

A GENERALIZATION OF THE AZ IDENTITY

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The identity discovered in [1] can be viewed as a sharpening of the LYM inequality ([3], [4], [5]). It was extended in [2] so that it covers also Bollobás' inequality [6]. Here we present a further generalization and demonstrate that it shares with its predecessors the usefulness for uniqueness proofs in extremal set theory.

1. Introduction

A few years ago Ahlswede and Zhang [1] found the following identity.

**Theorem AZ<sub>1</sub>.** For every family  $\mathcal{A} \subset 2^\Omega$  of non-empty subsets of  $\Omega = \{1, 2, \dots, n\}$

$$\sum_{X \subset \Omega} \frac{W_{\mathcal{A}}(X)}{|X| \binom{n}{|X|}} = 1, \text{ where } W_{\mathcal{A}}(X) = \left| \bigcap_{X \supset A \in \mathcal{A}} A \right|.$$

We associate with every  $\mathcal{E} \subset 2^\Omega$  the upset  $\mathcal{U}(\mathcal{E}) = \{U \subset \Omega : U \supset E \text{ for some } E \in \mathcal{E}\}$  and the downset  $\mathcal{D}(\mathcal{E}) = \{D \subset \Omega : D \subset E \text{ for some } E \in \mathcal{E}\}$ .

When  $\mathcal{A}$  is an antichain in the poset  $(2^\Omega, \supset)$ , then the identity becomes

$$(1) \quad \sum_{X \in \mathcal{A}} \frac{1}{\binom{n}{|X|}} + \sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{A}} \frac{W_{\mathcal{A}}(X)}{|X| \binom{n}{|X|}} = 1.$$

The LYM inequality is obtained by omission of the second summand, which by definition of  $W_{\mathcal{A}}$  can also be written in the form  $\sum_{X \notin \mathcal{D}(\mathcal{A})} \frac{W_{\mathcal{A}}(X)}{|X| \binom{n}{|X|}}$ . We call this the deficiency of the inequality.

More generally, in [2] the Bollobás inequality was lifted to an identity.

**Theorem AZ<sub>2</sub>.** For two families  $\mathcal{A} = \{A_1, \dots, A_N\}$  and  $\mathcal{B} = \{B_1, \dots, B_N\}$  of subsets of  $\Omega$  with the properties

- (a)  $A_i \subset B_i$  for  $i = 1, 2, \dots, N$
- (b)  $A_i \not\subset B_j$  for  $i \neq j$

$$(2) \quad \sum_{i=1}^N \frac{1}{\binom{n-|B_i \setminus A_i|}{|A_i|}} + \sum_{X \notin \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X)}{|X| \binom{n}{|X|}} = 1.$$

In [1] it was explained that Theorem AZ<sub>1</sub> gives immediately, what LYM does not, namely the uniqueness part in Sperner’s Theorem. In [2] the uniqueness of an optimal configuration of unrelated chains of subsets due to Griggs, Stahl and Trotter [7] was proved with the help of Theorem AZ<sub>2</sub>.

Recently, Körner and Simonyi [10] observed the LYM-type inequality:

For  $\mathcal{A} = \{A_1, \dots, A_N\}, \mathcal{B} = \{B_1, \dots, B_N\} \subset 2^\Omega$  with

$$A_i \cap B_i = \emptyset, A_i \not\subset A_j \cup B_j, B_i \not\subset A_j \cup B_j \text{ for } i \neq j$$

$$(3) \quad \sum_{i=1}^N \binom{n-|A_i|}{|B_i|}^{-1} + \binom{n-|B_i|}{|A_i|}^{-1} - \binom{n}{|A_i|+|B_i|}^{-1} \leq 1$$

and they asked (Problem 2) “Is this inequality ever tight?”.

This rather modest question was a challenging test of the power of the identities in [1], [2] or, more precisely, of the procedure to produce new identities described in [1].

The outcome is an Ahlswede–Zhang type identity (Theorem 1) which goes considerably beyond Theorem AZ<sub>2</sub>. From a special case of this identity we derive a *full characterization* of the cases with equality (Theorem 2) even for a generalized version of (3). In other words we characterize the cases with deficiency zero.

### 2. The identity

**Theorem 1.** Suppose that for a family  $\mathcal{B} = \{B_1, \dots, B_N\}$  of subsets of  $\Omega$  and a family  $\mathcal{A}^* = \{\mathcal{A}_1, \dots, \mathcal{A}_N\}$  of subsets of  $2^\Omega$ , where  $\mathcal{A}_i = \{A_i^t : t \in T_i\}$  for a finite index set  $T_i$ , we have the properties

- (a)  $A_i^t \subset B_i$  for  $t \in T_i$  and  $i = 1, 2, \dots, N$
- (b)  $A_i^t \not\subset B_j$  for  $t \in T_i$  and  $i \neq j$ .

Then with  $\mathcal{A} = \bigcup_{i=1}^N \mathcal{A}_i$

$$(4) \quad \sum_{i=1}^N \sum_{k=1}^{|T_i|} (-1)^{k-1} \sum_{S \subset T_i, |S|=k} \binom{n-|B_i - \bigcup_{t \in S} A_i^t|}{|\bigcup_{t \in S} A_i^t|}^{-1} + \sum_{X \notin \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X)}{|X| \binom{n}{|X|}} = 1.$$

The specialisation  $|T_i| = 1$  for  $i = 1, \dots, N$  gives Theorem AZ<sub>2</sub>. The proof goes again by counting chains. A key tool in [2] was

**Lemma 1.** For two sets  $A, B \subset \Omega$  with  $A \subset B$  exactly  $\frac{n!}{\binom{n-|B \setminus A|}{|A|}}$  maximal chains in  $(2^\Omega, \subset)$  meet  $\{X : A \subset X \subset B\}$ . ■

Using the principle of inclusion–exclusion this generalizes to

**Lemma 2.** For  $B \subset \Omega$  and  $\mathcal{C} \subset 2^\Omega$  with  $C \subset B$  for all  $C \in \mathcal{C}$  exactly

$$n! \sum_{k=1}^{|\mathcal{C}|} (-1)^{k-1} \sum_{\mathcal{C}' \subset \mathcal{C}, |\mathcal{C}'|=k} \binom{n - |B \setminus \cup_{C \in \mathcal{C}'} C|}{|\cup_{C \in \mathcal{C}'} C|}^{-1}$$

maximal chains in  $(2^\Omega, \subset)$  meet  $\{X : C \subset X \subset B \text{ for some } C \in \mathcal{C}\}$ . ■

**Proof of Theorem 1.** The number of maximal chains leaving  $\mathcal{U}(\mathcal{A})$  at  $U$  is

$$(n - |U|)! W_{\mathcal{A}}(U) (|U| - 1)!$$

Since the sets  $\mathcal{X}_i = \{X : A_i^t \subset X \subset B_i \text{ for some } t \in T_i\}$  ( $i = 1, 2, \dots, N$ ) are disjoint we have

$$\sum_{i=1}^N \sum_{X \in \mathcal{X}_i} (n - |X|)! W_{\mathcal{A}}(X) (|X| - 1)! + \sum_{X \in \mathcal{U}(\mathcal{A}) - \cup \mathcal{X}_i} (n - |X|)! W_{\mathcal{A}}(X) (|X| - 1)! = n!$$

By the definition of  $W_{\mathcal{A}}$  the last summand can be written in the form

$$\sum_{X \notin \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X) n!}{|X| \binom{n}{|X|}} \text{ and by Lemma 2}$$

$$\sum_{X \in \mathcal{X}_i} (n - |X|)! W_{\mathcal{A}}(X) (|X| - 1)! = n! \sum_{k=1}^{|T_i|} (-1)^{k-1} \sum_{S \subset T_i, |S|=k} \binom{n - |B_i \setminus \cup_{t \in S} A_i^t|}{|\cup_{t \in S} A_i^t|}^{-1}. \quad \blacksquare$$

### 3. On zero deficiency

We characterize here a case of zero deficiency, that is, the property

$$(5) \quad \sum_{X \notin \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X)}{|X| \binom{n}{|X|}} = 0.$$

**Theorem 2.** Under the assumptions of Theorem 1 and the additional conditions

- (c)  $A_i^t \cap A_i^{t'} = \emptyset$  for all  $i$  and  $t, t' \in T_i$  with  $t \neq t'$
- (d)  $|T_i| \geq 2$  and  $N \geq 2$

we have, that the identity

$$(6) \quad \sum_{i=1}^N \sum_{k=1}^{|T_i|} (-1)^{k-1} \sum_{S \subset T_i, |S|=k} \binom{n - |B_i - \cup_{t \in S} A_i^t|}{|\cup_{t \in S} A_i^t|}^{-1} = 1$$

holds exactly if

- (i)  $|A_i^t| = 1$  for all  $t \in T_i$  and all  $i$ .
- (ii)  $B_1 \setminus \cup_{t \in T_1} A_1^t = B_2 \setminus \cup_{t \in T_2} A_2^t = \dots = B_N \setminus \cup_{t \in T_N} A_N^t = B$ , say.
- (iii)  $\sum_{i=1}^N |T_i| = n - |B|$ .

In words, the  $B_i$  have a common part  $B$  and each  $B_i$  has a rest of singletons  $A_i^t$ . The  $B_i$ 's exhaust  $\Omega$ .

In the proof we use a well-known identity, which follows by iterative application of Pascal's identity.

**Lemma 3.**  $\sum_{k=1}^m (-1)^{k-1} \binom{M}{m-k} = \binom{M-1}{m-1}$ . ■

**Proof of Theorem 2.** From (i) and (ii) we derive in terms of  $\beta = |B|$

$$\begin{aligned} & \sum_{i=1}^N \sum_{k=1}^{|T_i|} (-1)^{k-1} \sum_{S \subset T_i, |S|=k} \binom{n - |B_i \setminus \cup_{t \in S} A_i^t|}{|\cup_{t \in S} A_i^t|}^{-1} \\ &= \sum_{i=1}^N \sum_{k=1}^{|T_i|} (-1)^{k-1} \binom{|T_i|}{k} \binom{n - (\beta + |T_i|) + k}{k}^{-1} \\ &= \sum_{i=1}^N \sum_{k=1}^{|T_i|} (-1)^{k-1} \frac{|T_i|!(n - (\beta + |T_i|))!}{(|T_i| - k)!(n - (\beta + |T_i|) + k)!} \\ &= \sum_{i=1}^N \frac{|T_i|!(n - (\beta + |T_i|))!}{(n - \beta)!} \sum_{k=1}^{|T_i|} (-1)^{k-1} \binom{n - \beta}{|T_i| - k} \\ &= \sum_{i=1}^N \frac{|T_i|!(n - (\beta + |T_i|))!}{(n - \beta)!} \frac{(n - \beta - 1)!}{(|T_i| - 1)!(n - (\beta + |T_i|))!}, \end{aligned}$$

by Lemma 3, and now by (iii)

$$(7) \quad = \sum_{i=1}^N \frac{|T_i|}{n - \beta} = 1.$$

We assume now that (6) holds and derive (i), (ii), and (iii). By Theorem 1 we have deficiency zero, that is,

$$(8) \quad W_{\mathcal{A}}(X) = 0 \text{ for all } X \notin \mathcal{D}(\mathcal{B}).$$

For the quantity

$$(9) \quad m = \min\{|A_i^t| : 1 \leq i \leq N, t \in T_i\}$$

we show first that it equals 1, then we establish (i) and (ii), and finally (iii).

**Step 1.** W.l.o.g. we can assume  $|A_1^1| = m$ . For any  $y \in \Omega \setminus B_1$  consider  $A_1^1 \cup \{y\}$ . Thus clearly  $A_1^1 \cup \{y\} \not\subset B_1$  and by condition (b) also  $A_1^1 \cup \{y\} \not\subset B_j$  for  $j \neq 1$ . Therefore  $A_1^1 \cup \{y\} \notin \mathcal{D}(\mathcal{B})$  and by (8)  $W_{\mathcal{A}}(A_1^1 \cup \{y\}) = 0$ . By the minimality of  $A_1^1$  in  $\mathcal{A}$  and the definition of  $W_{\mathcal{A}}$  every  $m$ -subset of  $A_1^1 \cup \{y\}$  must be in  $\mathcal{A}$ . In particular for any  $a \in A_1^1$  the set  $(A_1^1 \setminus \{a\}) \cup \{y\}$  is in  $\mathcal{A}$ . Since it is not in  $\mathcal{A}_1$  it must be in some  $\mathcal{A}_j$  with  $j \neq 1$ .

W.l.o.g. we can assume it to be  $A_2^1$ . Furthermore, since  $A_1^1 \neq A_1^2$  we can require the  $a$  chosen above to be from  $A_1^1 \setminus A_1^2$ . Also, since by (b)  $A_1^2 \not\subset B_2$  there is  $z \in A_1^2 \setminus B_2, z \neq a$ . As previously we conclude that  $A_2^1 \cup \{z\} \notin \mathcal{D}(\mathcal{B})$  and that the  $m$ -set

$$(A_2^1 \cup \{z\}) \setminus \{y\} = (A_1^1 \setminus \{a\}) \cup \{z\} \in \mathcal{A}.$$

However, we also have  $(A_1^1 \setminus \{a\}) \cup \{z\} \in \mathcal{A}_1$  and by (c)  $A_1^1 \cap ((A_1^1 \setminus \{a\}) \cup \{z\}) = \emptyset$ . This implies  $A_1^1 = \{a\}$  and  $m = 1$ .

**Step 2.** After relabelling we can assume now  $A_1^1 = \{1\}$  and  $B_1 = \{1, 2, \dots, \ell\}$ . By the arguments in Step 1 we get  $\{1, k\} \notin \mathcal{D}(\mathcal{B})$  and  $\{1, k\} \supset \{k\} \in \mathcal{A}$  whenever  $k > \ell$ . By (b) for all  $t \in T_i$  and  $i \geq 2$   $A_i^t$  has an element, say  $e$ , with  $e > \ell$ . However, since  $\{e\} \in \mathcal{A}$  by (a), (b) and (c) actually  $A_i^t$  must equal  $\{e\}$ . We thus know that  $A_i^t$  is a singleton for all  $i \geq 2$  and  $t \in T_i$ . Now we can let any  $i \geq 2$  take the role of 1 in the previous argument and get that all  $A_1^t$  are also singletons. We have proved (i).

Also we have arrived at the following configuration:  $B_i \supset A_i = \cup_{t \in T_i} A_i^t$  and  $B_i \cap A_j = \emptyset$  for  $i \neq j$ . We claim now that  $B_i = A_i \cup C$ , where  $C = \Omega \setminus \cup_{i=1}^N A_i$ . To see this, suppose that  $c \in C$  and  $c \notin B_i$ . Then for any  $a \in A_i$   $\{a, c\} \notin \mathcal{D}(\mathcal{B})$  and thus  $W_{\mathcal{A}}(\{a, c\}) = 0$ . This, however, contradicts  $W_{\mathcal{A}}(\{a, c\}) = |\{a\}| = 1$ .

We have established (ii) with  $B = C$ . (6), together with the equations leading to (7), give now also (iii). ■

Finally we present a consequence of Theorem 2, which in particular gives a positive answer to the question of Körner and Simonyi mentioned in the Introduction.

**Corollary.** *If we are given for  $t = 1, 2$  and  $i = 1, 2, \dots, N$  sets  $A_i^t \subset \Omega$  with  $A_i^1 \cap A_i^2 = \emptyset$  and  $A_i^t \not\subset A_j^1 \cup A_j^2$  for  $t = 1, 2$  and  $i \neq j$  then*

$$(10) \quad \sum_{i=1}^N \left( \binom{n - |A_i^1|}{|A_i^1|} \right)^{-1} + \left( \binom{n - |A_i^2|}{|A_i^2|} \right)^{-1} - \left( \binom{n}{|A_i^1| + |A_i^2|} \right)^{-1} = 1$$

exactly if

$$(i') \quad |A_i^t| = 1 \text{ for } t = 1, 2 \text{ and } i = 1, 2, \dots, N$$

(ii')  $n$  is even and  $N = \frac{n}{2}$ .

There is a direct proof of this Corollary which is shorter than the one via Theorem 2.

**Proof.** With the choice  $\mathcal{B}_i = \bigcup_{t \in T_i} A_i^t$  formula (6) takes the form

$$\sum_{i=1}^N \sum_{k=1}^{|T_i|} (-1)^{k-1} \sum_{S \subset T_i, |S|=k} \binom{n - \sum_{t \notin S} |A_i^t|}{\sum_{t \in S} |A_i^t|}^{-1} = 1$$

and if  $T_i = \{1, 2\}$  for all  $i$  this becomes (10). (i) specializes to (i'), (ii) is true by definition of  $B_i$  with  $B = \emptyset$ . (iii) specializes to  $N \cdot 2 = n$  and thus (ii'). ■

### 4. On general cloud antichains

A family  $\mathcal{A}^* = \{\mathcal{A}_1, \dots, \mathcal{A}_N\}$  of subsets of  $2^\Omega$  is a cloud-antichain, if

$$(1') \quad A_i \not\subset A_j \text{ for } A_i \in \mathcal{A}_i, A_j \in \mathcal{A}_j \text{ with } i \neq j.$$

They have been analyzed in [2] for  $N = 2$ . In particular, in the case  $|\mathcal{A}_i| = M$  for  $i = 1, \dots, N$  we are interested in the maximal length  $N(n, M)$  of these antichains.

Clearly, for  $\mathcal{A} = \bigcup_{i=1}^N \mathcal{A}_i$

$$(12) \quad W_{\mathcal{A}}(X) = W_{\mathcal{A}_i}(X) \text{ for } X \in \mathcal{A}_i$$

and therefore by Theorem AZ<sub>1</sub>

$$(13) \quad \sum_{i=1}^N \sum_{X \in \mathcal{A}_i} \frac{W_{\mathcal{A}_i}(X)}{|X| \binom{n}{|X|}} + \sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{A}} \frac{W_{\mathcal{A}}(X)}{|X| \binom{n}{|X|}} = 1.$$

Notice that  $n! \sum_{X \in \mathcal{B}} \frac{W_{\mathcal{B}}(X)}{|X| \binom{n}{|X|}}$  counts the number, say  $\sigma(\mathcal{B})$ , of saturated chains meeting a member of  $\mathcal{B}$ .

We can derive from (13) a bound on  $N(n, M)$ , if we have a bound or even exact result for the following seemingly basic quantity:

$$(14) \quad s(M, n) = \min\{\sigma(\mathcal{B}) : \mathcal{B} \subset 2^\Omega, |\mathcal{B}| = M\}.$$

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