathfrak{h}$.*

For a fixed $\mathfrak{h}$ one may decompose the complexification $\mathfrak{g}^{\mathbb{C}}$ into root spaces: $\mathfrak{g}^{\mathbb{C}}=\mathfrak{h}^{\mathbb{C}}\oplus\bigoplus_{\omega\in\Omega}\mathfrak{g}^{\mathbb{C}}_{i\omega}$ where Ω is a subset of $\mathfrak{h}^*=\operatorname{Hom}(\mathfrak{h},\mathbb{R})$ such that $\Omega=-\Omega$ and $0\notin\Omega$. This decomposition yields a decomposition of the real algebra $\mathfrak{g}$ along the set $\tilde{\Omega}:=\Omega/\{\pm1\}$ of equivalence classes of Ω: $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\tilde{\omega}\in\tilde{\Omega}}\mathfrak{g}_{\tilde{\omega}}$ where $\mathfrak{g}_{\tilde{\omega}}=(\mathfrak{g}^{\mathbb{C}}_{i\omega}\oplus\mathfrak{g}^{\mathbb{C}}_{-i\omega})\cap\mathfrak{g}$ if $\tilde{\omega}=\{-\omega,\omega\}$. Each $\omega\in\tilde{\omega}$ defines a complex structure J_ω on the real vector space $\mathfrak{g}_{\tilde{\omega}}$ by $[h,x]=\omega(h)\,J_\omega x$ for $h\in\mathfrak{h}$, $x\in\mathfrak{g}_{\tilde{\omega}}$. Clearly, $J_{-\omega}=-J_\omega$.

Theorem B (see [1, III.6.5, III.6.18./19. and III.6.22./23./24] or [3]) *Let* $\mathfrak{g}$ *be solvable Lie algebra permitting an invariant pointed generating cone,* $\mathfrak{n}$ *its nilradical and* $\mathfrak{z}$ *its center. Then* $\mathfrak{z}$ *is the center of* $\mathfrak{n}$, *and* $\mathfrak{n}/\mathfrak{z}$ *is abelian. For a fixed compactly embedded Cartan subalgebra* $\mathfrak{h}$ *and associated sets* Ω *and* $\tilde{\Omega}$ *the following assertions hold true.*

(i) $\mathfrak{g}_{\tilde{\omega}} \subset \mathfrak{n}$ *for* $\tilde{\omega} \in \tilde{\Omega}$.

(ii) $\mathfrak{n} = \mathfrak{z} \oplus \bigoplus_{\tilde{\omega} \in \tilde{\Omega}} \mathfrak{g}_{\tilde{\omega}}$.

(iii) $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$, $\mathfrak{h} \cap \mathfrak{n} = \mathfrak{z}$.

(iv) $[\mathfrak{g}_{\tilde{\omega}}, \mathfrak{g}_{\tilde{\omega}'}] = \begin{cases} \neq 0 & \text{if } \tilde{\omega} = \tilde{\omega}' \\ = 0 & \text{if } \tilde{\omega} \neq \tilde{\omega}' \end{cases}$ *for* $\tilde{\omega}, \tilde{\omega}' \in \tilde{\Omega}$.

The non-degeneracy stated in (iv) *can be sharpened using the complex structures* J_ω. *Each* J_ω *defines a map* $\mathfrak{g}_{\tilde{\omega}} \to \mathfrak{z}$ *by* $x \mapsto [x, J_\omega x]$ *for* $x \in \mathfrak{g}_{\tilde{\omega}}$.

(v) *If* $[x, J_\omega x]$ *is zero for an* $x \in \mathfrak{g}_{\tilde{\omega}}$ *then* x *is zero.*

Moreover, since $\mathrm{ad}(h)$ *is a derivation for all* $h \in \mathfrak{h}$ *and since all the commutators* $[x, y]$, $x, y \in \mathfrak{g}_{\tilde{\omega}}$ *are central the complex structures* J_ω *satisfy*

(vi) $[x, J_\omega y] + [J_\omega x, y] = 0$ *for all* $x, y \in \mathfrak{g}_{\tilde{\omega}}$.

The invariant cones in algebras as above are obtained in the following manner.

Theorem C (see [1, III.5.11/15. and III.7.10./11] or [3]) *Let* $\mathfrak{g}$ *be a solvable Lie algebra with the structural properties stated in Theorem* B, *in particular* $\mathfrak{h}$ *is a chosen compactly embedded Cartan algebra. Let a pointed generating cone* K *in the abelian algebra* $\mathfrak{h}$ *be given. There exists an invariant pointed generating cone* C *in* $\mathfrak{g}$ *such that* $C \cap \mathfrak{h} = K$ *if and only if* $\mathrm{ad}(x)^2(K) \subseteq K$ *for all* $x \in \mathfrak{g}_{\tilde{\omega}}$, $\tilde{\omega} \in \tilde{\Omega}$. *In this case* C *is uniquely determined by* K *and can be reconstructed as*

$$C = \cap \{\rho(K \oplus \bigoplus_{\tilde{\omega} \in \tilde{\Omega}} \mathfrak{g}_{\tilde{\omega}}) | \rho \in \mathrm{Inn}(\mathfrak{g})\}$$

where $\mathrm{Inn}(\mathfrak{g})$ *denotes the group of inner automorphisms of* $\mathfrak{g}$.

These results look conclusive and they are. But when I tried to use these three theorems as a device to construct examples I observed that still I had to do some work. Some of this work can be done quite general. And that's what I am going to present in this note. More precisely, I want to write down an explicit procedure for constructing (all) invariant pointed generating cones in solvable Lie algebras. The concepts will be, I hope, adequate for this constructive attitude. Normally, I don't like coordinates: in almost all cases one looses elegance, sometimes one gets lost completely. But in constructing examples of cones in finite dimensional spaces it seems to be the most efficient way to realize the space as $\mathbb{R}^n$ and to describe the cone therein by a set of inequalities. If a cone is invariant under a group, preferably the inequalities should be invariant under the group, i.e., the functions entering in the inequalities are invariant. This will turn out to be the case for invariant pointed generating cones in solvable Lie algebras.

The organization of this note is as follows. First we introduce the data from which an invariant pointed generating cone can be constructed and carry out the construction. Secondly, we show that our "list" is complete and give some remarks on the uniqueness of the data attached to a given cone. The article is finished by considering the case of two dimensional Cartan algebras.

The data. a) *On a finite dimensional real vector space* $\mathfrak{s}$ *there is given a finite collection* A *of non-zero real linear functionals such that* $L_{\mathbb{R}}(A)=\mathfrak{s}^*$ *and* $\{s\in\mathfrak{s}\,|\,\alpha(s)>0$ *for all* $\alpha\in A\}\neq\emptyset$. *In other words,* A *generates a pointed generating cone in* $\mathfrak{s}^*$.

b) *There is a natural number* d *such that for each* $\alpha\in A$ *there is given a complex vector space* V_α *endowed with* d *positive semidefinite hermitean scalar products* D_α^k, $1\leqq k\leqq d$, *such that the sum* $\sum_{k=1}^{d} D_\alpha^k$ *is positive definite.*

c) *In* $(\mathfrak{s}\oplus\mathbb{R}^d)^*=\mathfrak{s}^*\oplus\mathbb{R}^d$ *there is given a compact subset* Γ *such that:*

c0) $0\notin\Gamma$.

c1) *The cone* $C(\Gamma)$ *generated by* Γ *is pointed and generating.*

c2) *The set* A *is contained in* $C(\Gamma)$.

c3) *For each* $\gamma=(\gamma_0,\gamma_1,\ldots,\gamma_d)\in\Gamma\subset\mathfrak{s}^*\oplus\mathbb{R}^d$

and each $v\in V_\alpha$, $\alpha\in A$, *the inequality* $\sum_{k=1}^{d}\gamma_k D_\alpha^k(v,v)\geqq 0$ *holds true, i.e.,* $\sum_{k=1}^{d}\gamma_k D_\alpha^k$ *is positive semidefinite, too.*

The following remarks contain comments on the data. Some of them are a little premature in the sense that they can be understood much better *after* the construction.

Remark 1 To construct "concrete" examples one can always arrange that $\mathfrak{s}=\mathbb{R}^n=\mathfrak{s}^*$ and that A is contained in the "octant" $\mathbb{R}^n_+$.

Remark 2 If property c3) holds as stated, then it is clearly satisfied by all elements γ in $C(\Gamma)$. As we will see later the cone constructed out of the data does not really depend on Γ, but merely on $C(\Gamma)$. The only reason to introduce Γ is that in constructing examples (that is our point of view) one wants to describe the cones in question by a *minimal* set of inequalities. So, one should imagine Γ as the extremal points of a base of the cone $C(\Gamma)$.

Remark 3 If the data of a) and b) are given there always exists at least one subset Γ satisfying c0), c1), c2), c3), namely the union of A and the standard basis vectors of $\mathbb{R}^d$.

Remark 4 The case $\mathfrak{s}=0$, $A=\emptyset$ is not formally excluded even though this is, of course, not the situation we have in mind. It corresponds to (invariant) cones in abelian Lie algebras. More precisely, the cone we are going to construct out of the data is simply the dual cone of $C(\Gamma)$ in the abelian Lie algebra $\mathbb{R}^d$, see below. Also if $\mathfrak{s}$ is different from zero – which implies that the corresponding Lie algebra is non-abelian – it may happen that this algebra has an abelian factor. This could be excluded, if wanted, by requiring that $\{(D_\alpha^1(v_\alpha,v_\alpha),\ldots,D_\alpha^d(v_\alpha,v_\alpha))\,|\,\alpha\in A,\,v_\alpha\in V_\alpha\}$ spans $\mathbb{R}^d$.

Given the data one constructs a Lie algebra $\mathfrak{g}$. As a (real) vector space $\mathfrak{g}$ equals

$$\mathfrak{g}=\mathfrak{s}\oplus\bigoplus_{\alpha\in A}V_\alpha\oplus\mathbb{R}^d.$$

The bracket is given by

$$[(t, \sum_{\alpha\in A} v_\alpha, z_1, \ldots, z_d), (s, \sum_{\alpha\in A} u_\alpha, w_1, \ldots, w_d)]$$
$$=(0, \sum_{\alpha\in A} (i\alpha(t)\, u_\alpha - i\alpha(s)\, v_\alpha), \sum_{\alpha\in A} \operatorname{Im} D^1_\alpha(v_\alpha, u_\alpha), \ldots, \sum_{\alpha\in A} \operatorname{Im} D^d_\alpha(v_\alpha, u_\alpha))$$

where $s, t \in \mathfrak{s}$, $v_\alpha, u_\alpha \in V_\alpha$ and $z_k, w_k \in \mathbb{R}$.

It is not hard to see that $\mathfrak{g}$ is a Lie algebra. Observe that $\operatorname{Im} D^k_\alpha(v_\alpha, v_\alpha) = 0$ as D^k_α is hermitean. To construct the associated cone C one first considers the subset $\mathring{C}$ of $\mathfrak{g}$ consisting of all elements $(t, \sum_{\alpha\in A} v_\alpha, z_1, \ldots, z_d)$ which satisfy the inequalities $\alpha(t) > 0$ for all $\alpha \in A$, and $\gamma_0(t) + \sum_{k=1}^{d} \gamma_k \{z_k - \frac{1}{2} \sum_{\alpha\in A} \alpha(t)^{-1} D^k_\alpha(v_\alpha, v_\alpha)\} > 0$ for all $(\gamma_0, \gamma_1, \ldots, \gamma_d) \in \Gamma \subset \mathfrak{s}^* \oplus \mathbb{R}^d$.

As abbreviation we write occasionally $g_\gamma(t, \sum_{\alpha\in A} v_\alpha, z_1, \ldots, z_d)$ for the left hand side of the second inequality.

Theorem 1 *Let C be the closure of $\mathring{C}$. Then C is an invariant pointed generating cone in $\mathfrak{g}$. The cone C depends only on $C(\Gamma)$ rather than on the generating set Γ. The cones $C \cap (\mathfrak{s} \oplus \mathbb{R}^d)$ and $C(\Gamma)$ are dual to each other.*

Proof. Clearly, $\mathring{C}$ is invariant under multiplication with positive numbers. Hence C is stable under multiplication with nonnegative numbers. To see that $\mathring{C}$ (and hence C) is closed additively one takes two elements $(t, \sum_{\alpha\in A} v_\alpha, z_1, \ldots, z_d)$ and $(s, \sum_{\alpha\in A} u_\alpha, w_1, \ldots, w_d)$ in $\mathring{C}$. Clearly, $\alpha(s+t) > 0$ for all $\alpha \in A$. So, our claim is proved if we can show that

$$\gamma_0(s+t) + \sum_{k=1}^{d} \gamma_k \{z_k + w_k - \tfrac{1}{2} \sum_{\alpha\in A} \alpha(s+t)^{-1} D^k_\alpha(u_\alpha + v_\alpha, u_\alpha + v_\alpha)\}$$
$$\geqq \gamma_0(s) + \sum_{k=1}^{d} \gamma_k \{w_k - \tfrac{1}{2} \sum_{\alpha\in A} \alpha(s)^{-1} D^k_\alpha(u_\alpha, u_\alpha)\}$$
$$+ \gamma_0(t) + \sum_{k=1}^{d} \gamma_k \{z_k - \tfrac{1}{2} \sum_{\alpha\in A} \alpha(t)^{-1} D^k_\alpha(v_\alpha, v_\alpha)\}$$

for a fixed $\gamma = (\gamma_0, \gamma_1 \ldots, \gamma_d) \in \Gamma$.

The latter inequality follows if for each $\alpha \in A$ the inequality

$$\sum_{k=1}^{d} \gamma_k\, \alpha(s+t)^{-1} D^k_\alpha(u_\alpha + v_\alpha, u_\alpha + v_\alpha)$$
$$\leqq \sum_{k=1}^{d} \gamma_k\, \alpha(s)^{-1} D^k_\alpha(u_\alpha, u_\alpha) + \sum_{k=1}^{d} \gamma_k\, \alpha(t)^{-1} D^k_\alpha(v_\alpha, v_\alpha)$$

holds true. By c3), $D := \sum_{k=1}^{d} \gamma_k D_\alpha^k$ is a positive semidefinite hermitean form on V_α. And with $x := \alpha(s) > 0$, $y := \alpha(t) > 0$ the claim reduces to

$$\frac{1}{x+y} D(u+v, u+v) \leqq \frac{1}{x} D(u, u) + \frac{1}{y} D(v, v)$$

for all $u, v \in V_\alpha$. But this inequality follows from $D(\lambda u - \mu v, \lambda u - \mu v) \geqq 0$ if one chooses $\lambda = \left\{\frac{y}{x(x+y)}\right\}^{1/2}$ and $\mu = \left\{\frac{x}{y(x+y)}\right\}^{1/2}$.

Next we claim that $\mathring{C}$ (and hence C) is invariant under inner automorphisms. As I wrote already in the introduction even a sharper result is true, namely the functions defining $\mathring{C}$ are invariant under inner automorphisms. Clearly, the linear forms α (extended to $\mathfrak{g}$) are invariant because the $\mathfrak{s}$-component (not $\mathfrak{s}$ itself) is left pointwise fixed by all inner automorphisms, even by all members of the connected component $\mathrm{Aut}_0(\mathfrak{g})$. To investigate the g_γ, $\gamma \in \Gamma$, let $(s, \sum_{\alpha \in A} u_\alpha, w_1, \ldots, w_d)$ be any element in $\mathfrak{g}$ such that $\alpha(s) \neq 0$ for all α, otherwise g_γ is not defined. The automorphisms $\operatorname{Exp} \operatorname{ad}(t, 0, z_1, \ldots, z_d)$ applied to this element only multiply the u_α by complex numbers of modulus one. So, the values of g_γ are not changed.

It remains to consider for any $v_\alpha \in V_\alpha$ the element

$$\begin{aligned}
&\operatorname{Exp} \operatorname{ad}(0, \sum_{\alpha \in A} v_\alpha, 0)(s, \sum_{\alpha \in A} u_\alpha, w_1, \ldots, w_d) \\
&\quad = (s, \sum_{\alpha \in A} (u_\alpha - i\alpha(s) v_\alpha), w_1 + \sum_{\alpha \in A} \operatorname{Im} D_\alpha^1(v_\alpha, u_\alpha - \tfrac{1}{2} i\alpha(s) v_\alpha), \ldots, w_d \\
&\qquad + \sum_{\alpha \in A} \operatorname{Im} D_\alpha^d(v_\alpha, u_\alpha - \tfrac{1}{2} i\alpha(s) v_\alpha)).
\end{aligned}$$

Evaluating g_γ at this point gives

$$\begin{aligned}
&\gamma_0(s) + \sum_{k=1}^{d} \gamma_k \{w_k + \sum_{\alpha \in A} \operatorname{Im} D_\alpha^k(v_\alpha, u_\alpha - \tfrac{1}{2} i\alpha(s) v_\alpha) \\
&\quad - \tfrac{1}{2} \sum_{\alpha \in A} \alpha(s)^{-1} D_\alpha^k(u_\alpha - i\alpha(s) v_\alpha, u_\alpha - i\alpha(s) v_\alpha)\}.
\end{aligned}$$

But

$$\begin{aligned}
&\operatorname{Im} D_\alpha^k(v_\alpha, u_\alpha - \tfrac{1}{2} i\alpha(s) v_\alpha) - \tfrac{1}{2}\alpha(s)^{-1} D_\alpha^k(u_\alpha - i\alpha(s) v_\alpha, u_\alpha - i\alpha(s) v_\alpha) \\
&\quad = \operatorname{Im} D_\alpha^k(v_\alpha, u_\alpha) + \tfrac{1}{2}\alpha(s) \operatorname{Im} i D_\alpha^k(v_\alpha, v_\alpha) - \tfrac{1}{2}\alpha(s)^{-1} D_\alpha^k(u_\alpha, u_\alpha) \\
&\qquad - \tfrac{1}{2} i D_\alpha^k(u_\alpha, v_\alpha) + \tfrac{1}{2} i D_\alpha^k(v_\alpha, u_\alpha) - \tfrac{1}{2}\alpha(s) D_\alpha^k(v_\alpha, v_\alpha) \\
&\quad = -\tfrac{1}{2}\alpha(s)^{-1} D_\alpha^k(u_\alpha, u_\alpha)
\end{aligned}$$

for all $\alpha \in A$, $k = 1, \ldots, d$. We see that the v_α drop out. Hence the g_γ are invariant functions.

From the assumption that the cone $C(\Gamma)$ generated by Γ is pointed it follows immediately that $\mathring{C}$ is non-empty (one may even choose the u_α to be zero).

Using the compactness of Γ one sees that $\mathring{C}$ is open. In particular, C is generating. The question whether C is pointed is a little more delicate. One first has to derive a more direct description of C which is a productive exercise anyway.

We claim that C consists of those $(t, \sum_{\alpha\in A} u_\alpha, z_1, \ldots, z_d)$ which satisfy:

(1) $\alpha(t)\geqq 0$ for all $\alpha\in A$,
(2) $u_\alpha=0$ for $\alpha\in A_0=A_0(t):=\{\beta\in A\,|\,\beta(t)=0\}$, and
(3) $\gamma_0(t)+\sum_{k=1}^{d}\gamma_k\{z_k-\frac{1}{2}\sum_{\alpha\in A_+}\alpha(t)^{-1}D_\alpha^k(u_\alpha,u_\alpha)\}\geqq 0$ for all $(\gamma_0,\gamma_1,\ldots,\gamma_d)\in\Gamma$ where $A_+=A\setminus A_0$.

To this end, one first observes that there exists a finite collection of vectors γ^ι, $1\leqq\iota\leqq m$, in Γ, $\gamma^\iota=(\gamma_0^\iota,\gamma_1^\iota,\ldots,\gamma_d^\iota)$, such that

$$D_\alpha:=\sum_{\iota=1}^{m}\sum_{k=1}^{d}\gamma_k^\iota D_\alpha^k$$

is positive definite for all $\alpha\in A$. Indeed, since Γ generates $\mathfrak{s}^*\oplus\mathbb{R}^d$, for each ρ, $1\leqq\rho\leqq d$, there are elements $\gamma^{j,\rho}$ in Γ and real coefficients λ_j^ρ, $1\leqq j\leqq I_\rho$, such that

$$\sum_{j=1}^{I_\rho}\lambda_j^\rho\gamma_k^{j,\rho}=\delta_{k\rho}\qquad\text{for }\ 1\leqq k\leqq d.$$

For each $\alpha\in A$ the positive definite form $\sum_{k=1}^{d}D_\alpha^k$ can be written as

$$\sum_{k=1}^{d}D_\alpha^k=\sum_{k,\rho=1}^{d}\sum_{j=1}^{I_\rho}\lambda_j^\rho\gamma_k^{j,\rho}D_\alpha^k.$$

Let $M:=\max\{\lambda_j^\rho\,|\,\rho=1,\ldots,d, j=1,\ldots,I_\rho\}$. Since $\sum_{k=1}^{d}\gamma_k^{j,\rho}D_\alpha^k$ is positive semidefinite for each pair (j,ρ), one finds for all $\alpha\in A$ and $u\in V_\alpha$

$$\sum_{k=1}^{d}D_\alpha^k(u,u)=\sum_{\rho=1}^{d}\sum_{j=1}^{I_\rho}\lambda_j^\rho\sum_{k=1}^{d}\gamma_k^{j,\rho}D_\alpha^k(u,u)\leqq M\sum_{\rho=1}^{d}\sum_{j=1}^{I_\rho}\sum_{k=1}^{d}\gamma_k^{j,\rho}D_\alpha^k(u,u).$$

Hence the collection of vectors $\{\gamma^{j,\rho}\}$ does the job.

Now, let $(t, \sum_{\alpha\in A} u_\alpha, z_1, \ldots, z_d)$ be an element of C, i.e., there exists a sequence $(t^{(n)}, \sum_{\alpha\in A} u_\alpha^{(n)}, z_1^{(n)}, \ldots, z_d^{(n)})$, $n\in\mathbb{N}$, of elements in $\mathring{C}$ converging to this element. We claim that $(t, \sum_{\alpha\in A} u_\alpha, z_1, \ldots, z_d)$ has the properties (1), (2), (3). Clearly, $\alpha(t^{(n)})>0$

for all α, all n implies $\alpha(t)\geqq 0$. For each ι, $1\leqq\iota\leqq m$, and each n one has the inequality

$$\frac{1}{2}\sum_{k=1}^{d}\gamma_k^{\iota}\sum_{\alpha\in A_0}\alpha(t^{(n)})^{-1}\,D_\alpha^k(u_\alpha^{(n)},u_\alpha^{(n)})$$
$$<\gamma_0^{\iota}(t^{(n)})+\sum_{k=1}^{d}\gamma_k^{\iota}\{z_k^{(n)}-\tfrac{1}{2}\sum_{\alpha\in A_+}\alpha(t^{(n)})^{-1}\,D_\alpha^k(u_\alpha^{(n)},u_\alpha^{(n)})\}.$$

Summing over ι and taking the supremum over n of the right side (which is finite) one finds a constant M such that

$$\sum_{\alpha\in A_0}\alpha(t^{(n)})^{-1}\,D_\alpha(u_\alpha^{(n)},u_\alpha^{(n)})\leqq M$$

for all n. Hence $D_\alpha(u_\alpha^{(n)},u_\alpha^{(n)})\leqq\alpha(t^{(n)})\,M$ for all n and all $\alpha\in A_0$. As $\alpha(t^{(n)})$ tends to zero and D_α is non-degenerate, $(u_\alpha^{(n)})$ converges to 0, i.e., $u_\alpha=0$. The inequality $\gamma_0(t)+\sum_{k=1}^{d}\gamma_k\{z_k-\frac{1}{2}\sum_{\alpha\in A_+}\alpha(t)^{-1}\,D_\alpha(u_\alpha,u_\alpha)\}\geqq 0$ is an immediate consequence of the fact that all the $\sum_{k=1}^{d}\gamma_k\,D_\alpha^k$ are positive semidefinite.

On the other hand, let $x\in\mathfrak{g}$ satisfy (1), (2), (3). Take an arbitrary $c\in\mathring{C}$. The above proof that $\mathring{C}$ is closed additively can be used with only a slight modification to see that $x+\varepsilon c$ is in $\mathring{C}$ for all $\varepsilon>0$. But $x=\lim_{\varepsilon\to 0}x+\varepsilon c$.

Now it is very easy to see that C is pointed. If $x=(t,\sum_{\alpha\in A}u_\alpha,z_1,\dots,z_d)\in C$ has the property that all real multiples λx are in C, then $t=0$ by (1) and a), $u_\alpha=0$ for all α by (2), and $\sum_{k=1}^{d}\gamma_k z_k=0$ for all $(\gamma_0,\gamma_1,\dots,\gamma_d)\in\Gamma$ by (3). As Γ generates $\mathfrak{s}^*\oplus\mathbb{R}^d$ the latter equation gives $z_k=0$ for all k.

Next, we show that C only depends on $C(\Gamma)$, cf. also the above Remark 2. More precisely, C consists of those $(t,\sum_{\alpha\in A}u_\alpha,z_1,\dots,z_d)\in\mathfrak{g}$ which satisfy (1), (2) and

$$\text{(3')}\qquad \gamma_0(t)+\sum_{k=1}^{d}\gamma_k\{z_k-\tfrac{1}{2}\sum_{\alpha\in A_+(t)}\alpha(t)^{-1}\,D_\alpha^k(u_\alpha,u_\alpha)\}\geqq 0$$

for all $(\gamma_0,\gamma_1,\dots,\gamma_d)\in C(\Gamma)$.

One only has to prove that if a given element $x=(t,\sum_{\alpha\in A}u_\alpha,z_1,\dots z_d)\in\mathfrak{g}$ satisfies (3) then it satisfies (3'), too. But the subset $C(x)$ of *all* $\gamma=(\gamma_0,\gamma_1,\dots,\gamma_d)\in\mathfrak{s}^*\oplus\mathbb{R}^d$ which satisfy the inequality (3) for x, is a cone containing Γ, hence $C(\Gamma)$ is contained in $C(x)$.

The latter description of C shows that $C\cap(\mathfrak{s}\oplus\mathbb{R}^d)$ consists of all $x=(t,z_1,\dots z_d)$ such that $\alpha(t)\geqq 0$ for all $\alpha\in A$ and $\gamma(x)\geqq 0$ for all $\gamma\in C(\Gamma)$. But as A is contained in $C(\Gamma)$ by c2) it is evident that $C\cap(\mathfrak{s}\oplus\mathbb{R}^d)$ and $C(\Gamma)$ are dual to each other. This finishes the proof of Theorem 1.

Up to now our considerations were quite elementary, we didn't use any advanced theory, in particular not the Theorems A, B, C. Of course, we were guided by this ABC. These theorems are used explicitly in order to show that the constructed cones exhaust all possible invariant pointed generating cones in solvable Lie algebras. We will use the notations introduced in the beginning.

Theorem 2 *Let C be an invariant pointed generating cone in a solvable Lie algebra $\mathfrak{g}$. Then there are "data" $\mathfrak{s}$, A, Γ etc. such that there exists a Lie algebra isomorphism from $\mathfrak{g}$ onto the Lie algebra constructed from the data transforming the cone C onto the cone corresponding to the data.*

Proof. We fix a compactly embedded Cartan algebra $\mathfrak{h}$ in $\mathfrak{g}$ whose existence is guaranteed by Theorem A. As $C \cap \mathfrak{h}$ is a generating cone, $C \cap \mathfrak{h}$ contains a point c in the complement of $\bigcup_{\omega \in \Omega} \ker \omega$ where Ω denotes the set of (real) roots associated to $\mathfrak{h}$. Let $\Omega_+ := \{\omega \in \Omega \mid \omega(c) > 0\}$. Then Ω_+ is just one half of Ω, Ω is the disjoint union of Ω_+ and $-\Omega_+$. Actually, Ω_+ does not depend on the particular choice of c as we shall see soon. The most important property of Ω_+, used at several places in the sequel, is the following.

(*) For each $\omega \in \Omega_+$ and each non-zero x in $\mathfrak{g}_{\tilde{\omega}}$ the bracket $[J_\omega x, x]$ is a non-zero element of C, even of $C \cap \mathfrak{z}$, where J_ω is the complex structure associated with ω and $\mathfrak{z}$ is the center of $\mathfrak{g}$.

To prove (*) one first observes that Theorem B implies that $[J_\omega x, x]$ is a non-zero element of $\mathfrak{z}$ for all $\omega \in \Omega$. Theorem C gives that $\operatorname{ad}(x)^2(c)$, c as above, is an element of C. But

$$\operatorname{ad}(x)^2(c) = [x, -\omega(c) J_\omega x] = \omega(c)[J_\omega x, x].$$

As $\omega(x)$ is positive, $[J_\omega x, x]$ has to be in C.

Since C is pointed the property stated in (*) characterizes the roots in Ω_+, in particular Ω_+ does not depend on the choice of c.

The center $\mathfrak{z}$ of $\mathfrak{g}$ is just $\mathfrak{z} = \bigcap_{\omega \in \Omega} \ker \omega$. We choose an arbitrary vector space complement $\mathfrak{s}$ to $\mathfrak{z}$ in $\mathfrak{h}$, $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{z}$. The set A in $\mathfrak{s}^*$ is nothing but $A = \{\omega|_{\mathfrak{s}} \mid \omega \in \Omega_+\}$. Clearly, $\mathfrak{s}$ and A have the properties stated in a). The spaces V_α, $\alpha \in A$, are the $\mathfrak{g}_{\tilde{\omega}}$, $\alpha = \omega|_{\mathfrak{s}}$, endowed with the complex structure J_ω, $\omega \in \Omega_+$. Occasionally, we will write J_α instead of J_ω.

The cone $C \cap \mathfrak{z}$ in $\mathfrak{z}$ is pointed, possibly not generating. But anyway, there exists a basis $f_1, \ldots, f_d$ of $\mathfrak{z}$ such that $C \cap \mathfrak{z}$ is contained in $\left\{\sum_{k=1}^{d} \lambda_k f_k \mid \lambda_k \geqq 0\right\}$. Such a basis is used to identify $\mathfrak{z}$ with $\mathbb{R}^d$, $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{z} = \mathfrak{s} \oplus \mathbb{R}^d$. Next, the forms D_α^k on $V_\alpha = \mathfrak{g}_{\tilde{\omega}}$ are defined. For u_α, v_α in V_α the commutator $[u_\alpha, v_\alpha]$ is in $\mathfrak{z}$ and it can be developed in the chosen basis,

$$[u_\alpha, v_\alpha] = \sum_{k=1}^{d} \mu_\alpha^k(u_\alpha, v_\alpha) f_k.$$

Then D_α^k is defined by

$$D_\alpha^k(u_\alpha, v_\alpha) = \mu_\alpha^k(J_\alpha u_\alpha, v_\alpha) + i\,\mu_\alpha^k(u_\alpha, v_\alpha).$$

It is easy to check that D^k_α is hermitean using the skew-symmetry of the Lie bracket and Theorem B(vi): $[u, J_\alpha v]+[J_\alpha u, v]=0$. Concerning the (semi) positivity of D^k_α we first observe that $D^k_\alpha(u_\alpha, u_\alpha)=\mu^k_\alpha(J_\alpha u_\alpha, u_\alpha)$. By (*), $[J_\alpha u_\alpha, u_\alpha]$ is contained in $C\cap\mathfrak{z}\subset\left\{\sum_{k=1}^{d}\lambda_k f_k \mid \lambda_k\geqq 0\right\}$. Hence the coefficients $\mu^k_\alpha(J_\alpha u_\alpha, u_\alpha)$ are non-negative. Since $[J_\alpha u_\alpha, u_\alpha]\neq 0$ for $u_\alpha\neq 0$ the sum $\sum_{k=1}^{d} D^k_\alpha$ is non-degenerate.

We defined already all the data demanded in a) and b) starting from $\mathfrak{g}$ and C. It is easy to see using Theorem B that the obvious isomorphism from $\mathfrak{g}$ onto $\mathfrak{s}\oplus\bigoplus_{\alpha\in A} V_\alpha\oplus\mathbb{R}^d$ is a Lie algebra isomorphism, where the latter space is endowed with the bracket constructed in front of Theorem 1.

The compact set Γ can be chosen to be any base of the dual cone $(C\cap\mathfrak{h})^*=\{\xi\in\mathfrak{h}^* \mid \xi(c)\geqq 0 \text{ for all } c\in C\cap\mathfrak{h}\}$, $\Gamma=\{\xi\in(C\cap\mathfrak{h})^* \mid \xi(c_0)=1\}$ for a fixed element c_0 in the interior of $C\cap\mathfrak{h}$ (in $\mathfrak{h}$). Since $C\cap\mathfrak{h}$ is pointed and generating, $C(\Gamma)=(C\cap\mathfrak{h})^*$ is pointed and generating as well. Next we check property c2), i.e., $\Omega_+\subset(C\cap\mathfrak{h})^*$ or, equivalently, $\omega(c)\geqq 0$ for all $\omega\in\Omega_+$, $c\in C\cap\mathfrak{h}$. Again we use Theorem C. Take any non-zero x in $\mathfrak{g}_{\bar\omega}$. Then $\operatorname{ad}(x)^2(c)=\omega(c)[J_\omega x, x]$ is contained in C. Since $[J_\omega x, x]$ is a non-zero vector in C and C is pointed $\omega(c)$ has to be non-negative.

Clearly, Γ and $(C\cap\mathfrak{h})^*$ are identified with subsets of $\mathfrak{s}^*\oplus\mathbb{R}^d$. Concerning Γ it remains to show that $\sum_{k=1}^{d}\gamma_k D^k_\alpha(u_\alpha, u_\alpha)\geqq 0$ for any $\alpha=\omega|_\mathfrak{s}\in A$, $u_\alpha\in\mathfrak{g}_{\bar\omega}$ and $\gamma=(\gamma_0, \gamma_1, \ldots, \gamma_d)\in\Gamma\subset(C\cap\mathfrak{h})^*$. By (*) and by definition of $(C\cap\mathfrak{h})^*$ one gets

$$\begin{aligned}0\leqq\gamma([J_\alpha u_\alpha, u_\alpha])&=\gamma\left(\sum_{k=1}^{d}\mu^k_\alpha(J_\alpha u_\alpha, u_\alpha) f_\kappa\right)\\&=\sum_{k=1}^{d}\gamma_k\,\mu^k_\alpha(J_\alpha u_\alpha, u_\alpha)=\sum_{k=1}^{d}\gamma_k\, D^k_\alpha(u_\alpha, u_\alpha).\end{aligned}$$

Now, all the data are introduced. The proof of Theorem 2 is finished if we can show that the given cone C coincides (under the obvious identification) with the cone, say $C(\mathfrak{s}, A, \Gamma)$ to distinguish it from $C(\Gamma)$, corresponding in the sense of Theorem 1 to the just established data $\mathfrak{s}$, A etc. Since both cones are invariant, by the uniqueness part of Theorem C it is sufficient to prove that $C\cap\mathfrak{h}=\mathfrak{h}\cap C(\mathfrak{s}, A, \Gamma)$. But this is evident using that $C\cap\mathfrak{h}$ is the dual cone of $(C\cap\mathfrak{h})^*$ and that, by Theorem 1, $\mathfrak{h}\cap C(\mathfrak{s}, A, \Gamma)$ is the dual of $C(\Gamma)$.

Concerning uniqueness of the data attached to a given cone the proof of Theorem 2 shows that one should not expect too much. The vector space complement $\mathfrak{s}$ to $\mathfrak{z}$ in $\mathfrak{h}$ was completely arbitrary, and also the chosen basis $f_1, \ldots, f_d$ is far from being unique.

Theorem 3 *Suppose that an invariant pointed generating cone C in a solvable Lie algebra $\mathfrak{g}$ is realized by two collections of data $\mathfrak{s}$, A, V_α, d, D^k_α, Γ and $\mathfrak{t}$, B, W_β, d', E^k_β, Δ (I hope that the notation is self-explanatory). Then $d=d'$ and there exist a bijection $\beta\mapsto\hat\beta$ from B onto A, a linear isomorphism Φ: $\mathfrak{t}^*\to\mathfrak{s}^*$,* *a collection* σ^j, $1 \leqq j \leqq d$, *of vectors in* $\mathfrak{s}^*$, $\mathbb{C}$*-linear isomorphisms* Φ_β: $W_\beta \to V_{\hat\beta}$, $\beta \in B$, *and* $\psi \in \mathrm{GL}_d(\mathbb{R})$ *such that*

(i) $\Phi(\beta) = \hat\beta$ *for* $\beta \in B$.

(ii) *The forms* D^k_α *and* E^k_β *are related by* $E^j_\beta(v,u) = \sum_{k=1}^{d} \psi_{jk} D^k_{\hat\beta}(\Phi_\beta(v), \Phi_\beta(u))$ *for* $\beta \in B$, $1 \leqq j \leqq d$, *and* $v, u \in W_\beta$.

(iii) *The linear isomosphism* $\mathfrak{t}^* \oplus \mathbb{R}^d \to \mathfrak{s}^* \oplus \mathbb{R}^d$, *given by* $(\xi_0, \xi_1, \ldots, \xi_d) \mapsto \left(\Phi(\xi_0) + \sum_{j=1}^{d} \xi_j \sigma^j, \sum_{j=1}^{d} \xi_j \psi_{j1}, \ldots, \sum_{j=1}^{d} \xi_j \psi_{jd}\right)$, *transforms* $C(\Delta)$ *onto* $C(\Gamma)$.

Remark 5 The appearance of the σ^j reflects the arbitrary choice of the complement $\mathfrak{s}$ to $\mathfrak{z}$ in $\mathfrak{h}$. The appearance of the matrix ψ instead of the unit matrix reflects the choice of the bases $f_1, \ldots, f_d$ in proving Theorem 2.

Proof. Clearly, $d = \dim \mathfrak{z}(\mathfrak{g}) = d'$. By assumption, there is a Lie algebra isomorphism φ: $\mathfrak{s} \oplus \bigoplus_{\alpha \in A} V_\alpha \oplus \mathbb{R}^d \to \mathfrak{t} \oplus \bigoplus_{\beta \in B} W_\beta \oplus \mathbb{R}^d$ transforming the cones corresponding to the data onto each other. The proof consists of inspecting what that means. The isomorphism φ has to transform the center $\mathbb{R}^d$ onto $\mathbb{R}^d$, hence there is a matrix $\psi = (\psi_{jk}) \in \mathrm{GL}_d(\mathbb{R})$ such that $\varphi(x_1, \ldots, x_d) = \left(\sum_{j=1}^{d} \psi_{1j} x_j, \ldots, \sum_{j=1}^{d} \psi_{dj} x_j\right)$ for $(x_1, \ldots, x_d) \in \mathbb{R}^d$.

The compactly embedded Cartan algebra $\mathfrak{s} \oplus \mathbb{R}^d$ is transformed via φ onto a subalgebra of the same type. Since compactly embedded Cartan algebras are conjugate under inner automorphisms we may assume that $\varphi(\mathfrak{s} \oplus \mathbb{R}^d) = \mathfrak{t} \oplus \mathbb{R}^d$. Note that the potential change of φ by an inner automorphism does not disturb the *invariant* cones. The restriction of φ to $\mathfrak{s}$ is given by

$$\varphi(s) = (\varphi_w(s), \varphi_z(s)) \qquad \text{for } s \in \mathfrak{s}$$

where φ_w: $\mathfrak{s} \to \mathfrak{t}$ is a linear isomorphism and φ_z: $\mathfrak{s} \to \mathbb{R}^d$ is a linear map. The vectors σ^j of the theorem are the components of φ_z, $\varphi_z(s) = (\sigma^1(s), \ldots, \sigma^d(s))$; Φ is the transpose φ^*_w of φ_w.

So far, we have "computed" $\varphi|_{\mathfrak{s} \oplus \mathbb{R}^d}$. Since φ respects the Cartan algebras $\mathfrak{s} \oplus \mathbb{R}^d$ and $\mathfrak{t} \oplus \mathbb{R}^d$, it has to respect the root spaces (V_α) and (W_β) as well. Hence there is a bijective map $\alpha \mapsto \check\alpha$ from A onto B such that φ induces an $\mathbb{R}$-linear isomorphism φ_α from V_α onto $W_{\check\alpha}$. A closer inspection of the formula

$$\varphi[(s,0,0),(0,u,0)] = [\varphi(s,0,0),(0,\varphi_\alpha(u),0)]$$

for $s \in \mathfrak{s}$, $u \in V_\alpha$ shows that there is a sign $\varepsilon_\alpha \in \{\pm 1\}$ such that

$$\varphi_\alpha(iu) = \varepsilon_\alpha\, i\, \varphi_\alpha(u) \quad \text{and} \quad \alpha(s) = \varepsilon_\alpha\, \check\alpha(\varphi_w(s)),$$

i.e.

$$\alpha = \varepsilon_\alpha\, \Phi(\check\alpha).$$

Using that φ transforms the cones, one concludes that all the signs ε_α have to be $+1$. As a consequence all the φ_α are $\mathbb{C}$-linear. The map $\beta \to \hat\beta$ from B onto A is defined as the inverse of $\alpha \mapsto \check\alpha$, and (i) is clear.

Next, we evaluate the equation

$$\varphi[(0,v,0),(0,u,0)]=[(0,\varphi_\alpha(v),0),(0,\varphi_\alpha(u),0)]$$

for $\alpha\in A$ and $u, v\in V_\alpha$. One obtains

$$\sum_{k=1}^{d}\psi_{jk}\operatorname{Im} D_\alpha^k(v,u)=\operatorname{Im} E_\alpha^j(\varphi_\alpha(v),\varphi_\alpha(u))$$

for $1\leqq j\leqq d$ and $\alpha\in A$. Since the φ_α are $\mathbb{C}$-linear the latter equation implies

$$\sum_{k=1}^{d}\psi_{jk} D_\alpha^k(v,u)=E_\alpha^j(\varphi_\alpha(v),\varphi_\alpha(u)).$$

If for $\beta\in B$ we define Φ_β: $W_\beta\to V_{\hat\beta}$ to be the inverse of $\varphi_{\hat\beta}$ the equation takes the form stated in (ii).

It is easily checked that the transpose ω of the linear isomorphism $\varphi|_{\mathfrak{s}\oplus\mathbb{R}^d}$ from $\mathfrak{s}\oplus\mathbb{R}^d$ onto $\mathfrak{t}\oplus\mathbb{R}^d$ is given by

$$(\xi_0,\xi_1,\ldots,\xi_d)\mapsto\left(\Phi(\xi_0)+\sum_{j=1}^{d}\xi_j\sigma^j,\ \sum_{j=1}^{d}\xi_j\psi_{j1},\ \ldots,\ \sum_{j=1}^{d}\xi_j\psi_{jd}\right)\in\mathfrak{s}^*\oplus\mathbb{R}^d$$

for $(\xi_0,\xi_1,\ldots,\xi_d)\in\mathfrak{t}^*\oplus\mathbb{R}^d$.

Since φ transforms the cones intersected with $\mathfrak{s}\oplus\mathbb{R}^d$ and $\mathfrak{t}\oplus\mathbb{R}^d$, respectively, its transpose ω has to transform their duals which are $C(\Delta)$ and $C(\Gamma)$ by Theorem 1. But that is precisely (iii). Obviously, the quantities described in the theorem can be used to reconstruct a Lie algebra isomorphism φ which transforms the cones corresponding to the data.

As an illustration of the above construction let's consider invariant pointed generating cones in solvable Lie algebras $\mathfrak{g}$ with one-dimensional centers $\mathfrak{z}$ and two-dimensional (compactly embedded) Cartan algebras $\mathfrak{h}$. In terms of the data this means the following. For a suitable "orientation" of the one-dimensional space $\mathfrak{s}$, in $\mathfrak{s}=\mathbb{R}$ is given a finite collection A of positive numbers. As $d=1$ each V_α is simply an ordinary complex Hilbert space with scalar product D_α. Also the orthogonal sum $V=\bigoplus_{\alpha\in A}V_\alpha$ is a complex Hilbert space with scalar product D, say. In $\mathbb{R}^2=\mathfrak{s}\oplus\mathbb{R}$ there don't exist too many pointed generating cones $C(\Gamma)$, in particular as we know by c3) that $C(\Gamma)$ has to be contained in $\{(x,y)\in\mathbb{R}^2\,|\,y\geqq 0\}$ and that by c2) the positive x-axis is contained in $C(\Gamma)$. This means that we may choose Γ to be

$$\Gamma=\{(1,0),(\xi,1)\}\qquad\text{with some }\ \xi\in\mathbb{R}.$$

The corresponding algebra $\mathfrak{g}$ is $\mathfrak{g}=\mathbb{R}\oplus V\oplus\mathbb{R}$ with the bracket

$$\begin{aligned}&[(t,\sum_{\alpha\in A}v_\alpha,z),(s,\sum_{\alpha\in A}u_\alpha,w)]\\&\quad=(0,\sum_{\alpha\in A}i\alpha(tu_\alpha-sv_\alpha),\operatorname{Im} D(\sum_{\alpha\in A}v_\alpha,\sum_{\alpha\in A}u_\alpha)).\end{aligned}$$

The corresponding $\mathring{C}$ consists of all $(t, \sum_{\alpha \in V} v_\alpha, z)$ satisfying $t>0$ and

$$0<\xi t+z-\tfrac{1}{2}\sum_{\alpha \in A} \alpha^{-1} t^{-1} D_\alpha(v_\alpha, v_\alpha) \text{ or, equivalently,}$$

$$\begin{aligned} 0&<\xi t^2+zt-\tfrac{1}{2}\sum_{\alpha \in A} \alpha^{-1} D_\alpha(v_\alpha, v_\alpha) \\ &=\xi t^2+zt-\tfrac{1}{2} D\Big(\sum_{\alpha \in A} \alpha^{-1/2} v_\alpha, \sum_{\alpha \in A} \alpha^{-1/2} v_\alpha\Big). \end{aligned}$$

Clearly, to obtain the closure C one only has to allow equality in these inequalities. As we have shown above our construction yields invariant inequalities. This means that the quadratic form $\xi t^2+zt-\frac{1}{2} D(\sum_{\alpha \in A} \alpha^{-1/2} v_\alpha, \sum_{\alpha \in A} \alpha^{-1/2} v_\alpha)$ on the *real* vector space $\mathfrak{g}$ is invariant. Evidently the form is Lorentzian. So, we have met old friends, invariant Lorentzian forms and their associated cones! All Lie algebras permitting an invariant Lorentzian form are already classified, see [2, 5, 6]. Our results may be used to provide another proof for this classification in the solvable case: If the solvable Lie algebra L carries an invariant Lorentzian form q then q defines an invariant pointed generating cone $C=C_q$. Hence (L, C) must be in our "list". A small consideration shows that for the associated data one has to have $d=1=\dim \mathfrak{s}$. – But I think this is the wrong way to look at our results. One had better view the cones we constructed as genuine generalizations of the Lorentzian cones. By the way, even further generalizations are possible. One may allow A to be infinite, even uncountable if one has a measure on A and uses direct integrals of Hilbert spaces.

In [4], the authors ask whether it is possible that the group $\operatorname{Inn}(\mathfrak{g})$ of inner automorphisms is not closed in $GL(\mathfrak{g})$ in case that $\mathfrak{g}$ allows an invariant pointed generating cone. The potential non-closedness causes some trouble in the considerations of [4]. Our results show in particular that for solvable algebras $\mathfrak{g}$ the group $\operatorname{Inn}(\mathfrak{g})$ is almost never closed – whatever that means. Anyway, to see such an algebra and to produce a really concrete example what is in the spirit of this note we specialize the data even further. In addition to $d=1=\dim \mathfrak{s}$ we assume that $\# A=2$ and $\dim_{\mathbb{C}} V_\alpha=1$, i.e., $V_\alpha=\mathbb{C}$. The bracket on $\mathfrak{g}=\mathbb{R}\oplus\mathbb{C}^2\oplus\mathbb{R}$ is given by

$$\begin{aligned} &[(t, v_1, v_2, z), (s, u_1, u_2, w)] \\ &\quad=(0, i\alpha_1(tu_1-sv_1), i\alpha_2(tu_2-sv_2), \operatorname{Im}(v_1\overline{u_1}+v_2\overline{u_2})) \end{aligned}$$

where α_1, α_2 are positive numbers. This is an one-dimensional extension of the five-dimensional Heisenberg algebra $\mathbb{C}^2\oplus\mathbb{R}$. An invariant Lorentzian form on $\mathfrak{g}$ is given by (corresponding to $\xi=0$)

$$q(t, v_1, v_2, z)=2zt-\frac{1}{\alpha_1}|v_1|^2-\frac{1}{\alpha_2}|v_2|^2.$$

The group $\operatorname{Inn}(\mathfrak{g})$ is closed if and only if the numbers α_1, α_2 are linearly dependent over the rationals – which is almost never the case.

By the way, a nice matrix representation of $\mathfrak{g}$ is

$$\mathfrak{g}=\left\{\begin{bmatrix} 0 & v_1 & v_2 & i\mu \\ 0 & i\lambda\alpha_1 & 0 & \bar{v}_2 \\ 0 & 0 & i\lambda\alpha_2 & \bar{v}_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| v_j\in\mathbb{C}, \lambda, \mu\in\mathbb{R}\right\}.$$

This algebra and its corresponding group in $GL_4(\mathbb{C})$ leave invariant a particular hermitean form on $\mathbb{C}^4$. I don't know whether that is a special case of a general (and noteworthy) phenomenon. I didn't consider this question.

Let me finish this paper with three remarks. First, we have seen in this paper that the Theorems A, B, C are indeed conclusive. Only using them we could establish a constructive procedure to obtain all invariant cones in solvable Lie algebras. Secondly, I want to emphasize once more (cf. Remark 3 above) that one only needs the data given in a) and b) in order to construct a solvable Lie algebra which admits at least one invariant pointed generating cone. So, all those algebras are known. The final remark could be used to write this paper all over again – in a different light. The above invariant functions g_γ clearly have an origin which has nothing to do with a particular linear functional γ. Suppose that the Lie algebra $\mathfrak{g}$ possesses a compactly embedded Cartan algebra $\mathfrak{h}$. Let $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\tilde{\omega}\in\tilde{\Omega}}\mathfrak{g}_{\tilde{\omega}}$ be the decomposition into root spaces, cf. the introduction for notations. Suppose further that $[\mathfrak{g}_{\tilde{\omega}},\mathfrak{g}_{\tilde{\omega}'}]=0$ for $\tilde{\omega}'\neq\tilde{\omega}$ and that $[\mathfrak{g}_{\tilde{\omega}},\mathfrak{g}_{\tilde{\omega}}]$ is contained in the center $\mathfrak{z}$ of $\mathfrak{g}$. These assumption imply, of course, that $\mathfrak{g}$ is solvable (of length ≤ 3). Let A be *any* representative set for the set $\tilde{\Omega}$ of equivalence classes, i.e., $\Omega=A\,\dot{\cup}\,-A$. Each $\omega\in A$ defines a complex structure J_ω on $\mathfrak{g}_{\tilde{\omega}}$ such that $[h,x]=\omega(h)\,J_\omega x$ for $\omega\in A$, $x\in\mathfrak{g}_{\tilde{\omega}}$. Given A we define a function f on a subset of $\mathfrak{g}$ with values in $\mathfrak{h}$. Decompose $x\in\mathfrak{g}$ into $x=h+\sum_{\omega\in A}u_\omega$, $h\in\mathfrak{h}$, $u_\omega\in\mathfrak{g}_{\tilde{\omega}}$. If $\omega(h)\neq 0$ for all ω, i.e., if h is regular, then

$$f(x)=h-\tfrac{1}{2}\sum_{\omega\in A}\omega(h)^{-1}\,[J_\omega u_\omega, u_\omega].$$

In [1] and [3], the authors study a similar function, but with $\omega(h)^{-1}$ replaced by $\omega(h)$. The above function has the advantage that it is homogeneous under multiplication by scalars – which is crucial when considering cones. The function is independent of A because the transition from ω to $-\omega$ changes the signs of $\omega(h)^{-1}$ *and* of J_ω. Even better, one can write down a more "closed form" of f. For regular h, $\mathrm{ad}(h)$ induces a linear automorphism of $\mathfrak{h}^\perp:=\bigoplus_{\tilde{\omega}\in\tilde{\Omega}}\mathfrak{g}_{\tilde{\omega}}$. It follows from the assumptions that f may be written as

$$f(x)=h+\tfrac{1}{2}[\mathrm{ad}(h)^{-1}(x'), x']$$

if

$$x=h+x', \qquad h\in\mathfrak{h}, \qquad x'\in\mathfrak{h}^\perp.$$

One can show using the first description of f that f is invariant under inner automorphisms of $\mathfrak{g}$ – that's what we have done in the proof of Theorem 1 when showing that $\mathring{C}$ is invariant, only in another terminology. There are also invariance properties under arbitrary automorphisms φ of $\mathfrak{g}$. Let's first assume that $\varphi(\mathfrak{h})=\mathfrak{h}$. Then using the second description of f one can show that

$f(\varphi(x)) = \varphi(f(x))$ for all $x \in \mathfrak{g}$ where f is defined. Since all compactly embedded Cartan subalgebras are conjugate under inner automorphisms an arbitrary automorphism ψ of $\mathfrak{g}$ may be written as $\psi = \iota\varphi$ where $\iota \in \mathrm{Inn}(\mathfrak{g})$, and $\varphi \in \mathrm{Aut}(\mathfrak{g})$ satisfies $\varphi(\mathfrak{h}) = \mathfrak{h}$. Putting together these two pieces of informations one obtains

$$f(\psi(x)) = f(\iota\varphi(x)) = f(\varphi(x)) = \varphi(f(x)).$$

The latter observation might be useful when considering invariant cones in general Lie algebras.

The connection between f and the above functions g_γ is that $g_\gamma(x) = \langle \gamma, f(x) \rangle$ for all x in the domain of f.

Addendum. When I circulated this article as a preprint I was kindly informed by K.-H. Neeb that V.M. Gichev in his paper "Invariant Orderings in Solvable Lie Groups", Sib. Math. J. **30**, 44–53 (1989), has determined explicitly the semigroups corresponding to the invariant cones in solvable Lie algebras.

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