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# THE THEOREM OF LIE AND HYPERPLANE
# SUBALGEBRAS OF LIE ALGEBRAS

ABSTRACT. First is presented a proof of Lie's theorem on solvable Lie algebras based on the non-existence of the Heisenberg commutation relation. This is used to construct an effective procedure for finding all quotients of a given Lie algebra $\mathfrak{g}$ which are isomorphic to the non-abelian two-dimensional algebra. As a byproduct one gets that the ideal $\Delta(\mathfrak{g})$ recently introduced by K. H. Hofmann is characteristic if the characteristic of the ground field is zero.

## 1. INTRODUCTION

In [5], Tits determined the one-codimensional subalgebras of finite dimensional Lie algebras $\mathfrak{g}$ over fields $\Phi$ of characteristic 0. All of them contain an ideal $\mathfrak{c}$ such that $\mathfrak{g}/\mathfrak{c}$ is isomorphic to $\Phi$, to $\mathfrak{sl}_2(\Phi)$ or to the two-dimensional non-abelian Lie algebra $\mathfrak{s}_2$. Clearly, the one-codimensional subalgebras of $\Phi$, $\mathfrak{s}_2$ and $\mathfrak{sl}_2$ are very well understood. In [3], K. H. Hofmann studied systematically how one can actually compute these subalgebras in a given Lie algebra. In this context, he introduced and explored the ideal $\Delta(\mathfrak{g})$ which is the intersection of all one-codimensional subalgebras. Except for $\Phi = \mathbb{R}, \mathbb{C}$, one question was left, namely, whether $\Delta(\mathfrak{g})$ is a characteristic ideal. By Tits' result, $\Delta(\mathfrak{g})$ is the intersection of $[\mathfrak{g}, \mathfrak{g}]$, $\Delta_{\mathrm{aff}}(\mathfrak{g})$ and $\Delta_s(\mathfrak{g})$ where $\Delta_{\mathrm{aff}}$, respectively, $\Delta_s$ denotes the intersection of all ideals $\mathfrak{c}$ such that $\mathfrak{g}/\mathfrak{c}$ is isomorphic to $\mathfrak{s}_2$, respectively, to $\mathfrak{sl}_2$. Clearly, $[\mathfrak{g}, \mathfrak{g}]$ is characteristic. Also $\Delta_s(\mathfrak{g})$ is characteristic because $\Delta_s(\mathfrak{g})$ contains the solvradical $\mathfrak{r}$ (which is characteristic) and each derivation of $\mathfrak{g}/\mathfrak{r}$ is inner. The fact that $\Delta_{\mathrm{aff}}(\mathfrak{g})$ is characteristic follows from general theorems in algebraic group theory. (Obviously, $\Delta_{\mathrm{aff}}(\mathfrak{g})$ is invariant under the linear algebraic group $\mathrm{Aut}(\mathfrak{g})$, the automorphism group of $\mathfrak{g}$. The Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ consists of all derivations of $\mathfrak{g}$; each subspace which is invariant under a linear algebraic group is invariant under its Lie algebra. For the latter two facts see, for instance, [1, Exer. 2, p. 93 and Th. 9.1, p. 60].) But we will present a very elementary proof which also gives a slightly different method to compute all non-abelian two-dimensional quotients and the corresponding one-codimensional subalgebras. Our approach will be to view hyperplanes through the origin as points of the linear dual $\mathfrak{g}^*$ or, more precisely, its associated projective space.

Thinking about this problem led me to consider Lie's theorem on representations of solvable Lie algebras because the one-codimensional subalgebras corresponding to $\mathfrak{s}_2$-quotients are given by certain eigenvectors.

I asked myself why this theorem is wrong for fields of positive characteristic. I found that the (non-)existence of the Heisenberg commutation relation $[X, Y] = \mathrm{id}$ in finite dimensional spaces is responsible for the failure (truth) of Lie's theorem. Clearly, if $[X, Y] = \mathrm{id}$ the operators $X$ and $Y$ cannot have a common eigenvector. For each prime $p$ one can produce matrices over fields of characteristic $p$ such that $[X, Y] = \mathrm{id}$, for instance

$$X = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & & \\ & & & p-1 & \\ & & & & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & & & \\ & & & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

Using these matrices there will be constructed a Lie algebra $\mathfrak{g}$ such that $\Delta_{\mathrm{aff}}(\mathfrak{g})$ and $\Delta(\mathfrak{g})$ are not characteristic.

On the other hand, I will give below a proof of Lie's theorem only using the non-existence of the Heisenberg commutation relation. This proof differs from the ones I found in the standard textbooks. Altogether, my proof might not be shorter but I find it conceptually clearer.

By the way, the Heisenberg commutation relation cannot be realized in any Banach algebra; an elegant elementary proof is due to H. Wielandt.

Let me finish this introduction with the remark that the content of this article is elementary. The material could be presented in a course on Lie algebras after introducing the first basic concepts. It has the advantage that these concepts are employed and that it uses very concrete computations in Lie algebras. The same applies to Tits' theorem mentioned above.

## 2. ANOTHER PROOF OF LIE'S THEOREM

Let us start with two simple consequences (1) and (2) of the non-existence of the Heisenberg commutation relation.

(1) Let $V$ be a finite dimensional vector space over a field $\Phi$ of characteristic 0, and let $A$ and $B$ be endomorphisms of $V$ such that $[A, B] = B$.

(a) $B$ is nilpotent.

(b) If $V$ is simple as a $\mathscr{L}_\Phi(A, B)$-module, then $B = 0$.

*Proof.* By taking simple subquotients, (a) follows from (b). Concerning (b), let $W := \{v \in V; Bv = 0\}$. The relation $[A, B] = B$ implies that $W$ is also $A$-invariant, hence $W = 0$ or $W = V$. If $W = V$ we are done. In the other case $B$

is invertible. Putting $X = AB^{-1}$ one computes $[X, B] = \mathrm{id}$ which is impossible (compute the trace). □

More general than (1)(a) is

(2) Let $V$ be as above and let $\mathfrak{q} \subset \mathrm{End}(V)$ be a linear space of commuting endomorphisms. If $D \in \mathrm{End}(V)$ satisfies $[D, \mathfrak{q}] \subset \mathfrak{q}$, then $[D, \mathfrak{q}]$ consists of nilpotent endomorphisms.

*Proof.* Extending the field of scalars we may assume that $\Phi$ is algebraically closed. Again, taking simple subquotients of the $\Phi D + \mathfrak{q}$-module $V$ we may assume that $V$ itself is simple. Define $\mathrm{ad}(D): \mathfrak{q} \to \mathfrak{q}$ as $\mathrm{ad}(D)(Q) = [D, Q]$. If $\mathrm{ad}(D) = 0$ nothing is to show (actually, under our assumptions this will turn out to be the case). It remains to consider the following two cases.

CASE N: $\mathrm{ad}(D)$ is a non-zero nilpotent operator.

Then there exists $X, Y \in \mathfrak{q}$ such that $[D, Y] = 0$ and $[D, X] = Y \neq 0$. As $Y$ commutes with everything it has to be invertible. But $[D, X] = Y$ implies the Heisenberg relation $[D, X Y^{-1}] = \mathrm{id}$!

CASE S: There exists $\gamma \in \Phi$, $\gamma \neq 0$, and a non-zero $X \in \mathfrak{q}$ such that $[D, X] = \gamma X$.

By (1)(a) $X$ is nilpotent, hence the kernel $W$ of $X$ is non-zero. From $[D, X] = \gamma X$ and $[\mathfrak{q}, X] = 0$ it follows that $W$ is $\Phi D + \mathfrak{q}$-invariant, hence $W = V$, contradicting $X \neq 0$. □

LEMMA 1. *Let $\mathfrak{g}$ be a Lie algebra over $\Phi$, char $\Phi = 0$, let $D$ be a derivation on $\mathfrak{g}$, and let $V$ be a finite dimensional module over the semidirect sum $\Phi D \ltimes \mathfrak{g} =: \tilde{\mathfrak{g}}$. Then the eigenspace $V_\alpha := \{v \in V; Xv = \alpha(X)v, \forall X \in \mathfrak{g}\}$ for $\alpha \in \mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \Phi)$ is $D$-invariant.*

*Proof.* The eigenspace $V_\alpha$ is contained in the space $W = V^{[\mathfrak{g},\mathfrak{g}]}$ of $[\mathfrak{g}, \mathfrak{g}]$-fixed points which is $\tilde{\mathfrak{g}}$-invariants as $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\tilde{\mathfrak{g}}$. Actually, $W$ is a module over $\Phi D \ltimes \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. By (2), $D(\mathfrak{g}) \subset \mathfrak{g}$ acts nilpotently on $W$, hence $\alpha(D(\mathfrak{g})) = 0$ if $V_\alpha \neq 0$. The latter fact implies immediately that $V_\alpha$ is $D$-invariant. □

From Lemma 1 one can deduce in the usual manner the theorem of Lie and its corollary.

THEOREM (Lie). *Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field $\Phi$ of characteristic 0. If $V$ is a finite dimensional $\mathfrak{g}$-module then there exists $\alpha \in \mathfrak{g}^*$ such that $V_\alpha := \{v \in V; Xv = \alpha(X)v, \forall X \in \mathfrak{g}\}$ is different from zero.*

*Proof.* Take an ideal $\mathfrak{h}$ of codimension 1 in $\mathfrak{g}$ and $D \in \mathfrak{g} \backslash \mathfrak{h}$, hence $\mathfrak{g} = \Phi D \ltimes \mathfrak{h}$. By induction there exists $\alpha' \in \mathfrak{h}^*$ such that $W = \{v \in V; Xv = \alpha'(X)v, \forall X \in \mathfrak{h}\} \neq 0$. Then choose any $D$-eigenvector in $W$. □

COROLLARY. *Let $\mathfrak{g}$ be a solvable Lie algebra over a field $\Phi$ of characteristic 0 and let $V$ be a finite dimensional $\mathfrak{g}$-module.*

(a) *$[\mathfrak{g}, \mathfrak{g}]$ acts on $V$ by nilpotent transformations.*

(b) *If, in addition, $V$ is a simple $\mathfrak{g}$-module then $[\mathfrak{g}, \mathfrak{g}]$ annihilates $V$.*

*Proof.* (a) Extend the scalars to an algebraic closure and apply the theorem to the simple subquotients.

(b) Suppose that $\mathfrak{g}$ acts faithfully on $V$ and assume (contrary to the assertion) that $[\mathfrak{g}, \mathfrak{g}]$ contains a non-zero abelian ideal $\mathfrak{a}$. By (a) the fixed point space $V^{\mathfrak{a}}$ is non-zero. Note that for abelian $\mathfrak{a}$ this is an elementary fact, so we are *not* using Engel's theorem. As $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ the space $V^{\mathfrak{a}}$ is $\mathfrak{g}$-invariant, hence $V^{\mathfrak{a}} = V$, a contradiction to the assumption that $\mathfrak{g}$ acts faithfully. $\quad\square$

## 3. A DESCRIPTION OF $\Delta_{\mathrm{aff}}(\mathfrak{g})$

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an arbitrary field $\Phi$. In this section we shall describe all pairs $(\mathfrak{c}, \mathfrak{h})$ such that

(i) $\mathfrak{c}$ is an ideal in $\mathfrak{g}$ with $\mathfrak{g}/\mathfrak{c} \cong \mathfrak{s}_2$, the two-dimensional non-abelian Lie algebra,

(ii) $\mathfrak{h} \supset \mathfrak{c}$, $\mathfrak{h}/\mathfrak{c}$ is one-dimensional,

(iii) $\mathfrak{h}$ is not an ideal, i.e. $\mathfrak{h} + [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

As an application we get a description of $\Delta_{\mathrm{aff}}(\mathfrak{g}) = \bigcap \mathfrak{c}$, $\mathfrak{c}$ as above. This ideal turns out to be characteristic if the characteristic of $\Phi$ is zero, while an example shows that the corresponding assertion is wrong in finite characteristic. We start with an easy (and well-known) lemma.

LEMMA 2. *If $\mathfrak{q}$ is a finite dimensional Lie algebra with one-dimensional non-central commutator algebra $[\mathfrak{q}, \mathfrak{q}]$ then there exists $A, B \in \mathfrak{q}$ such that $[A, B] = B$, $\mathfrak{q} = \mathcal{L}_{\Phi}(A, B) \oplus \mathfrak{z}\mathfrak{q}$ and $\Phi B \oplus \mathfrak{z}\mathfrak{q}$ is the centralizer of $[\mathfrak{q}, \mathfrak{q}] = \Phi B$. Moreover, $\mathfrak{z}\mathfrak{q}$ is the unique ideal $\mathfrak{c}$ such that $\mathfrak{q}/\mathfrak{c}$ is isomorphic to $\mathfrak{s}_2$.*

*Proof.* Choose any non-zero $B$ in $[\mathfrak{q}, \mathfrak{q}]$ and let $\mathfrak{k}$ be the centralizer of $B$. By assumption, $\mathfrak{k}$ is of codimension 1 and there exists $A \in \mathfrak{q}$ such that $[A, B] = B$. Let $\mathfrak{h}$ be the centralizer of $A$, by assumption $\mathfrak{q} = \mathfrak{h} \oplus \Phi B$. For $X, Y \in \mathfrak{h}$ let $B' = [X, Y] \in [\mathfrak{q}, \mathfrak{q}] = \Phi B$. Then $B' = [A, B'] = [[A, X], Y] + [X, [A, Y]] = 0$, hence $\mathfrak{h}$ is abelian. From $\mathfrak{h} = \Phi A \oplus (\mathfrak{h} \cap \mathfrak{k})$ one gets the decomposition $\mathfrak{q} = \Phi A \oplus \Phi B \oplus (\mathfrak{h} \cap \mathfrak{k})$. Since $\mathfrak{h} \cap \mathfrak{k}$ is abelian, from this decomposition one can read off all the brackets and one can conclude the first assertions. As the center of $\mathfrak{s}_2$ is trivial any homomorphism from $\mathfrak{q}$ onto $\mathfrak{s}_2$ has to annihilate $\mathfrak{z}\mathfrak{q}$. $\quad\square$

The Lie algebra $\mathfrak{g}$ acts on the dual $\mathfrak{g}^*$ and on the dual $[\mathfrak{g}, \mathfrak{g}]^*$ by

$$(Xf)(Y) = f([Y, X]) \quad \text{and} \quad (Xg)(Z) = g([Z, X])$$

for $X, Y \in \mathfrak{g}$, $Z \in [\mathfrak{g}, \mathfrak{g}]$, $f \in \mathfrak{g}^*$ and $g \in [\mathfrak{g}, \mathfrak{g}]^*$. For $g \in [\mathfrak{g}, \mathfrak{g}]^*$ we define $\mathfrak{g}^g = \{X \in \mathfrak{g}; g([\mathfrak{g}, X]) = 0\}$; $\mathfrak{g}^g$ is nothing but the stabilizer algebra of any of the linear extensions of $g$ to $\mathfrak{g}$. In the following statements (A)–(D) we assume that $\alpha \in \mathfrak{g}^*$ is non-zero and $g \in [\mathfrak{g}, \mathfrak{g}]^*$ is a (non-zero) $\alpha$-eigenvector, i.e. $Xg = \alpha(X)g$ for all $X \in \mathfrak{g}$. In particular $\alpha = 0$ on $[\mathfrak{g}, \mathfrak{g}]$.

(A) $\mathfrak{g}^g$ is an ideal in $\mathfrak{g}$, $\mathfrak{g}/\mathfrak{g}^g$ is isomorphic to $\mathfrak{s}_2$.

(B) If $\mathfrak{k}$ denotes the kernel of $\alpha$ then $\mathfrak{k} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{g}^g$.

(C) There is a unique $h \in \mathfrak{k}^*$ such that $h|_{[\mathfrak{g},\mathfrak{g}]} = g$ and $h$ is an $\alpha$-eigenvector (note that $\mathfrak{g}$ acts on $\mathfrak{k}^*$ as well because $\mathfrak{k}$ is an ideal in $\mathfrak{g}$). Indeed, $h$ is given by $h = 0$ on $\mathfrak{g}^g$; $\mathfrak{g}^g$ is precisely the kernel of $h$.

(D) If $f \in \mathfrak{g}^*$ is any extension of $h$ then the pair $(\mathfrak{g}^g, \ker f)$ satisfies (i)–(iii).

*Proof.* Clearly, $\ker g$ is an ideal in $\mathfrak{g}$. Let $\mathfrak{q} = \mathfrak{g}/\ker g$, denote by $\varphi: \mathfrak{g} \to \mathfrak{q}$ the quotient map and by $q \in [\mathfrak{q}, \mathfrak{q}]^*$ the functional induced by $g$. But $\mathfrak{q}$ satisfies the assumptions of Lemma 2. Everything follows from the known structure of $\mathfrak{q}$. To be more specific, $\mathfrak{q}^q$ is just $\mathfrak{z}\mathfrak{q}$, hence $\mathfrak{g}^g = \varphi^{-1}(\mathfrak{q}^q)$ is an ideal with $\mathfrak{g}/\mathfrak{g}^g \cong \mathfrak{s}_2$, i.e. (A). The space $\mathfrak{k}$ is the preimage of the centralizer of $[\mathfrak{q}, \mathfrak{q}]$ which is $[\mathfrak{q}, \mathfrak{q}] + \mathfrak{z}\mathfrak{q}$, hence $\mathfrak{k}$ equals $[\mathfrak{g}, \mathfrak{g}] + \varphi^{-1}(\mathfrak{z}\mathfrak{q}) = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{g}^g$. (C) follows from the fact that $q$ allows a unique linear extension to $[\mathfrak{q}, \mathfrak{q}] + \mathfrak{z}\mathfrak{q}$ as an $\alpha$-eigenvector, the extension annihilates $\mathfrak{z}\mathfrak{q}$. (D) is now obvious. □

On the other hand, if a pair $(\mathfrak{c}, \mathfrak{h})$ satisfying (i)–(iii) is given, choose any $f \in \mathfrak{g}^*$ with $\ker f = \mathfrak{h}$. Then $g = f|_{[\mathfrak{g},\mathfrak{g}]} \neq 0$ is an eigenvector for a certain non-zero $\alpha \in \mathfrak{g}^*$ and $f|_{\ker \alpha}$ is an $\alpha$-eigenvector, too. Thus we find

THEOREM. *Starting from all possible common $\mathfrak{g}$-eigenvectors $g$ in $[\mathfrak{g}, \mathfrak{g}]^*$ corresponding to non-zero eigenfunctionals the procedure described in (A) through (D) leads to all possible pairs $(\mathfrak{c}, \mathfrak{h})$ satisfying (i)–(iii).* □

Altogether, one has an algorithm to compute all possible pairs $(\mathfrak{c}, \mathfrak{h})$: Compute all common $\mathfrak{g}$-eigenvectors $g$ in $[\mathfrak{g}, \mathfrak{g}]^*$. The $\mathfrak{c}$'s are obtained as $\mathfrak{g}^g$, the solution set of a certain system of homogeneous linear equations. To get $\mathfrak{h}$ form $\mathfrak{k} = \mathfrak{g}^g + [\mathfrak{g}, \mathfrak{g}]$, define $h \in \mathfrak{k}^*$ by $h = 0$ on $\mathfrak{g}^g$ and $h|_{[\mathfrak{g},\mathfrak{g}]} = g$ and extend $h$ to $f \in \mathfrak{g}^*$. Then $\ker f$ gives all possible $\mathfrak{h}$'s. Alternatively, one may take any non-zero $\alpha \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$, determine $\ker \alpha$ and then compute all $\alpha$-eigenvectors $h \in (\ker \alpha)^*$. If again $f$ denotes a linear extension of $h$ then $(\ker, h, \ker f)$ gives all possible pairs $(\mathfrak{c}, \mathfrak{h})$.

The following corollary is an easy consequence of the theorem and of Lemma 1.

COROLLARY. $\Delta_{aff}(\mathfrak{g})$ *is a characteristic ideal for finite dimensional Lie algebras* $\mathfrak{g}$ *over fields of characteristic* $0$. *Hence also* $\Delta(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \cap \Delta_{aff}(\mathfrak{g}) \cap \Delta_s(\mathfrak{g})$ *is characteristic.*

*Proof.* For $\alpha \in \mathfrak{g}^*$, $\alpha \neq 0$, let $V_\alpha = \{g \in [\mathfrak{g}, \mathfrak{g}]^*; \; Xg = \alpha(X)g, \forall X \in \mathfrak{g}\}$. The theorem implies that $\Delta_{aff}(\mathfrak{g})$ is the intersection of all $\mathfrak{g}^g$, $g \in M :=$ $\bigcup \{V_\alpha; \alpha \in \mathfrak{g}^* \setminus \{0\}\}$; clearly, we may include $g = 0$ because this yields $\mathfrak{g}^g = \mathfrak{g}$. We have to show that for any derivation $D$ of $\mathfrak{g}$, any $g \in M$ and any $X \in \bigcap_{k \in M} \mathfrak{g}^k$ the value $DX$ is contained in $\mathfrak{g}^g$. By Lemma 1, applied to the $\Phi D \ltimes \mathfrak{g}$-module $[\mathfrak{g}, \mathfrak{g}]^*$ the functional $Dg$, defined by $Dg(Y) = g(-DY)$, is contained in $M$. To check that $DX \in \mathfrak{g}^g$ take any $Z \in \mathfrak{g}$ and compute

$$g([DX, Z]) = g(D[X, Z]) - g([X, DZ]) \in Dg([X, \mathfrak{g}]) + g([X, \mathfrak{g}]).$$

The latter set is zero as $g, Dg \in M$ and $X \in \bigcap_{k \in M} \mathfrak{g}^k$.                       □

3.1. *An Example*

Before studying an example for finite characteristic as mentioned in the introduction we remark that a weaker form of Tits' theorem is true for any field.

REMARK. For any Lie subalgebra $\mathfrak{h}$ of codimension 1 in a finite dimensional Lie algebra $\mathfrak{g}$ one of the following conditions holds:

either    (a) $\mathfrak{h}$ contains the solvradical $\mathfrak{r}$ of $\mathfrak{g}$,

or        (b) $\mathfrak{h}$ contains $[\mathfrak{g}, \mathfrak{g}]$,

or        (c) $\mathfrak{h}$ contains an ideal $\mathfrak{c}$ such that $\mathfrak{g}/\mathfrak{c} \cong \mathfrak{s}_2$ and $\mathfrak{h} + [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

*Proof.* Let $\mathfrak{g}$ be a given Lie algebra with solvradical $\mathfrak{r}$ and assume inductively that the remark is true for all algebras of lower dimension. If $\mathfrak{r} = 0$ nothing is to show, case (a) happens. If $\mathfrak{r} \neq 0$ there exists a non-zero abelian ideal $\mathfrak{a}$ in $\mathfrak{g}$. Then either $\mathfrak{a} + \mathfrak{h} = \mathfrak{h}$ or $\mathfrak{a} + \mathfrak{h} = \mathfrak{g}$. In the first case we apply the induction hypothesis to the pair $(\mathfrak{h}/\mathfrak{a}, \mathfrak{g}/\mathfrak{a})$. So, assume that $\mathfrak{a} + \mathfrak{h} = \mathfrak{g}$. Then $\mathfrak{a} \cap \mathfrak{h}$ is an ideal in $\mathfrak{g}$. Again, by induction we may assume that $\mathfrak{a} \cap \mathfrak{h} = 0$, i.e. $\mathfrak{a}$ is one-dimensional, $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$. There are two cases, $[\mathfrak{h}, \mathfrak{a}] = 0$ or $[\mathfrak{h}, \mathfrak{a}] = \mathfrak{a}$. If $[\mathfrak{h}, \mathfrak{a}] = 0$ then $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{h}, \mathfrak{h}]$ is contained in $\mathfrak{h}$, case (b) happens. If $[\mathfrak{h}, \mathfrak{a}] = \mathfrak{a}$ let $\mathfrak{k}$ be the centralizer of $\mathfrak{a}$. Then $\mathfrak{k}$ is an ideal of codimension 1 which is different from $\mathfrak{h}$. Choosing $\mathfrak{c} = \mathfrak{h} \cap \mathfrak{k}$ one easily discovers case (c).                       □

To see that (a) is possible for simple Lie algebras $\mathfrak{g}$, not isomorphic to $\mathfrak{s}l_2$, we

write down an example in characteristic 5. Similar examples exist for higher characteristic, the so-called $p$-dimensional Witt algebras; the choice of the number 5 has the advantage that the matrices can be written 'without dots'.

Let $\mathfrak{s}$ consist of all matrices

$$\begin{pmatrix} -x_1 & y & 0 & 0 & 0 \\ 2x_2 & 0 & 2y & 0 & 0 \\ 3x_3 & x_2 & x_1 & 3y & 0 \\ 4x_4 & 2x_3 & 0 & 2x_1 & 4y \\ 0 & 3x_4 & x_3 & -x_2 & 3x_1 \end{pmatrix}$$

in $\mathfrak{gl}_5(\Phi)$ where $y, x_1, x_2, x_3, x_4 \in \Phi$ and $\Phi$ is any field of characteristic 5. By direct computation one checks that $\mathfrak{s}$ is a subalgebra of $\mathfrak{gl}_5(\Phi)$. It is not hard to see that $\mathfrak{s}$ is simple using, for instance, the fact that the spectrum of

$$\mathrm{ad}(X) : \mathfrak{s} \to \mathfrak{s}, \quad X = \begin{pmatrix} -1 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 2 & \\ & & & & 3 \end{pmatrix},$$

consists of $\{0, 1, 2, 3, 4\}$ which implies that each potential ideal is spanned by some of the (easily computed) eigenvectors of $\mathrm{ad}(X)$. Finally, the subset $\mathfrak{h}$ of $\mathfrak{s}$ where $y = 0$ forms a subalgebra of codimension 1.

Now let $\Phi$ be any field of finite characteristic. The matrices written down in the introduction show that there exists a finite dimensional vector space $V$ over $\Phi$ and endomorphisms $X$ and $Y$ on $V$ such that

  (I) $[X, Y] = \mathrm{id}$,
  (II) $W := Y(V)$ is of codimension 1 in $V$,
  (III) $Y + \beta\, \mathrm{id}$ is invertible for any non-zero $\beta \in \Phi$.

Using $Y$ we shall construct a metabelian Lie algebra $\mathfrak{g}$ over $\Phi$ for which it is easy to compute $\Delta_{\mathrm{aff}}(\mathfrak{g})$ and $\Delta(\mathfrak{g})$. Using $X$ we shall construct a derivation $D$ on $\mathfrak{g}$ such that neither $\Delta_{\mathrm{aff}}(\mathfrak{g})$ nor $\Delta(\mathfrak{g})$ is $D$-invariant.

As a vector space, $\mathfrak{g}$ is $\Phi^2 \oplus V$ where the commutator is given by

$$[(b, c, v), (b', c', v')] = (0, 0, b\,Yv' + cv' - b'\,Yv - c'v)$$

for $b, b', c, c' \in \Phi$ and $v, v' \in V$.

It is easy to check that $\mathfrak{g}$ is a Lie algebra and that $[\mathfrak{g}, \mathfrak{g}]$ equals $V$.

Likewise one verifies without difficulty using (I) that $D: \mathfrak{g} \to \mathfrak{g}$ defined by

$$D(b, c, v) = (0, b, Xv)$$

is a derivation on $\mathfrak{g}$.

To compute $\Delta_{\mathrm{aff}}(\mathfrak{g})$ we use the algorithm described above. Suppose that $\alpha \in \mathfrak{g}^*$ vanishes on $V = [\mathfrak{g}, \mathfrak{g}]$, i.e. $\alpha(b, c, v) = b\beta + c\gamma$ for some $\beta, \gamma \in \Phi$ and suppose that $g \in V^*$ is an (non-zero) $\alpha$-eigenvector, i.e.

$$g(-[(b, c, v), (0, 0, v')]) = \alpha(b, c, v)g(v')$$

for all $b, c \in \Phi$ and $v, v' \in V$. Evaluating this equation gives

$$g(Yv') = -\beta g(v') \quad \text{and} \quad g(v') = -\gamma g(v')$$

for all $v' \in V$, hence $\gamma = -1$. Using (II) one obtains $\beta = 0$; $g$ vanishes on the one-codimensional space $W = Y(V)$. Therefore,

$$\Delta_{\mathrm{aff}}(\mathfrak{g}) = \mathfrak{g}^g = \{(b, 0, w); \ b \in \Phi, \ w \in W\}.$$

As $\mathfrak{g}$ is solvable the above remark implies that $\Delta(\mathfrak{g})$, the intersection of all subalgebras of codimension 1, equals $[\mathfrak{g}, \mathfrak{g}] \cap \Delta_{\mathrm{aff}}(\mathfrak{g})$, hence

$$\Delta(\mathfrak{g}) = W.$$

Neither $\Delta(\mathfrak{g})$ nor $\Delta_{\mathrm{aff}}(\mathfrak{g})$ is $D$-invariant because $D(W) = X(W)$ is not contained in $W$: $[X, Y] = \mathrm{id}$ implies that $XYv \equiv v \bmod W$ for all $v \in V$; $XY(V) = X(W) \subset W$ would give $v \equiv 0 \bmod W$ for all $v$.

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