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Cocycles on Abelian Groups and Primitive Ideals in Group C*–Algebras of Two Step Nilpotent Groups and Connected Lie Groups

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Introduction

In the article [27] Moore and Rosenberg proved that in the primitive ideal space Priv (G) of the group C^*–algebra $C^*(G)$ of a connected Lie group G each one–point–set is locally closed. This means that for each $\mathcal{I} \in$ Priv (G) the quotient $C^*(G)/\mathcal{I}$ contains a unique simple ideal, say $M(\mathcal{I})$. The structure of $M(\mathcal{I})$ was determined in [31]. It turned out that except for the case that $\mathcal{I}$ is of finite codimension where, of course, $C^*(G)/\mathcal{I} = M(\mathcal{I})$ is a matrix algebra, the algebra $M(\mathcal{I})$ is isomorphic to the C^*–tensor product of the algebra of compact operators on a separable Hilbert space and a noncommutative torus in a certain dimension n. There the problem was reduced to the study of primitive (= simple in that case) quotients of group C^*–algebras of compactly generated two step nilpotent groups, which actually have a similar structure. For K–theoretic reasons, see [14], the number n is an invariant of the primitive quotient in question. But in both cases, for connected Lie groups and for two step nilpotent groups, it was not clear at all in [31], how to relate directly the number n to the given primitive ideal $\mathcal{I}$. The present article is devoted to the study of this question. Most of the ideas presented here were already laid down in the first author's doctoral dissertation, [25].

The article is divided into three parts. In the first one we study (non–degenerate) skew–symmetric bicharacters on locally compact abelian groups with values in the one–dimensional torus $\mathbb{T}$. As is well known, see [22], each measurable cocycle β yields by antisymmetrization, $(x,y) \mapsto \beta(x,y)\beta(y,x)^{-1}$, a skew–symmetric bicharacter. This bicharacter determines the cohomology class of β. Dividing out the kernel $\{x|\beta(x,y) = \beta(y,x) \text{ for all } y\}$ of the bicharacter, one obtains what in the sequel is called a quasi–symplectic space, see (1.1) below. Similarily, each unitary character on the center of a two step nilpotent group leads in a canonical fashion to a quasi–symplectic space. In the case of, say,

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two step nilpotent connected Lie groups, in order to obtain representations out of symplectic spaces one has to polarize these spaces. Hence we try to polarize quasi–symplectic spaces. It turns out that in a strict sense this is impossible in general: a discrete obstruction–group appears. In the compactly generated case this obstruction is closely related to the number n above, which shows that the obstruction is an immanent quantity. For non compactly generated groups the situation is a little more subtle: one is led to a certain equivalence class of discrete abelian groups.

In the second part we determine the structure of the primitive quotients of $C^*(\mathcal{G})$ for an *arbitrary* locally compact two step nilpotent group $\mathcal{G}$, thus generalizing the results of [31]. However, the most interesting point of the new approach is that in the compactly generated case the results of the first part can be used to compute the above number n directly in terms of the structure of the group $\mathcal{G}$. For later purposes in forthcoming articles we include the following results: If π is an irreducible continuous representation of $\mathcal{G}$ such that $C^*(\mathcal{G})/\ker\pi$ is isomorphic to the algebra of compact operators then there exists a continuous function f on $\mathcal{G}$, integrable against each weight function w on $\mathcal{G}$, such that $\pi(f)$ is an orthogonal projection of rank one. Moreover, we determine the set of primitive ideals in Beurling algebras $L^1_w(\mathcal{G})$ on $\mathcal{G}$, and we show the existence of rank one operators for algebraically irreducible representations of such algebras — in case that there is any hope for their existence, i.e., in the "type I case".

In the final part we consider connected Lie groups. The above number n is computed in terms of the first homotopy groups of certain subsets of primitive ideal spaces associated with a given primitive ideal, see (3.9). In the solvable case, where the primitive ideal space can be parametrized according to [37], we give an explicit expression for n in terms of the parameters, see (3.13). As a particular case one obtains the well known Auslander–Kostant criterion for a connected solvable Lie group to be of type I.

§ 1 Quasi–symplectic Spaces

In this section we study skew–symmetric continuous bicharacters γ on locally compact abelian groups G with values in $\mathbb{T}$. Skew–symmetry means here that $\gamma(x,x)=1$ for all $x\in G$, which implies that $\gamma(x,y)=\gamma(y,x)^{-1}$ for all $x,y\in G$ (but not the other way around). Mainly we are interested in the structure of non–degenerate γ's where non–degeneracy means that $\gamma(x,G)=1$ implies $x=0$. In particular we look for "polarizations" of (G,γ) in an appropriate sense, see (1.1) below. – In our investigations we learnt a lot from the study of the article [2]. Indeed, quite a few of our arguments were already used in that paper.

For any subset W of G we denote by $W^\perp$ the closed subgroup $W^\perp=\{x\in G|\gamma(x,y)=1,\forall y\in W\}$. The reader should observe that even for a non–degenerate γ and a closed subgroup W it may happen that W is a proper subgroup of $\left(W^\perp\right)^\perp$. If W is a closed subgroup of G the bicharacter γ induces a continuous homomorphism $\psi_W : G \to \widehat{W}$ given by $\psi_W(x)(u) = \gamma(x,u)$. The kernel of ψ_W is $W^\perp$, hence ψ_W induces a homomorphism from $G/W^\perp$ into $\widehat{W}$, occasionally denoted by ψ_W, too. Furthermore γ induces a continuous homomorphism $\varphi_W : W \to \widehat{G}$ given by $\varphi_W(u)(x) = \gamma(x,u)$. This homomorphism takes its values in $\left(G/W^\perp\right)^\wedge$. The homomorphisms $\psi_W : G/W^\perp \to \widehat{W}$ and $\varphi_W : W \to \left(G/W^\perp\right)^\wedge$ are dual to each other. For non–degenerate γ they are injective with dense image. — The best known example of such an γ is the case where G is a vector group and γ is of the form $\gamma(x,y) = e^{iB(x,y)}$ with a real symplectic form B on G. Each non–degenerate γ on a vector group is of that type. These spaces will be called ordinary symplectic spaces. The more general pairs (G,γ) are called quasi–symplectic spaces.

Definition 1.1. A *quasi–symplectic space* (G,γ) is a locally compact abelian group G endowed with a non–degenerate skew–symmetric continuous bicharacter γ on $G \times G$ with values in $\mathbb{T}$. If (G,γ) is a quasi–symplectic space, a closed subgroup P of G is called a *prepolarization* if $P \subset P^\perp$ and $\varphi_P : P \to \left(G/P^\perp\right)^\wedge$ is an isomorphism of topological groups (or, by duality, $\psi_P : G/P^\perp \to \widehat{P}$ is an isomorphism). A closed subgroup P of G is called a *quasi–polarization* if it is a prepolarization and if $P^\perp/P$ is discrete. ■

Example 1.2. An almost ideal quasi–symplectic space is obtained as follows. Let A be any locally compact abelian group with Pontryagin dual $\widehat{A}$. Let $H = \widehat{A} \times A$ and define γ on $H \times H$ by $\gamma\left((\chi,a),(\chi',a')\right) = \chi(a')\chi'(a)^{-1}$. In this case $P = A$ and $P = \widehat{A}$ are quasi–polarizations with $P^\perp = P$. Slightly more intrinsically, a quasi–symplectic space (G,γ) is of this particular sort, if G can be decomposed as $G = A \times B$ such that γ is trivial on A and on B and that γ yields an isomorphism from A, resp. B, onto the Pontryagin dual of B, resp. A. Clearly, each ordinary symplectic space is isomorphic to a space of that type. ■

For later use we recall some consequences of the well–known structure of compactly generated locally compact abelian groups, see [34], and introduce two notations.

1.3. *For a locally compact abelian group H the following properties are equivalent:*

(i) *The connected component H_0 is open in H, and H_0 is a vector group.*

(ii) *Each compact subgroup of H is finite.*

(iii) *H is isomorphic to a direct product of a vector group and a discrete group.*

Such locally compact abelian groups will be called *essentially compact–free.* ■

1.4 *Every locally compact abelian group H contains a compact subgroup K such that H/K is essentially compact–free.*

Such subgroups K are called *large compact* subgroups. If K and L are large compact subgroups then $K \cap L$ is of finite index in $K + L$. ■

The next easy lemma provides a sufficient criterion for a quasi–symplectic space to split orthogonally; it will be used several times in the sequel.

Lemma 1.5. *Let γ be a skew–symmetric continuous bicharacter on the locally compact abelian group H. Suppose that W is a closed subgroup of H such that the restriction of ψ_W induces an isomorphism from W onto $\widehat{W}$. Then the map $(u,x) \mapsto u + x$ from $W \times W^\perp$ into H is an isomorphism of topological groups.*

Proof. The injectivity of ψ_W on W implies that $W \cap W^\perp = 0$. For each $y \in H$ there exists a unique $w \in W$ with $\psi_W(y) = \psi_W(w)$; moreover, by assumption w depends continuously on y. Then $y = w + (y - w)$ and $y - w$ is in $\ker \psi_W = W^\perp$. ∎

In the first theorem we investigate how close we can come in the compactly generated case to the almost ideal situation described in (1.2).

Theorem 1.6. *Let (G,γ) be a quasi–symplectic space with compactly generated G. Then exists a decomposition $G = G_{\mathrm{I}} \oplus G_{\mathrm{II}}$ with the following properties:*

(a) $\gamma(G_{\mathrm{I}}, G_{\mathrm{II}}) = 1$, *i.e., the decomposition is orthogonal w.r.t. γ.*

(b) *The connected component $(G_{\mathrm{II}})_0$ is a vector group, G_{II} is a direct product of $(G_{\mathrm{II}})_0$ and a finitely generated discrete group, and γ is trivial on $(G_{\mathrm{II}})_0$.*

(c) *The group G_{I} allows an γ-orthogonal decomposition $G_{\mathrm{I}} = G_{\mathrm{I}}^t \oplus G_{\mathrm{I}}^v$ such that*

(i) *G_{I}^t is isomorphic to the direct product of a torus T and its dual group $\widehat{T}$, and the bicharacter γ is given, under this identification, by*

$$\gamma\left((t_1,\chi_1),(t_2,\chi_2)\right) = \chi_1(t_2)\,\chi_2(t_1)^- .$$

(ii) *G_{I}^v is isomorphic to the direct product of a vector group V and its dual group $\widehat{V}$, and γ is given by $\gamma\left((v_1,\chi_1),(v_2,\chi_2)\right) = \chi_1(v_2)\,\chi_2(v_1)^-$.*

Remark 1.7. Assertion (c) (ii) simply tells that G_{I}^v is an ordinary symplectic space. It is well known that such a space can be decomposed into an orthogonal sum of two–dimensional spaces, so–called hyperbolic planes. A similar remark applies to G_{I}^t. This space can be decomposed into an orthogonal sum of spaces of the form $\mathbb{T} \times \mathbb{Z}$ where the bicharacter is given in the usual manner by the duality between $\mathbb{T}$ and $\mathbb{Z}$. Actually, in the proof of the theorem we will inductively construct such a decomposition.

Clearly, G_{II} contains a largest finite subgroup. One might hope to split off this subgroup orthogonally. But this is in general impossible, for instance in the case of the bicharacter γ on $\mathbb{Z} \times \mathbb{Z} \times \{1,-1\}$ given by $\gamma(a,b,\varepsilon;a',b',\varepsilon') = z^{ab'-a'b}\varepsilon^{-a'}\varepsilon'^{a}$ for any fixed $z \in \mathbb{T}$ of infinite order.

Observe that the existence of a non–degenerate skew–symmetric bicharacter γ on the compactly generated group G forces G to be a Lie group. This follows at once from the injectivity of $\psi_G : G \to \widehat{G}$ and the structure theorem for compactly generated abelian groups. ∎

Proof of Theorem 1.6. As we just remarked G is a Lie group, in particular the maximal compact subgroup K of the connected component G_0 is a torus. Using the following lemma the assertion is reduced to the case that K is trivial.

Lemma 1.8. *Let (H,γ) be a quasi–symplectic space, and let T be a closed subgroup of H, isomorphic to a torus of a certain dimension. Then there exist a locally compact abelian group R and an isomorphism from H onto $T \times \widehat{T} \times R$ such that, under this identification, $T \times \widehat{T}$ and R are γ–orthogonal, and γ is given on $T \times \widehat{T}$ by*

$$\gamma\left(\left(t_1,\chi_1\right),\left(t_2,\chi_2\right)\right) = \chi_1\left(t_2\right)\chi_2\left(t_1\right)^{-}.$$

Proof of the Lemma. *First we note that T is contained in $T^{\perp}$ because tori don't allow non–trivial continuous bicharacters (the connected subgroup $\psi_T(T)$ of the discrete group $\widehat{T}$ must be trivial). The proof proceeds by induction on* $\dim T$. *Let $T_1 \subset T$ be an one–dimensional subtorus. Via ψ_{T_1} the group $H/T_1^{\perp}$ is isomorphic to $\widehat{T}_1$ which is an infinite cyclic group. We choose a discrete infinite cyclic subgroup Z_1 of H such that H is as a topological group the direct product of Z_1 and $T_1^{\perp}$. Since T_1 is contained in $T_1^{\perp}$, also the sum $T_1 + Z_1 =: W$ is direct. Using that γ is trivial on Z_1 as Z_1 is cyclic one concludes that the isomorphism $\alpha : W \to T_1 \times \widehat{T}_1$ given by $\alpha(t+z) = (t, \psi_{T_1}(z))$ for $t \in T_1$ and $z \in Z_1$ has the property that*

$$\gamma\left(\alpha^{-1}\left(t_1,\chi_1\right),\alpha^{-1}\left(t_2,\chi_2\right)\right) = \chi_1\left(t_2\right)\chi_2\left(t_1\right)^{-}$$

for t_1, $t_2 \in T_1$ and χ_1, $\chi_2 \in \widehat{T}_1$.

Then clearly W satisfies the assumption of (1.5), hence H is isomorphic to the γ–orthogonal direct sum $W \oplus W^{\perp}$. Since T is contained in $T_1 + W^{\perp}$ the torus T is the direct product of T_1 and $T_r \overset{\text{def}}{=} T \cap W^{\perp}$. The induction hypothesis applied to $W^{\perp}$ with toroidal subgroup T_r gives the lemma. ■

Proof of Theorem 1.6, continued. We may now assume that the (open) connected component G_0 of G is isomorphic to a vector group. Then the kernel of the restriction of γ to G_0, which is $G_0 \cap (G_0)^{\perp}$, is a vector subspace of G_0. If G_{I}^{v} is any chosen vector space complement to $G_0 \cap (G_0)^{\perp}$ in G_0 then G_{I}^{v} is an ordinary symplectic space. Clearly, $W = G_{\mathrm{I}}^{v}$ satisfies the assumptions of (1.5). Hence G is an orthogonal sum of G_{I}^{v} and $G_{\mathrm{II}} \overset{\text{def}}{=} (G_{\mathrm{I}}^{v})^{\perp}$. By construction γ is trivial on $(G_{\mathrm{II}})_0$. The claimed structure of the topological group G_{II} is obvious.■

From (1.6) one can easily draw consequences on the structure of cocycles on an abelian group.

Corollary 1.9. *Let β be a measurable cocycle on the locally compact abelian group H. The bicharacter $(x,y) \mapsto \beta(x,y)\beta(y,x)^{-1}$ induces the structure γ of a quasi–symplectic space on $G = H/C_\beta$ where $C_\beta = \{x \in H | \beta(x,y) = \beta(y,z)$ for all $y \in H\}$. Suppose that G is compactly generated. Then there exist*

- *vector groups V and U,*
- *a locally compact abelian group A,*
- *discrete finitely generated free abelian groups D and E,*
- *a dense homomorphism $\tau : E \to U^{\wedge}$,*
- *a skew–symmetric bicharacter $\delta : E \times E \to \mathbb{T}$,*

- *a closed embedding* $\iota : D \to \widehat{A}$,

and an isomorphism from $V \times V^\wedge \times A \times D \times U \times E$ *onto a closed subgroup* H' *of* H *of finite index, containing* C_β*, such that under this identification the restriction* β' *of* β *to* H' *is cohomologous to the cocycle* α *on* $V \times V^\wedge \times A \times D \times U \times E$ *given by*

$$\alpha(v, \chi, a, d, u, e, ;\ v', \chi', a', d', u', e') = \chi(v')\iota(d)(a')\tau(e)(u')\delta(e, e').$$

Proof. Decompose (G, γ) according to (1.6). Let F be the (finite) torsion subgroup of G_{II} and let H' be the preimage of $F^\perp$ under the natural map $\nu : H \to G$, which is a subgroup of finite index. The quasi–symplectic space (G', γ') associated with (H', β') possesses a decomposition according to (1.6) where now G'_{II} is torsion–free. Therefore, we may assume from now on that G_{II} is torsion–free.

Let $V \subset G_{\mathrm{I}}^v$ be as in (1.6), let $U = (G_{\mathrm{II}})_0$, let $A = \nu^{-1}(T)$ where $T \subset G_{\mathrm{I}}^t$ is as in (1.6), let $D = \widehat{T}$ and let E be any complement to U in G_{II}. The homomorphism $\tau : E \to U^\wedge$ is defined by $\tau(e)(u) = \gamma(e, u)$, the closed embedding $\iota : D \to \widehat{A}$ is the transpose of the canonical surjection $A \to T$, and the skew–symmetric bicharacter δ on E is chosen such that $\delta(e, e')^2 = \gamma(e, e')$ for all $e, e' \in E$; observe that E is a free group.The image of τ has to be dense because otherwise γ wouldn't be non–degenerate.

Identifying G with $V \times V^\wedge \times T \times D \times U \times E$ and using that $V \times V^\wedge \times D \times U \times E$ is a projective locally compact abelian group one concludes that there is an identification of H with $V \times V^\wedge \times A \times D \times U \times E$ such that ν corresponds to the identity on $V \times V^\wedge \times D \times U \times E$ and to the canonical surjection $A \to T$ on A. It is easily checked that under these identifications the antisymmetrizations of α as defined in the corollary and of β coincide. Hence by [22] the cocycles α and β are cohomologous. ∎

Next, we collect some elementary facts on prepolarizations in general quasi–symplectic spaces.

Lemma 1.10. *Let* (G, γ) *be a quasi–symplectic space.*

(i) *If* K *is a compact subgroup of* G *with* $K \subset K^\perp$ *then* K *is a prepolarization.*

(ii) *If* K *is a compact subgroup of* G *then* $K^\perp$ *is open in* G *and* $K^\perp \cap K$ *is of finite index in* K*.*

(iii) *There exist large compact subgroups* K *in* G *which are prepolarizations.*

Proof. Obviously, $\varphi_K : K \to \left(G/K^\perp\right)^\wedge$ is an isomorphism of topological groups, whence (i). Also (ii) is an immediate consequence of the fact that φ_K is an isomorphism. Concerning (iii) let L be any large compact subgroup of G. Then $K \overset{\text{def}}{=} L \cap L^\perp$ is of finite index in L, hence it is large, too. By (i), K is a prepolarization. ∎

Lemma 1.11. *Let* P *be a prepolarization in the quasi–symplectic space* (G, γ)*.*

(i) *If* A *is a closed subgroup of* P *then* A *is a prepolarization.*

(ii) *The subgroup* $\left(P^{\perp}\right)^{\perp}$ *equals* P. *Hence* γ *induces the structure of a quasi–symplectic space on* $P^{\perp}/P$.

(iii) *If* Q *is a closed subgroup of* G *with* $P \subset Q \subset P^{\perp}$ *then* Q *is a prepolarization of* G *iff* Q/P *is a prepolarization of* $P^{\perp}/P$.

Proof. Evidently, $A \subset P$ implies $A \subset P \subset P^{\perp} \subset A^{\perp}$. Since φ_A is obtained from φ_P by restriction, it is an isomorphism as well. Concerning (ii) we only have to show that $\left(P^{\perp}\right)^{\perp}$ is contained in P. If x is in $\left(P^{\perp}\right)^{\perp}$ then the character $\varphi_G(x)$ annihilates $P^{\perp}$, i.e., $\varphi_G(x)$ is contained in $\left(G/P^{\perp}\right)^{\wedge}$. As φ_P is an isomorphism from P onto $\left(G/P^{\perp}\right)^{\wedge}$ there exists $p \in P$ such that $\varphi_G(x) = \varphi_P(p)$ which gives $x = p \in P$. The easy proof of (iii) is omitted. ■

One of the main goals of this section is to show that quasi–polarizations P always exist and that the quotient $P^{\perp}/P$ is in some sense independent of the choice of P. More precisely, if P_1 and P_2 are quasi–polarizations in a quasi–symplectic space (G, γ) then it will turn out that $P_1^{\perp}/P_1$ and $P_2^{\perp}/P_2$ are equivalent in the following sense.

Lemma and Definition 1.12. Two (discrete) abelian groups D_1 and D_2 are called equivalent if there exist an abelian group A and homomorphisms $\alpha_j : A \to D_j$, $j = 1, 2$, such that $D_j/\alpha_j(A)$ is finite for $j = 1, 2$, and $\ker \alpha_1$ and $\ker \alpha_2$ are finitely generated groups of the same rank. This is an equivalence relation, denoted $D_1 \sim D_2$. Moreover D_1 is equivalent to D_2 iff there exist a group B and homomorphisms $\beta_j : D_j \to B$ such that $B/\beta_j(D_j)$ is finite for $j = 1, 2$, and $\ker \beta_1$ and $\ker \beta_2$ are finitely generated groups of the same rank. The equivalence class of a group D is denoted by [D].

Proof. Clearly the defined relation is reflexive and symmetric, it remains to show that it is transitive. So, let $\alpha_j : A \to D_j$, $j = 1, 2$, and let $\delta_j : C \to D_j$, $j = 2, 3$, be homomorphisms with the above properties. Then let $R = \{(a, c) \in A \times C | \alpha_2(a) = \delta_2(c)\}$ and define $\varepsilon_j : R \to D_j$, $j = 1, 3$, by $\varepsilon_1(a, c) = \alpha_1(a)$ and $\varepsilon_3(a, c) = \delta_2(c)$. One verifies immediately that $(\varepsilon_1, \varepsilon_3)$ establishes an equivalence between D_1 and D_3. Concerning the alternate description of the relation assume that $\alpha_j : A \to D_j$, $j = 1, 2$, establish an equivalence between D_1 and D_2. Then define $B = D_1 \times D_2/\{(\alpha_1(a), \alpha_2(a)) \,|a \in A\}$ and define $\beta_j : D_j \to B$ as the inclusion of D_j in $D_1 \times D_2$ followed by the quotient map $D_1 \times D_2 \to B$. If on the other hand $\beta_j : D_j \to B$, $j = 1, 2$, are given then let $A = \{(d_1, d_2) \in D_1 \times D_2 | \beta_1(d_1) = \beta_2(d_2)\}$ and $\alpha_j(d_1, d_2) = d_j$. The verifications are simple. ■

Examples 1.13. The groups equivalent to $\mathbb{Z}^n$ are precisely the groups of the form $\mathbb{Z}^n \times F$ with finite F. This applies in particular to $n = 0$. But also the groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z} \times \mathbb{Z}$ are equivalent. ■

The strategy for proving $\left[P_1^{\perp}/P_1\right] = \left[P_2^{\perp}/P_2\right]$ for any two quasi–polarizations P_1 and P_2 in a quasi–symplectic space (G, γ) will be to associate with (G, γ) an equivalence class of discrete groups, called $\mathrm{Inv}(G)$, and to show that $\mathrm{Inv}(G) = \mathrm{Inv}\left(P^{\perp}/P\right)$ for each prepolarization P in G. If $P^{\perp}/P$ is discrete, i.e., if P is a quasi–polarization, then it will turn out that $\mathrm{Inv}\left(P^{\perp}/P\right) = [P^{\perp}/P]$ which gives $\mathrm{Inv}\,(G) = [P^{\perp}/P]$, whence the independence. To define $\mathrm{Inv}(G)$ we associate with each continuous homomorphism $\rho : G \to H$ between locally compact abelian groups G and H a discrete group or, more precisely, an equivalence class of groups. Choose a large compact subgroup K in G and a large compact subgroup L in H such that $\rho(K) \subset L$. Then form $(L/\rho(K))^\wedge$. The equivalence class of this group is independent of the choice of K and L. This is fairly easy. Now suppose that (G, γ) is a quasi–symplectic space. Then γ induces an injective dense continuous homomorphism $\psi = \psi_G : G \to G^\wedge$. The group associated with ψ can be computed as follows: Choose a large compact subgroup K in G with $K \subset K^\perp$. Such K exists and it is a prepolarization, see (1.10). Once K is chosen then $U \supset K$ is defined by $(G/K)_0 = U/K$. Then U is open in G and U is contained in $K^\perp$, because every continuous homomorphism from K in $(U/K)^\wedge$, which is a vector group, is trivial. It is easy to see that $(G/U)^\wedge$ is a large compact subgroup in $G^\wedge$ with $\psi_G(K) \subset (G/U)^\wedge$. Hence one of the discrete groups associated with ψ_G is $\{(G/U)^\wedge/\psi_G(K)\}^\wedge$ which is isomorphic to $K^\perp/U$.

Definition 1.14. Let (G, γ) be a quasi–symplectic space. Then the equivalence class of groups associated with ψ_G is denoted by $\mathrm{Inv}(G, \gamma)$ or $\mathrm{Inv}(G)$ for short. By the preceding remarks $\mathrm{Inv}(G, \gamma) = [K^\perp/U]$ if K and U are as above. In particular, if G is discrete then $\mathrm{Inv}(G) = [G]$. ∎

Theorem 1.15. *Let (G, γ) be a quasi–symplectic space. Then there exists a quasi–polarization P in G. Moreover, P may be chosen such that $\mathrm{Inv}(G) = \mathrm{Inv}\left(P^\perp/P\right) = [P^\perp/P]$ where, of course, $P^\perp/P$ is endowed with the induced bicharacter, see* (1.11).

Proof. The equation $\mathrm{Inv}\left(P^\perp/P\right) = [P^\perp/P]$ was already observed in (1.14). To construct P with the required properties choose K and U as above, i.e., K is a large compact subgroup of G such that $K \subset K^\perp$ and U is defined by $U/K = (G/K)_0$. Since K is a prepolarization in G and since evidently $\mathrm{Inv}\left(K^\perp/K\right) = \mathrm{Inv}(G)$, by (1.11) it is sufficient to construct a quasi–polarization in $K^\perp/K$ with the required property. Hence from now on we assume that G is essentially compact–free.

Let $V = G_0$ be the (open) connected component of G. The kernel of the restriction of γ to V, i.e. $V \cap V^\perp$, is a subspace of the vector space V. If W denotes any vector space complement to $V \cap V^\perp$ in V then $(W, \gamma|_W)$ is an ordinary symplectic space. In particular, W satisfies the assumptions of (1.5), hence $G = W \oplus W^\perp$. Evidently, $\mathrm{Inv}(G) = \mathrm{Inv}\left(W^\perp\right)$. Since $(W, \gamma|_W)$ allows a polarization in the usual sense, i.e., a prepolarization Q with $Q = Q^\perp$, it is sufficient to construct a quasi–polarization for $W^\perp$ with the required property.

In other words, in addition to G being essentially compact–free we may assume that $\gamma|_{V\times V}$ is trivial. Since $\psi_V(G)$ is a dense subgroup of $\widehat{V}$ it contains in particular a lattice of $\widehat{V}$. As $V^\perp \supset V$ is open in G there exists a discrete group D in G (free of finite rank) such that ψ_V induces an isomorphism from D onto a lattice in $\widehat{V}$. Then ψ_D maps V onto the torus $\widehat{D}$. Let Γ be the kernel of this map, i.e., $\Gamma = V \cap D^\perp$, which is a lattice in V. Let $H = V + D$. One easily verifies that $H^\perp \cap H = \Gamma$: Obviously, Γ is contained in $H^\perp \cap H$. If $x = v + d$ where $v \in V$, $d \in D$, is contained in $H^\perp = V^\perp \cap D^\perp$ then

$1 = \psi_V(v+d) = \psi_V(d)$, hence $d = 0$. Therefore, $x \in V \cap D^\perp = \Gamma$.

We conclude that γ induces a non–degenerate skew–symmetric bicharacter on $H/\Gamma = (V+D)/\Gamma$, say δ. Applying (1.8) we find a discrete subgroup E of $V+D$ with $E \supset \Gamma$ such that H/Γ is a direct sum of the torus V/Γ and E/Γ, that δ is trivial on E/Γ and that δ induces an isomorphism from V/Γ onto $(E/\Gamma)^\wedge$. As E/Γ is free we may choose a subgroup F of E such that E is a direct sum of Γ and F. Then clearly $H = V+D = V+F$ and γ is trivial on F, i.e., $F \subset F^\perp$. We claim that F is a quasi–polarization. First, ψ_F maps G onto $\widehat{F}$, even V is mapped onto $\widehat{F}$, and it is easy to see that ψ_F induces an isomorphism from $G/F^\perp$ onto $F^\wedge$. Secondly, $F^\perp \cap H = F^\perp \cap (V+F) = \Gamma + F = E$, hence $F^\perp \cap V = \Gamma$. As V is open in G, in the present case $F^\perp$ itself is discrete in G.

It remains to show that the discrete groups G/V and $F^\perp/F$ are equivalent in the sense of (1.12); observe that $G/V \in \mathrm{Inv}(G)$ by (1.14). The desired equivalence is established by the canonical homomorphisms $\alpha : F^\perp \to F^\perp/F$ and $\beta : F^\perp \to G/V$. Concerning the images of α and β we only have to show that the image of β is cofinite. As we saw above $G/F^\perp$ is isomorphic to $\widehat{F}$, hence compact. Therefore, $G/(F^\perp + V)$ is compact, too. Being discrete in addition it has to be finite, which gives that the image of β is cofinite. Concerning the ranks of the kernels of α and β we observe that the ranks of $F \cong E/\Gamma \cong (V/\Gamma)^\wedge$ and $V \cap F^\perp = \Gamma$ coincide with the dimension of V. ∎

Remark 1.16. The proof gives a little more than explicitly stated in the theorem, namely some information on the structure of particular quasi–polarizations. To this end, let's first introduce one more invariant of a quasi–symplectic space (G,γ). If K is any large compact subgroup of G with $K \subset K^\perp$ and if U, as usual, is defined as $U/K = (G/K)_0$ then the bicharacter γ induces a form on the vector space U/K, which has a kernel of a certain dimension, say z, i.e., $z = \dim\left(U \cap U^\perp\right)/K$. It is easy to see that $z = z(G,\gamma)$ does not depend on the choice of K. The proof of (1.15) shows that there always exist quasi–polarizations P which are isomorphic to $K \times \mathbb{R}^a \times \mathbb{Z}^z$, where $2a + z = \dim U/K$; in particular such an P is compactly generated. On the other hand, we claim that if Q is any compactly generated quasi–polarization, hence Q is isomorphic to $L \times \mathbb{R}^b \times \mathbb{Z}^c$ with a compact subgroup L, then $c \geq z$. If $c = z$ then L is a large compact subgroup of G and $2b + z = \dim V/L$ where $V/L = (G/L)_0$.

Proof. Let Q be as above. Choose a large compact subgroup K such that $L \subset K \subset K^\perp$. Since $Q^\perp/Q$ is discrete, the quotient $(K \cap Q^\perp)/(K \cap Q)$ is finite. Moreover, L equals $K \cap Q$, hence L is of finite index in $K \cap Q^\perp$. The homomorphism ψ_Q induces an isomorphism from $K/K \cap Q^\perp$ onto a subgroup of $\widehat{Q} \cong \widehat{L} \times \mathbb{R}^b \times \mathbb{T}^c$. Therefore, $K/K \cap Q^\perp$ and K/L are compact Lie groups. Substituting, if necessary, the group K by the preimage of the connected component in K/L we may assume that K/L is a torus.

The group $Q' \stackrel{\text{def}}{=} Q + (Q^\perp \cap K)$ is another quasi–polarization, it is isomorphic to $L' \times \mathbb{R}^b \times \mathbb{Z}^c$ where $L' = L + (Q^\perp \cap K)$ is the largest compact subgroup of Q'. The form γ induces the structure of a quasi–symplectic space on the group $G_1 \stackrel{\text{def}}{=} (K \cap Q^\perp)^\perp/(K \cap Q^\perp)$. Clearly, one has $z(G_1) = z(G)$.

The subgroup $K_1 \overset{\text{def}}{=} K/(K\cap Q^\perp)$ of G_1 is a large compact subgroup of G_1, it is isomorphic to a torus. The subgroup $Q_1 \overset{\text{def}}{=} Q'/(K\cap Q^\perp)$ of G_1 is a quasi–polarization of G_1, it is isomorphic to $\mathbb{R}^b\times\mathbb{Z}^c$.

Moreover, the homomorphism ψ_{K_1} from G_1 onto $\widehat{K}_1$ maps Q_1 onto $\widehat{K}_1$, because the intersection $K_1\cap Q_1^\perp$ is trivial. Since $\widehat{K}_1$ is a free abelian group there exists a discrete, free abelian subgroup D of Q_1, whose rank equals $\dim K_1 = \dim(K/L)$, such that ψ_{K_1} induces an isomorphism from D onto $\widehat{K}_1$. Let $W = D + K_1$. It is easy to see that the restriction of ψ_W to W induces a homeomorphism from W onto $\widehat{W}$. Hence, by (1.5), the quasi–symplectic space G_1 is the orthogonal sum of W and $W^\perp$. The quasi–polarization Q_1 decomposes accordingly, $Q_1 = (W\cap Q_1)\oplus(W^\perp\cap Q_1)$, because $W\cap D^\perp = D$, and $W\cap Q_1$ equals D. The intersection $Q_2 \overset{\text{def}}{=} W^\perp\cap Q_1$ is a quasi–polarization of the quasi–symplectic space $G_2 \overset{\text{def}}{=} W^\perp$, it is isomorphic to $\mathbb{R}^b\times\mathbb{Z}^{c'}$ where $c = c' + \dim(K/L)$. Moreover, $z(G_2) = z(G_1) = z(G)$. The connected component $(G_2)_0$ is a vector group because $K_1\subset W$ is a large compact subgroup of G_1. By asumption, ψ_{Q_2} induces an isomorphism from $G_2/Q_2^\perp$ onto $Q_2^\wedge$. In particular, $G_2/Q_2^\perp$ is connected, hence ψ_{Q_2} maps $(G_2)_0$ onto $Q_2^\wedge$. It follows that

$$\dim(G_2)_0 = b + c' + \dim\left((G_2)_0\cap Q_2^\perp\right)_0 .$$

But as $Q_2^\perp/Q_2$ is discrete, one has $((G_2)_0\cap Q_2^\perp)_0 = ((G_2)_0\cap Q_2)_0 = (Q_2)_0$ which is isomorphic to $\mathbb{R}^b$. Hence $\dim(G_2)_0 = 2b + c'$. Let T be the z–dimensional kernel of the form on $(G_2)_0$, and let $m = \frac{1}{2}\dim(G_2)_0/T$. Since $\psi_{(Q_2)_0}$ maps $(G_2)_0$ onto $((Q_2)_0)^\wedge$ we conclude that the intersection $T\cap(Q_2)_0$ is trivial. As $(Q_2)_0$ is a portion of a quasi–polarization, it is an isotropic subspace of $(G_2)_0$. These two informations imply $b = \dim(Q_2)_0 \le m$. Then from $\dim(G_2)_0 = 2b + c' = 2m + z$ one deduces $z \le c'$, whence $z\le c$.

In case $z = c$ one has $z = c'$, $b = m$ and $\dim K/L = 0$. In particular, $L = K$ is a large compact subgroup of G. As $Q^\perp\cap K = L\subset Q$, the above group Q' equals Q. Moreover, $G_1 = G_2 = L^\perp/L$ and $Q_1 = Q_2 = Q/L$. If V/L is, as introduced in this remark, the connected component of G/L, then $V/L = (G_2)_0$ and $\dim V/L = \dim(G_2)_0 = 2m + z = 2b + z$. ■

Theorem 1.17. *Let (G,γ) be a quasi–symplectic space. If P is any prepolarization in G then $\mathrm{Inv}(G) = \mathrm{Inv}\left(P^\perp/P\right)$. In particular, $\left[P_1^\perp/P_1\right] = \left[P_2^\perp/P_2\right]$ for every pair of quasi–polarizations P_1, P_2 in G.*

Proof. The strategy will be to reduce to the following basic situation.

(B) Let Q be a prepolarization in the essentially compact–free quasi–symplectic space (H,δ). If $Q^\perp$ is discrete then $\mathrm{Inv}(H) = \left[Q^\perp/Q\right]$.

In the reduction we will use the following results in particular cases.

(1) Let W be a prepolarization in the quasi–symplectic space (H,δ). If W is either compact or a vector group then $\mathrm{Inv}(H) = \mathrm{Inv}\left(W^\perp/W\right)$.

(2) Let Q be a prepolarization in the quasi–symplectic space (H,δ) and let M be a compact subgroup contained in $Q^\perp\cap M^\perp$. Then $Q' \overset{\text{def}}{=} Q + M$ is a prepolarization in H and $\mathrm{Inv}\left(Q^\perp/Q\right) = \mathrm{Inv}({Q'}^\perp/Q')$.

(3) Let Q be a discrete prepolarization in the quasi–symplectic space (H,δ). Suppose that there exists a compact subgroup L in H such that $L\subset L^\perp$, $H=L+Q^\perp$ and $L\cap Q^\perp=0$. Then $\mathrm{Inv}(H)=\mathrm{Inv}\left(Q^\perp/Q\right)$.

Next, we prove the assertions (B), (1), (2) and (3).

ad (B): This is more or less a repetition of the final part of the proof of (1.15). We have to show that the discrete groups H/H_0 and $Q^\perp/Q$ are equivalent. The equivalence will be established by the canonical homomorphisms $Q^\perp\to Q^\perp/Q$ and $Q^\perp\to H/H_0$. As $\widehat{Q}\cong H/Q^\perp$ is compact the groups $Q^\perp$ and $Q^\perp+H_0$ are cocompact in H. Hence $Q^\perp+H_0$ is of finite index in H. Concerning the kernels of the canonical homomorphisms we observe that the compact group $H/Q^\perp$ contains $\left(Q^\perp+H_0\right)/Q^\perp\cong H_0/Q^\perp\cap H_0$ as a cofinite subgroup. As $Q^\perp\cap H_0$ is discrete it is a free abelian group, whose rank equals $\dim H_0=\dim H_0/H_0\cap Q^\perp$. Since $\widehat{Q}$ is up to a finite extension the same as $H_0/H_0\cap Q^\perp$, the group Q has to be finitely generated of rank $\dim H_0$.

ad (1): This is trivial for compact W. We remarked and used this fact already in the first part of the proof of (1.15). So, let's suppose that W is a vector group. The quotient $H/W^\perp$ is a vector group being isomorphic to $\widehat{W}$. Choose a large compact subgroup K of H. Then K is contained in $W^\perp$. Let L be a large compact subgroup of $\widehat{H}$ such that $\psi_H(K)\subset L$. Then L is contained in $(H/W)^\wedge$ because $\widehat{H}/(H/W)^\wedge$ is a vector group. Let $H'=W^\perp/W$, let $K'=(K+W)/W\subset H'$, and let $L'\subset(H')^\wedge$ be the image of L under the canonical homomorphism $(H/W)^\wedge\to(H')^\wedge=\left(W^\perp/W\right)^\wedge$ obtained by restriction. We claim that K' and L' are large compact subgroups of H' and $(H')^\wedge$, respectively, that $\psi'(K')\subset L'$ where $\psi':H'\to(H')^\wedge$ denotes the ψ–map of the quasi–symplectic space H', and that $L/\psi_H(K)$ and $L'/\psi'(K')$ are isomorphic. Clearly, this claim gives (1) by definition of $\mathrm{Inv}(H)$. To see that K' is large in H' we observe that $H/(W+K)$ is essentially compact–free because H/K is essentially compact–free and the kernel of $H/K\to H/(W+K)$ is a vector group. But H'/K' is a subgroup of $H/(W+K)$. The argument for the largeness of L' is similar as again the kernel of the quotient map $(H/W)^\wedge\to\left(W^\perp/W\right)^\wedge$ is a vector group. Since evidently $\psi'(K')\subset L'$ we are left to show that the kernel of the canonical map $L\to L'/\psi'(K')$ equals $\psi_H(K)$. The kernel in question is given as $\{\chi\in L|\ \chi|_{W^\perp}\in\psi_H(K)|_{W^\perp}\}=:S$. Obviously, $\psi_H(K)\subset S$. If $\chi\in S$ then exists $k\in K$ such that $\chi-\psi_H(k)\in\left(H/W^\perp\right)^\wedge$, hence $\chi-\psi_H(k)\in L\cap\left(H/W^\perp\right)^\wedge$. But $L\cap\left(H/W^\perp\right)^\wedge$ is trivial because $\left(H/W^\perp\right)^\wedge$ is a vector group.

ad (2): By assumption Q' is contained in $Q^\perp$ and Q'/Q is a compact subgroup of $Q^\perp/Q$ with $(Q'/Q)^\perp\supset Q'/Q$. Hence by part (i) of (1.10) the group Q'/Q is a prepolarization in $Q^\perp/Q$, by part (iii) of (1.11) Q' is a prepolarization in H. Applying (1) to the quasi–symplectic space $Q^\perp/Q$ with compact prepolarization $Q'/Q(=W)$ one gets $\mathrm{Inv}\left(Q^\perp/Q\right)=\mathrm{Inv}(Q'^\perp/Q')$.

ad (3): Let $W=L+Q$. We claim first that the restriction α of ψ_W defines an isomorphism from W onto $\widehat{W}$ such that $\alpha(L)=(W/L)^\wedge$ and $\alpha(Q)=(W/Q)^\wedge$. To see that α is injective let $x\in\ker\alpha=W\cap W^\perp$, i.e., there exist $\ell\in L$

and $q \in Q$ such that $x = \ell + q \in L^{\perp} \cap Q^{\perp}$. Then $\ell = x - q \in L \cap Q^{\perp} = 0$, hence $x \in L^{\perp} \cap Q$ which is zero as $0 = H^{\perp} = L^{\perp} \cap \left(Q^{\perp}\right)^{\perp} = L^{\perp} \cap Q$ by part (ii) of (1.11). Since $L \subset L^{\perp}$ and $Q \subset Q^{\perp}$ one obtains that $\alpha(L)$ and $\alpha(Q)$ are contained in $(W/L)^{\wedge}$ and $(W/Q)^{\wedge}$, respectively. Since φ_Q maps Q isomorphically onto $\left(H/Q^{\perp}\right)^{\wedge}$ and since $H/Q^{\perp}$ is canonically isomorphic to W/Q, one concludes that α maps Q onto $(W/Q)^{\wedge}$. The proof for L is similar: ψ_Q maps H onto $\widehat{Q}$; as ψ_Q vanishes on $Q^{\perp}$ the image of $H = L + Q^{\perp}$ is $\psi_Q(L)$; but Q is canonically isomorphic to W/L. ¿From (1.5) we conclude that H is the δ–orthogonal direct sum of W and $W^{\perp}$. Since $\psi_W : W \to \widehat{W}$ is an isomorphism, the invariant of W is zero and hence $\mathrm{Inv}(H) = \mathrm{Inv}\left(W^{\perp}\right)$. But the inclusion of $W^{\perp}$ in $Q^{\perp}$ induces an isomorphism from $W^{\perp}$ onto $Q^{\perp}/Q$: The canonical projection $H = W \oplus W^{\perp} \to W^{\perp}$, restricted to $Q^{\perp}$, factors through $Q^{\perp} \to Q^{\perp}/Q$ and yields the inverse map; observe that $Q^{\perp} \cap W = Q$. This isomorphism $W^{\perp} \to Q^{\perp}/Q$ is not only an isomorphism of groups, but also of quasi–symplectic spaces, whence $\mathrm{Inv}\left(W^{\perp}\right) = \mathrm{Inv}\left(Q^{\perp}/Q\right)$.

Now, let (G, γ) be the quasi–symplectic space of the theorem and let P be a prepolarization in G.

Step 1. There exist a quasi–symplectic space (G_1, γ_1), a prepolarization P_1 in G_1 and a large compact subgroup K_1 of G_1 such that $K_1 \subset K_1^{\perp}$, $P_1^{\perp} \cap K_1 = 0$, $\mathrm{Inv}(G_1) = \mathrm{Inv}(G)$ and $\mathrm{Inv}\left(P^{\perp}/P\right) = \mathrm{Inv}\left(P_1^{\perp}/P_1\right)$.

Choose a large compact subgroup K in G with $K \subset K^{\perp}$. Then put $P' \overset{\text{def}}{=} P + \left(P^{\perp} \cap K\right)$, which is by (2) a prepolarization in G with $\mathrm{Inv}(P'^{\perp}/P') = \mathrm{Inv}(P^{\perp}/P)$. Put $G_1 = \left(P^{\perp} \cap K\right)^{\perp}/P^{\perp} \cap K$, $K_1 = K/P^{\perp} \cap K$ and $P_1 = P'/P^{\perp} \cap K$. Then $\mathrm{Inv}(G_1) = \mathrm{Inv}(G)$ by (1). Evidently, K_1 is a large compact subgroup of G_1, and from the properties of P' it follows that P_1 is a prepolarization with $\mathrm{Inv}\left(P^{\perp}/P\right) = \mathrm{Inv}\left(P_1^{\perp}/P_1\right)$. The equation $P_1^{\perp} \cap K_1 = 0$ is obvious.

Step 2. Given $(G_1, \gamma_1, P_1, K_1)$ as above there exist an essentially compact–free quasi–symplectic space (G_2, γ_2) and a prepolarization P_2 in G_2 such that $\mathrm{Inv}(G_1) = \mathrm{Inv}(G_2)$ and $\mathrm{Inv}\left(P_1^{\perp}/P_1\right) = \mathrm{Inv}\left(P_2^{\perp}/P_2\right)$.

Let $P_1' = P_1 \cap K_1^{\perp}$ and $P_1'' = P_1' + K_1$. From (2), applied to $Q = P_1'$ and $M = K_1$, we conclude that P_1'' is a prepolarization with $\mathrm{Inv}(P_1''^{\perp}/P_1'') = \mathrm{Inv}(P_1'^{\perp}/P_1')$. We want to know that $\mathrm{Inv}(P_1''^{\perp}/P_1'') = \mathrm{Inv}\left(P_1^{\perp}/P_1\right)$. It suffices to prove $\mathrm{Inv}(P_1^{\perp}/P_1) = \mathrm{Inv}(P_1'^{\perp}/P_1')$. To this end we first show that $P_1'^{\perp} = K_1 + P_1^{\perp}$. Since P_1 is a prepolarization the form γ induces an isomorphism $\alpha \overset{\text{def}}{=} \varphi_{P_1}$ from P_1 onto $\left(G_1/P_1^{\perp}\right)^{\wedge}$. It is easy to see that $\alpha(P_1')$ is precisely the subgroup $\left(G_1/K_1 + P_1^{\perp}\right)^{\wedge}$ of $\left(G_1/P_1^{\perp}\right)^{\wedge}$ which gives $P_1'^{\perp} = K_1 + P_1^{\perp}$. Now apply (3) to $H = P_1'^{\perp}/P_1'$, $Q = P_1/P_1'$ and $L = (K_1 + P_1')/P_1'$. The assumptions of (3) are easily verified; note that Q is discrete as $K_1^{\perp}$ is open in G_1. From (3) we obtain $\mathrm{Inv}(P_1^{\perp}/P_1) = \mathrm{Inv}(P_1'^{\perp}/P_1')$.

Put $G_2 = K_1^{\perp}/K_1$ and $P_2 = P_1''/K_1 = \left(\left(P_1 \cap K_1^{\perp}\right) + K_1\right)/K_1$. Again by (1), $\mathrm{Inv}(G_2) = \mathrm{Inv}(G_1)$. Note that G_2 is essentially compact–free, and $\mathrm{Inv}\left(P_2^{\perp}/P_2\right) = \mathrm{Inv}\left(P_1^{\perp}/P_1\right)$ by what we saw above.

Final step. Let (G_2, γ_2, P_2) be as above. By (1.15) applied to the quasi–

symplectic space $P_2^\perp/P_2$ there exists a quasi–polarization Q_3 in $P_2^\perp/P_2$ such that $\mathrm{Inv}\left(P_2^\perp/P_2\right) = \mathrm{Inv}\left(Q_3^\perp/Q_3\right) = \left[Q_3^\perp/Q_3\right]$. By (1.11), Q_3 is of the form $Q_3 = P_3/P_2$ where P_3 is a quasi–polarization in G_2 with $P_2 \subset P_3 \subset P_3^\perp \subset P_2$. Clearly, $P_3^\perp/P_3 = Q_3^\perp/Q_3$. Let W be the connected component in P_3. As G_2 and hence P_3 is essentially compact–free, W is open in P_3, and W is a vector group. Moreover, being a closed subgroup of a prepolarization W is a prepolarization as well, and P_3/W is a prepolarization in the quasi–symplectic space $W^\perp/W$. As $(P_3/W)^\perp = P_3^\perp/W$ is discrete and $W^\perp/W$ is essentially compact–free we know from (B) that $\mathrm{Inv}\left(W^\perp/W\right) = \mathrm{Inv}\left(P_3^\perp/P_3\right) = \left[P_3^\perp/P_3\right]$. Using (1), applied to W in G_2, we conclude that $\mathrm{Inv}(G_2) = \mathrm{Inv}\left(W^\perp/W\right) = \mathrm{Inv}\left(P_3^\perp/P_3\right) = \mathrm{Inv}\left(Q_3^\perp/Q_3\right) = \mathrm{Inv}\left(P_2^\perp/P_2\right)$. This finishes the proof of (1.17) as $\mathrm{Inv}(G) = \mathrm{Inv}(G_2)$ and $\mathrm{Inv}\left(P^\perp/P\right) = \mathrm{Inv}\left(P_2^\perp/P_2\right)$. ■

Let's summarize. With each quasi–symplectic space (G, γ) there is associated an equivalence class $\mathrm{Inv}(G)$ of discrete abelian groups, see (1.14) and (1.12). For any quasi–polarization P the group $P^\perp/P$ is a member of this class. In particular the equivalence class of $P^\perp/P$ does not depend on the choice of P. We don't know whether a sharper result is possible in the sense that there exist a finer equivalence relation (finer than the one defined above) on the class of discrete abelian groups with the property that whenever Q and P are quasi–polarizations in a quasi–symplectic space then $Q^\perp/Q$ and $P^\perp/P$ are equivalent w.r.t. this finer relation. We constructed an example where $Q^\perp/Q = \mathbb{Q}$ and $P^\perp/P = \mathbb{Q}/\mathbb{Z} \times \mathbb{Z}$.

Before studying special cases of invariant groups $\mathrm{Inv}(G, \gamma)$ let's briefly discuss the case of finite spaces where, of course, explicit results are possible and well–known.

Proposition 1.18. *Let (G, γ) be a finite quasi–symplectic space. Then there exists a finite abelian group A and an isomorphism $G \to A \times \widehat{A}$ which carries the bicharacter γ into a bicharacter γ' on $A \times \widehat{A}$ given by $\gamma'((a_1, \chi_1), (a_2, \chi_2)) = \chi_1(a_2)\chi_2(a_1)^-$, i.e., (G, γ) is of the form as discussed in (1.2).*

Proof. Of course, we proceed by induction on $\#G$. Let X be a cyclic subgroup of G such that there exists a complementary subgroup Y of G, i.e., G is the direct sum of X and Y. By the main theorem on finite abelian groups such a pair X and Y exists. As $G^\perp = 0 = X^\perp \cap Y^\perp$ one concludes that G is also the direct sum of $X^\perp$ and $Y^\perp$ by counting $X, Y, X^\perp$ and $Y^\perp$. Moreover, as X is cyclic, $\gamma|_{X \times X}$ is trivial. From $X \subset X^\perp$ and $G = X \oplus Y$ we deduce $X^\perp = X \oplus (Y \cap X^\perp)$ and $G = X^\perp \oplus Y^\perp = X \oplus (Y \cap X^\perp) \oplus Y^\perp$. Let $D = X \oplus Y^\perp$. Then $D^\perp = X^\perp \cap (Y^\perp)^\perp = X^\perp \cap Y$ and $D \cap D^\perp = (X + Y^\perp) \cap (X^\perp \cap Y) = 0$. Hence $G = D \oplus D^\perp$ is an orthogonal decomposition and by induction we are done if this decomposition is non–trivial. So, let's suppose that $G = D = X \oplus Y^\perp = X \oplus Y$. In this case all the cardinalities of $X, X^\perp, Y$ and $Y^\perp$ coincide, in particular follows $X = X^\perp$. As γ induces an isomorphism from X onto $(G/X^\perp)^\wedge = (G/X)^\wedge$, which is isomorphic to $Y^\wedge$, the groups $Y^\wedge$ and Y are cyclic. Hence γ is trivial on $Y, Y = Y^\perp$. Now it is evident that γ has the structure as described in the proposition. ■

Theorem 1.19. *Let (G, γ) be a quasi–symplectic space and let n be a non–*

negative integer. Then the following properties are equivalent.

(i) $\mathrm{Inv}(G) = [\mathbb{Z}^n]$.

(ii) *For each large compact subgroup L of G with $L \subset L^\perp$ the group $L^\perp/V$ is a finitely generated discrete group of rank n where V/L is the connected component of G/L.*

(iii) *For each quasi–polarization Q of G the group $Q^\perp/Q$ is a finitely generated discrete group of rank n.*

(iv) *There exists a quasi–polarization P of G such that $P^\perp/P$ is isomorphic to $\mathbb{Z}^n$.*

(v) *There exists a large compact subgroup K in G with $K \subset K^\perp$ such that the quasi–symplectic space $K^\perp/K$ is isomorphic to an orthogonal sum of an ordinary symplectic space S and a space R where R, as a topological group, is isomorphic to the direct product of an z–dimensional vector group W, z as in* (1.16), *and of $\mathbb{Z}^n$; moreover the bicharacter is trivial on W.*

Remark 1.20. In (v) the group $W \times \mathbb{Z}^n$ carries a non–degenerate quasi–symplectic structure which forces $z \leq n$, and $z < n$ as soon as $z > 0$.

Remark 1.21. In (v) the group K is, of course, the largest compact subgroup of $K^\perp$, and $K^\perp$ is isomorphic as a topological group to $K \times S \times W \times \mathbb{Z}^n$.

Remark 1.22. If (G, γ) is any quasi–symplectic space and if K is any large compact subgroup of G with $K \subset K^\perp$ then (G, γ) satisfies the equivalent conditions of the theorem for a certain n iff $K^\perp$ is compactly generated.

Proof of Theorem 1.19. The equivalence of (i) and (ii) is more or less the definition of $\mathrm{Inv}(G)$. The equivalence of (i) and (iii) is an immediate consequence of (1.17). Trivially, (iv) implies (iii).

Let Q as in (iii) be given. To conclude (iv) we have to dispose of the (potential) torsion part of $Q^\perp/Q$. To this end let A be a subgroup, $Q \subset A \subset Q^\perp$, such that A/Q is a finite cyclic subgroup of $Q^\perp/Q$. Then A is a quasi–polarization in G by (1.10) and (1.11), and if A/Q is non–trivial the torsion group of $A^\perp/A$ is strictly smaller than the torsion group of $Q^\perp/Q$. After finitely many steps we find a quasi–polarization as claimed in (iv).

Let K as in (v) be given. Then $\mathrm{Inv}(G) = \mathrm{Inv}(K^\perp/K)$ by (1.10) and (1.17). From the structure of $K^\perp/K$ as described in (v) one easily deduces that $\mathrm{Inv}(K^\perp/K) = [\mathbb{Z}^n]$, whence (i).

It remains to show that (ii) implies (v). Choose any large compact subgroup L with $L \subset L^\perp$. By assumption $L^\perp/L$ is isomorphic as a topological group to the direct product of $\mathbb{Z}^n$, of the vector group V/L, and of a finite group. By enlarging L we want to dispose of the torsion part of $L^\perp/L$. Again let A be a subgroup, $L \subset A \subset L^\perp$, such that A/L is a finite cyclic group. Then A is another large compact subgroup with $A \subset A^\perp$. After finitely many steps we find a large compact subgroup K with $K \subset K^\perp$ such that $K^\perp/K$ is isomorphic to $\mathbb{Z}^n \times U/K$ where $U/K = (G/K)_0$ is an open vector group. If we split off a maximal ordinary symplectic subspace S of U/K as in the proof of (1.6), we find the asserted structure of the quasi–symplectic space $K^\perp/K$. ∎

Because of the importance of the case $n=0$ we reformulate the previous theorem for that case and add two further conditions.

Theorem 1.23. *For a quasi–symplectic space (G,γ) the following properties are equivalent:*

(i) $\mathrm{Inv}(G)$ *is trivial.*

(ii) *For each large compact subgroup L of G with $L \subset L^{\perp}$ the group $L^{\perp}/V$ is finite where V/L is the connected component of G/L.*

(iii) *For each quasi–polarization Q of G the group $Q^{\perp}/Q$ is finite.*

(iv) *There exists a quasi–polarization P of G such that $P^{\perp}=P$.*

(v) *There exists a large compact subgroup K in G with $K \subset K^{\perp}$ such that $K^{\perp}/K$ is an ordinary symplectic space.*

(vi) *The canonical homomorphism $\psi=\psi_G : G \to \widehat{G}$ is an isomorphism of topological groups.*

(vii) *The canonical homomorphism $\psi : G \to \widehat{G}$ is bijective.*

Remark 1.24. Given a subgroup K as in (v) it is very easy to find quasi–polarizations P with $P^{\perp}=P$: One simply polarizes the ordinary symplectic space $K^{\perp}/K$ in the usual manner.

Proof of Theorem 1.23. The equivalence of (i) through (v) follows from the preceding theorem and from (1.20). Next we show that (iv) implies (vi). Let P be a quasi–polarization with $P=P^{\perp}$. The canonical homomorphism ψ_P from $G/P^{\perp}=G/P$ into $\widehat{P}$ is an isomorphism. The restriction $\alpha : P \to (G/P)^{\wedge}$ of ψ is an isomorphism, too, because $\alpha(p)=\varphi_P(p)^{-1}$ for all $p \in P$. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & G & \longrightarrow & G/P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_P} & & \\ 0 & \longrightarrow & (G/P)^{\wedge} & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{P} & \longrightarrow & 0 \end{array}$$

gives the claim (vi).

Since (vi) $\Rightarrow$ (vii) is obvious we are done if we can show that (vii) implies (iii). Let Q be a given quasi–polarization in G. It is pretty easy to see that the quasi–symplectic space $Q^{\perp}/Q$ also satisfies (vii). Hence we have to show that any discrete quasi–symplectic space, again denoted by G, which satisfies (vii), is finite. To this end, choose a subgroup X of G, which is maximal w.r.t. the property $X \subset X^{\perp}$, a "maximal quasi–isotropic subspace". Clearly, by Zorn's lemma such an X exists. The maximality implies that actually $X=X^{\perp}$ (which does not a priori mean that X is a quasi–polarization).

Using $X^{\perp}=X$ the hypothesis (vii) readily implies that the canonical homomorphisms $\psi_X : G/X \to \widehat{X}$ and $\varphi_X : X \to (G/X)^{\wedge}$ are bijective. It is enough to show that $\widehat{X}$ is finite because then also G/X and X are finite. Let $\chi : \widehat{X} \to \mathbb{T}$ be any homomorphism, not necessarily continuous. Then $\chi \circ \psi_X$ is a character of the discrete group G/X, hence of the form $\chi \circ \psi_X = \varphi_X(x)$ for some $x \in X$. One concludes that $\chi(\eta)=\eta(x)$ for $\eta \in \widehat{X} = \mathrm{Hom}\,(X,\mathbb{T})$, which shows in particular that χ is continuous. But if $\widehat{X}$ were infinite it would allow discontinuous characters: Take any countable subgroup A of $\widehat{X}$. Then A is a proper subgroup of its closure $\overline{A}$ because otherwise A were a countable compact group which is incompatible with the existence of an invariant probability measure. Clearly, there exist homomorphisms $\chi : \widehat{X} \to \mathbb{T}$ with $\chi = 1$ on A, but $\chi \neq 1$ on $\overline{A}$. Such χ's are discontinuous. ■

We continue with some observations on compactly generated quasi–symplectic spaces (G, γ). In this case, $\mathrm{Inv}(G) = [\mathbb{Z}^n]$ for a certain n. Indeed, in the notations of (1.6) the number n is nothing but the rank of the discrete group $G_{\mathrm{II}}/(G_{\mathrm{II}})_0$. The invariant z introduced in (1.16) is just $\dim(G_{\mathrm{II}})_0$. In the particular case $n = 0$ we know by (1.20) that $z = 0$, hence G_{II} is finite. Applying (1.18) to the finite space G_{II} we find using (1.6):

Remark 1.25. Let (G, γ) be a compactly generated quasi–symplectic space. If $\mathrm{Inv}(G)$ is trivial then there is a locally compact abelian group A, isomorphic to the product of a vector group, a torus and a finite group, such that the quasi–symplectic space (G, γ) is isomorphic to $A \times \widehat{A}$ where the latter group is endowed with the standard skew–symmetric bicharacter. ■

Still in the particular case $n = 0$ the structure of cocycles given in (1.9) simplifies considerably.

Remark 1.26. Let β be a measurable cocycle on the locally compact abelian group H, and let $G = H/C_\beta$ and γ be as in (1.9). Suppose that G is compactly generated and that $\mathrm{Inv}(G)$ is trivial. Then there exist

- a vector group V,
- a locally compact abelian group A,
- a discrete finitely generated free abelian group D,
- a closed embedding $\iota : D \to \widehat{A}$,

and an isomorphism from $V \times V^\wedge \times A \times D$ onto a closed subgroup H' of H of finite index, containing C_β, such that under this identification the restriction β' of β to H' is cohomologous to the cocycle α on $V \times V^\wedge \times A \times D$ given by

$$\alpha(v, \chi, a, d\,;\, v', \chi', a', d') = \chi(v')\iota(d)(a'). \qquad ■$$

Returning to the case of a general compactly generated quasi–symplectic space (G, γ) we now observe that $\mathrm{Inv}(G, \gamma)$ is actually independent of γ, it merely depends on the structure of G. Indeed, we have seen above that $\mathrm{Inv}(G) = [\mathbb{Z}^n]$ where n is the rank of $G_{\mathrm{II}}/(G_{\mathrm{II}})_0$. But this rank is equal to the difference of the rank of G/G_0 and the dimension of the maximal torus T of G! In other words, the number n is equal to the difference of $\mathrm{rk}\,(G/G_0)$ and $\mathrm{rk}\,\pi_1(G)$. Since in the case at hand $\mathrm{rk}\pi_0(G) = \mathrm{rk}\pi_1(\widehat{G})$, one may also write $n = \mathrm{rk}\pi_1(\widehat{G}) - \mathrm{rk}\pi_1(G) = \mathrm{rk}\pi_0(G) - \mathrm{rk}\pi_0(\widehat{G})$. The assignment $G \mapsto \mathrm{rk}\,(G/G_0) - \mathrm{rk}\,\pi_1(G)$ can be extended to a much wider class of Lie groups. This generalization is motivated by the fact that later, in the third section, we have to compute the invariant of quasi–symplectic spaces (G, γ) where G appears as the quotient group of a noncommutative Lie group. To this end, we recall that a discrete group H is called polycyclic if it allows a composition series with cyclic factors. The number of infinite factors in such a series is an invariant of the group, the so–called rank $\mathrm{rk}\,(H)$.

Definition 1.27. Denote by $\mathfrak{C}$ the class of Lie groups G such that the (discrete) group G/G_0 is polycyclic. For $G \in \mathfrak{C}$ define a "rank" $r(G)$ by $r(G) = \mathrm{rk}\,(G/G_0) - \mathrm{rk}\,\pi_1(G_0)$. ■

In view of the foregoing discussion we have the following proposition.

Proposition 1.28. *Let G be a compactly generated abelian Lie group.*

(i) *The groups G and $\widehat{G}$ are in $\mathfrak{C}$, and $r(G) = -r(\widehat{G})$.*

(ii) *If γ is a quasi–symplectic structure on G then* $\mathrm{Inv}(G,\gamma) = [\mathbb{Z}^n]$ *where* $n = r(G) = \mathrm{rk}\pi_1(\widehat{G}) - \mathrm{rk}\pi_1(G)$. ■

The "rank" r can be characterized by four properties. Before doing so we prove that the class $\mathfrak{C}$ is stable under extensions.

Lemma 1.29. *If N is a closed normal subgroup in a topological group G then the following statements are equivalent.*

(a) *The group G belongs to $\mathfrak{C}$.*

(b) *The groups N and G/N belong to $\mathfrak{C}$.*

Proof. Suppose (b). First note that then G is a Lie group. This follows for instance from the characterization of Lie groups as locally compact groups without "small subgroups", see [26]. In particular, G_0 is open in G. Hence the image of G_0 under the quotient map $G \mapsto G/N$, i.e., G_0N/N, is open and connected in G/N, it is the connected component of G/N. As G/N belongs to $\mathfrak{C}$, the quotient $G/G_0N \cong (G/N)/(G/N)_0$ is polycyclic. To see that G/G_0 is polycyclic it is therefore enough to show that G_0N/G_0 is polycyclic. But G_0N/G_0 is isomorphic to $N/N \cap G_0$, which is a quotient of N/N_0 and hence polycyclic.

Now suppose (a). Then it is well known that N and G/N are Lie groups. As we have seen above the quotient $(G/N)/(G/N)_0$ is isomorphic to G/G_0N, in particular, it is a quotient of G/G_0, hence polycyclic. To see that N/N_0 is polycyclic the short exact sequence

$$1 \to (N \cap G_0)/N_0 \to N/N_0 \to N/(N \cap G_0) \to 1$$

shows that it suffices to prove that $(N \cap G_0)/N_0$ and $N/(N \cap G_0)$ are polycyclic. But $N/(N \cap G_0)$ is isomorphic to the subgroup NG_0/G_0 of G/G_0, hence polycyclic, and $(N \cap G_0)/N_0$ is a discrete normal subgroup of the connected Lie group G_0/N_0, hence finitely generated abelian. ■

Proposition 1.30. *There is a unique function $s : \mathfrak{C} \to \mathbb{Z}$, namely $s = r$, satisfying the following properties:*

(i) *If G_1, $G_2 \in \mathfrak{C}$ are isomorphic topological groups then $s(G_1) = s(G_2)$.*

(ii) $s(\mathbb{Z}) = 1$.

(iii) *If G is a finite cyclic group or if G is a simply connected connected Lie group then $s(G) = 0$.*

(iv) *If $G \in \mathfrak{C}$ and if N is a closed normal subgroup in G, then*

$$s(G) = s(N) + s(G/N).$$

Proof. If s satisfies (i) through (iv) then it is pretty obvious that $s(G) = r(G)$ for all $G \in \mathfrak{C}$. Also it is clear that r satisfies (i) through (iii). To show (iv) let N be a closed normal subgroup of the group G in the class $\mathfrak{C}$. Putting $H = G/N$ there is an exact sequence of polycyclic groups

$$1 \to \pi_1(N) \to \pi_1(G) \to \pi_1(H) \to N/N_0 \to G/G_0 \to H/H_0 \to 1.$$

It is easily verified that for any exact sequence

$$1 \to A_1 \to A_2 \to \cdots \to A_k \to 1$$

of polycyclic groups one has

$$\sum_{j=1}^{k} (-1)^j \mathrm{rk}\,(A_j) = 0.$$

This formula applied to the above exact sequence yields property (iv) of the function r. ■

As an illustration of (iv) and (1.28) we collect some special cases appearing in the next chapters.

Examples 1.31 (a) Let β be the any measurable cocycle on the compactly generated locally compact abelian Lie group H. As in (1.9), β induces the structure γ of a quasi–symplectic space on $G = H/C_\beta$. The invariant of (G, γ) is $[\mathbb{Z}^n]$ where $n = r(H) - r(C_\beta)$.

(b) Assume in addition that H is isomorphic to a direct product of a vector group and a discrete free group. Then the first homotopy groups of H and C_β vanish, hence one has $n = r(G) = \mathrm{rk}(H/H_0) - \mathrm{rk}(C_\beta/(C_\beta)_0)$. In particular, the quasi–symplectic space (G, γ) satisfies the equivalent conditions of (1.23) if and only if $\mathrm{rk}(H/H_0) = \mathrm{rk}(C_\beta/(C_\beta)_0)$.

(c) Let $\mathcal{G}$ be a locally compact two step nilpotent group, and let $\mathcal{L}$ be a closed central subgroup of $\mathcal{G}$ containing the commutator subgroup such that $\mathcal{G}/\mathcal{L}$ is a compactly generated Lie group. Each $\lambda \in \mathcal{L}^\wedge$ yields a bicharacter ε on $\mathcal{G}$, defined by $\varepsilon(x, y) = \lambda([x, y])$, and ε leads to the structure of a quasi–symplectic space on $G = \mathcal{G}/\mathcal{G}_\lambda$, where $\mathcal{G}_\lambda/\ker\lambda$ is by definition the center of $\mathcal{G}/\ker\lambda$. The "rank" $n = r(G)$ can be computed as $r(\mathcal{G}/\mathcal{L}) - r(\mathcal{G}_\lambda/\mathcal{L})$. ■

Next, we discuss the behaviour of r under dense injective homomorphisms. This is motivated by the fact that the invariant Inv (G) of a quasi–symplectic space measures the deviation from ψ_G being an isomorphism; note that $r(G) - r(\widehat{G}) = 2r(G)$ for compactly generated abelian Lie groups by (1.28). Recall from [32, Theorem 1.9] the following.

Proposition 1.32. *Let H and G be connected Lie groups, and let $f : H \to G$ be an injective continuous homomorphism with dense image. Then the homomorphism $\pi_1(f) : \pi_1(H) \to \pi_1(G)$ is injective, and the quotient $\pi_1(G)/\mathrm{im}\pi_1(f)$, where $\mathrm{im}\pi_1(f)$ denotes the image of $\pi_1(f)$, is a free abelian group of finite rank. This rank is zero if and only if f is an isomorphism.* ■

Proposition 1.33. *Let $H, G \in \mathfrak{C}$ and let $f : H \to G$ be an injective continuous homomorphism with dense image. Then $r(H) \geq r(G)$. Actually, the difference $r(H) - r(G)$ composes of three nonnegative quantities, namely*

$$r(H) - r(G) = \{rk\, \pi_1(\overline{f\,(H_0)}) - rk\, \pi_1\,(H_0)\} \\ + rk\,(f^{-1}\,(G_0)\,/H_0) + rk\, \pi_1(G_0/\overline{f\,(H_0)}),$$

the first one being nonnegative because of (1.32). Moreover $r(H) = r(G)$ iff f is an isomorphism of topological groups.

Proof. First one notes that $f(H_0)^-$ is normal in G as $f(H)$ is dense in G. Applying (iv) of (1.30) three times one obtains

$$r(H) - r(G) = r(H_0) + r(H/H_0) - r(f(H_0)^-) - r(G/f(H_0)^-) =$$
$$= r(H_0) - r(f(H_0)^-) + r(H/H_0) - r(G_0/f(H_0)^-) - r(G/G_0).$$

The exact sequence

$$1 \to f^{-1}(G_0)/H_0 \to H/H_0 \to G/G_0 \to 1$$

yields, by another application of (iv) of (1.30), that $r(H/H_0) - r(G/G_0) = r\left(f^{-1}(G_0)/H_0\right) = \mathrm{rk}\,\left(f^{-1}(G_0)/H_0\right)$. Using the definition of r for the connected groups H_0, $f(H_0)^-$ and $G_0/f(H_0)^-$ the asserted equality follows.

Concerning the characterization of f being an isomorphism we first observe that f induces an homomorphism $f_0 : f^{-1}(G_0) \to G_0$ and an isomorphism of the discrete groups G/G_0 and $H/f^{-1}(G_0)$. The diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & f^{-1}(G_0) & \longrightarrow & H & \longrightarrow & H/f^{-1}(G_0) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 0 & \longrightarrow & G_0 & \longrightarrow & G & \longrightarrow & G/G_0 & \longrightarrow & 1 \end{array}$$

shows that f_0 is a dense embedding as well, that the differences $r(H) - r(G)$ and $r\left(f^{-1}(G_0)\right) - r(G_0)$ coincide (this follows, of course, also from the above expression for $r(H) - r(G)$), and that f is an isomorphism iff f_0 is.

As we observed already in (1.30), if f is an isomorphism then clearly $r(H) = r(G)$. So let's assume that $r(H) = r(G)$. Then the three components in the above formula have to be zero, in particular

$$\mathrm{rk}\, \pi_1\left(f(H_0)^-\right) - \mathrm{rk}\, \pi_1(H_0) = 0 = \mathrm{rk}\,(f^{-1}(G_0)/H_0).$$

The second equation implies that $f^{-1}(G_0)/H_0$ is finite. Since f_0 is a dense embedding one deduces that $f(H_0)^- = G_0$. The first equation yields, applying (1.32) to the dense embedding $H_0 \to f(H_0)^-$, that $f(H_0)$ is closed. Altogether one obtains $f(H_0) = G_0$. Hence f_0 is an isomorphism from $f^{-1}(G_0) = H_0$ onto G_0, whence f is an isomorphism.

The reader might wonder why we didn't use that $\operatorname{rk}\pi_1(G_0/f(H_0)^-)$ is zero. Actually, if $\operatorname{rk} f^{-1}(G_0)/H_0$ is zero, i.e., if H_0 is of finite index in $f^{-1}(G_0)$, then $f(H_0)^- = G_0$. In general, there is no inequality between the ranks of $f^{-1}(G_0)/H_0$ and of $\pi_1(G_0/f(H_0)^-)$. For instance, it may happen that $\operatorname{rk}(f^{-1}(G_0)/H_0)$ is one while $\operatorname{rk}\ \pi_1(G_0/f(H_0)^-)$ is arbitrarily large. ∎

In the final chapter we shall need some information on homotopy groups of quasi–orbits. This information is supplied by the next theorem, which may be considered as a generalization of (1.32). Recall that if (G, X) is any transformation group then the G–quasi–orbit through $x \in X$ consists of all $y \in X$ such that $(Gx)^- = (Gy)^-$.

Theorem 1.34. *Let G and Q be connected Lie groups, let $\varphi : G \to Q$ be a continuous homomorphism such that the image of φ contains the commutator subgroup of Q, and let D be a closed subgroup of Q. Via φ the quotient space Q/D is a G–space. Let $\mathcal{B} \subset Q/D$ be the G–quasi–orbit through b, and denote by α the canonical map from $\mathcal{A} \stackrel{\text{def}}{=} G/G_b$ into $\mathcal{B}$.*

(i) *Then $\pi_1(\alpha)$ is injective, the image $\mathrm{im}\pi_1(\alpha)$ is normal in $\pi_1(\mathcal{B})$, and the quotient group $\pi_1(\mathcal{B})/\mathrm{im}\pi_1(\alpha)$ is abelian and torsionfree. The latter group is of finite rank for instance when D is compactly generated. Moreover, $\pi_1(\mathcal{B})/\mathrm{im}\pi_1(\alpha)$ is zero if and only if α is bijective, i.e., the quasi–orbit is an orbit.*

Let E be another closed subgroup of Q. Assume that $D \subset E$, that E/D is compact and that $\varphi^{-1}(E) = \varphi^{-1}(D)$. Actually, these assumptions imply that D is coabelian in E, whence E/D is a compact abelian Lie group. Also Q/E may be viewed as a G–space, and the quotient map $\nu : Q/D \to Q/E$ is a G–map.

(ii) *Then ν maps any G–quasi–orbit in Q/D onto a G–quasi–orbit in Q/E. Let $\mathcal{C} = \nu(\mathcal{B})$ be the G–quasi–orbit through $\nu(b)$, let $\beta : \mathcal{B} \to \mathcal{C}$ be the restriction of ν, and let $\gamma = \beta \circ \alpha : \mathcal{A} \to \mathcal{C}$.*

(iii) *Then any fiber F of β is homeomorphic to the underlying space of a compact abelian Lie group, and there is an exact sequence*

$$0 \to \pi_1(F) \to \pi_1(\mathcal{B})/\mathrm{im}\pi_1(\alpha) \to \pi_1(\mathcal{C})/\mathrm{im}\pi_1(\gamma) \to \pi_0(F) \to 0.$$

In addition, if one of the maps α or γ is bijective then all three maps α, β, γ are homeomorphisms.

Proof. W.l.o.g. we may assume that Q is simply connected and that G is a (closed, coabelian, simply connected) subgroup of Q. Furthermore we may assume that b equals the standard base point $b = eD$. The G–quasi–orbit through b is just $(GD)^-/D$. Denoting by Q' the connected component of $(GD)^-$ it is easily checked that

$$(GD)^- = Q'D \text{ and } Q' = (GD')^-$$

where $D' = Q' \cap D$. Evidently $(D')_0 = D_0$ and $G \cap D' = G \cap D$.

The coset space $\mathcal{B} = Q'/D'$ is as a G–space isomorphic to $(GD)^-/D$. Hence we are left to consider the canonical map $\alpha : \mathcal{A} \stackrel{\text{def}}{=} G/(G \cap D) \to Q'/D'$. The homomorphism $\pi_1(\alpha)$ can be identified with the obvious homomorphism $G \cap D/(G \cap D)_0 \to D'/(D')_0$. The image $\mu(D_0)$ of D_0 under the quotient homomorphism $\mu : Q \to Q/G$ is simply connected as Q/G is a vector group. Hence $\ker\mu \cap D_0 = G \cap D_0$ is connected, and it coincides with $(G \cap D)_0 = (G \cap D')_0 = G \cap (D')_0$. Therefore, $\pi_1(\alpha)$ is injective. Since $G \cap D$ is coabelian in D the image of $\pi_1(\alpha)$ is coabelian.

Because of $(G(D')_0) \cap D' = (D')_0(G \cap D)$ the quotient group $D'/(G \cap D)(D')_0$ is isomorphic to a subgroup of the vector group $Q/G(D')_0$, which shows that $\pi_1(\mathcal{B})/\mathrm{im}\pi_1(\alpha)$ is torsionfree. Clearly, if α is bijective then $\pi_1(\mathcal{B})/\mathrm{im}\pi_1(\alpha)$ vanishes. On the other hand, the vanishing of this group means that $G \cap D/(G \cap D)_0 \to D'/(D')_0$ is surjective, i.e., $D' = (D')_0(G \cap D) \subset (D')_0 G$. Hence $Q' = (GD')^- = (G(D')_0)^- = G(D')_0$ because the connected coabelian subgroup $G(D')_0$ of Q is closed. The surjectivity of α is an immediate consequence of this equation. If D is compactly generated, also the subgroup $D'/(G \cap D)(D')_0 = D'/(G \cap D)D_0$ of the abelian group $D/(G \cap D)D_0$ is compactly (finitely) generated. Hence $\pi_1(\mathcal{B})/\mathrm{im}\pi_1(\alpha)$ is free of finite rank in this case.

The second assertion (ii) follows immediately from the fact that $(GD)^-E$ is closed because E/D is compact. Hence ν maps $(GD)^-/D$ onto $(GD)^-E/E = (GE)^-/E$. Moreover, one has $(GE)^- = (GD)^-E = Q'DE = Q'E$. Therefore, the G–quasi–orbit $\mathcal{C} = (GE)^-/E$ can be identified, as a G–space, with Q'/E' where $E' \stackrel{\text{def}}{=} E \cap Q'$.

Claim (iii) could be proved by using the exact homotopy sequence of the fibration $F \to \mathcal{B} \to \mathcal{C}$ while identifying $\pi_2(\mathcal{B})$ and $\pi_2(\mathcal{C})$ appropriately. From a technical point of view this is more or less the same as we shall do in the following more direct proof not using Puppe's sequence explicitly, but mainly showing that E'_0/D_0 is simply connected.

According to the notations of the theorem we have homomorphisms $\pi_1(\alpha) : \pi_1(G/(G \cap D)) \to \pi_1(\mathcal{B})$, $\pi_1(\beta) : \pi_1(\mathcal{B}) \to \pi_1(\mathcal{C})$, and $\pi_1(\gamma) : \pi_1(G/(G \cap D)) \to \pi_1(\mathcal{C})$ with $\pi_1(\gamma) = \pi_1(\beta) \circ \pi_1(\alpha)$. Since $\mathrm{im}\pi_1(\beta)$ contains $\mathrm{im}\pi_1(\gamma)$, also $\mathrm{im}\pi_1(\beta)$ is normal and coabelian in $\pi_1(\mathcal{C})$. As $\pi_1(\gamma)$ is injective by (i) and as $\varphi^{-1}(D) = \varphi^{-1}(E)$ by assumption, elementary group theory shows that there is an exact sequence

$$0 \to \ker \pi_1(\beta) \to \pi_1(\mathcal{B})/\mathrm{im}\pi_1(\alpha) \to \pi_1(\mathcal{C})/\mathrm{im}\pi_1(\gamma) \to \pi_1(\mathcal{C})/\mathrm{im}\pi_1(\beta) \to 0$$

where the middle homomorphism is induced by $\pi_1(\beta)$. It remains to identify $\pi_1(\beta)$ appropriately.

Each fiber of $\beta : \mathcal{B} \to \mathcal{C}$ is homeomorphic to $E'/D' =: F$, which is isomorphic to a closed subgroup of E/D. Therefore, F is a compact abelian Lie group. The fundamental groups of $\mathcal{B}$ and $\mathcal{C}$ can be identified with $D'/D'_0 = D'/D_0$ and E'/E'_0, respectively. The homomorphism $\pi_1(\beta)$ corresponds to the canonical homomorphism $D'/D_0 \to E'/E'_0$. Hence the quotient $\pi_1(\mathcal{C})/\mathrm{im}\pi_1(\beta)$ is isomorphic to $E'/D'E'_0$. But as F_0 equals E'_0D'/D', the groups F/F_0 and $E'/D'E'_0$ are isomorphic.

For the final claim $\ker\pi_1(\beta)\cong\pi_1(F)$ we first observe that E_0'/D_0 is a vector group: Like in the beginning of the proof $\mu(E_0')$ is a vector subgroup of Q/G, hence $E_0'/(E_0'\cap G)$ is a vector group, and $E_0'\cap G=(E_0'\cap G)_0$ is connected. Using $E_0'\cap G\subset E\cap G=D\cap G\subset D$ one concludes that $E_0'\cap G$ is contained in D_0, which readily implies that E_0'/D_0 is a vector group.

The kernel of $\pi_1(\beta)$ can be identified with $(D'\cap E_0')/D_0$. The exact sequence

$$0\to(E_0'\cap D')/D_0\to E_0'/D_0\to DE_0'/D'\to 0$$

shows that E_0'/D_0 is the universal covering group of the torus $DE_0'/D'=F_0$, hence $(E_0'\cap D')/D_0\cong\pi_1(F_0)=\pi_1(F)$.

§ 2 Representations of Two Step Nilpotent Groups

First we shall investigate the structure of the primitive quotients of $C^*(\mathcal{G})$, where $\mathcal{G}$, as always in this section, denotes a two step nilpotent locally compact group. Let $\mathcal{K}$ be the closure of the commutator subgroup of $\mathcal{G}$ which is central in $\mathcal{G}$. Any unitary character $\lambda\in\mathcal{K}^\wedge$ defines a map $\mathcal{G}\times\mathcal{G}\to\mathbb{T}$, $(x,y)\mapsto\lambda([x,y])$. This map defines a quasi–symplectic structure $\gamma=\gamma_\lambda$ on the locally compact abelian group $G_\lambda=\mathcal{G}/\mathcal{G}_\lambda$ where $\mathcal{G}_\lambda/\ker\lambda$ is the center of $\mathcal{G}/\ker\lambda$. Let $\mathcal{J}$ be a primitive ideal in $C^*(\mathcal{G})$, say $\mathcal{J}=\ker_{C^*(\mathcal{G})}\pi$ for a continuous irreducible unitary representation π of $\mathcal{G}$. Then π is scalar on $\mathcal{K}$, $\pi(z)=\lambda(z)\mathrm{Id}$ for some $\lambda\in\mathcal{K}^\wedge$. Since π factors through $\mathcal{G}\to\mathcal{G}/\ker\lambda$, repeating the argument yields a unitary character μ on the (possibly non–abelian) group $\mathcal{G}_\lambda$ with $\pi(z)=\mu(z)\mathrm{Id}$ for $z\in\mathcal{G}_\lambda$. Clearly, μ is an extension of λ, and the pair (λ,μ) depends only on the ideal $\mathcal{J}$. The next lemma says that the pair (λ,μ) determines $\mathcal{J}$ completely. This lemma is known, compare e.g. Lemma 2 in [20]. For the convenience of the reader we include its short and simple proof. For an L^1–version of the lemma compare (2.12) below.

Lemma 2.1. *Let $\mathcal{J}$ be a primitive ideal in $C^*(\mathcal{G})$ where $\mathcal{G}$ is a locally compact two step nilpotent group. If (λ,μ) are the associated unitary characters as above then $\mathcal{J}=\ker_{C^*(\mathcal{G})}\mathrm{ind}_{\mathcal{G}_\lambda}^{\mathcal{G}}\mu=C^*(\mathcal{G})*\ker_{C^*(\mathcal{G}_\lambda)}\mu$. In particular, $\mathcal{J}$ is maximal in the set of closed ideals in $C^*(\mathcal{G})$.*

Proof. Let again $\mathcal{J}=\ker_{C^*(\mathcal{G})}\pi$ with a continuous irreducible unitary representation π of $\mathcal{G}$. For $x\in\mathcal{G}$ define π^x by $\pi^x(y)=\pi(xyx^{-1}),y\in\mathcal{G}$. The representation π^x is unitarily equivalent to π, hence $\ker_{C^*(\mathcal{G})}\pi^x=\mathcal{J}$. On the other hand, $\pi^x(y)=\pi(xyx^{-1}y^{-1}y)=\lambda([x,y])\pi(y)$. Hence π^x is unitarily equivalent to $\pi\otimes\psi(x)$, where $\psi(x)\in(\mathcal{G}/\mathcal{G}_\lambda)^\wedge=(G_\lambda)^\wedge$ is defined as in the first section. In particular $\mathcal{J}=\ker_{C^*(\mathcal{G})}\pi\otimes\psi(x)$ for all $x\in\mathcal{G}$. Since the image of ψ is dense in $(G_\lambda)^\wedge$ it follows $\mathcal{J}=\bigcap_{\chi\in(G_\lambda)^\wedge}\ker_{C^*(\mathcal{G})}\pi\otimes\chi=\ker_{C^*(\mathcal{G})}\pi\otimes\rho$ where ρ denotes the regular representation of $\mathcal{G}$ in $L^2(G_\lambda)$. But $\pi\otimes\rho$ is unitarily equivalent to $\mathrm{ind}_{\mathcal{G}_\lambda}^{\mathcal{G}}\mu$, hence $\mathcal{J}=\ker_{C^*(\mathcal{G})}\mathrm{ind}_{\mathcal{G}_\lambda}^{\mathcal{G}}\mu$. The rest is an obvious consequence of the theory of C^*–algebras. Clearly, the closed two–sided ideal $\mathcal{J}_{\lambda,\mu}\overset{\mathrm{def}}{=}C^*(\mathcal{G})*\ker_{C^*(\mathcal{G}_\lambda)}\mu$ is contained in $\ker_{C^*(\mathcal{G})}\mathrm{ind}_{\mathcal{G}_\lambda}^{\mathcal{G}}\mu$. Moreover,

$\mathcal{J}_{\lambda,\mu}$ is the intersection of all primitive ideals $\mathcal{J}'$ in $C^*(\mathcal{G})$ containing $\mathcal{J}_{\lambda,\mu}$. But the parameters (λ',μ') corresponding to $\mathcal{J}'$ have to coincide with (λ,μ), hence all the $\mathcal{J}'$ are equal to the given $\mathcal{J}$. This also shows that $\mathcal{J}$ is maximal in the set of closed ideals in $C^*(\mathcal{G})$. ■

Remark 2.2. The above lemma says that the *set* Priv $(\mathcal{G})$ of primitive ideals in $C^*(\mathcal{G})$ can be parametrized by the set $\{(\lambda,\mu)|\ \lambda\in\widehat{\mathcal{K}}, \mu\in\mathcal{G}_\lambda^\wedge$ is an extension of $\lambda\}$. In [3], BAGGETT and PACKER proved some results on the *topology* of Priv $(\mathcal{G})$ in terms of the parameters. For later use we note that the subset Priv $_\lambda(\mathcal{G})$ of all primitive ideals with the same first parameter λ forms a closed Hausdorff subspace; actually it is easy to see that this space is homeomorphic to $(\mathcal{G}_\lambda/\mathcal{K})^\wedge$, which acts simply transitively on the corresponding set of parameters. ■

In preparation of the next theorem let again $\mathcal{J}$ be a primitive ideal in $C^*(\mathcal{G})$, let (λ,μ) be the associated unitary characters on $\mathcal{K}$ and $\mathcal{G}_\lambda$, respectively, and let $(\mathcal{G}/\mathcal{G}_\lambda,\gamma_\lambda)$ be the associated quasi–symplectic space. Since λ will be fixed we write $(G,\gamma)=(\mathcal{G}/\mathcal{G}_\lambda,\gamma_\lambda)$ for short. Let P be any quasi–polarization for (G,γ) and $\mathcal{P}$ be the preimage of P under the canonical homomorphism $\mathcal{G}\to G$. The group $\mathcal{P}^\perp$ is defined accordingly. Since $\mathcal{P}/\ker\lambda$ is abelian there exists a unitary character ν on $\mathcal{P}$ with $\nu|\mathcal{G}_\lambda=\mu$. Let $\tilde{\nu}$ be a continuous projective extension of ν to $\mathcal{P}^\perp$, i.e., $\tilde{\nu}$ is a continuous map from $\mathcal{P}^\perp$ into $\mathbb{T}$ with $\tilde{\nu}(xp)=\tilde{\nu}(x)\nu(p)$ for all $x\in\mathcal{P}^\perp$, $p\in\mathcal{P}$. Such an $\tilde{\nu}$ can be constructed as follows. Take any set R of representatives for the (open !) $\mathcal{P}$–cosets in $\mathcal{P}^\perp$ and define $\tilde{\nu}(rp)=\nu(p)$ for $r\in R$ and $p\in\mathcal{P}$. Any chosen $\tilde{\nu}$ defines a map $m:\mathcal{P}^\perp\times\mathcal{P}^\perp\to\mathbb{T}$, $m(x,y)=\tilde{\nu}(xy)\tilde{\nu}(x)^{-1}\tilde{\nu}(y)^{-1}$ which factors through $\mathcal{P}^\perp\to\mathcal{P}^\perp/\mathcal{P}$ and hence delivers a map $\mathcal{P}^\perp/\mathcal{P}\times\mathcal{P}^\perp/\mathcal{P}\to\mathbb{T}$, also denoted by m. Clearly, m is a cocycle on the discrete abelian group $\mathcal{P}^\perp/\mathcal{P}$ which is canonically isomorphic to $P^\perp/P$.

Theorem 2.3. *Let $\mathcal{G},\mathcal{J},\lambda,\mu,\mathcal{P},\mathcal{P}^\perp,\nu,\tilde{\nu}$ and m be as above. Then the quotient algebra $C^*(\mathcal{G})/\mathcal{J}$ is isomorphic to the C^*–tensor product of $C^*(\mathcal{P}^\perp/\mathcal{P},m)$ and the algebra of compact operators on $L^2(\mathcal{G}/\mathcal{P}^\perp)$. Here $C^*(\mathcal{P}^\perp/\mathcal{P},m)$ denotes the C^*–completion of the twisted convolution algebra $\ell^1(\mathcal{P}^\perp/\mathcal{P},m)$ where the multiplication and the involution are given by*

$$(f*g)(x)=\sum_{y\in\mathcal{P}^\perp/\mathcal{P}} f(xy)g(y^{-1})m(xy,y^{-1}), f^*(x)=f(x^{-1})^- m(x^{-1},x)^{-1}.$$

The antisymmetrization $\alpha=\alpha_m$ of m, $\alpha(x,y)=m(x,y)m(y,x)^{-1}$ for $x,y\in\mathcal{P}^\perp/\mathcal{P}$, and hence the cohomology class of m is independent of the choice of ν and $\tilde{\nu}$. Actually, $\alpha(x,y)=\gamma(x,y)=\lambda([x,y])$ for $x,y\in\mathcal{P}^\perp$. In particular, the algebras $\ell^1(\mathcal{P}^\perp/\mathcal{P},m)$ (and their C^*–completions) are isomorphic for different choices of $\nu,\tilde{\nu}$.

Remarks 2.4. The Hilbert space $L^2(\mathcal{G}/\mathcal{P}^\perp)$, whose algebra of compact operators appears in the theorem is finite–dimensional if and only if $\mathcal{G}/\mathcal{G}_\lambda$ is discrete.

The theorem shows that the isomorphism class of a primitive quotient of $C^*(\mathcal{G})$ depends only on λ and not on μ, because for a given λ one may choose

P simultaneously for all extensions μ of λ and because also the cohomology class of the cocycle m only depends on λ. We conjecture that at least in case $\mathrm{Inv}(G,\gamma_\lambda) = [\mathbb{Z}^n]$ the quotient of $C^*(\mathcal{G})$, which corresponds to λ, is stably isomorphic to a tensor product of a commutative algebra and of $C^*(\mathcal{P}^\perp/\mathcal{P}, m)$.

The quotient group $\mathcal{P}^\perp/\mathcal{P}$, let alone the cocycle m living on it, is not uniquely determined by a given primitive ideal, but different choices of P lead to quotients which are equivalent in the sense of (1.12). This gives rise to the question: Let m_1 and m_2 be cocycles on the discrete abelian groups D_1 and D_2, respectively. Suppose that m_1 and m_2 are non–degenerate in the sense that their antisymmetrizations define quasi–symplectic structures on D_1 and D_2. Is it true that if $C^*(D_1,m_1)$ and $C^*(D_2,m_2)$ are (stably) isomorphic, the groups D_1 and D_2 are equivalent in the sense of (1.12)? The answer is yes for finitely generated groups because in this case by the results of [14] the rank of D_1 is an invariant of the (stable) isomorphism class of $C^*(D_1,m_1)$.

Now suppose that $\mathrm{Inv}(G,\gamma)$ contains $\mathbb{Z}^n$ for a certain n, in (1.22) we formulated a criterion for this situation. Then one may choose the quasi–polarization P such that $\mathcal{P}^\perp/\mathcal{P}$ is isomorphic to $\mathbb{Z}^n$, see (1.19). Hence the corresponding primitive quotients of $C^*(\mathcal{G})$ are isomorphic to tensor products of the algebra of compact operators and of noncommutative tori. This generalizes [31, Theorem 1]. There is an extensive literature on noncommutative tori, in the references we give a sample, namely [6,9,14,41,42,43,44]. Theorem (1.19) characterizes the number n in terms of the quasi–symplectic space (G,γ). In case that G is compactly generated the number n depends only on the structure of the topological group G, see (1.28); for the computation of n cf. also (1.31).■

Proof of Theorem 2.3. Let's first consider the antisymmetrization α. For $x,y\in\mathcal{P}^\perp$ one has

$$\begin{aligned}\alpha(x,y) =&\tilde\nu(xy)\tilde\nu(x)^{-1}\tilde\nu(y)^{-1}\tilde\nu(yx)^{-1}\tilde\nu(x)\tilde\nu(y) =\\ =&\tilde\nu(yxx^{-1}y^{-1}xy)\tilde\nu(yx)^{-1} = \tilde\nu(yx)\nu(x^{-1}y^{-1}xy)\tilde\nu(yx)^{-1} =\\ =&\lambda([x^{-1},y^{-1}]) = \lambda([x,y]).\end{aligned}$$

Denote by $L^1(\mathcal{G})_\mu$ the Banach space of all measurable functions $f:\mathcal{G}\to\mathbb{C}$ with $f(xz)=f(x)\mu(z)^{-1}$ for all $x\in\mathcal{G}$, $z\in\mathcal{G}_\lambda$ such that $\|f\| = \int\limits_{\mathcal{G}/\mathcal{G}_\lambda} |f|(x)\,dx$ is finite. The space $L^1(\mathcal{G})_\mu$ is an involutive Banach algebra with the usual involution, and the convolution is given by

$$(f*g)(x) = \int_{\mathcal{G}/\mathcal{G}_\lambda} f(xy)g(y^{-1})\,dy.$$

The C^*–hull of $L^1(\mathcal{G})_\mu$ is denoted by $C^*(\mathcal{G})_\mu$. By (2.1) the quotient $C^*(\mathcal{G})/\mathcal{J}$ is isomorphic to $C^*(\mathcal{G})_\mu$. Similarily, one may form $L^1(\mathcal{P})_\mu$ and $C^*(\mathcal{P})_\mu$. Since $C^*(\mathcal{P})_\mu$ is a quotient of $C^*(\mathcal{P}/\ker\lambda)$ and the latter algebra is commutative, the algebra $C^*(\mathcal{P})_\mu$ is commutative, too. Its structure space $C^*(\mathcal{P})^\wedge_\mu$ can be identified with the unitary characters on $\mathcal{P}$ which are extensions of μ, i.e., with the coset $\nu(\mathcal{P}/\mathcal{G}_\lambda)^\wedge$ in $\mathcal{P}^\wedge$. The group $\mathcal{G}$ acts by conjugation on $L^1(\mathcal{P})$, $f^x(p) =$

$f(xpx^{-1})$ for $x \in \mathcal{G}$, $p \in \mathcal{P}$. This action factors through $L^1(\mathcal{P}) \to L^1(\mathcal{P})_\mu$, $f \mapsto f'$, $f'(x) = \int_{\mathcal{G}_\lambda} f(xz)\mu(z)\,dz$, and induces an action of $\mathcal{G}$ on $L^1(\mathcal{P})_\mu$ and on $C^*(\mathcal{P})_\mu$ from the right and an action on $C^*(\mathcal{P})^\wedge_\mu = (\mathcal{P}/\mathcal{G}_\lambda)^\wedge \nu$ from the left. Explicitly, for $x \in \mathcal{G}$, $p \in \mathcal{P}$ and $\nu' \in C^*(\mathcal{P})^\wedge_\mu$ one has $(x\nu')(p) = \nu'(x^{-1}px) = \nu'(pp^{-1}x^{-1}px) = \nu'(p)\lambda([p,x])$, i.e., $x\nu' = \nu'\psi(x)^{-1}$ where $\psi = \psi_P$ denotes the map introduced in the first section. Since P is in particular a prepolarization the map ψ induces an isomorphism from $G/P^\perp \cong \mathcal{G}/\mathcal{P}^\perp$ onto $(\mathcal{P}/\mathcal{G}_\lambda)^\wedge$. By means of the chosen base point ν the space $C^*(\mathcal{P})^\wedge_\mu$ can be identified with $\mathcal{G}/\mathcal{P}^\perp$, and the $\mathcal{G}$–action becomes just translation. Hence the $\mathcal{G}$–algebra $C^*(\mathcal{P})_\mu$ can be identified with $C_\infty(\mathcal{G}/\mathcal{P}^\perp)$ where $\mathcal{G}$ acts by translations.

In the sense of [16, 17] the C^*–algebra $C^*(\mathcal{G})_\mu = C^*(\mathcal{G})/\mathcal{J}$ may be viewed as the twisted covariance algebra $C^*(\mathcal{G}, C^*(\mathcal{P})_\mu, \tau)$ where the action of $\mathcal{G}$ on $C^*(\mathcal{P})_\mu$ is as above and the twist τ is based on the translations of $\mathcal{P}$ on $C^*(\mathcal{P})_\mu$. In the transformed picture $C_\infty(\mathcal{G}/\mathcal{P}^\perp) \cong C_\infty(G/P^\perp)$ of $C^*(\mathcal{P})_\mu$ the twist is given by $\tau(p)f = \nu(p)\varphi_P(p)f$ for $p \in \mathcal{P}$ and $f \in C_\infty(G/\mathcal{P}^\perp)$, say, where $\varphi_P(p) \in (G/P^\perp)^\wedge$ is as in the first section. In the third part of this article we shall use a slightly modified definition of twisted covariance algebras; this will be explained there in greater detail. Since the $\mathcal{G}$–space $C^*(\mathcal{P})^\wedge_\mu$ was determined above explicitly, the twisted covariance algebra at hand is a particularly easy special case of the situations studied in [17, Theorem 2.13 and 3.1]. These theorems give at once that $C^*(\mathcal{G})/\mathcal{J} = C^*(\mathcal{G}, C^*(\mathcal{P})_\mu, \tau)$ has the structure as claimed — in case that $\mathcal{G}$ is second countable. To avoid this assumption one has to use the cross sections constructed by Kehlet, [21]. Kehlet has even explicitly formulated one of Green's isomorphism theorems, namely in the case of transformation group C^*–algebras, which is good enough for our purposes. Observe that $C^*(\mathcal{G}, C^*(\mathcal{P})_\mu, \tau)$ is by its very definition a quotient of the untwisted covariance algebra $C^*(\mathcal{G}, C^*(\mathcal{P})_\mu)$; one has to divide out some relations due to the twist. But as we have seen above $C^*(\mathcal{G}, C^*(\mathcal{P})_\mu)$ is isomorphic to the transformation group C^*–algebra $C^*(\mathcal{G}, C_\infty(\mathcal{G}/\mathcal{P}^\perp))$, which is by [21] isomorphic to the tensor product of $C^*(\mathcal{P}^\perp)$ and the algebra $\mathfrak{K}$ of compact operators on $L^2(\mathcal{G}/\mathcal{P}^\perp)$. Dividing out the twist gives that $C^*(\mathcal{G}, C^*(\mathcal{P})_\mu, \tau)$ is isomorphic to the tensor product of $\mathfrak{K}$ with $C^*(\mathcal{P}^\perp)_\nu$. And $C^*(\mathcal{P}^\perp)_\nu$ is isomorphic to $C^*(\mathcal{P}^\perp/\mathcal{P}, m)$. ∎

In the following comments we retain the above notations. To get all continuous irreducible unitary representation of $\mathcal{G}$ whose C^*–kernel coincides with the given $\mathcal{J}$ one may proceed as follows. Take any irreducible unitary representation σ of $\mathcal{P}^\perp$ with $\sigma|_\mathcal{P} = \nu \mathrm{Id}$ and form $\mathrm{ind}_{\mathcal{P}^\perp}^{\mathcal{G}} \sigma$. Hence the determination of the unitary dual $\mathcal{G}^\wedge$ is as difficult as in the discrete case — and this is difficult enough.

The quasi–polarization P can be used to write down a factor representation π of $\mathcal{G}$ with $\mathcal{J} = \ker_{C^*} \pi$, namely $\pi = \mathrm{ind}_{\mathcal{P}}^{\mathcal{G}} \nu$. We are going to show that this factor representation has some particular properties. The space $\mathfrak{H}$ of π consists of measurable functions $\xi : \mathcal{G} \to \mathbb{C}$ such that $\xi(xp) = \nu(p)^{-1}\xi(x)$ for $p \in \mathcal{P}$ and $x \in \mathcal{G}$ and that $\int_{\mathcal{G}/\mathcal{P}} |\xi(x)|^2\,dx = \|\xi\|^2 < \infty$. The representation π is given by $(\pi(x)\xi)(y) = \xi(x^{-1}y)$ for $x, y \in \mathcal{G}$. By induction in stages π is unitarily equivalent to $\mathrm{ind}_{\mathcal{P}^\perp}^{\mathcal{G}} \beta$, where $\beta = \mathrm{ind}_{\mathcal{P}}^{\mathcal{P}^\perp} \nu$ acts similarly in a space $\mathfrak{M}$

of (measurable) functions on $\mathcal{P}^\perp$. The group $\mathcal{P}^\perp$ acts also 'from the right' on the functions in $\mathfrak{H}$ and in $\mathfrak{M}$

$$(\kappa(q)\xi)(x) = \xi(xq)$$
$$(\rho(q)\eta)(a) = \eta(aq)$$

for $a, q \in \mathcal{P}^\perp$, $x \in \mathcal{G}$, $\xi \in \mathfrak{H}$ and $\eta \in \mathfrak{M}$. The group $\mathcal{P}^\perp$ behaves relative to $(\mathcal{P}, \nu)$ very much like a discrete group with infinite conjugacy classes, i.e., the associated von Neumann algebras $\beta(\mathcal{P}^\perp)''$ and $\rho(\mathcal{P}^\perp)''$ are finite factors (the type I case occurs only when $\mathcal{P}^\perp/\mathcal{P}$ is finite) and they are commutants of each other. The trace of $A \in \beta(\mathcal{P}^\perp)''$ is given as the scalar product $(A\eta_0, \eta_0)$ where $\eta_0 \in \mathfrak{M}$ is defined by $\eta_0(p) = \nu(p)^{-1}$ for $p \in \mathcal{P}$ and $\eta_0(\mathcal{P}^\perp \setminus \mathcal{P}) = 0$. These facts can be proved exactly as in the case of discrete groups with infinite conjugacy classes, see e.g. [45, section 4.2].

A slight modification gives that the commutant $\pi(\mathcal{G})'$ equals $\kappa(\mathcal{P}^\perp)''$. Moreover, $\kappa(\mathcal{P}^\perp)''$ is canonically isomorphic to $\rho(\mathcal{P}^\perp)''$, hence $\pi(\mathcal{G})'$ is a finite factor of type I or II_1. The isomorphism is given 'pointwise', for $A \in \rho(\mathcal{P}^\perp)''$ define $\widetilde{A}$ on the space of $\mathrm{ind}_{\mathcal{P}^\perp}^{\mathcal{G}}\beta$ by $(\widetilde{A}\xi)(x) = A(\xi(x))$. Actually, the equation $\pi(\mathcal{G})' = \kappa(\mathcal{P}^\perp)''$ is proved along this line: If $T \in \pi(\mathcal{G})'$, where again π is thought of being $\pi = \mathrm{ind}_{\mathcal{P}^\perp}^{\mathcal{G}}\beta$, then $(T\xi)(x) = A(\xi(x))$ for some operator A on $\mathfrak{M}$ which has necessarily to be in $\beta(\mathcal{P}^\perp)' = \rho(\mathcal{P}^\perp)''$.

Clearly, in general $\pi(\mathcal{G})''$ is not a finite factor. One has the following easy characterizations.

Remark 2.5. *$\pi(\mathcal{G})''$ is a finite factor of type* I *iff $G = \mathcal{G}/\mathcal{G}_\lambda$ is finite,*

$\pi(\mathcal{G})''$ is an infinite factor of type I *iff $\mathcal{P}^\perp/\mathcal{P}$ is finite and $\mathcal{G}/\mathcal{G}_\lambda$ is infinite. This means in particular that the invariant* Inv *of the associated quasi–symplectic space (G, γ) is trivial, compare* (1.23).

$\pi(\mathcal{G})''$ is a factor of type II_1 *iff $\mathcal{G}/\mathcal{P}^\perp$ is finite and* $\mathrm{Inv}(G,\gamma)$ *is non–trivial iff G is discrete and infinite.*

In all other cases, i.e., G is not discrete and $\mathrm{Inv}(G,\gamma)$ *is non–trivial, $\pi(\mathcal{G})''$ is a factor of type* II_∞. ∎

For the needs in a forthcoming paper, [33], on operators of finite rank in the image of L^1–algebras on connected Lie groups under suitable representations we now discuss under which conditions $\sigma(L^1(\mathcal{G}))$ contains non–zero finite rank operators for a continuous irreducible unitary representation σ. Certainly, it is necessary that $\sigma(C^*(\mathcal{G}))$ contains the compact operators. This condition will turn out to be sufficient. But first we draw the following easy consequence from (2.3).

Corollary 2.6. *Let σ be a continuous irreducible unitary representation of the locally compact two step nilpotent group $\mathcal{G}$. Let (λ, μ) be the unitary characters on the groups $[\mathcal{G},\mathcal{G}]^-$ and $\mathcal{G}_\lambda$, respectively, associated with the primitive ideal $\ker_{C^*(\mathcal{G})}\sigma$ and let $(G,\gamma) = (\mathcal{G}/\mathcal{G}_\lambda, \gamma_\lambda)$ be the corresponding quasi–symplectic space. Then the following conditions are equivalent:*

(a) *The image of $C^*(\mathcal{G})$ under σ contains the compact operators.*

(b) *The image of $C^*(\mathcal{G})$ under σ is equal to the algebra of compact operators.*
(c) *The invariant* $\mathrm{Inv}(G,\gamma)$ *is trivial, in particular there exists a quasi–polariza tion P of (G,γ) with $P^\perp = P$.*

Under these conditions let P be any quasi–polarization as in (c), let again $\mathcal{P}$ be the preimage of P under the canonical map $\mathcal{G} \to G$ and let ν be a unitary character of $\mathcal{P}$ with $\nu|_{\mathcal{G}_\lambda} = \mu$. Then σ is unitarily equivalent to $\mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}\nu$.

Remark 2.7. In view of (1.23), condition (c) says that the canonical map $\psi_G : G \to \widehat{G}$ is an isomorphism. This is the classical Baggett–Kleppner criterion for a twisted convolution C^*–algebra on a locally compact abelian group to be of type I, see [2].

Proof of Corollary 2.6. The equivalence of (a) and (b) follows from the facts that $\ker_{C^*(\mathcal{G})}\sigma$ is maximal in the set of closed ideals in $C^*(\mathcal{G})$ and that the compact operators form a closed two–sided ideal in the algebra of all bounded operators. Let $\mathcal{P}$ be the preimage of any quasi–polarization in (G,γ) and let m be as in front of (2.3). Then $C^*(\mathcal{G})/\ker_{C^*(\mathcal{G})}\sigma$ is isomorphic to the algebra of compact operators iff $C^*(\mathcal{P}^\perp/\mathcal{P}, m)$ is isomorphic to the algebra of compact operators. Since $C^*(\mathcal{P}^\perp/\mathcal{P}, m)$ has a unit this is equivalent to the finiteness of $\mathcal{P}^\perp/\mathcal{P} \cong P^\perp/P$. But the latter property is one of the equivalent conditions of (1.23).

If $\mathcal{P}$ and ν are as in the corollary then $\mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}\nu$ is irreducible. Moreover, the C^*–kernels of $\mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}\nu$ and σ coincide, namely with the unique primitive ideal associated to the parameters (λ,μ). Since the algebra of compact operators has up to equivalence only one irreducible representation the representations σ and $\mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}\nu$ are unitarily equivalent. ∎

If σ is given as above we want to construct a function f on $\mathcal{G}$ such that $\sigma(f)$ is an orthogonal projection of rank one and that f is not only integrable, but even integrable when multiplied by any weight function on $\mathcal{G}$. Recall, cf. [40, pp. 83] that a weight (function) is a measurable function $w : \mathcal{G} \to \mathbb{R}$ such that

(2.8)
$$w(x) \geq 1, \quad w(xy) \leq w(x)w(y) \text{ for all } x,y \in \mathcal{G}\ ,$$
$$\text{and } w \text{ is bounded on every compact subset of } \mathcal{G}$$

With each weight function w can be canonically associated an upper semi–continuous weight function $\widetilde{w}$, namely $\widetilde{w}(x) = \inf\limits_{V} \sup\limits_{y\in V} w(xy)$ where V ranges over the compact neighborhoods of the identity. The weights w and $\widetilde{w}$ are related by $w(x) \leq \widetilde{w}(x) \leq \widetilde{w}(e)w(x)$ for $x \in \mathcal{G}$.

Theorem 2.9. *Let σ be a continuous irreducible unitary representation of the locally compact two step nilpotent group $\mathcal{G}$ such that $\sigma(C^*(\mathcal{G}))$ contains the compact operators. Then there exists a continuous function f on $\mathcal{G}$ such that*
(a) $\int_{\mathcal{G}} |f(x)|w(x)dx < \infty$ *for all weight functions w on $\mathcal{G}$,*
(b) $\sigma(f)$ *is an orthogonal projection of rank one,*
(c) $\sup\limits_{x\in\mathcal{G}} |f(x)|w(x) < \infty$ *for all weight functions w on $\mathcal{G}$.*

Proof. As usual denote by (λ,μ) the unitary characters on $\mathcal{K} = [\mathcal{G},\mathcal{G}]^-$ and $\mathcal{G}_\lambda$, resp., associated with $\ker_{C^*(\mathcal{G})}\sigma$, and by $(G,\gamma) = (\mathcal{G}/\mathcal{G}_\lambda, \gamma_\lambda)$ the corresponding quasi–symplectic space. By (2.6) the invariant $\mathrm{Inv}(G,\gamma)$ is trivial, hence by (1.23) there exists a compact subgroup L of G with the properties that $L \subset L^\perp$, $U \stackrel{\mathrm{def}}{=} L^\perp$ is open in G and U/L is an ordinary symplectic space. Choose a polarization P/L of the symplectic space U/L and denote by $\mathcal{L},\mathcal{P}$ and $\mathcal{U}$ the preimages of L, P and U, respectively, under the quotient homomorphism $\mathcal{G} \to G$. Let ν be any unitary character of $\mathcal{P}$ with $\nu|_{\mathcal{G}_\lambda} = \mu$. We may assume that σ is equal to $\mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}\nu$. The intersection $\mathcal{L} \cap \ker\nu$ is normal in $\mathcal{U}$ as $L^\perp = U$. The quotient group $\mathcal{U}/(\mathcal{L} \cap \ker\nu)$ is isomorphic to a Heisenberg group with compact center.

To define the function f we have to introduce some further notation. There are one–parameter groups $\alpha'_1,\dots,\alpha'_n : \mathbb{R} \to \mathcal{U}/(\mathcal{L}\cap\ker\nu)$ and $\beta'_1,\dots,\beta'_n : \mathbb{R} \to \mathcal{U}/(\mathcal{L}\cap\ker\nu)$, $n = \dim P/L$, with the following properties. If $\alpha',\beta' : \mathbb{R}^n \to \mathcal{U}/(\mathcal{L}\cap\ker\nu)$ are defined by

$$\alpha'(t) = \alpha'(t_1,\dots,t_n) = \alpha'_n(t_n)\cdot\ldots\cdot\alpha'_1(t_1)$$

and

$$\beta'(s) = \beta'(s_1,\dots,s_n) = \beta'_n(s_n)\cdot\ldots\cdot\beta'_1(s_1)$$

then α' and β' are homomorphisms, $\alpha'(\mathbb{R}^n)\mathcal{L}/(\mathcal{L}\cap\ker\nu) = \mathcal{P}/(\mathcal{L}\cap\ker\nu)$ and the map

$$\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{L}/(\mathcal{L}\cap\ker\nu) \ni (s,t,l) \mapsto \beta'(s)\alpha'(t)l \in \mathcal{U}/(\mathcal{L}\cap\ker\nu)$$

is a homeomorphism.

The one–parameter groups α'_j,β'_j can be lifted against the quotient map $\rho : \mathcal{U} \to \mathcal{U}/(\mathcal{L}\cap\ker\nu)$, i.e., there exist continuous homomorphism $\alpha_j,\beta_j : \mathbb{R} \to \mathcal{U}$, $1 \le j \le n$, such that $\rho\circ\alpha_j = \alpha'_j$ and $\rho\circ\beta = \beta'_j$. If $\alpha : \mathbb{R}^n \to \mathcal{U}$ and $\beta : \mathbb{R}^n \to \mathcal{U}$ are defined by

$$\alpha(t) = \alpha_n(t_n)\cdot\ldots\cdot\alpha_1(t_1) \text{ and } \beta(s) = \beta_n(s_n)\cdot\ldots\cdot\beta_1(s_1)$$

$$\text{then } \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{L} \ni (s,t,l) \mapsto \beta(s)\alpha(t)l \in \mathcal{U}$$

is a homeomorphism.

The continuous map $\kappa : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{K} \subset \mathcal{G}_\lambda$ is defined by

$$\kappa(x,t) = \alpha(t)^{-1}\beta(s)^{-1}\alpha(t)\beta(s) = [\alpha(t)^{-1},\beta(s)^{-1}] = [\alpha(t),\beta(s)].$$

The map $(s,t) \mapsto \nu(\kappa(s,t)) = \nu'([\alpha'(t),\beta'(s)])$, if $\nu' : \mathcal{L}/(\mathcal{L}\cap\ker\nu) \to \mathbb{T}$ denotes the induced character, is a biadditive continuous map from $\mathbb{R}^n\times\mathbb{R}^n$ into $\mathbb{T}$. If $(\ ,\ )$ denotes the standard scalar product on $\mathbb{R}^n$ then there is a unique real non–singular n by n matrix Δ with

$$\nu(\kappa(s,t)) = e^{2\pi i(s,\Delta t)}$$

for $s,t \in \mathbb{R}^n$. Denote by δ the absolute value of the determinant of Δ.

Next we normalize the Haar measures on the various groups appropriately. Starting with an arbitrarily chosen Haar measure on $\mathcal{G}_\lambda$ choose Haar measures on $\mathcal{L}$ and $\mathcal{L}/\mathcal{G}_\lambda$ such that Weil's formula holds true w.r.t. $(\mathcal{G}_\lambda, \mathcal{L})$ and that $\mathcal{L}/\mathcal{G}_\lambda$ has total mass one. Then choose Haar measures on $\mathcal{U}$ and $\mathcal{U}/\mathcal{L}$ such that Weil's formula holds true w.r.t. $(\mathcal{U}, \mathcal{L})$ and that the measure on $\mathcal{U}/\mathcal{L}$ corresponds to the Lebesgue measure on $\mathbb{R}^{2n}$ under the above identification. Finally, extend the Haar measure on the open subgroup $\mathcal{U}$ to the whole of $\mathcal{G}$ in the most obvious way. To define the induced representation σ one needs an invariant measure on $\mathcal{G}/\mathcal{P}$. This is fixed again by requiring that Weil's formula is true w.r.t. $(\mathcal{G}, \mathcal{P})$ and that $\mathcal{P}$ carries the product measure under the identification $\mathbb{R}^n \times \mathcal{L} \ni (t, l) \mapsto \alpha(t)l \in \mathcal{P}$.

With these notations we define $h : \mathcal{U} \to \mathbb{C}$ by

$$h(\beta(s)\alpha(t)l) = \delta\nu(\alpha(t)l)^{-1} e^{-\frac{\pi}{2}(s,s)} e^{-\frac{\pi}{2}(\Delta t, \Delta t)} e^{\pi i (\Delta t, s)}$$

where $s, t \in \mathbb{R}^n$, $l \in \mathcal{L}$.

Finally we choose a compactly supported continuous function b on $\mathcal{L}$ such that

$$\int_{\mathcal{G}_\lambda} dz\, b(lz)\nu(lz) = 1$$

for all $l \in \mathcal{L}$. The existence of such a function b follows immediately from the existence of Bruhat functions for the extension $\mathcal{L} \to \mathcal{L}/\mathcal{G}_\lambda$, cf. [40, p. 163]. Then we define $f : \mathcal{U} \to \mathbb{C}$ by

$$f(\beta(s)\alpha(t)l) = h(\beta(s)\alpha(t))b(l)$$

for $s, t \in \mathbb{R}^n$ and $l \in \mathcal{L}$, and extend f in the most obvious way to the whole $\mathcal{G}$, i.e., by requiring that $f(\mathcal{G} \setminus \mathcal{U}) = 0$.

We claim that f has the properties (a), (b), (c) of the theorem. Concerning (a) and (c) we note that from the submultiplicativity of w it follows that

$$w(\beta(s)\alpha(t)l) \le w(l)w(\beta_n(s_n)) \cdot \ldots \cdot w(\beta_1(s_1))w(\alpha_n(t_n)) \cdot \ldots \cdot w(\alpha_1(t_1))$$

Since for each j the function $t_j \mapsto w(\alpha_j(t_j))$ is a weight function on the real line there exist positive constants C_j and W_j such that $w(\alpha_j(t_j)) \le C_j e^{W_j |t_j|}$ for all $t_j \in \mathbb{R}$. The same applies to $s_j \mapsto w(\beta_j(s_j))$. Hence altogether there are positive constants C, W such that

$$w(\beta(s)\alpha(t)l) \le C\, w(l) e^{W(|t_1| + \cdots + |t_n| + |s_1| + \cdots + |s_n|)}$$

Claims (a) and (c) follow immediately from the structure of f, in particular from the fact that b is compactly supported and that w is bounded on compact sets.

Concerning (b) we shall compute $\sigma(f)\xi$ for a vector ξ in the space of $\sigma = \mathrm{ind}_{\mathcal{P}}^{\mathcal{G}} \nu$, i.e., ξ is a function $G \to \mathbb{C}$ with $\xi(xp) = \nu(p)^{-1}\xi(x)$ for $x \in \mathcal{G}$, $p \in \mathcal{P}$, and w.l.o.g. ξ is continuous and compactly supported modulo $\mathcal{P}$.

Using Weil's formula one obtains

$$\sigma(f)\xi = \int_{\mathcal{U}/\mathcal{G}_\lambda} dx \int_{\mathcal{G}_\lambda} dz\, f(xz)\sigma(xz)\xi = \int_{\mathcal{U}/\mathcal{G}_\lambda} dx \int_{\mathcal{G}_\lambda} dz\, f(xz)\mu(z)\sigma(x)\xi$$

For $x \in \mathcal{U}$, $x = \beta(s)\alpha(t)l$ with $s, t \in \mathbb{R}^n$, $l \in \mathcal{L}$, the inner integral equals

$$\int_{\mathcal{G}_\lambda} dz\, h(\beta(s)\alpha(t))b(lz)\mu(z) = \int_{\mathcal{G}_\lambda} dz\, h(\beta(s)\alpha(t))\nu(l)^{-1}b(lz)\nu(l)\mu(z) = h(x)$$

by the choice of b and the definition of h. Therefore, again by Weil's formula

$$\begin{aligned}\sigma(f)\xi &= \int_{\mathcal{U}/\mathcal{G}_\lambda} dx\, h(x)\sigma(x)\xi = \int_{\mathcal{U}/\mathcal{L}} dx \int_{\mathcal{L}/\mathcal{G}_\lambda} dl\, h(xl)\sigma(xl)\xi = \\ &= \int_{\mathcal{U}/\mathcal{L}} dx \int_{\mathcal{L}/\mathcal{G}_\lambda} dl\, h(x)\nu(l)^{-1}\sigma(xl)\xi = \\ &= \int_{\mathcal{U}/\mathcal{L}} dx\, h(x)\sigma(x) \int_{\mathcal{L}/\mathcal{G}_\lambda} dl\, \nu(l)^{-1}\sigma(l)\xi\end{aligned}$$

For $x \in \mathcal{U}$ the inner integral evaluated at $y \in \mathcal{G}$ yields

$$\begin{aligned}\int_{\mathcal{L}/\mathcal{G}_\lambda} dl\, \nu(l)^{-1}\xi(l^{-1}y) &= \int_{\mathcal{L}/\mathcal{G}_\lambda} dl\, \nu(l)^{-1}\xi(yy^{-1}l^{-1}y) = \\ &= \int_{\mathcal{L}/\mathcal{G}_\lambda} dl\, \nu(l)^{-1}\nu(y^{-1}ly)\xi(y) = \int_{\mathcal{L}/\mathcal{G}_\lambda} dl\, \gamma(y,l)^{-1}\xi(y)\end{aligned}$$

Because of $U = L^\perp$, if $y \notin \mathcal{U}$ then $l \mapsto \gamma(y,l)^{-1}$ is a non–trivial character on $\mathcal{L}/\mathcal{G}_\lambda$, hence the integral over l is zero. If $y \in \mathcal{U}$ then $l \mapsto \gamma(y,l)^{-1}$ is identical to one, hence the integral is one. We conclude that $\int_{\mathcal{L}/\mathcal{G}_\lambda} dl\, \nu(l)^{-1}\sigma(l)\xi =: \xi'$ is the restriction of ξ to $\mathcal{U}$. So far we obtained that

$$\sigma(f)\xi = \int_{\mathcal{U}/\mathcal{L}} dx\, h(x)\sigma(x)\xi'$$

In particular, $\sigma(f)\xi$ is supported by $\mathcal{U}$, too.

The vector $\sigma(f)\xi$ is known if all the values $(\sigma(f)\xi)(\beta(s_0))$, $s_0 \in \mathbb{R}^n$, are known. Using the chosen representatives for the $\mathcal{L}$–cosets in $\mathcal{U}$ one obtains

$$(\sigma(f)\xi)(\beta(s_0)) = \int_{\mathbb{R}^n} ds \int_{\mathbb{R}^n} dt\, h(\beta(s)\alpha(t))\xi'(\alpha(t)^{-1}\beta(s)^{-1}\beta(s_0)) \ .$$

Since $\beta(s)^{-1}\beta(s_0) \equiv \beta(s_0 - s) \mod \mathcal{L} \cap \ker \nu$ one has

$$\begin{aligned}&\xi'(\alpha(t)^{-1}\beta(s)^{-1}\beta(s_0)) = \xi'(\alpha(t)^{-1}\beta(s_0 - s)) = \\ &= \xi'(\beta(s_0-s)\beta(s_0-s)^{-1}\alpha(t)^{-1}\beta(s_0-s)\alpha(t)\alpha(t)^{-1}) = \\ &= \nu(\alpha(t))\nu([\beta(s_0-s)^{-1}, \alpha(t)^{-1}]^{-1}\xi'(\beta(s_0-s)) = \\ &= \nu(\alpha(t))\nu(\kappa(s_0-s,t))\xi'(\beta(s_0-s)) = \\ &= \nu(\alpha(t))e^{2\pi i(s_0-s,\Delta t)}\xi'(\beta(s_0-s))\end{aligned}$$

With $s_0 - s$ as new variable of integration one gets

$$
\begin{aligned}
&(\sigma(f)\xi)(\beta(s_0)) = \\
&\qquad = \int_{\mathbb{R}^n} ds \int_{\mathbb{R}^n} dt\, h(\beta(s_0 - s)\alpha(t))\nu(\alpha(t))e^{2\pi i(s,\Delta t)}\xi'(\beta(s)) = \\
&\qquad = \int_{\mathbb{R}^n} ds \int_{\mathbb{R}^n} dt\, \delta\nu(\alpha(t))^{-1} e^{-\frac{\pi}{2}(s_0-s,s_0-s)} \\
&\qquad\qquad\qquad e^{-\frac{\pi}{2}(\Delta t,\Delta t)} e^{\pi i(s_0-s,\Delta t)} \nu(\alpha(t)) e^{2\pi i(s,\Delta t)}\xi'(\beta(s)) = \\
&\qquad = \int_{\mathbb{R}^n} ds \int_{\mathbb{R}^n} dt\, \delta e^{-\frac{\pi}{2}(s_0-s,s_0-s)} e^{-\frac{\pi}{2}(\Delta t,\Delta t)} e^{\pi i(s_0+s,\Delta t)}\xi'(\beta(s))\ .
\end{aligned}
$$

Substituting $t' = \Delta t$ and writing $t \overset{\text{def}}{=} t'$ again one obtains

$$
(\sigma(f)\xi)(\beta(s_0)) = \int_{\mathbb{R}^n} ds \int_{\mathbb{R}^n} dt\, e^{-\frac{\pi}{2}(s_0-s,s_0-s)} e^{-\frac{\pi}{2}(t,t)} e^{\pi i(s_0+s,t)}\xi'(\beta(s))
$$

The integration over t can be carried out, and one finds

$$
\begin{aligned}
(\sigma(f)\xi)(\beta(s_0)) &= \int_{\mathbb{R}^n} ds\, 2^{\frac{n}{2}} e^{-\frac{\pi}{2}(s+s_0,s+s_0)} e^{-\frac{\pi}{2}(s_0-s,s_0-s)}\xi'(\beta(s)) = \\
&= 2^{\frac{n}{4}} e^{-\pi(s_0,s_0)} \int_{\mathbb{R}^n} ds\, 2^{\frac{n}{4}} e^{-\pi(s,s)}\xi'(\beta(s))
\end{aligned}
$$

Defining ξ_0 in the space of $\sigma = \mathrm{ind}_{\mathcal{P}}^{\mathcal{G}}\nu$ by $\xi_0(\mathcal{G}\setminus\mathcal{U}) = 0$ and $\xi_0(\beta(s)p) = \nu(p)^{-1}2^{\frac{n}{4}} e^{-\pi(s,s)}$ for $s \in \mathbb{R}^n$, $p \in \mathcal{P}$ one readily verifies that ξ_0 is a unit vector. The above formula for $\sigma(f)\xi$ shows that

$$
\sigma(f)\xi = \langle \xi', \xi_0\rangle \xi_0 = \langle \xi, \xi_0\rangle \xi_0
$$

Hence $\sigma(f)$ is an orthogonal projection of rank one.

For a better understanding of the construction we include the following observation. If $k \in L^1(\mathcal{G})_\mu$ has the property that $\sigma(k)$ is an orthogonal projection of rank one, say $\sigma(k) = \langle -, \eta\rangle\eta$, then k is unique determined by η, namely $k(x) = \langle \eta, \sigma(x)\eta\rangle$. One could begin the proof by showing that the function $x \mapsto \langle \xi_0, \sigma(x)\xi_0\rangle$, ξ_0 as above, is given by the formula for h. ∎

In preparation of the study of primitive ideals in Beurling algebras on two step nilpotent groups we prove the following Hahn–Banach–type lemma.

Lemma 2.10. *Let w be a weight function on a locally compact two step nilpotent group $\mathcal{G}$. Suppose that for each element x in the closure $\mathcal{K}$ of the commutator subgroup of $\mathcal{G}$ there exist a constant $C = C_x$ and a natural number $k = k_x$ such that $w(x^n) \le Cn^k$ for all $n \in \mathbb{N}$. Define $w' : \mathcal{G} \to \mathbb{R}$ by $w'(y) = \inf_{n\in\mathbb{N}} w(y^n)^{\frac{1}{n}}$.*

(i) *For each $y \in \mathcal{G}$ the sequence $w(y^n)^{\frac{1}{n}}$, $n \in \mathbb{N}$, converges to $w'(y)$.*

(ii) *The function w' is submultiplicative, and w' is constant on $\mathcal{K}$–cosets.*

(iii) *If $\mathcal{A}$ is a subgroup of $\mathcal{G}$ containing $\mathcal{K}$ and if φ is a homomorphism from $\mathcal{A}$ into the multiplicative group $\mathbb{R}_+$ of positive real numbers with the property that there exists a constant E such that $\varphi(y) \leq Ew(y)$ for all $y \in \mathcal{A}$ then there exists a continuous homomorphism $\widetilde{\varphi} : \mathcal{G} \to \mathbb{R}_+$ such that $\widetilde{\varphi}|_{\mathcal{A}} = \varphi$ and $\widetilde{\varphi}(x) \leq w'(x) \leq w(x)$ for all $x \in \mathcal{G}$.*

Proof. Claim (i) follows in the usual manner from the submultiplicativity of w – take any textbook on normed algebras and look for the proof that $\inf_{n\in\mathbb{N}} \|b^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|b^n\|^{\frac{1}{n}}$ for an element b in a normed algebra. The proof will apply to the present situation, in particular claim (i) is independent of the assumption on the polynomial growth of w on $\mathcal{K}$.

Concerning (ii) one first observes that $(xy)^n = x^n y^n [y,x]^{\binom{n}{2}}$ for all $x, y \in \mathcal{G}$ and $n \in \mathbb{N}$. This is easily checked by induction on n using that $\mathcal{K}$ is central in $\mathcal{G}$. To verify the submultiplicativity of w' let $x, y \in \mathcal{G}$ be given. For any $n \in \mathbb{N}$ one has

$$\begin{aligned} w'(xy) &\leq w((xy)^n)^{\frac{1}{n}} = w(x^n y^n [y,x]^{\binom{n}{2}})^{\frac{1}{n}} \leq \\ &\leq w(x^n)^{\frac{1}{n}} w(y^n)^{\frac{1}{n}} w([y,x]^{\binom{n}{2}})^{\frac{1}{n}} \\ &\leq w(x^n)^{\frac{1}{n}} w(y^n)^{\frac{1}{n}} \{C\tbinom{n}{2}^k\}^{\frac{1}{n}} \end{aligned}$$

using the assumption for $[y,x] \in \mathcal{K}$. Passing to the limit one obtains $w'(xy) \leq w'(x)w'(y)$ by means of (i) The polynomial growth of w on $\mathcal{K}$ implies immediately that w' is identical to one on $\mathcal{K}$. Using the submultiplicativity of w' one readily deduces that w' is constant on $\mathcal{K}$–cosets.

Claim (iii) is proved by a "discrete Hahn–Banach–argument". First one observes that $\varphi(x^n) \leq Ew(x^n)$ for all $x \in \mathcal{A}$, $n \in \mathbb{N}$, whence $\varphi(x) = \varphi(x^n)^{\frac{1}{n}} \leq E^{\frac{1}{n}} w(x^n)^{\frac{1}{n}}$ and, therefore, $\varphi(x) \leq w'(x)$. In particular, φ is trivial on $\mathcal{K}$. As usual in such a context Zorn's lemma shows that there exists a maximal extension ψ of φ on a subgroup $\mathcal{B} \supset \mathcal{A}$ subject to the condition that $\psi(x) \leq w'(x)$ for all $x \in \mathcal{B}$. We claim that $\mathcal{B} = \mathcal{G}$. Suppose to the contrary that $c \in \mathcal{G} \setminus \mathcal{B}$. Let's first consider the case that there exists $m \in \mathbb{N}$ with $c^m \in \mathcal{B}$. Each element in the group $\langle c, \mathcal{B}\rangle$ generated by c and $\mathcal{B}$ is of the form $c^j b$ with $j \in \mathbb{Z}$ and $b \in \mathcal{B}$. Define $\widetilde{\psi}$ on $\langle c, \mathcal{B}\rangle$ by $\widetilde{\psi}(c^j b) = \psi(c^{jm})^{\frac{1}{m}} \psi(b)$. It is readily verified that $\widetilde{\psi}$ is a well–defined homomorphism. The desired inequality $\widetilde{\psi}(x) \leq w'(x)$ for all $x \in \langle c, \mathcal{B}\rangle$ is equivalent to

$$\psi(c^{jm} b^m) \leq w'(c^j b)^m$$

for all $j \in \mathbb{Z}$, $b \in \mathcal{B}$. But $\psi(c^{jm} b^m) \leq w'(c^{jm} b^m) = w'((c^j b)^m)$ by assumption (and because w' factors through $\mathcal{G}/\mathcal{K}$). From (i) follows that $w'(x^n) = w'(x)^n$ for all $x \in \mathcal{G}$, $n \in \mathbb{N}$. In particular, $w'((c^j b)^m) = w'(c^j b)^m$, whence the claim.

Now consider the case that $c^j \notin \mathcal{B}$ for all non–zero integers j. Then each $x \in \langle c, \mathcal{B}\rangle$ can be written as $x = c^j b$ with uniquely determined $j \in \mathbb{Z}$ and $b \in \mathcal{B}$. If ρ is any positive real number then $\widetilde{\psi} : \langle c, \mathcal{B}\rangle \to \mathbb{R}_+$ defined by $\widetilde{\psi}(c^j b) = \rho^j \psi(b)$ is a homomorphism. To get the desired inequality for $\widetilde{\psi}$ one has to choose ρ appropriately. To this end, one observes that for all $n, j \in \mathbb{N}$ and $x, y \in \mathcal{B}$ one has

$$(*) \qquad \{w'(c^{-n}x)^{-1}\psi(x)\}^{\frac{1}{n}} \leq \{w'(c^j y)\psi(y)^{-1}\}^{\frac{1}{j}}$$

The inequality $(*)$ is equivalent to

$$\psi(x)^j \psi(y)^n \leq w'(c^j y)^n w'(c^{-n}x)^j$$

The latter inequality is true because $\psi(x)^j\psi(y)^n = \psi(x^j y^n) \leq w'(x^j y^n) = w'(c^{-nj}x^j c^{nj} y^n) \leq w'(c^{-nj}x^j)w'(c^{nj}y^n)$ by (ii), and $w'(c^{-nj}x^j) = w'(c^{-n}x)^j$, $w'(c^{nj}y^n) = w'(c^j y)^n$.

¿From $(*)$ follows that there exists a positive ρ such that

$$\{w'(c^{-n}x)^{-1}\psi(x)\}^{\frac{1}{n}} \leq \rho \leq \{w'(c^j y)\psi(y)^{-1}\}^{\frac{1}{j}}$$

for all $n, j \in \mathbb{N}$ and $x, y \in \mathcal{B}$.

If ρ is chosen that way, it is easily verified that $\widetilde{\psi}(z) \leq w'(z)$ for all $z \in \langle c, \mathcal{B}\rangle$. We obtained a contradiction, hence $\mathcal{B} = \mathcal{G}$, and ψ is defined on the whole of $\mathcal{G}$.

Finally, it is claimed that ψ is automatically continuous because it is dominated by w'. Since ψ and w' factor through $\mathcal{G} \to \mathcal{G}/\mathcal{K}$ we may assume that $\mathcal{G}$ is abelian. Then $\mathcal{G}$ contains an open subgroup isomorphic to the direct product of a compact group L and some vector group $\mathbb{R}^d$. It is sufficient to know the continuity of ψ on this open subgroup, hence we may suppose that $\mathcal{G} = L \times \mathbb{R}^d$. Since w and, therefore, w' is bounded on the compact set L one finds a constant $F > 0$ such that $w'(x)^n = w'(x^n) \leq F$ for all $n \in \mathbb{N}$, $x \in L$. One deduces that w' is identically one on L which implies that ψ is identically one on L. Hence ψ factors through $\mathcal{G} \to \mathcal{G}/L$. But w' being submultiplicative it also factors through $\mathcal{G} \to \mathcal{G}/L$. We are reduced to $\mathcal{G} = \mathbb{R}^d$. But in this case the continuity of ψ (at zero) follows readily from the boundedness of w' on compact neighborhoods of zero in $\mathbb{R}^d$ and from the unique divisibility of $\mathbb{R}^d$. ■

Let still w be a weight function on the locally compact two step nilpotent group $\mathcal{G}$ with the property as stated in (2.10), let λ be any element in the Pontryagin dual $\mathcal{K}^\wedge$, $\mathcal{K}$ as above, and let, as usual, $\mathcal{G}_\lambda = \{z \in \mathcal{G} | \lambda([x, z]) = 1 \text{ for all } x \in \mathcal{G}\}$. But now let μ be a continuous homomorphism on $\mathcal{G}_\lambda$ with values in the multiplicative group $\mathbb{C}^\times$ of $\mathbb{C}$ satisfying $\mu|_\mathcal{K} = \lambda$ and $|\mu(x)| \leq E w(x)$ for all $x \in \mathcal{G}_\lambda$ with some constant E. With each pair (λ, μ) we shall associate a two–sided ideal $\mathcal{I}^w_{\lambda,\mu}$ in the Beurling algebra $L^1_w(\mathcal{G})$, which is the subalgebra of $L^1(\mathcal{G})$ consisting of all $f \in L^1(\mathcal{G})$ such that $\|f\|_{1,w} = \int_\mathcal{G} |f(x)| w(x) dx$ is finite, compare e.g. [40].

The homomorphism μ defines a regular maximal ideal

$$\ker_{L^1_w(\mathcal{G}_\lambda)} \mu = \{\varphi \in L^1_w(\mathcal{G}_\lambda) | \int_{\mathcal{G}_\lambda} \varphi(z)\mu(z) dz = 0\}$$

in the Beurling algebra $L^1_w(\mathcal{G}_\lambda)$. Since $L^1_w(\mathcal{G}_\lambda)$ acts by convolution on $L^1_w(\mathcal{G})$ we may form $\ker_{L^1_w(\mathcal{G}_\lambda)} \mu * L^1_w(\mathcal{G})$. The space $\mathcal{I}^w_{\lambda,\mu}$ is defined as the closure in

$L^1_w(\mathcal{G})$ of (the span of) this set. It is readily verified that $\mathcal{I}^w_{\lambda,\mu}$ is a two–sided ideal. Alternatively, $\mathcal{I}^w_{\lambda,\mu}$ may be defined as

$$I^w_{\lambda,\mu} = \{f \in L^1_w(\mathcal{G})| \int_{\mathcal{G}_\lambda} f(xz)\mu(z)dz = 0 \text{ for almost all } x \in \mathcal{G}\}$$

or as the closed linear span of the set $\{\varepsilon_z * f - \mu(z)f | z \in \mathcal{G}_\lambda\,,\ f \in L^1_w(\mathcal{G})\}$ where ε_z denotes the point measure at z.

The linear dual space $L^1_w(\mathcal{G})'$ of $L^1_w(\mathcal{G})$ can be identified via

$$\langle \varphi, f\rangle = \int_{\mathcal{G}} \varphi(x)f(x)dx\,,\ f \in L^1(\mathcal{G}),$$

with the space of measurable functions $\varphi : G \to \mathbb{C}$ such that $\varphi\frac{1}{w}$ is essentially bounded. The annihilator $(\mathcal{I}^w_{\lambda,\mu})^\perp$ of $\mathcal{I}^w_{\lambda,\mu}$ in $L^\infty(\mathcal{G}, \frac{1}{w})$ consists of all φ such that for all $z \in \mathcal{G}$ the identity $\varphi(xz) = \mu(z)\varphi(x)$ is true for almost all $x \in \mathcal{G}$. Of course, this description of $(\mathcal{I}^w_{\lambda,\mu})^\perp$ could also serve as a definition of $\mathcal{I}^w_{\lambda,\mu}$.

There is another useful characterization of $\mathcal{I}^w_{\lambda,\mu}$.

Remark 2.11. *Let $\mu = \mu_r\mu_c$ be the decomposition into an $\mu_c \in \mathcal{G}_\lambda^\wedge$ and a homomorphism $\mu_r : \mathcal{G}_\lambda \to \mathbb{R}_+$ (being trivial on $\mathcal{K}$). Let $\widetilde{\mu} : \mathcal{G} \to \mathbb{R}_+$ be any homomorphic extension of μ_r with $\widetilde{\mu}(x) \le w(x)$ for all $x \in \mathcal{G}$, see (2.10), and let π be any continuous unitary representation of $\mathcal{G}$ in a Hilbert space $\mathfrak{H}$ such that $\pi(z) = \mu_c(z)Id_{\mathfrak{H}}$ for $z \in \mathcal{G}_\lambda$. Then $\pi \otimes \widetilde{\mu}$ delivers a representation of $L^1_w(\mathcal{G})$ in $\mathfrak{H}$,*

$$(\pi \otimes \widetilde{\mu})(f) = \int_{\mathcal{G}} f(x)\widetilde{\mu}(x)\pi(x)\,dx,$$

and

$$\ker_{L^1_w(\mathcal{G})} \pi \otimes \widetilde{\mu} = \mathcal{I}^w_{\lambda,\mu}.$$

Proof. If $z \in \mathcal{G}_\lambda$ and $f \in L^1_w(\mathcal{G})$ then $(\pi\otimes\widetilde{\mu})(\varepsilon_z * f) = \mu(z)(\pi\otimes\widetilde{\mu})(f)$. Hence $\pi\otimes\widetilde{\mu}$ annihilates $\varepsilon_z*f-\mu(z)f$ which implies by one of the above characterizations that $\mathcal{I}^w_{\lambda,\mu}$ is contained in $\ker_{L^1_w(\mathcal{G})} \pi \otimes \widetilde{\mu}$. As we shall see in the next theorem the ideals $\mathcal{I}^w_{\lambda,\mu}$ are maximal. Hence $\mathcal{I}^w_{\lambda,\mu} = \ker_{L^1_w(\mathcal{G})} \pi \otimes \widetilde{\mu}$. ∎

Theorem 2.12. *Let $\mathcal{G}$ be a locally compact two step nilpotent group, and let w be a weight function on $\mathcal{G}$ with the property as stated in (2.10).*

a) *If λ and μ are as above then $\mathcal{I}^w_{\lambda,\mu}$ is maximal in the set of closed two–sided ideals in $L^1_w(\mathcal{G})$.*

b) *The ideals $\mathcal{I}^w_{\lambda,\mu}$ are primitive.*

c) *Each primitive ideal in $L^1_w(\mathcal{G})$ is of the form $\mathcal{I}^w_{\lambda,\mu}$ for a certain unique pair (λ, μ).*

In short, Priv $L^1_w(\mathcal{G}) = \{\mathcal{I}^w_{\lambda,\mu} | \lambda \in \widehat{\mathcal{K}}\,, \mu \in \mathrm{Hom}(\mathcal{G}_\lambda, \mathbb{C}^\times)\,, \mu|_{\mathcal{K}} = \lambda,$
$|\mu(z)| \le w(z)$ *for all $x \in \mathcal{G}_\lambda\} \subset$ Max $L^1_w(\mathcal{G})$*

Remarks 2.13. *In general, Max $L^1_w(\mathcal{G})$ will be strictly larger than* Priv $L^1_w(\mathcal{G})$. *Also the set of closed prime ideals might be strictly larger than* Priv $L^1_w(\mathcal{G})$, *for* *instance it may happen that the closure of* $L^1_w(\mathcal{G}) * \ker_{L^1_w(\mathcal{K})} \lambda$ *is prime while* $\mathcal{K}$ *is strictly smaller than* $\mathcal{G}_\lambda$. *Most difficulties of this sort result from the complicated ideal theory of the (almost commutative) Beurling algebras* $L^1_w(\mathcal{G}_\lambda)$. *For instance, maximal ideals in* $L^1_w(\mathcal{G}_\lambda)$ *"lying over* λ*" need not to be regular; here regular means to be of codimension one. For more information on this circle of questions see [13, 47] and the references given there.*

If the weight w *grows harmless, for instance if for each* $x \in \mathcal{G}$ *there exist* $C > 0$ *and* $k \in \mathbb{N}$ *such that* $w(x^n) \leq C\,n^k$ *for all* $n \in \mathbb{N}$, *the situation is much better. Then the sets of primitive ideals, of closed maximal ideals and of closed prime ideals coincide. Furthermore in this case the* μ*'s are necessarily unitary characters, and Wiener's theorem holds true in the following sense: Each closed two–sided ideal in* $L^1_w(\mathcal{G})$ *is contained in a primitive ideal.* ■

Proof of Theorem 2.12. Concerning a) let $\mathcal{J}$ be any closed proper two–sided ideal in $L^1_w(\mathcal{G})$ containing $\mathcal{I}^w_{\lambda,\mu}$. Then the annihilator $\mathcal{J}^\perp \subset L^\infty(\mathcal{G}, \frac{1}{w})$ is non–zero, it even contains a non–zero continuous function φ (as $C_c(\mathcal{G}) * \mathcal{J}^\perp$ is not zero). Let f be any function in $\mathcal{J}$. We claim that f is contained in $\mathcal{I}^w_{\lambda,\mu}$. It is enough to show that $\alpha * f \in \mathcal{I}^w_{\lambda,\mu}$ for all $\alpha \in C_c(\mathcal{G})$. Hence we may assume that f is continuous. By construction one has $\langle \varphi, \varepsilon_a * \varepsilon_x * f * \varepsilon_{x^{-1}} \rangle = 0$ for all $a, x \in \mathcal{G}$, or $\langle \varepsilon_{x^{-1}} * \varepsilon_{a^{-1}} * \varphi * \varepsilon_x, f \rangle = 0$. But $(\varepsilon_{x^{-1}} * \varepsilon_{a^{-1}} * \varphi * \varepsilon_x)(y) = (\varepsilon_{a^{-1}} * \varphi)(xyx^{-1}) = (\varepsilon_{a^{-1}} * \varphi)(y)\lambda([x,y])$ for $a, x, y \in \mathcal{G}$ as $\varepsilon_{\alpha^{-1}} * \varphi$ is contained in $(\mathcal{I}^w_{\lambda,\mu})^\perp$. For $x \in \mathcal{G}$ let $\psi_x \in (\mathcal{G}/\mathcal{G}_\lambda)^\wedge$ be given by $\psi_x(y) = \lambda([x,y])$, compare the first section.

If we define $T : L^1(\mathcal{G}) \to L^1(\mathcal{G}/\mathcal{G}_\lambda)$ by

$$(Tg)(y) = \int_{\mathcal{G}_\lambda} g(yz)\, dz,$$

the equation $\langle \varepsilon_{x^{-1}} * \varepsilon_{a^{-1}} * \varphi * \varepsilon_x, f \rangle = 0$ takes the form

$$\{T((\varepsilon_{a^{-1}} * \varphi) f)\}^\wedge(\psi_x^{-1}) = 0.$$

Since $\{\psi_x^{-1} | x \in \mathcal{G}\}$ is dense in $(\mathcal{G}/\mathcal{G}_\lambda)^\wedge$, one concludes that $\{T((\varepsilon_{a^{-1}} * \varphi)f)\}^\wedge$ is zero, hence $T((\varepsilon_{a^{-1}} * \varphi)f)$ is identically zero because the functions in question are continuous. Therefore, for all $a, y \in \mathcal{G}$ one has

$$0 = \int_{\mathcal{G}_\lambda} (\varepsilon_{a^{-1}} * \varphi)(yz) f(yz) dz = \int_{\mathcal{G}_\lambda} \varphi(ayz) f(yz) dz = \varphi(ay) \int_{\mathcal{G}_\lambda} \mu(z) f(yz)\, dz.$$

For any given $y \in \mathcal{G}$ there is an $a \in \mathcal{G}$ such that $\varphi(ay) \neq 0$, hence

$$\int_{\mathcal{G}_\lambda} \mu(z) f(yz) dz = 0 \quad \text{for all } y \in \mathcal{G}$$

which means that f is contained in $\mathcal{I}^w_{\lambda,\mu}$.

Concerning b) it is sufficient to show that $L^1_w(\mathcal{G})/\mathcal{I}^w_{\lambda,\mu}$ is not a radical Banach algebra, for this notion see e.g. [5], because then there exist algebraically irreducible representations of this quotient algebra, which are automatically faithful in view of a). Decompose $\mu = \mu_r \mu_c$ as in (2.11), and as there choose $\widetilde{\mu} : \mathcal{G} \to \mathbb{R}_+$ and a continuous unitary representation π of $\mathcal{G}$ in $\mathfrak{H}$ with $\pi(z) = \mu_c(z)\mathrm{Id}_{\mathfrak{H}}$ for $z \in \mathcal{G}_\lambda$. Then $\pi \otimes \widetilde{\mu}$ yields a representation of $L^1_w(\mathcal{G})$ in $\mathfrak{H}$, whose kernel coincides with $\mathcal{I}^w_{\lambda,\mu}$. Of course, there exists an $g = g^* \in C_c(\mathcal{G})$ such that $\pi(g)$ is a non–zero (self–adjoint) operator in $\mathfrak{H}$. The function $f \stackrel{\mathrm{def}}{=} \widetilde{\mu}^{-1} g$ is in $L^1_w(\mathcal{G})$, and $(\pi \otimes \widetilde{\mu})(f) = \pi(g)$ has a non–zero spectrum. Therefore, $L^1_w(\mathcal{G})/\mathcal{I}^w_{\lambda,\mu}$ is not a radical Banach algebra.

To prove c) let a simple (=algebraically irreducible) $L^1_w(\mathcal{G})$–module E be given. For some properties of such modules see [5] or [30]. In particular, E is a Banach space, and there is a (unique) strongly continuous homomorphism ρ from $\mathcal{G}$ into the group of invertible bounded operators on E such that $\|\rho(x)\| \le C\, w(x)$ for all $x \in \mathcal{G}$ with some positive constant C, and that

$$f\xi = \rho(f)\xi = \int_{\mathcal{G}} f(x)\rho(x)\xi \, dx$$

for $f \in L^1_w(\mathcal{G})$ and $\xi \in E$.

Since each $a \in \mathcal{K}$ is central in $\mathcal{G}$ the operator $\rho(a)$ commutes with all $\rho(x)$, $x \in \mathcal{G}$, and all $\rho(f)$, $f \in L^1_w(\mathcal{G})$. Schur's lemma implies that $\rho(a)$ is scalar, $\rho(a) = \lambda(a)\mathrm{Id}_E$ with some continuous homomorphism $\lambda : \mathcal{K} \to \mathbb{C}^\times$. For all $a \in \mathcal{K}$ and all $n \in \mathbb{N}$ one has $|\lambda(a^n)| = \|\rho(a^n)\| \le C\, w(a^n)$, whence $|\lambda(a)| \le \inf_{n \in \mathbb{N}} C^{\frac{1}{n}} w(a^n)^{\frac{1}{n}}$, which is one by assumption, compare (2.10). Therefore, λ is contained in the Pontryagin dual $\mathcal{K}^\wedge$. The group representation ρ factors through $\mathcal{G} \to \mathcal{G}/\ker\lambda$. As $\mathcal{G}_\lambda/\ker\lambda$ is central in $\mathcal{G}/\ker\lambda$, another application of Schur's lemma yields a continuous homomorphism $\mu : \mathcal{G}_\lambda \to \mathbb{C}^\times$ such that $|\mu(z)| \le C\, w(z)$ and

$$\rho(z) = \mu(z)\mathrm{Id}_E$$

for all $z \in \mathcal{G}_\lambda$. It is evident that $\{\varepsilon_z * f - \mu(z) f | z \in \mathcal{G}_\lambda\,,\ f \in L^1_w(\mathcal{G})\}$ is annihilated by the representation ρ of $L^1_w(\mathcal{G})$, hence $\mathcal{I}^w_{\lambda,\mu}$ is contained in $\ker_{L^1_w(\mathcal{G})} \rho$. Part a) gives $\mathcal{I}^w_{\lambda,\mu} = \ker_{L^1_w(\mathcal{G})} \rho$. ∎

In general, it will be difficult to find, for a given pair (λ, μ), an explicit simple $L^1_w(\mathcal{G})$–module E realizing $\mathcal{I}^w_{\lambda,w}$ as annihilator, let alone to find all those simple $L^1_w(\mathcal{G})$–modules. This is typical for non–type I situations. The above proof was not constructive at all. But in "type I situations", more precisely, in case that the canonical form γ on $\mathcal{G}/\mathcal{G}_\lambda$ associated with λ satisfies the equivalent conditions of (1.23), compare also (2.6) and (2.9), something can be said. To this end, we need a little lemma which is probably known, at least partly, see e.g. [10, Theorème 2].

Lemma 2.14. *Let $\mathcal{A}$ be a Banach algebra, let $\mathfrak{H}$ be a Banach space, and let $\sigma : \mathcal{A} \to \mathcal{B}(\mathfrak{H})$ be a topologically irreducible representation. Suppose that $\sigma(\mathcal{A})$ contains non–zero operators of finite rank, and let $\mathfrak{a}$ be the two–sided ideal of all $a \in \mathcal{A}$ such that $\sigma(a)$ is an operator of finite rank.*

If E denotes the span of $\{\sigma(a)\xi | a \in \mathfrak{a},\, ,\ \xi \in \mathfrak{H}\}$ then the following assertions are true.

(i) *The space* E *is* $\mathcal{A}$*–invariant, i.e., it is an* $\mathcal{A}$*–module, and* E *is dense in* $\mathfrak{H}$.

(ii) E *is the smallest non–zero* $\mathfrak{a}$*–invariant subspace of* $\mathfrak{H}$, *in particular* E *is a simple* $\mathcal{A}$*–submodule of* $\mathfrak{H}$, *and it is the only one.*

(iii) *The annihilator of* E *in* $\mathcal{A}$ *equals* $\ker\sigma$, *and, up to isomorphism,* E *is the unique simple* $\mathcal{A}$*–module, whose annihilator coincides with* $\ker\sigma$.

Proof. The $\mathcal{A}$–invariance of E is an immediate consequence of the fact that $\mathfrak{a}$ is an ideal in $\mathcal{A}$. Since $E \neq 0$ and since σ is topologically irreducible, E has to be dense in $\mathfrak{H}$. Concerning (ii) we first observe that $\mathfrak{H}_0 \stackrel{\text{def}}{=} \{\xi \in \mathfrak{H} | \sigma(a)\xi = 0 \text{ for all } a \in \mathfrak{a}\}$ is zero. As $\mathfrak{a}$ is an ideal in $\mathcal{A}$, the space $\mathfrak{H}_0$ is $\mathcal{A}$–invariant. If $\mathfrak{H}_0 \neq 0$ then $\mathfrak{H}_0$ were dense in $\mathfrak{H}$, hence for each $a \in \mathfrak{a}$ the operator $\sigma(a)$ would annihilate the whole of $\mathfrak{H}$, in contrary to our assumptions. Now, let any $\mathfrak{a}$–invariant subspace F of $\mathfrak{H}$ be given. It is sufficient to show that $\sigma(b)\mathfrak{H}$ is contained in F for each $b \in \mathfrak{a}$. Pick any non–zero $\xi \in F$. Since $\sigma(\mathfrak{a})\xi$ is non–zero (as $\mathfrak{H}_0$ is zero) and since $\sigma(\mathfrak{a})\xi$ is $\mathcal{A}$–invariant it follows that $\sigma(\mathfrak{a})\xi$ is dense in $\mathfrak{H}$. Hence $\sigma(b)\mathfrak{H} = \sigma(b)\sigma(\mathfrak{a})\xi$ is contained in F. The other assertions in (ii) are immediate consequences.

Concerning (iii) we show first that there exists an $p \in \mathfrak{a}$ such that $\sigma(p)$ is an idempotent and that $\sigma(p)\mathfrak{H}$ is one–dimensional. Since E is a simple $\mathfrak{a}$–module, for each chosen non–zero $\xi \in E$ there exists an $a \in \mathfrak{a}$ such that $\sigma(a)\xi = \xi$. In the finite–dimensional space $\sigma(a)\mathfrak{H}$ choose a complementary space K to the one–dimensional subspace $\mathbb{C}\xi$: $\sigma(a)\mathfrak{H} = \mathbb{C}\xi \oplus K$. By [5, Corollary 5, p. 128], which is essentially Jacobson's density theorem, there exists an $b \in \mathcal{A}$ with $\sigma(b)\xi = \xi$ and $\sigma(b)K = 0$. If one puts $p = ba \in \mathfrak{a}$, it is readily checked that p has the claimed properties.

Since E is dense in $\mathfrak{H}$ it is clear that the annihilator of E in $\mathcal{A}$ coincides with $\ker\sigma$. Let another simple $\mathcal{A}$–module F with $\mathrm{Ann}_{\mathcal{A}}F = \mathrm{Ann}_{\mathcal{A}}E = \ker\sigma$ be given. W.l.o.g. we may assume that $\ker\sigma = 0$. Choose p as above, $\sigma(p)E = \mathbb{C}\xi \neq 0$. Then $p\mathcal{A}p = \mathbb{C}p$ is one–dimensional. Hence pF is one–dimensional, say $pF = \mathbb{C}\eta$. Define $T : E \to F$ by $T(\sigma(a)\xi) = a\eta$ for $a \in \mathcal{A}$. It is easily checked that T is well–defined and that it is an $\mathcal{A}$–linear isomorphism.■

Theorem 2.15. *Let* w *be a weight function on the locally compact two step nilpotent group* $\mathcal{G}$ *satisfying the assumptions of (2.10). Let* $\mathcal{K}$ *be the closure of the commutator subgroup of* $\mathcal{G}$, *and let* $\lambda \in \widehat{\mathcal{K}}$. *Suppose that the associated form* $\gamma = \gamma_\lambda$ *on* $\mathcal{G}/\mathcal{G}_\lambda$ *satisfies the equivalent conditions of (1.23). Moreover, let* $\mu : \mathcal{G}_\lambda \to \mathbb{C}^\times$ *be a continuous homomorphism with* $\mu|_{\mathcal{K}} = \lambda$ *and with the property that there exists a constant* C *such that* $|\mu(z)| \leq C\,w(z)$ *for all* $z \in \mathcal{G}_\lambda$. *Then there exists a unique, up to isomorphism, algebraically irreducible representation* ρ *of* $L^1_w(\mathcal{G})$ *in a (Banach) space* E *such that* $\ker\rho = \mathcal{I}^w_{\lambda,\mu}$. *Furthermore, there exists* $p \in L^1_w(\mathcal{G})$ *such that* $\rho(p)$ *is an idempotent and that* $\rho(p)E$ *is one–dimensional.*

Proof. Decompose $\mu = \mu_r\mu_c$ as in (2.11), in particular, $\mu_c \in \mathcal{G}^\wedge_\lambda$. Let π be a continuous irreducible unitary representation of $\mathcal{G}$ in the Hilbert space $\mathfrak{H}$ with $\pi(z) = \mu_c(z)\mathrm{Id}_{\mathfrak{H}}$ for $z \in \mathcal{G}_\lambda$; such a π is unique up to unitary equivalence. Choose $\widetilde{\mu} : \mathcal{G} \to \mathbb{R}_+$ according to (2.10) as an extension of μ_r, compare also

(2.11). Then $\pi \otimes \widetilde{\mu}$ yields, by integration, a topologically irreducible representation σ of $L^1_w(\mathcal{G})$ in $\mathfrak{H}$ with $\ker \sigma = \mathcal{I}^w_{\lambda,\mu}$, see (2.11). From (2.9) it follows that there exists a continuous function g on $\mathcal{G}$, integrable against each weight function on $\mathcal{G}$, such that $\pi(g)$ is an orthogonal projection of rank one. In particular, g is integrable against the weight $(\widetilde{\mu}+\widetilde{\mu}^{-1})w$, hence $f = \widetilde{\mu}^{-1}g$ is in $L^1_w(\mathcal{G})$, and $\sigma(f)$ is an orthogonal projection of rank one. Now, the theorem readily follows from (2.14). ■

§ 3 Twisted Covariance Algebras and Primitive Ideals in Group C^*–algebras of Connected Lie Groups

Using the results of the foregoing sections we now want to obtain information on the structure of certain subsets of primitive ideals in group C^*–algebras of connected Lie groups and on the structure of the corresponding primitive quotients. The following results extend and generalize those of [31]. More specifically, we give a new proof of the fact that a simple subquotient of $C^*(G)$, G connected Lie, is stably isomorphic to a noncommutative torus in a certain dimension n. The most important achievement in the new approach is that we are now able to compute this number n, see (3.9) and (3.11) below. An explicit formula for n in terms of the Pukanszky parameters will be given in the case of solvable Lie groups, (3.13). Some of our auxiliary results might be of independent interest. Occasionally they are formulated and proved more generally than actually needed for the present purposes. Since at a crucial point we will use the generalized Effros–Hahn conjecture from now on we assume that

everything is separable,

i.e., all treated groups and C^*–algebras are separable. Our approach uses the machinery of twisted covariance algebras. As there are several (equivalent) definitions in the literature it seems to us reasonable to recall the definition we are going to use.

Definition 3.1. A twisted covariance system is a quintuple $(G, N, \mathcal{A}, T, \tau)$ consisting of a (separable) locally compact group G a closed normal subgroup N of G, a (separable) C^*–algebra $\mathcal{A}$, a strongly continuous action $T : G \to \mathrm{Aut}(\mathcal{A})$ and a strictly continuous homomorphism τ from N into the group of unitaries in the multiplier algebra $\mathcal{A}^\flat$ of $\mathcal{A}$. The "action" T and the "twist" are related by

$$\tau(n)a\tau(n)^{-1} = T_n(a) \text{ and } \tau(xnx^{-1}) = T_x(\tau(n))$$

for $a \in \mathcal{A}$, $x \in G$ and $n \in N$. Of course, in the expression $T_x(\tau(n))$ we mean the canonical extension of T_x to $\mathcal{A}^\flat$. Occasionally a^x is written instead of $T_{x^{-1}}(a)$. ■

For a motivation of this notion see [16] or [31]. With each twisted covariance system $(G, N, \mathcal{A}, T, \tau)$ there is associated a C^*–algebra $C^*(G, N, \mathcal{A}, T, \tau)$, a so–called twisted covariance C^*–algebra, in the following fashion. Let $L^1(G, N, \mathcal{A}, T, \tau)$ be the space of (equivalence classes of) measurable functions $f : G \to \mathcal{A}$ such that

$$f(xn) = \tau(n)^{-1} f(x)$$

for all $x \in G$, $n \in N$ and that $\|f\|_{\mathcal{A}}$ is integrable w.r.t. a left invariant measure dx on G/N. By the way, all our Haar measures will be *left* invariant, in contrary to the normalizations in [16].

The Banach space $L^1(G, N, \mathcal{A}, T, \tau)$ is made into an involutive Banach algebra by means of the following operations

$$(f * g)(x) = \int_{G/N} f(xy)^{y^{-1}} g(y^{-1})\, d\dot{y}$$
$$f^*(x) = \Delta_{\dot{G}}(x)^{-1} f(x^{-1})^{*\,x}.$$

where $\Delta_{\dot{G}}$ denotes the modular function of $\dot{G} = G/N$. Observe that because of the transformation property of f and of the relation (3.1) between T and τ, the integrand is a function on $\dot{G}$. The twisted covariance C^*–algebra is defined as the C^*–completion of this involutive Banach algebra $L^1(G, N, \mathcal{A}, T, \tau)$.

The algebra $L^1(G, N, \mathcal{A}, T, \tau)$ is a quotient of the ordinary untwisted covariance algebra $L^1(G, \mathcal{A}, T)$ via the map $f \mapsto f^\natural$,

$$f^\natural(x) = \int_N \tau(n) f(xn) dn$$

Hence also $C^*(G, N, \mathcal{A}, T, \tau)$ is a quotient of the covariance algebra $C^*(G, \mathcal{A}, T)$. This point of view is taken in [16]. Green describes the kernel of $C^*(G, \mathcal{A}, T) \to C^*(G, N, \mathcal{A}, T, \tau)$ in terms of the twist and defines $C^*(G, N, \mathcal{A}, T, \tau)$ as the corresponding quotient.

With each involutive representation π of $C^*(G, N, \mathcal{A}, T, \tau)$ in a Hilbert space $\mathfrak{H}$ there are associated a strongly continuous group representation π_G of G and an involutive representation $\pi_{\mathcal{A}}$ of $\mathcal{A}$, which are related by $\pi_{\mathcal{A}}(T_x(a)) = \pi_G(x)\pi_{\mathcal{A}}(a)\pi_G(x)^{-1}$ and $\pi_G(n) = \pi_{\mathcal{A}}(\tau(n))$ for $a \in \mathcal{A}$, $x \in G$ and $n \in N$, such that

$$\pi(f)\xi = \int_{G/N} \pi_G(x)\pi_{\mathcal{A}}(f(x))\xi\, d\dot{x}$$

holds for $\xi \in \mathfrak{H}$ and $f \in L^1(G, N, \mathcal{A}, T, \tau)$. Conversely, each pair $(\pi_G, \pi_{\mathcal{A}})$ satisfying the above properties delivers a representation of $C^*(G, N, \mathcal{A}, T, \tau)$.

The twisted covariance algebra $C^*(G, N, \mathcal{A}, T, \tau)$ can be formed “in stages”. To this end, let M be a closed *normal* subgroup of G containing N. Clearly, one may form the twisted covariance algebra $C^*(M, N, \mathcal{A}, T, \tau)$ with the restricted action of M on $\mathcal{A}$. The group G acts on $L^1(M, N, \mathcal{A}, T, \tau)$ by $(T^M_x f)(m) = \delta(x) f(x^{-1}mx)^{x^{-1}}$ for $x \in G$, $m \in M$ and $f \in L^1(M, N, \mathcal{A}, T, \tau)$, where δ denotes the modular function of the action of G on M/N.

For $m \in M$ there is a twist $\tau^M(m)$ on $L^1(M, N, \mathcal{A}, T, \tau)$ given by $(\tau^M(m)f(m') = f(m^{-1}m')$. Both, the action T^M and the twist τ^M extend to $C^*(M, N, \mathcal{A}, T, \tau)$. Hence one may form the twisted covariance C^*–algebra $C^*(G, M, C^*(M, N, \mathcal{A}, T, \tau),\ T^M, \tau^M)$. It is easy to see that this iterated twisted covariance C^*–algebra is canonically isomorphic to $C^*(G, N, \mathcal{A}, T, \tau)$.

The constructions of the “induced action” T^M and the “induced twist” τ^M are similar to the construction of induced representations, which we now describe in the context of twisted covariance systems. Let $(G, N, \mathcal{A}, T, \tau)$ be such a system, and let M be a closed subgroup of G containing N. For simplicity we assume that G/M has an G–invariant measure μ; no other case will appear in the sequel. Let ρ be a representation of $C^*(M, N, \mathcal{A}, T, \tau)$ in $\mathfrak{H}$, given as above by a unitary group representation ρ_M and an involutive representation $\rho_{\mathcal{A}}$ of $\mathcal{A}$. Then let $\pi_G = \mathrm{ind}_M^G \rho_M$ be the ordinary induced representation realized in the usual manner in the space of measurable functions $\xi : G \to \mathfrak{H}$ satisfying $\xi(xm) = \rho_M(m)^{-1}\xi(x)$ for $x \in G$, $m \in M$ and $\int_{G/M} \|\xi(x)\|^2 d\mu(x) < \infty$. The representation $\pi_{\mathcal{A}}$ of $\mathcal{A}$ in this space is defined by $(\pi_{\mathcal{A}}(a)\xi)(x) = \rho_{\mathcal{A}}(a^x)(\xi(x))$. It is easy to verify that the pair $(\pi_G, \pi_{\mathcal{A}})$ satisfies $\pi_{\mathcal{A}}(\tau(n)) = \pi_G(n)$ and $\pi_{\mathcal{A}}(T_x(a)) = \pi_G(x)\pi_{\mathcal{A}}(a)\pi_G(x)^{-1}$ for $a \in \mathcal{A}$, $n \in N$ and $a \in \mathcal{A}$. Hence $(\pi_G, \pi_{\mathcal{A}})$ delivers a representation π of $C^*(G, N, \mathcal{A}, T, \tau)$, the *induced representation* of ρ, $\pi = \mathrm{ind}_M^G \rho$.

In the following we will also use the notion of *induced ideals.* If $\mathcal{F}$ is a closed two–sided ideal in $C^*(M, N, \mathcal{A}, T, \tau)$ we choose a representation ρ with $\ker\rho = \mathcal{F}$ and define $\mathrm{ind}_M^G \mathcal{F}$ as $\ker(\mathrm{ind}_M^G \rho)$, which can be seen to be independent of the choice of ρ.

Our results on primitive quotients and primitive ideals spaces apply only to very special classes of twisted covariance algebras, which are defined now. Their primitive ideal spaces can be computed easily, see (3.4) below, – as soon as one is acquainted with the somewhat cumbersome terminology.

Definition 3.2. A twisted covariance system $(H, N, \mathcal{A}, T, \tau)$ is *regular*, if $\mathcal{A}$ is of type I, which means that the dual $\widehat{\mathcal{A}}$ is canonically homeomorphic to the primitive ideal space $\mathrm{Priv}(\mathcal{A})$ of $\mathcal{A}$, and if for each $\mathcal{J} \in \mathrm{Priv}(\mathcal{A})$ the map $x \mapsto x\mathcal{J} \stackrel{\text{def}}{=} T_x(\mathcal{J})$ from H into $\mathrm{Priv}(\mathcal{A})$ is surjective and open.

A twisted covariance system $(G, N, \mathcal{A}, T, \tau)$ is *centrally regularizable* if there exist a locally compact group H containing G and an extension $T : H \to \mathrm{Aut}(\mathcal{A})$ of T, also denoted by T, such that $(H, N, \mathcal{A}, T, \tau)$ is a regular twisted covariance system and that G/N is central in H/N. Observe that in particular G/N is abelian. In this case $(H, N, \mathcal{A}, T, \tau)$ is called a regularization of $(G, N, \mathcal{A}, T, \tau)$.

In other words, a centrally regularizable system is nothing but a certain subsystem of a regular system. The terminology reflects our point of view that we consider centrally regularizable system as our object of study and the regular systems as tools. ■

Let a centrally regularizable twisted covariance system $(G, N, \mathcal{A}, T, \tau)$ with regularization $(H, N, \mathcal{A}, T, \tau)$ be given, and let $\mathcal{J}$ be a primitive ideal in $\mathcal{A}$. The stabilizer $H_{\mathcal{J}}$ of $\mathcal{J}$ in H, i.e., the set of $h \in H$ such that $T_h(\mathcal{J}) = \mathcal{J}$, may depend on $\mathcal{J}$, while the stabilizer $G_{\mathcal{J}} = G \cap H_{\mathcal{J}}$ of $\mathcal{J}$ in G is independent of $\mathcal{J}$, because G/N is central. The quotient algebra $\dot{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{A}/\mathcal{J}$ is isomorphic to the algebra of compact operators, hence the multiplier algebra $(\dot{\mathcal{A}})^\flat$ is isomorphic to the algebra of all bounded operators on some Hilbert space and its group of unitaries is isomorphic to the group of unitary operators on that Hilbert space. The twist τ induces a twist $\dot{\tau}$ on $\dot{\mathcal{A}}$, i.e., a homomorphism into the group of unitary operators. The action T induces an action $\dot{T}$ of $H_{\mathcal{J}}$ on $\dot{\mathcal{A}}$.

From the structure of $\dot{\mathcal{A}}$ it follows that for each $h \in H_{\mathcal{J}}$ there exists an $u(h)$ in the unitaries of $(\dot{\mathcal{A}})^\flat$, unique up to a scalar of modulus one, such that $\dot{T}_h(a) = u(h)au(h)^{-1}$ for all $a \in \dot{\mathcal{A}}$. Choose for each $h \in H_{\mathcal{J}}$ any such $u(h)$; this is not necessarily a measurable choice. The commutator $[u(h_1), u(h_2)] = u(h_1)u(h_2)u(h_1)^{-1}u(h_2)^{-1}$ is independent of this choice. Since G/N is central in H/N for each $h \in H_{\mathcal{J}}$ and each $s \in G_{\mathcal{J}}$ the commutator $[h, s]$ is contained in N. The operator $\dot{\tau}([h, s])$ is a multiple of $[u(h), u(s)]$, say

$$\dot{\tau}([h, s]) = \zeta_{\mathcal{J}}(h)(s)[u(h), u(s)],$$

because $[u(h), u(s)]a[u(h), u(s)]^{-1} = \dot{T}_{[h,s]}(a)$ for all $a \in \dot{\mathcal{A}}$.

The scalar $\zeta_{\mathcal{J}}(h)(s)$ depends only on the N–cosets of h and s, and it is a routine matter to check that we obtain a *continuous homomorphism* $\zeta_{\mathcal{J}}$ from $H_{\mathcal{J}}$ (or from $H_{\mathcal{J}}/N$) into $(G_{\mathcal{J}}/N)^\wedge$. Clearly, If $\mathcal{J}$ varies the homomorphism $\zeta_{\mathcal{J}}$ varies in a controlled way. Indeed, If $\mathcal{J}_1$ and $\mathcal{J}_2$ are any two primitive ideals in $\mathcal{A}$, say $\mathcal{J}_2 = T_h(\mathcal{J}_1)$ with $h \in H$, then $h^{-1}H_{\mathcal{J}_2}h = H_{\mathcal{J}_1}$ and $\zeta_{\mathcal{J}_2}$ can be computed in terms of $\zeta_{\mathcal{J}_1}$ by

$$\zeta_{\mathcal{J}_2}(x)(s) = \zeta_{\mathcal{J}_1}(h^{-1}xh)(h^{-1}sh) = \zeta_{\mathcal{J}_1}(h^{-1}xh)(s)$$

for $x \in H_{\mathcal{J}_2}$ and $s \in G_{\mathcal{J}_1} = G_{\mathcal{J}_2}$.

For $s_1, s_2 \in G_{\mathcal{J}}$ we define $\alpha : G_{\mathcal{J}} \times G_{\mathcal{J}} \to \mathbb{T}$ by

$$\alpha(s_1, s_2) = \zeta_{\mathcal{J}}(s_1)(s_2)$$

The map α is a skew–symmetric bicharacter on $G_{\mathcal{J}}$, or rather on $G_{\mathcal{J}}/N$, which actually is independent of $\mathcal{J}$ (and of the central regularization), again because G/N is central and because of the above relation between $\zeta_{\mathcal{J}_1}$ and $\zeta_{\mathcal{J}_2}$; it is called the *Mackey bicharacter* associated with the twisted covariance system $(G, N, \mathcal{A}, T, \tau)$.

The group H acts on $C^*(G, N, \mathcal{A}, T, \tau)$ via the induced action,

$$(T_h^G f)(x) = f(h^{-1}xh)^{h^{-1}}$$

for f in the dense subspace $L^1(G, N, \mathcal{A}, T, \tau)$, see above; as G/N is central in H/N no modular function appears. In the present case the action may also be written as

$$(T_h^G f)(x) = \tau([x^{-1}, h])(f(x)^{h^{-1}})$$

This can be seen by writing $h^{-1}xh = xx^{-1}h^{-1}xh = x[x^{-1}, h^{-1}]$ and using the transformation property of f w.r.t. $[x^{-1}, h^{-1}] \in N$.

Also the Pontryagin dual $(G/N)^\wedge$ acts on $C^*(G, N, \mathcal{A}, T, \tau)$ via multiplication:

$$(\chi \cdot f)(x) = \chi(x)^{-1} f(x)$$

for $\chi \in (G/N)^\wedge$ and $f \in L^1(G, N, \mathcal{A}, T, \tau)$. These two actions commute, therefore, the direct product $H \times (G/N)^\wedge$ acts on $C^*(G, N, \mathcal{A}, T, \tau)$ and on its primitive ideal space $\mathrm{Priv}\,(G, N, \mathcal{A}, T, \tau)$, explicitly, $(h, \chi) \cdot \mathcal{I} = T_h^G(\chi \cdot$

$\mathcal{I}$). The normalization of the action is arranged that way that on the level of representations one has $((h,\chi)\cdot\rho)(f)=(\chi\otimes\rho)(T^G_{h^{-1}}f)$.

Each primitive ideal $\mathcal{I}$ in $C^*(G,N,\mathcal{A},T,\tau)$ defines by 'restriction' a G–quasi–orbit $Q(\mathcal{I})$ in $\mathrm{Priv}(\mathcal{A})$: The restriction $\mathcal{I}|_{\mathcal{A}}$ is given by $\{a\in\mathcal{A}|a*f\in\mathcal{I}, \forall f\in C^*(G,N,\mathcal{A},T,\tau)\}$, where $a*f$ is defined by $(a*f)(x)=a^xf(x)$ for f in the dense subspace $L^1(G,N,\mathcal{A},T,\tau)$. And $Q(\mathcal{I})$ is determined by requiring that $\mathcal{I}|_{\mathcal{A}}=\cap\{\mathcal{J}|\mathcal{J}\in Q(\mathcal{I})\}$. ¿From the following lemma one can deduce some information on the G–quasi–orbits in $\mathrm{Priv}(\mathcal{A})$; its simple proof is omitted.

Lemma 3.3. *Let X be a homogeneous space for the acting locally compact group $\dot H$, and let $\dot G$ be a (closed) central subgroup of $\dot H$. For $x,y\in X$ the following properties are equivalent:*

(i) $(\dot Gx)^-=(\dot Gy)^-$,

(ii) $y\in(\dot Gx)^-$,

(iii) $y\in(\dot G\dot H_x)^-x$,

where, of course, $\dot H_x$ denotes the stabilizer of x in $\dot H$. If these conditions are satisfied the stabilizer groups $\dot H_x$ and $\dot H_y$ coincide. ∎

The lemma says that the $\dot G$–quasi–orbits in X coincide with the closures of the $\dot G$–orbits, and that they are orbits of a certain group. The space of $\dot G$–quasi–orbits in X can be identified with the coset space $\dot H/(\dot G\dot H_{x_0})^-$ in the obvious manner, where x_0 is any chosen point in X.

Since N acts trivially on $\mathrm{Priv}(\mathcal{A})$ we may view $\mathrm{Priv}(\mathcal{A})$ as a homogeneous space for the acting group $\dot H=H/N$ with central subgroup $\dot G=G/N$, hence the lemma applies.

The subgroup $\Gamma_{\mathcal{J}}$ of $H\times(G/N)^\wedge$ is defined by $\Gamma_{\mathcal{J}}\overset{\text{def}}{=}\{(h,\chi)\in H_{\mathcal{J}}\times(G/N)^\wedge|\zeta_{\mathcal{J}}(h)=\chi|_{G_{\mathcal{J}}}\}$. The group $\Gamma_{\mathcal{J}}$ depends only on the G–quasi–orbit through $\mathcal{J}$. Indeed, if $\mathcal{J}_1$ and $\mathcal{J}_2$ are on the same G–quasi–orbit then $H_{\mathcal{J}_1}=H_{\mathcal{J}_2}$ and $\mathcal{J}_2=T_h(\mathcal{J}_1)$ with $h\in(GH_{\mathcal{J}_1})^-$ by (3.3). The closed subgroup $\{k\in H|kH_{\mathcal{J}_1}k^{-1}=H_{\mathcal{J}_1}$ and $\zeta_{\mathcal{J}_1}(k^{-1}xk)=\zeta_{\mathcal{J}_1}(x), \forall x\in H_{\mathcal{J}_1}\}$ contains G and $H_{\mathcal{J}_1}$ as well because $\zeta_{\mathcal{J}_1}$ is a homomorphism into an abelian group. Hence $(GH_{\mathcal{J}_1})^-$, in particular the above element h, is contained in this group which proves that $\zeta_{\mathcal{J}_2}=\zeta_{\mathcal{J}_1}$ in view of the above general relation between $\zeta_{\mathcal{J}_1}$ and $\zeta_{\mathcal{J}_2}$.

Using the introduced notations we can now formulate the first theorem of this section.

Theorem 3.4. *Let $(G,N,\mathcal{A},T,\tau)$ be a centrally regularizable twisted covariance system with regularization $(H,N,\mathcal{A},T,\tau)$. Then $\mathrm{Priv}(G,N,\mathcal{A},T,\tau)$ is a locally compact Hausdorff space, homogeneous for the action of $H\times(G/N)^\wedge$. If $\mathcal{I}$ is any primitive ideal in $C^*(G,N,\mathcal{A},T,\tau)$ and if $\mathcal{J}$ is any point in the G–quasi-orbit $Q(\mathcal{I})\subset\mathrm{Priv}(\mathcal{A})$ then the stabilizer of $\mathcal{I}$ in $H\times(G/N)^\wedge$ is equal to $(G\Gamma_{\mathcal{J}})^-$. Hence $\mathrm{Priv}(G,N,\mathcal{A},T,\tau)$ is homeomorphic to $H\times(G/N)^\wedge/(G\Gamma_{\mathcal{J}})^-$.*

Corollary 3.5. *Let in addition M be a closed (normal) subgroup of G containing N. Then the following are equivalent:*

(i) *The inducing process* ind_M^G *yields a surjection from* $\mathrm{Priv}(M, N, \mathcal{A}, T, \tau)$ *onto* $\mathrm{Priv}(G, N, \mathcal{A}, T, \tau)$.

(ii) *The group* $(G/M)^\wedge$ *stabilizes each point in* $\mathrm{Priv}(G, N, \mathcal{A}, T, \tau)$ *or, equivalently, one of those.*

(iii) *The group* $(G/M)^\wedge$ *is contained in the closure* $(G\Gamma_{\mathcal{J}})^-$ *of* $G\Gamma_{\mathcal{J}}$ *in* $H \times (G/N)^\wedge$ *for any* $\mathcal{J} \in \mathrm{Priv}(\mathcal{A})$.

Under these circumstances the map ind_M^G *yields a homeomorphism from the space of* G*–quasi–orbits in* $\mathrm{Priv}\,(M, N, \mathcal{A}, T, \tau)$ *onto* $\mathrm{Priv}\,(G, N, \mathcal{A}, T, \tau)$.

Criterion (iii) is for instance satisfied if M *contains the kernel* Z_α *of the Mackey bicharacter, i.e.,* $Z_\alpha = \{s \in G_{\mathcal{J}} | \alpha(s,t) = 1\,, \forall t \in G_{\mathcal{J}}\}$. *Also there is a smallest subgroup* M *satisfying (i) through (iii), which is characterized by* $(G/M)^\wedge = (G/N)^\wedge \cap (G\Gamma_{\mathcal{J}})^-$ *for any (or all)* $\mathcal{J} \in \mathrm{Priv}(\mathcal{A})$. *This minimal* M *can be computed as soon as* $\zeta_{\mathcal{J}}$ *is known.*

Proof of Theorem 3.4. *The theorem and its proof are greatly influenced by the remarks on the last two pages of [15]. Actually, the proof consists of an application of the Effros–Hahn–conjecture and of a very precise description of the algebra* $C^*(G_{\mathcal{J}}, N, \dot{\mathcal{A}}, \dot{T}, \dot{\tau})$ *for any* $\mathcal{J} \in Priv(\mathcal{A})$, *for notations, e.g.* $\dot{\mathcal{A}} = \mathcal{A}/\mathcal{J}$, *see above. The structure of the latter algebra was also determined in [17] using cross sections. Here we can avoid those, mainly by introducing the group*

$$\widetilde{G}_{\mathcal{J}} \stackrel{\text{def}}{=} \{(s,v) \in G_{\mathcal{J}} \times (\dot{\mathcal{A}})^\flat | v^* = v^{-1} \text{ and } \dot{T}_s(a) = vav^*\,, \forall a \in \dot{\mathcal{A}}\}$$

This is a subgroup of the direct product of $G_{\mathcal{J}}$ *and the unitary group* $U((\dot{\mathcal{A}})^\flat)$ *of* $(\dot{\mathcal{A}})^\flat$. *Endow* $U((\dot{\mathcal{A}})^\flat)$ *with the strong operator topology (recall that* $U((\dot{\mathcal{A}})^\flat)$ *is nothing but the unitary group of a certain Hilbert space), and endow* $\widetilde{G}_{\mathcal{J}}$ *with the relative topology of the product space* $G_{\mathcal{J}} \times U((\dot{\mathcal{A}}^\flat))$. *Clearly, the first projection* $\widetilde{G}_{\mathcal{J}} \to G_{\mathcal{J}}$ *is surjective and continuous; from the fact that* $\dot{T}$ *is a strongly continuous action of* $G_{\mathcal{J}}$ *on* $\dot{\mathcal{A}}$, *one readily deduces that this homomorphism is also open. As the kernel of this homomorphism is compact, namely isomorphic to* $\mathbb{T}$, *one concludes that* $\widetilde{G}_{\mathcal{J}}$ *is a locally compact group.*

The group $\widetilde{G}_{\mathcal{J}}$ *contains a distinguished normal subgroup* $\widetilde{N} \stackrel{\text{def}}{=} \{(s,v) \in \widetilde{G}_{\mathcal{J}} | s \in N\} = \{(n, \dot{\tau}(n)z) | n \in N, z \in \mathbb{T}\}$. *The formula*

$$\widetilde{\tau}(n, \dot{\tau}(n)z) = z^{-1} \text{ or } \widetilde{\tau}(s,v)Id \;= v^{-1}\dot{\tau}(s) = \dot{\tau}(s)v^{-1}$$

yields a unitary character $\widetilde{\tau}$ *on* $\widetilde{N}$. *The character* $\widetilde{\tau}$ *can be used to define the twisted covariance algebra* $C^*(\widetilde{G}_{\mathcal{J}}, \widetilde{N}, \mathbb{C}, \widetilde{\tau})$ *with trivial action, which is nothing but the* C^**–algebra* $C^*(\widetilde{G}_{\mathcal{J}})_{\widetilde{\tau}}$ *in the terminology of the proof of (2.3). Also one may form* $C^*(\widetilde{G}_{\mathcal{J}}, \widetilde{N}, \dot{\mathcal{A}}, \widetilde{\tau})$ *with trivial action, which is the tensor product of* $C^*(\widetilde{G}_{\mathcal{J}})_{\widetilde{\tau}}$ *and* $\dot{\mathcal{A}}$.

We claim that $C^*(G_{\mathcal{J}}, N, \dot{\mathcal{A}}, \dot{T}, \dot{\tau})$ *is isomorphic to* $C^*(\widetilde{G}_{\mathcal{J}}, \widetilde{N}, \dot{\mathcal{A}}, \widetilde{\tau})$. *Indeed, it is readily checked that for* $f \in L^1(G_{\mathcal{J}}, N, \dot{\mathcal{A}}, \dot{T}, \dot{\tau})$ *the function* $\widetilde{f} : \widetilde{G}_{\mathcal{J}} \to \dot{\mathcal{A}}$, *given by*

$$\widetilde{f}(s,v) = vf(s),$$

is contained in $L^1(\widetilde{G}_{\mathcal{J}}, \widetilde{N}, \dot{\mathcal{A}}, \widetilde{\tau})$, *and that* $f \mapsto \widetilde{f}$ *defines an isometric* $*$*–isomorphism of the involutive Banach algebras in question.*

We shall also need the induced action of $H_{\mathcal J}$ *on* $C^*(G_{\mathcal J}, N, \dot{\mathcal A}, \dot T, \dot\tau)$ *and the corresponding action of* $H_{\mathcal J}$ *in the transformed picture* $C^*(\widetilde G_{\mathcal J}, \widetilde N, \dot{\mathcal A}, \widetilde\tau)$. *Recall that* $h \in H_{\mathcal J}$ *acts on* $f \in L^1(G_{\mathcal J}, N, \dot{\mathcal A}, \dot T, \dot\tau)$ *by*

$$(\dot T_h^{G_{\mathcal J}} f)(s) = \dot T_h(f(h^{-1}sh)) = \dot\tau([s^{-1}, h])\dot T_h(f(s)),$$

compare the above formula for T^G.

The transformed action $\widetilde T_h$, $h \in H_{\mathcal J}$, *is given by* $\widetilde T_h(\widetilde f) = (T_h^{H_{\mathcal J}} f)^\sim$, *hence*

$$\begin{aligned}\widetilde T_h(\widetilde f)(s,v) &= v\dot\tau([s^{-1}, h])\dot T_h(f(s)) = \\ &= v\dot\tau([s^{-1}, h])\dot T_h(v^{-1}\widetilde f(s,v)).\end{aligned}$$

Using the above chosen element $u(h) \in U(\dot{\mathcal A}^\flat)$ *we get*

$$\begin{aligned}\widetilde T_h(\widetilde f)(s,v) &= v\dot\tau([s^{-1}, h])u(h)v^{-1}u(h)^{-1}\widetilde f(s,v)^{h^{-1}} = \\ &= v\dot\tau([s^{-1}, h])v^{-1}[v, u(h)]\widetilde f(s,v)^{h^{-1}}.\end{aligned}$$

Conjugating $\dot\tau([s^{-1}, h])$ *by* v *is the same as conjugating the argument by* s, *hence*

$$v\dot\tau([s^{-1}, h])v^{-1} = \dot\tau(s[s^{-1}, h]s^{-1}) = \dot\tau([h, s]).$$

The commutator $[v, u(h)]$ *is equal to* $[u(s), u(h)]$. *Using the defining equation for* $\zeta_{\mathcal J}$, $\dot\tau([h,s]) = \zeta_{\mathcal J}(h)(s)[u(s), u(h)]$, *we finally obtain*

$$\begin{aligned}\widetilde T_h(\widetilde f)(s,v) &= \zeta_{\mathcal J}(h)(s)[u(h), u(s)][u(s), u(h)]\widetilde f(s,v)^{h^{-1}} = \\ &= \zeta_{\mathcal J}(h)(s)\widetilde f(s,v)^{h^{-1}}.\end{aligned}$$

In view of the decomposition $C^*(\widetilde G_{\mathcal J}, \widetilde N, \dot{\mathcal A}, \widetilde\tau) = C^*(\widetilde G_{\mathcal J})_{\widetilde\tau} \otimes \dot{\mathcal A}$ *the formula means that* $h \in H_{\mathcal J}$ *acts diagonally, by* $\dot T_h$ *on the second factor and by multiplication with* $\zeta_{\mathcal J}(h)$ *on the first one.*

The primitive ideal space of $C^*(G_{\mathcal J}, N, \dot{\mathcal A}, \dot T, \dot\tau)$ *can be identified with the primitive ideal space of* $C^*(\widetilde G_{\mathcal J})_{\widetilde\tau}$, *and we also know how* $H_{\mathcal J} \times (G/N)^\wedge$ *acts on the latter space, namely* (h, χ) *acts by multiplication with* $\zeta_{\mathcal J}(h)\chi^{-1}|_{G_{\mathcal J}}$, *where we tacitly identified* $\widetilde G_{\mathcal J}/\widetilde N$ *with* $G_{\mathcal J}/N$.

The representations of $C^*(\widetilde G_{\mathcal J})_{\widetilde\tau}$ *correspond to the group representations of* $\widetilde G_{\mathcal J}$ *which are equal to* $\widetilde\tau$ *on* $\widetilde N$. *Hence they may be viewed as representations of* $\widetilde G_{\mathcal J}/\ker\widetilde\tau$ *which is a two step nilpotent group. By the results of § 2, in particular (2.1), the primitive ideals in* $C^*(\widetilde G_{\mathcal J})_{\widetilde\tau}$ *correspond to those unitary characters of the center of* $\widetilde G_{\mathcal J}/\ker\widetilde\tau$, *which are extensions of* $\widetilde\tau$. *To describe this center we compute* $\widetilde\tau([(s_1, v_1), (s_2, v_2)])$ *for* $(s_1, v_1), (s_2, v_2) \in \widetilde G_{\mathcal J}$:

$$\begin{aligned}&\widetilde\tau([(s_1, v_1), (s_2, v_2)])Id = \dot\tau([s_1, s_2])[v_1, v_2]^{-1} = \\ &= \dot\tau([s_1, s_2])[u(s_1), u(s_2)]^{-1} = \alpha(s_1, s_2)[u(s_1), u(s_2)][u(s_1), u(s_2)]^{-1},\end{aligned}$$

hence $\widetilde{\tau}([(s_1,v_1),(s_2,v_2)] = \alpha(s_1,s_2)$.

Therefore, the center of $\widetilde{G}_{\mathcal{J}}/\ker\widetilde{\tau}$ *is equal to* $\widetilde{Z}_\alpha/\ker\widetilde{\tau}$ *where* $\widetilde{Z}_\alpha = \{(s,v)\in\widetilde{G}_{\mathcal{J}} \mid s\in Z_\alpha\}$. *We conclude that the stabilizer in* $H_{\mathcal{J}}\times(G/N)^\wedge$ *of any point in Priv* $(G_{\mathcal{J}},N,\dot{\mathcal{A}},\dot{T},\dot{\tau})$ *is equal to* $\{(h,\chi)\in H_{\mathcal{J}}\times(G/N)^\wedge \mid \zeta_{\mathcal{J}}(h)|_{Z_\alpha} = \chi|_{Z_\alpha}\}$.

Now it is easy to determine the space Priv $(G_{\mathcal{J}},N,\mathcal{A},T,\tau)$. *Each* $\mathcal{F}$ *in that space restricts to a primitive ideal in* $\mathcal{A}$, $\mathcal{F}|_{\mathcal{A}}$. *The map* $\mathcal{F}\mapsto\mathcal{F}|_{\mathcal{A}}$ *is continuous and* $H\times(G/N)^\wedge$*-equivariant, where* $(G/N)^\wedge$ *acts trivially on* Priv $(\mathcal{A})$ *(and on* $\mathcal{A}$*) and* H *acts via the induced action* $T^{G_{\mathcal{J}}}$ *on* $C^*(G_{\mathcal{J}},N,\mathcal{A},T,\tau)$ *and accordingly on its primitive ideal space. Each fiber* $\{\mathcal{F}\mid \mathcal{F}|_{\mathcal{A}}=\mathcal{J}\}$ *for a given* $\mathcal{J}\in$ Priv $(\mathcal{A})$ *can be identified with* Priv $(G_{\mathcal{J}},N,\mathcal{A}/\mathcal{J},\dot{T},\dot{\tau})$. *Since* H *acts transitively on* Priv $(\mathcal{A})$ *and* $(G/N)^\wedge$ *acts transitively on* Priv $(G_{\mathcal{J}},N,\mathcal{A}/\mathcal{J},\dot{T},\dot{\tau})$ *we conclude that* $H\times(G/N)^\wedge$ *acts transitively on* Priv $(G_{\mathcal{J}},N,\mathcal{A},T,\tau)$. *The above considerations also show that the stabilizer in* $H\times(G/N)^\wedge$ *of any point* $\mathcal{F}$ *is equal to* $\{(h,\chi)\in H_{\mathcal{J}}\times(G/N)^\wedge|\zeta_{\mathcal{J}}(h)|_{Z_\alpha}=\chi|_{Z_\alpha}\}$ *if* $\mathcal{J}=\mathcal{F}|_{\mathcal{A}}$. *We want to show that the corresponding coset space is homeomorphic to* Priv $(G_{\mathcal{J}},N,\mathcal{A},T,\tau)$ *as a* $H\times(G/N)^\wedge$*-space. In particular, we have to have that* Priv $(G_{\mathcal{J}},N,\mathcal{A},T,\tau)$ *is Hausdorff. To this end we shall use the following lemma, which is probably known, but by lack of a suitable reference we include its short proof.*

Lemma 3.6. *Let* Q *be a Hausdorff topological group, let* X *and* Y *be* Q*-spaces, and let* $p: X\to Y$ *be a continuous* Q*-equivariant map. Suppose that* Y *is Hausdorff, that each fiber* $p^{-1}(y), y\in Y$, *is Hausdorff, and that for every* $y\in Y$ *the map* $Q\ni q\mapsto qy\in Y$ *is open. Then* X *is Hausdorff.*

Proof. Take two different points x_1,x_2 in X. If $p(x_1)\neq p(x_2)$ they can be separated by the Hausdorff property of Y. Suppose that $y \stackrel{\text{def}}{=} p(x_1)=p(x_2)$. By the assumption on the fibers there exists open subsets U_1,U_2 in X such that $x_j\in U_j$ and $U_1\cap U_2\cap p^{-1}(y)=\emptyset$. Since X is a Q–space one finds an open symmetric neighborhood W of the identity in Q and open subsets U_j' in X such that $x_j\in U_j'\subset U_j$ and $WU_j'\subset U_j$ for $j=1,2$. By assumption Wy is open in Y, hence $p^{-1}(Wy)$ is an open subset of X containing x_1 and x_2. Substituting U_j' by $U_j'\cap p^{-1}(Wy)$ we may assume in addition that U_j' is contained in $p^{-1}(Wy)$. The proof is finished if we can show that $U_1'\cap U_2'$ is empty. Suppose that x is in this intersection. Then $p(x)$ is of the form $p(x)=wy$, $w\in W$. The point $w^{-1}x$ lies in $p^{-1}(y)$. From $x\in U_j'$ and the symmetry of W we conclude that $w^{-1}x$ is also in U_j. Hence $w^{-1}x\in p^{-1}(y)\cap U_1\cap U_2$, a contradiction. ∎

Proof of Theorem 3.4, continued. Of course, we apply the lemma to $X=$ Priv $(G_{\mathcal{J}},N,\mathcal{A},T,\tau)$, $Q=H\times(G/N)^\wedge$, $Y=$ Priv $(\mathcal{A})$, and $p(\mathcal{F})=\mathcal{F}|_{\mathcal{A}}$. The Q–space Priv $(\mathcal{A})$ satisfies the conditions of the lemma by assumption. As we observed above the fiber of p over $\mathcal{J}$ can be identified, also topologically, with Priv $(G_{\mathcal{J}},N,\mathcal{A}/\mathcal{J},\dot{T},\dot{\tau})$ which is homeomorphic to Priv $C^*(\widetilde{G}_{\mathcal{J}})_{\widetilde{\tau}}$ and hence Hausdorff as we remarked in (2.2). Actually, Priv $(G_{\mathcal{J}},N,\mathcal{A}/\mathcal{J},\dot{T},\dot{\tau})$ is homeomorphic to

$$(Z_\alpha/N)^\wedge \cong H_{\mathcal{J}}\times(G/N)^\wedge/\{(h,\chi)\in H_{\mathcal{J}}\times(G/N)^\wedge|\zeta_{\mathcal{J}}(h)|_{Z_\alpha}=\chi|_{Z_\alpha}\}.$$

Since each primitive ideal space is locally quasi–compact we conclude that $X = \mathrm{Priv}\,(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$ is a locally compact Hausdorff space. As the *separable* group $H \times (G/N)^\wedge$ acts transitively on X the usual Baire argument shows that X is homeomorphic to a coset space; recall that the stabilizer groups were already computed.

Soon we shall need the structure of the space of G–quasi–orbits in X. Again (3.3) is applicable as G/N is central in $H/N \times (G/N)^\wedge$ and as only the latter group acts. We conclude that the G–quasi-orbits coincide with the G–orbit closures and that the space of G–quasi–orbits in $\mathrm{Priv}\,(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$ is homeomorphic to the coset space of $H \times (G/N)^\wedge$ modulo the closure of

$$G\{(h, \chi) \in H_{\mathcal{J}} \times (G/N)^\wedge | \zeta_{\mathcal{J}}(h)|_{Z_\alpha} = \chi|_{Z_\alpha}\}$$

for any chosen $\mathcal{J} \in \mathrm{Priv}\,(\mathcal{A})$. We claim that this closure is equal to $(G\Gamma_{\mathcal{J}})^-$. Of course, this reduces to showing that each pair $(h, \chi) \in H_{\mathcal{J}} \times (G/N)^\wedge$ with $\zeta_{\mathcal{J}}(h)|_{Z_\alpha} = \chi|_{Z_\alpha}$ is contained in $(G\Gamma_{\mathcal{J}})^-$. Take any extension $\eta \in (G/N)^\wedge$ of $\zeta_{\mathcal{J}}(h) \in (G_{\mathcal{J}}/N)^\wedge$. Multiplying (h, χ) with $(h^{-1}, \eta^{-1}) \in \Gamma_{\mathcal{J}}$ reduces the problem to $h = 1$ and $\chi \in (G/Z_\alpha)^\wedge$. Since α induces a non–degenerate form on $G_{\mathcal{J}}/Z_\alpha$, one readily concludes that $(G/Z_\alpha)^\wedge$ is contained in $(G_{\mathcal{J}}\Gamma_{\mathcal{J}})^- \subset (G\Gamma_{\mathcal{J}})^-$. To summarize, we have proved that the space of G–quasi–orbits in $\mathrm{Priv}\,(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$ is as a $H \times (G/N)^\wedge$–space homeomorphic to $H \times (G/N)^\wedge/(G\Gamma_{\mathcal{J}})^-$ for any chosen $\mathcal{J} \in \mathrm{Priv}\,(\mathcal{A})$.

Now we are ready to treat the proper theme of the theorem, namely the structure of $\mathrm{Priv}\,(G, N, \mathcal{A}, T, \tau)$. First we claim that the procedure of inducing yields a (continuous) surjective map ind: $\mathrm{Priv}\,(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau) \to \mathrm{Priv}\,(G, N, \mathcal{A}, T, \tau)$. If $\mathcal{I}$ in $\mathrm{Priv}\,(G, N, \mathcal{A}, T, \tau)$ is given, by the Effros–Hahn conjecture, see [15], there exists $\mathcal{F} \in \mathrm{Priv}\,(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$ such that $\mathcal{I} = \mathrm{ind}_{G_{\mathcal{J}}}^G \mathcal{F}$; note that $G/G_{\mathcal{J}}$ is abelian, hence amenable. Using that $\mathrm{ind}_{G_{\mathcal{J}}}^G$ is clearly $H \times (G/N)^\wedge$–equivariant one concludes that each primitive ideal in $C^*(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$ leads to a primitive ideal in $C^*(G, N, \mathcal{A}, T, \tau)$, and that $\mathrm{Priv}\,(G, N, \mathcal{A}, T, \tau)$ is a transitive $H \times (G/N)^\wedge$–space.

If $\mathcal{I} = \mathrm{ind}_{G_{\mathcal{J}}}^G \mathcal{F}$ as above then the restriction of $\mathcal{I}$ to $C^*(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$ is $\bigcap_{x \in G} \mathcal{F}^x$, i.e., the ideal corresponding in the hull–kernel sense to the closed subset $(G\mathcal{F})^-$ of $\mathrm{Priv}\,(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$, which is, as we have seen, the G–quasi–orbit through $\mathcal{F}$. In other words, we find a canonical continuous map Res from $\mathrm{Priv}\,(G, N, \mathcal{A}, T, \tau)$ into (actually onto) the space $\mathrm{Priv}\,(G, N, \mathcal{A}, T, \tau)/G$ of G–quasi–orbits such that the diagram

$$\begin{array}{ccc} \mathrm{Priv}\,(G_{\mathcal{J}}, N, A, T, \tau) & \longrightarrow & \mathrm{Priv}\,(G, N, A, T, \tau) \\ \searrow & & \swarrow \mathrm{Res} \\ & \mathrm{Priv}\,(G_{\mathcal{J}}, N, A, T, \tau)/G & \end{array}$$

commutes, where the unnamed arrow represents the natural map. The diagram also shows that Res is an open map because the natural map is an open map (under the present circumstances, $\mathrm{Priv}\,(G_{\mathcal{J}}, N.\mathcal{A}, T, \tau)/G$ is a coset space).

Comparing the claim of the theorem and the above derived description of $\mathrm{Priv}\,(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)/G$ we are left to show that Res is injective. To this end we observe the following. Let ρ be a representation of $C^*(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$ given by a pair $(\rho_{G_{\mathcal{J}}}, \rho_{\mathcal{A}})$, let $\pi = \mathrm{ind}_{G_{\mathcal{J}}}^G \rho = (\pi_G, \pi_{\mathcal{A}})$, and let $\rho' = (\rho'_{G_{\mathcal{J}}}, \rho'_{\mathcal{A}})$ be the restriction of π to $C^*(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$, i.e., $\rho'_{G_{\mathcal{J}}} = \pi_G|_{G_{\mathcal{J}}}$ and $\rho'_{\mathcal{A}} = \pi_{\mathcal{A}}$. Then $\pi' \overset{\text{def}}{=} \mathrm{ind}_{G_{\mathcal{J}}}^G \rho' = (\pi'_G, \pi'_{\mathcal{A}})$ is unitarily equivalent to a multiple of π. Indeed, if $\mathfrak{H}$ denotes the representation space of ρ then π' acts in the space $\mathfrak{H}_{\pi'}$ of measurable functions $\xi : G \times G \to \mathfrak{H}$ satisfying

$$\xi(xu, y) = \xi(x, uy),$$
$$\xi(x, yu) = \rho_{G_{\mathcal{J}}}(u)^{-1}\xi(x, y)$$

for $x, y \in G$, $u \in G_{\mathcal{J}}$ and

$$\int_{G/G_{\mathcal{J}}} \int_{G/G_{\mathcal{J}}} \|\xi(x, y)\|^2 \, d\dot{x} \, d\dot{y} < \infty.$$

The representation π' is given by

$$(\pi'_G(z)\xi)(x, y) = \xi(z^{-1}x, y) \text{ and } (\pi'_{\mathcal{A}}(a)\xi)(x, y) = \rho_{\mathcal{A}}(a^{xy})(\xi(x, y))$$

for $x, y, z \in G$ and $a \in \mathcal{A}$.

With each $\xi \in \mathfrak{H}_{\pi'}$ associate another function $\xi' : G \times G \to \mathfrak{H}$ by

$$\xi'(x, y) = \xi(xy, y^{-1}).$$

It is easily checked that $\xi \mapsto \xi'$ is a unitary operator from $\mathfrak{H}_{\pi'}$ onto the space of measurable functions $\xi' : G \times G \to \mathfrak{H}$ satisfying

$$\xi'(xu, y) = \xi'(x, y)$$
$$\xi'(x, yu) = \rho_{G_{\mathcal{J}}}(u)^{-1}\xi'(x, y)$$

for $x, y \in G$, $u \in G_{\mathcal{J}}$ and

$$\int_{G/G_{\mathcal{J}}} \int_{G/G_{\mathcal{J}}} \|\xi'(x, y)\|^2 d\dot{x} d\dot{y} < \infty.$$

Transforming the operators $\pi'_G(z)$ and $\pi'_{\mathcal{A}}(a)$ along $\xi \mapsto \xi'$ one obtains operators, say $\pi''_G(z)$ and $\pi''_{\mathcal{A}}(a)$, which are given by

$$(\pi''_G(z)\xi')(x, y) = \xi'(x, z^{-1}y)$$

and

$$(\pi''_{\mathcal{A}}(a)\xi')(x, y) = \rho_{\mathcal{A}}(a^y)(\xi'(x, y)).$$

In other words, the space of those ξ's is a tensor product of $L^2(G/G_{\mathcal{J}})$ and the representation space of $\pi = \mathrm{ind}\rho$, $\pi''_G(z)$ is the tensor product of $\mathrm{Id}_{L^2(G/G_{\mathcal{J}})}$ and $\pi_G(z)$, and $\pi''_{\mathcal{A}}(a)$ is the tensor product of $\mathrm{Id}_{L^2(G/G_{\mathcal{J}})}$ and $\pi_{\mathcal{A}}(a)$.

The preceding considerations are of course well known in group representation theory. One sees that the same arguments apply to twisted covariance algebras. Note that the only assumption we have used is the normality of $G_{\mathcal{J}}$.

From this observation we conclude the injectivity of Res as follows. Any given $\mathcal{I} \in \mathrm{Priv}\,(G, N, \mathcal{A}, T, \tau)$ may be written as $\mathcal{I} = \ker(\pi = \mathrm{ind}_{G_{\mathcal{J}}}^G \rho)$ for an appropriate representation ρ. In the above terminology we also have $\mathcal{I} = \ker(\pi' = \mathrm{ind}_{G_{\mathcal{J}}}^G \rho') = \mathrm{ind}_{G_{\mathcal{J}}}^G \ker \rho'$ by definition. But $\ker \rho'$ is nothing but the restriction of $\mathcal{I}$ to $C^*(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$ which is the kernel $k(\mathrm{Res}\,(\mathcal{I}))$ in the hull–kernel sense of $\mathrm{Res}\,(\mathcal{I}) \subset \mathrm{Priv}\,(G_{\mathcal{J}}, N, \mathcal{A}, T, \tau)$. Hence $\mathcal{I} = \mathrm{ind}_{G_{\mathcal{J}}}^G k(\mathrm{Res}\,(\mathcal{I}))$ which shows that $\mathcal{I}$ is completely determined by $\mathrm{Res}\,(\mathcal{I})$, whence Res is injective. ∎

Proof of Corollary 3.5. The implication (i) $\Rightarrow$ (ii) is obvious because ideals, which are induced from M, are $(G/M)^\wedge$–invariant. The equivalence of (ii) and (iii) is an immediate consequence of the theorem. We are left with the implication (ii) $\Rightarrow$ (i). Let $\mathcal{I} \in \operatorname{Priv}(G, N, \mathcal{A}, T, \tau)$ be given. As $\mathcal{I}$ is invariant under $(G/M)^\wedge$, and hence under $C_\infty(G/M)$, from a C^*-version of the imprimitivity theorem it follows that $\mathcal{I}$ is of the form $\mathcal{I} = \operatorname{ind}_M^G \mathcal{Q}$ for some ideal $\mathcal{Q}$ in $C^*(M, N, \mathcal{A}, T, \tau)$. Take any primitive ideal $\mathcal{F}$ in $C^*(M, N, \mathcal{A}, T, \tau)$ containing $\mathcal{Q}$. Then $\mathcal{I} = \operatorname{ind}_M^G \mathcal{Q} \subset \operatorname{ind}_M^{\mathcal{Q}} \mathcal{F}$ and hence $\mathcal{I} = \operatorname{ind}_M^G \mathcal{F}$ as $\mathcal{I}$ is maximal. The rest follows as in the corresponding part of the proof of the theorem from the $H \times (G/N)^\wedge$–equivariance of ind_M^G and the fact that both, $\operatorname{Priv}(M, N, \mathcal{A}, T, \tau)$ and $\operatorname{Priv}(G, N, \mathcal{A}, T, \tau)$, are transitive $H \times (G/N)^\wedge$–spaces.

Also the fact that $\operatorname{Priv}(G, N, \mathcal{A}, T, \tau)$ is homeomorphic to the space of G–quasi–orbits in $\operatorname{Priv}(M, N, \mathcal{A}, T, \tau)$ can be shown mutatis mutandis as in the proof of the theorem where we discussed the case $M = G_{\mathcal{J}}$. The claimed homeomorphy can also be achieved by describing the $H \times (G/N)^\wedge$–spaces $\operatorname{Priv}(G, N, \mathcal{A}, T, \tau)$ and $\operatorname{Priv}(M, N, \mathcal{A}, T, \tau)$ by means of (3.4); to this end observe that $(G/M)^\wedge \subset (G\Gamma_{\mathcal{J}})^-$ for $\mathcal{J} \in \operatorname{Priv}(\mathcal{A})$ implies $(G/M_{\mathcal{J}})^\wedge \subset (G\Gamma_{\mathcal{J}})^-$ because $(G/G_{\mathcal{J}})^\wedge$ is contained in $(G\Gamma_{\mathcal{J}})^-$. ■

Of course, the description of $\operatorname{Priv}(G, N, \mathcal{A}, T, \tau)$ simplifies if the twisted covariance system is already regular (and if G/N is abelian). In this case one can also easily describe the corresponding primitive quotients. The next proposition is a combination of one of Green's isomorphism theorems and of our insights in the representation theory of two step nilpotent groups.

Proposition 3.7. *Let $(H, N, \mathcal{A}, T, \tau)$ be a regular twisted covariance system with abelian quotient H/N. Let $\mathcal{J}$ be any point in $\operatorname{Priv}(\mathcal{A})$ with associated Mackey bicharacter $\alpha: H_{\mathcal{J}} \times H_{\mathcal{J}} \to \mathbb{T}$, and let $Z_\alpha = \{x \in H_{\mathcal{J}} | \alpha(x, y) = 1$ for all $y \in H_{\mathcal{J}}\}$ be its kernel.*

Then $\operatorname{Priv}(H, N, \mathcal{A}, T, \tau)$ is a transitive $(H/N)^\wedge$–space for the canonical action, and each stabilizer is equal to $(H/Z_\alpha)^\wedge$. Hence $\operatorname{Priv}(H, N, \mathcal{A}, T, \tau)$ is homeomorphic to $(Z_\alpha/N)^\wedge$. For any quasi–polarization P of the quasi–symplectic space $H_{\mathcal{J}}/Z_\alpha$, endowed with the form induced by α, there exists a cocycle m on $P^\perp/P$ such that $m(x, y)m(y, x)^{-1} = \alpha(x, y)$ for all $x, y \in P^\perp$ and such that each primitive quotient of $C^*(H, N, \mathcal{A}, T, \tau)$ is isomorphic to the tensor product of the twisted convolution algebra $C^*(P^\perp/P, m)$ and the algebra of compact operators.

The latter factor is finite–dimensional if and only if $H_{\mathcal{J}}/Z_\alpha$ is discrete and $\mathcal{A}$ is finite–dimensional. Moreover, $C^*(H, N, \mathcal{A}, T, \tau)$ is of type I if and only if the image of $\zeta_{\mathcal{J}} : H_{\mathcal{J}} \to (H_{\mathcal{J}}/N)^\wedge$ is closed, namely equal to $(H_{\mathcal{J}}/Z_\alpha)^\wedge$.

If in addition H/N is a compactly generated Lie group, one may choose P such that $P^\perp/P$ is isomorphic to $\mathbb{Z}^n$. Hence the primitive quotients of $C^*(H, N, \mathcal{A}, T, \tau)$ are stably isomorphic to noncommutative tori. The number n is given by $n = r((Z_\alpha/N)^\wedge) + r(H/N) - r(H/H_{\mathcal{J}}) = r((Z_\alpha/N)^\wedge) + r(H_{\mathcal{J}}/N)$; for the definition of r compare (1.27) and (1.28).

Proof. The first assertions on the structure of the space $\operatorname{Priv}(H, N, \mathcal{A}, T, \tau)$ follow immediately from (3.4), because for any chosen $\mathcal{J} \in \operatorname{Priv}(\mathcal{A})$ the space $(H \times (H/N)^\wedge)/(H\Gamma_{\mathcal{J}})^-$ is canonically homeomorphic to $(H/N)^\wedge/(H/Z_\alpha)^\wedge \cong$

$(Z_\alpha/N)^\wedge$. By [17] the twisted covariance algebra $C^*(H,N,\mathcal{A},T,\tau)$ is isomorphic to the tensor product of $C^*(H_{\mathcal{J}},N,\mathcal{A}/\mathcal{J},\dot{T},\dot{\tau})$ and the algebra of compact operators on $L^2(H/H_{\mathcal{J}})$. As we have seen in the proof of (3.4), the algebra $C^*(H_{\mathcal{J}},N,\mathcal{A}/\mathcal{J},\dot{T},\dot{\tau})$ is isomorphic to $C^*(\widetilde{H}_{\mathcal{J}})_{\widetilde{\tau}}\otimes\mathcal{A}/\mathcal{J}$. By the results of § 2, in particular (2.3), the primitive quotients of $C^*(\widetilde{H}_{\mathcal{J}})_{\widetilde{\tau}}$, which is a quotient of the C^*–algebra of a two step nilpotent group, are isomorphic to the tensor product of a twisted convolution C^*–algebra on a discrete abelian group as described in the corollary and the algebra $\mathfrak{K}$ of compact operators on a certain Hilbert space. Hence the primitive quotients of $C^*(H,N,\mathcal{A},T,\tau)$ are isomorphic to $C^*(P^\perp/P,m)\otimes\mathfrak{K}\otimes\mathcal{A}/\mathcal{J}\otimes\mathfrak{K}(L^2(H/H_{\mathcal{J}}))$. The factor $\mathfrak{K}\otimes\mathcal{A}/\mathcal{J}\otimes\mathfrak{K}(L^2(H/H_{\mathcal{J}}))$ is finite–dimensional if and only if $\mathfrak{K}$ is finite–dimensional, $H/H_{\mathcal{J}}$ is finite, and $\mathcal{A}/\mathcal{J}$ is finite–dimensional. The first condition means that $H_{\mathcal{J}}/Z_\alpha$ is discrete, compare (2.3) and (2.4), the latter two conditions are equivalent to $\mathcal{A}$ being finite–dimensional.

The algebra $C^*(H,N,\mathcal{A},T,\tau)$ is of type I if and only if $C^*(\widetilde{H}_{\mathcal{J}})_{\widetilde{\tau}}$ is of type I. By (2.6) this means that the canonical map $\psi: H_{\mathcal{J}}/Z_\alpha\to(H_{\mathcal{J}}/Z_\alpha)^\wedge$ of § 1 is an isomorphism (or merely bijective), which in the present terminology is nothing but the image of $\zeta_{\mathcal{J}}$ being equal to $(H_{\mathcal{J}}/Z_\alpha)^\wedge$.

The addition follows easily from the results of § 1, cf. also (2.4). By (1.19) the group P may be chosen as claimed. The number n equals the “rank” $r(H_{\mathcal{J}}/Z_\alpha)$ of the quasi–symplectic space $H_{\mathcal{J}}/Z_\alpha$. From the exact sequence

$$0\to H_{\mathcal{J}}/Z_\alpha\to H/Z_\alpha\to H/H_{\mathcal{J}}\to 0$$

one gets $n=r(H/Z_\alpha)-r(H/H_{\mathcal{J}})$ by (1.30). The exact sequence

$$0\to Z_\alpha/N\to H/N\to H/Z_\alpha\to 0$$

yields $r(H/Z_\alpha)=r(H/N)-r(Z_\alpha/N)$. Using $r((Z_\alpha/N)^\wedge)=-r(Z_\alpha/N)$, compare (1.28), one finally obtains $n=r(H/N)-r(Z_\alpha/N)-r(H/H_{\mathcal{J}})=r(H/N)+r((Z_\alpha/N)^\wedge)-r(H/H_{\mathcal{J}})$. ■

For general centrally regularizable systems we don't know how to determine the structure of the primitive quotients. But under different additional hypotheses we can do it. One such hypothesis is formulated in [25, Lemma 2], the corresponding proof uses computations similar to those in [31, Theorem 2]. Here we shall assume another hypothesis, which is also satisfied when studying C^*–algebras of connected Lie groups. This latter hypothesis will allow us to reduce to regular systems via Takai duality.

Proposition 3.8. *Let $(G,N,\mathcal{A},T,\tau)$ be a centrally regularizable twisted covariance system with regularization $(H,N,\mathcal{A},T,\tau)$. Suppose that H/N is abelian and that H is a semidirect product of G and a closed (abelian) subgroup W, $H=W\ltimes G$. Then $C^*(G,N,\mathcal{A},T,\tau)$ is stably isomorphic to the C^*–algebra of a regular twisted covariance system $(Q,N',\mathcal{A},T',\tau')$ described in the proof below. As Q/N' is abelian, (3.7) is applicable and gives the primitive quotients of $C^*(G,N,\mathcal{A},T,\tau)$ up to stable isomorphism.*

Proof. The group W acts on $\varphi \in L^1(G,N,\mathcal{A},T,\tau)$ via $\varphi^w(x) = T_{w^{-1}}(\varphi(wxw^{-1}))$ for $w \in W$, $x \in G$, where $T_{w^{-1}}$ denotes the extended action of $w^{-1} \in W \subset H$ on $\mathcal{A}$. Clearly, this action gives rise to an ordinary covariance system $(W,\mathcal{B})$ where we put $\mathcal{B} \overset{\text{def}}{=} C^*(G,N,\mathcal{A},T,\tau)$. The Pontryagin dual $\widehat{W}$ acts on $L^1(W,\mathcal{B})$, namely

$$f^\chi(w) = \chi(w)f(w)$$

for $\chi \in \widehat{W}$, $w \in W$ and $f \in L^1(W,\mathcal{B})$.

Also this action extends to $C^*(W,\mathcal{B})$, hence we may form the covariance algebra $C^*(\widehat{W},C^*(W,\mathcal{B}))$. By Takai duality, see [46], the latter algebra is canonically isomorphic to the tensor product of $\mathcal{B}$ with the algebra of compact operators on $L^2(W)$.

It remains to show that $C^*(\widehat{W},C^*(W,\mathcal{B}))$ is isomorphic to the C^*–algebra of a regular twisted covariance system. To this end, we form the group $Q = \widehat{W} \ltimes (H \times \mathbb{T})$ where the multiplication is given by

$$(\chi,h,t)(\chi',h',t') = (\chi\chi',hh',tt'\chi'(h)).$$

Here, of course, $\chi' \in \widehat{W}$ is considered as an element in $(H/G)^\wedge \subset \widehat{H}$. As $\widehat{W} \ltimes \mathbb{T} = \widehat{W} \times \mathbb{T}$ is normal in Q, the group H is a quotient of Q in a canonical manner. Hence the action of H on $\mathcal{A}$ yields an action of Q on $\mathcal{A}$, denoted by T'. The group $N' \overset{\text{def}}{=} N \times \mathbb{T}$ is normal in Q. We define a twist τ' on N' with values in $\mathcal{A}^\flat$ by $\tau'(n,t)(a) = t\tau(n)(a)$ for $a \in \mathcal{A}$, $n \in N$, $t \in \mathbb{T}$.

It is easy to check that T' and τ' are compatible, i.e., $(Q,N',\mathcal{A},T',\tau')$ is a twisted covariance system. Indeed it is a regular one, and $Q/N' \cong \widehat{W} \times (H/N)$ is abelian. The proof is finished by observing that $C^*(\widehat{W},C^*(W,\mathcal{B}))$ and $C^*(Q,N',\mathcal{A},T',\tau')$ are isomorphic. Actually, if $f : \widehat{W} \times W \times G \to \mathcal{A}$ is a measurable function with $f(\chi,w,xn) = \tau(n)^{-1}f(\chi,w,x)$, whose norm is integrable modulo N, then f may be considered as an element of $C^*(\widehat{W},C^*(W,\mathcal{B}))$. With such an f we associate the function $f' : Q \to \mathcal{A}$ defined by $f'(\chi,wx,t) = t^{-1}f(\chi,w,x)$. This assignment yields the desired isomorphism. ■

Now we are ready to study connected Lie groups. Let G be such a group and assume in addition that G is simply connected. Let N be the derived group of G. This group is known to be locally algebraic, hence it is a type I group. By $\mathfrak{g}$ and $\mathfrak{n}$ we denote the Lie algebras of G and N, respectively. According to [33, (8.1) and (8.3)] there exists an abelian Lie algebra $\mathfrak{a}$ acting on $\mathfrak{g}$ by derivations such that $[\mathfrak{a},\mathfrak{g}] \subset \mathfrak{n}$ and that each connected Lie group with algebra $\mathfrak{a} \ltimes \mathfrak{g}$ has an almost algebraic adjoint group. In particular, this applies to the simply connected group $A \ltimes G$ where A is a simply connected group with algebra $\mathfrak{a}$, i.e., A is an $\mathbb{R}^d$. Then also the restriction of $\mathrm{Ad}(A \ltimes G)$ to $\mathfrak{n}$ is an almost algebraic group. From results of Pukanszky, see [36], we conclude that each $A \ltimes G$–orbit through a point $\mathcal{J} \in \mathrm{Priv}\, C^*(N)$ is locally closed and homeomorphic to $(A \ltimes G)/(A \ltimes G)_{\mathcal{J}}$. Since the stabilizer $(A \ltimes G)_{\mathcal{J}}$ contains N the latter space is actually a connected abelian Lie group.

Let now $\mathcal{I}$ be a given primitive ideal in $C^*(G)$ and let $\mathfrak{X}$ be the corresponding G–quasi–orbit in Priv (N). If $\mathcal{J}$ is any point in $\mathfrak{X}$ then $\mathfrak{X}$ is the relative closure of the G–orbit $G\mathcal{J}$ in $(A \ltimes G)\mathcal{J}$, which implies in particular that $\mathfrak{X}$ is locally closed. It is easy to see that this relative closure is equal to $(W \ltimes G)\mathcal{J}$ for a suitable closed connected subgroup W of A, which is a vector group as well. Put $H \stackrel{\text{def}}{=} W \ltimes G$.

We collect some well known facts. The closed subsets $\overline{\mathfrak{X}}$ and $\overline{\mathfrak{X}} \setminus \mathfrak{X}$ of Priv $C^*(N)$ define ideals $k(\overline{\mathfrak{X}})$ and $k(\overline{\mathfrak{X}} \setminus \mathfrak{X})$ in $C^*(N)$. The group C^*–algebra $C^*(G)$ may be viewed as a twisted covariance algbra $C^*(G, N, C^*(N), T, \tau)$ where $(T_x f)(z) = \delta(x) f(x^{-1}zx)$ and $(\tau(y)f)(z) = f(y^{-1}z)$ for $x \in G, f \in L^1(N)$ and $y, z \in N$; here δ denotes the appropriate modular function turning T_x into a $\| \, \|_1$ – preserving action.

The subquotient $C^*(G) * k(\overline{\mathfrak{X}} \setminus \mathfrak{X})/C^*(G) * k(\overline{\mathfrak{X}})$ of $C^*(G)$ is isomorphic to $C^*(G, N, \mathcal{A}, T, \tau)$ where $\mathcal{A} \stackrel{\text{def}}{=} k(\overline{\mathfrak{X}} \setminus \mathfrak{X})/k(\overline{\mathfrak{X}})$. Its primitive ideal space is canonically homeomorphic to the subset $\mathfrak{Y}$ of Priv $C^*(G)$ consisting of all $\mathcal{I}'$ such that $\mathcal{I}'|_{C^*(N)} = k(\overline{\mathfrak{X}})$. Alternatively, $\mathfrak{Y}$ can be characterized as the set of primitive ideals $\mathcal{I}'$ in $C^*(G)$ such that $\mathcal{I}'$ contains $C^*(G) * k(\overline{\mathfrak{X}})$, but $C^*(G) * k(\overline{\mathfrak{X}} \setminus \mathfrak{X})$ is not contained in $\mathcal{I}'$. This shows that $\mathfrak{Y}$ is a locally closed subset of Priv $C^*(G)$.

Theorem 3.9. *Let G be a simply connected Lie group, and let N be its commutator subgroup. Let $\mathcal{I}$ be a given primitive ideal in $C^*(G)$, let $\mathfrak{X}$ be the corresponding G–quasi–orbit in* Priv $C^*(N)$, *and let $\mathfrak{Y}$ be the set of primitive ideals in $C^*(G)$ lying over $\mathfrak{X}$ as explained above. Then $\mathfrak{X}$ and $\mathfrak{Y}$ are locally closed subsets of Priv $C^*(N)$ and* Priv $C^*(G)$, *respectively. Moreover, they are homeomorphic to the underlying spaces of connected abelian Lie groups. The primitive quotient $C^*(G)/\mathcal{I}$ contains a unique simple ideal $\mathcal{M}(\mathcal{I})$, namely $(\mathcal{I} + C^*(G) * k(\overline{\mathfrak{X}} \setminus \mathfrak{X}))/\mathcal{I}$, which is isomorphic to the simple quotient $(C^*(G) * k(\overline{\mathfrak{X}} \setminus \mathfrak{X}))/(C^*(G) * k(\overline{\mathfrak{X}} \setminus \mathfrak{X})) \cap \mathcal{I}$ of $C^*(G) * k(\overline{\mathfrak{X}} \setminus \mathfrak{X})/C^*(G) * k(\overline{\mathfrak{X}})$. The C^*–algebra $\mathcal{M}(\mathcal{I})$ is stably isomorphic to a noncommutative torus in dimension n, where n is given by*

$$n = rk\pi_1(\mathfrak{X}) - rk\pi_1(\mathfrak{Y}).$$

Actually, it is isomorphic to the tensor product of such a noncommutative torus with the algebra of compact operators on a separable Hilbert space except for the case that $\mathcal{I}$ is of finite codimension, which implies that $\mathfrak{X}$ reduces to a closed one point set $\{\ker\rho\}$, and that ρ can be extended to a representation of G.

Proof. We remarked already in front of (3.9) that $\mathfrak{X}$ and $\mathfrak{Y}$ are locally closed subsets. In the sequel, we shall use the notations $\mathcal{A}$ and $H = W \ltimes G$ introduced there, again $\mathcal{J}$ denotes a chosen point in $\mathfrak{X}$. The set $\mathfrak{X}$ is homeomorphic to $H/H_{\mathcal{J}}$, which is a connected abelian Lie group. The group H delivers a regularization of the twisted covariance system $(G, N, \mathcal{A}, T, \tau)$. Because of the semidirect decomposition of H we can construct as in the proof (3.8) the regular twisted covariance system $(Q, N', \mathcal{A}, T', \tau')$. The subquotient $C^*(G) * k(\overline{\mathfrak{X}} \setminus \mathfrak{X})/C^*(G) * k(\overline{\mathfrak{X}}) \cong C^*(G, N, \mathcal{A}, T, \tau)$ is by (3.8) stably isomorphic to $C^*(Q, N', \mathcal{A}, T', \tau')$. Hence $\mathfrak{Y}$ is homeomorphic to the primitive ideal space of the latter algebra, which was computed in (3.7). Let α denote the Mackey bicharacter on $Q_{\mathcal{J}}$ associated with $\mathcal{J}$ and the system $(Q, N', \mathcal{A}, T', \tau')$, and let Z_α be its kernel. Observe that $Q_{\mathcal{J}} = \widehat{W} \ltimes (H_{\mathcal{J}} \times \mathbb{T})$ and that $Q/Q_{\mathcal{J}} \cong H/H_{\mathcal{J}}$. By (3.7) the space $\mathfrak{Y}$ is homeomorphic to $(Z_\alpha/N')^\wedge$. Since Z_α/N' is isomorphic to a closed subgroup of the vector group $Q/N' \cong \widehat{W} \times (H/N)$, its Pontryagin dual is a connected Lie group.

Clearly, the ideal $\mathcal{M}(\mathcal{I})$ as defined in the theorem is isomorphic to the primitive quotient as given there. But the primitive quotients of $C^*(G) * k(\overline{\mathfrak{X}} \setminus \mathfrak{X})/C^*(G) * k(\overline{\mathfrak{X}})$ are simple by (3.4) and stably isomorphic to primitive quotients of $C^*(Q, N', \mathcal{A}, T', \tau')$, which by (3.7) are stably isomorphic to noncommutative tori in dimension n, where n is given by

$$n = r((Z_\alpha/N')^\wedge) + r(Q/N') - r(Q/Q_{\mathcal{J}}).$$

Since Q/N' is a vector group the middle term vanishes. The groups $(Z_\alpha/N')^\wedge$ and $Q/Q_{\mathcal{J}}$ being connected one has $r((Z_\alpha/N')^\wedge) = -\mathrm{rk}\pi_1((Z_\alpha/N')^\wedge)$ and $r(Q/Q_{\mathcal{J}}) = -\mathrm{rk}\pi_1(Q/Q_{\mathcal{J}})$. As we remarked earlier the spaces $(Z_\alpha/N')^\wedge$ and $Q/Q_{\mathcal{J}}$ are homeomorphic to $\mathfrak{Y}$ and $\mathfrak{X}$, respectively. Hence we obtain the claimed formula $n = \mathrm{rk}\pi_1(\mathfrak{X}) - \mathrm{rk}\pi_1(\mathfrak{Y})$.

Concerning the assertion on isomorphy rather than stable isomorphy we just refer to [31, Theorem 2]. It also follows from (3.12) below. ■

Remarks 3.10. The theorem says that the quasi–orbit $\mathfrak{X}$ in Priv $C^*(N)$ is homeomorphic to $\mathbb{R}^r \times \mathbb{T}^a$, while the set $\mathfrak{Y}$ of primitive ideals in $C^*(G)$ lying over $\mathfrak{X}$ is homeomorphic to $\mathbb{R}^s \times \mathbb{T}^{a-n}$, i.e., the dimension n of the noncommutative torus corresponding to $\mathcal{I}$ reduces the compact part of $\mathfrak{Y}$. We have not claimed any relation between r and s.

Our approach allows, at least in principle, to compute the cocycles m on $P^\perp/P \cong \mathbb{Z}^n$, which define the noncommutative torus corresponding to a given primitive ideal.

If G is a connected Lie group, not necessarily simply connected, then clearly the theorem gives that also in this case each primitive quotient $C^*(G)/\mathcal{I}$ of $C^*(G)$ contains a unique simple ideal $\mathcal{M}(\mathcal{I})$, which is stably isomorphic to a noncommutative torus. Its dimension $n = n_{\mathcal{I}}$ is an invariant of the ideal $\mathcal{I}$, hence we obtain a map $\mathcal{I} \mapsto n_{\mathcal{I}}$ from Priv $C^*(G)$ into the set of nonnegative integers. A corresponding statement is true for arbitrary connected locally compact groups. This is seen by the usual trick: Each irreducible unitary representation of such a group factors through a Lie quotient. As examples show, see (3.16) below, the map $\mathcal{I} \mapsto n_{\mathcal{I}}$ has in general no continuity properties at all. ■

The next proposition gives an alternative way of computing $n_{\mathcal{I}}$ without referring explicitly to the commutator subgroup.

Proposition 3.11. *Let G be a connected Lie group, and let $\mathcal{I}$ be a primitive ideal in $C^*(G)$. The set of unitary characters η on G with $\eta \cdot \mathcal{I} = \mathcal{I}$ is a closed subgroup of* Hom $(G, \mathbb{T})$*, hence it coincides with $(G/M)^\wedge$ for a certain closed coabelian subgroup $M = M^{\mathcal{I}}$ of G. Let $\mathfrak{V}$ be the G–quasi–orbit in* Priv $C^*(M)$ *corresponding to $\mathcal{I}$, i.e., $\mathfrak{V}$ is the set of primitive ideals $\mathcal{F}$ in $C^*(M)$ such that the closure $(G\mathcal{F})^-$ of the G–orbit through $\mathcal{F}$ coincides with the hull of*

$\mathcal{I}|_{C^*(M)}$ in Priv $C^*(M)$ or, alternatively , such that $\mathrm{ind}_M^G\mathcal{F} = \mathcal{I}$. Then $\mathfrak{V}$ is homeomorphic to the underlying space of a connected abelian Lie group, and one has

$$n_{\mathcal{I}} = rk\pi_1(\mathfrak{V}) - rk\pi_1(G/M).$$

Proof. The assertions can be easily deduced from the corresponding assertions on the universal covering of G, hence we assume in the sequel that G is simply connected. We shall use the notations of (3.9), in particular N is the commutator subgroup of G, $\mathfrak{X}$ is the G–quasi–orbit in Priv $C^*(N)$ corresponding to $\mathcal{I}$, which is an H–orbit for some group of the form $H = W \ltimes G$, and $\mathcal{A} \stackrel{\mathrm{def}}{=} k(\overline{\mathfrak{X}} \setminus \mathfrak{X})/k(\overline{\mathfrak{X}})$. The ideal $\mathcal{I}$ defines a primitive ideal in the subquotient $C^*(G) * k(\overline{\mathfrak{X}} \setminus \mathfrak{X})/C^*(G) * k(\overline{\mathfrak{X}})$, which is the C^*–algebra of the twisted covariance system $(G, N, \mathcal{A}, T, \tau)$. Indeed, this system is centrally regularizable, a regularization is given by the group H.

The space $\mathfrak{Y} \subset$ Priv (G) of primitive ideals lying over $\mathfrak{X}$ can be identified with the spectrum of $C^*(G, N, \mathcal{A}, T, \tau)$, hence it can be computed by (3.4). The group $M = M^{\mathcal{I}}$ is precisely the minimal group M in the sense of (3.5). The space $\mathfrak{V}$ of the proposition is homeomorphic to a G–quasi–orbit $\mathfrak{V}'$ in Priv $(M, N, \mathcal{A}, T, \tau)$. If $\mathcal{F}'$ is any chosen point in $\mathfrak{V}'$ the restriction $\mathcal{J} \stackrel{\mathrm{def}}{=} \mathcal{F}'|_{\mathcal{A}}$ lies in Priv $(\mathcal{A}) \stackrel{\sim}{=} \mathfrak{X}$ as M is contained in the kernel of the Mackey bicharacter associated with $(G, N, \mathcal{A}, T, \tau)$. From (3.4) we deduce that Priv $(M, N, \mathcal{A}, T, \tau)$ is a transitive $H \times (G/N)^\wedge$–space, and that, in the notations of (3.4), the stabilizer of $\mathcal{F}'$ is equal to

$$\{(h, \chi) | h \in H_{\mathcal{J}}, \chi|_M = \zeta_{\mathcal{J}}(h)|_M\},$$

which coincides with $\Gamma_{\mathcal{J}}(G/M)^\wedge$ as M is contained in $G_{\mathcal{J}}$. Hence Priv $(M, N, \mathcal{A}, T, \tau)$ can be identified with $H \times (G/N)^\wedge/\Gamma_{\mathcal{J}}(G/M)^\wedge$. The G–quasi–orbit $\mathfrak{V}'$ in Priv $(M, N, \mathcal{A}, T, \tau)$ is homeomorphic to

$$\{G\Gamma_{\mathcal{J}}(G/M)^\wedge\}^-/\Gamma_{\mathcal{J}}(G/M)^\wedge.$$

Since by (3.5) the group $(G/M)^\wedge$ is contained in $(G\Gamma_{\mathcal{J}})^-$, the space $\mathfrak{V}'$, and hence $\mathfrak{V}$, is homeomorphic to $(G\Gamma_{\mathcal{J}})^-/\Gamma_{\mathcal{J}}(G/M)^\wedge$, which is a connected abelian Lie group.

The basic formula of (3.9) gives $n_{\mathcal{I}} = \mathrm{rk}\,\pi_1(\mathfrak{X}) - \mathrm{rk}\,\pi_1(\mathfrak{Y})$. Since $\mathfrak{X}$ is homeomorphic to $H/H_{\mathcal{J}}$ one has $\mathrm{rk}\,\pi_1(\mathfrak{X}) = r(H_{\mathcal{J}})$. As by (3.4) the space $\mathfrak{Y}$ can be identified with $(H \times (G/N)^\wedge)/(G\,\Gamma_{\mathcal{J}})^-$ the rank of $\pi_1(\mathfrak{Y})$ coincides with $r((G\,\Gamma_{\mathcal{J}})^-)$. Hence $n_{\mathcal{I}} = r(H_{\mathcal{J}}) - r((G\,\Gamma_{\mathcal{J}})^-)$.

On the other hand

$$\begin{aligned}\mathrm{rk}\,\pi_1(\mathfrak{V}) - \mathrm{rk}\,\pi_1(G/M) &= -r((G\,\Gamma_{\mathcal{J}})^-/\Gamma_{\mathcal{J}}(G/M)^\wedge) + r(G/M) \\ &= -r((G\,\Gamma_{\mathcal{J}})^-) + r(\Gamma_{\mathcal{J}}(G/M)^\wedge) + r(G/M).\end{aligned}$$

To compute $r(\Gamma_{\mathcal{J}}(G/M)^\wedge)$ we consider the exact sequence

$$0 \to (G/M)^\wedge \longrightarrow \Gamma_{\mathcal{J}}(G/M)^\wedge \longrightarrow \Gamma_{\mathcal{J}}/(\Gamma_{\mathcal{J}} \cap (G/M)^\wedge) \to 0.$$

The kernel of the map $\Gamma_{\mathcal{J}} \ni (h,\chi) \longmapsto h \in H_{\mathcal{J}}$ coincides with $\Gamma_{\mathcal{J}} \cap (G/M)^\wedge$, hence $\Gamma_{\mathcal{J}}/(\Gamma_{\mathcal{J}} \cap (G/M)^\wedge)$ is isomorphic to $H_{\mathcal{J}}$. Using (1.30) we conclude $r(\Gamma_{\mathcal{J}}(G/M)^\wedge) = r((G/M)^\wedge) + r(H_{\mathcal{J}})$.

Therefore, $\operatorname{rk}\pi_1(\mathfrak{V}) - \operatorname{rk}\pi_1(G/M) = -r((G\Gamma_{\mathcal{J}})^-) + r((G/M)^\wedge) + r(H_{\mathcal{J}}) + r(G/M) = -r((G\Gamma_{\mathcal{J}})^-) + r(H_{\mathcal{J}})$ by (1.28), which is equal to $n_{\mathcal{I}}$ as we have seen above. ■

Until now we were mainly concerned with the computation of $n_{\mathcal{I}}$ of a given primitive ideal $\mathcal{I}$ in the group C^*–algebra of a connected Lie group. In the derivation of our results we were forced to gain some insights in the structure of subquotients corresponding to quasi–orbits in the spectrum of the commutator subgroup. These insights are made explicit in the next theorem, where we change our view point, now starting with a given quasi–orbit in $\widehat{N}$ rather than with a given ideal $\mathcal{I}$.

Theorem 3.12. *Let G be a simply connected Lie group with derived group N, and let $\mathfrak{X}$ be a G–quasi–orbit in* $\operatorname{Priv}(N) = \check{N}$. *Then either $\mathfrak{X}$ is a closed point, $\mathfrak{X} = \{\ker\rho\}$, ρ is finite–dimensional and extendible to G, and $C^*(G)/C^*(G) * k(\mathfrak{X})$ is isomorphic to a tensor product of an abelian algebra and a matrix algebra in $\dim\rho$ dimensions or there exists a compactly generated nilpotent Lie group $\mathcal{G}$ of step 1 or 2, a closed central subgroup $\mathcal{L}$ of $\mathcal{G}$ containing $[\mathcal{G},\mathcal{G}]$ and $\lambda \in \mathcal{L}^\wedge$ such that $C^*(G)*k(\overline{\mathfrak{X}}\setminus\mathfrak{X})/C^*(G)*k(\overline{\mathfrak{X}})$ is isomorphic to the tensor product of the algebra of compact operators on an infinite–dimensional separable Hilbert space and the algebra $C^*(\mathcal{G})_\lambda = C^*(\mathcal{G},\mathcal{L},\mathbb{C},\lambda)$ with trivial action and twist λ.*

Proof. This proof generalizes and simplifies the considerations on stability in [31]. First we treat the case that $\mathfrak{X} = \{\mathcal{J}\}$, but not necessarily closed. Then $G = G_{\mathcal{J}}$, $\mathcal{A} = k(\overline{\mathfrak{X}}\setminus\mathfrak{X})/k(\overline{\mathfrak{X}})$ is isomorphic to the algebra of compact operators and $C^*(G) * k(\overline{\mathfrak{X}}\setminus\mathfrak{X})/C^*(G) * k(\overline{\mathfrak{X}})$ is isomorphic to the C^*–algebra of the regular system $(G,N,\mathcal{A},T,\tau)$ with the obvious action T and twist τ as in front of (3.9). Such an algebra was studied in the proof of (3.4). It turned out that $C^*(G,N,\mathcal{A},T,\tau)$ is isomorphic to $C^*(\widetilde{G})_{\widetilde{\tau}} \otimes \mathcal{A}$ in the notations introduced there. Put $\mathcal{G} = \widetilde{G}/\ker\widetilde{\tau}$, $\mathcal{L} = \widetilde{N}/\ker\widetilde{\tau}$, and let $\lambda \in \mathcal{L}^\wedge$ be the character induced by $\widetilde{\tau}$. As $C^*(\widetilde{G})_{\widetilde{\tau}}$ is isomorphic to $C^*(\mathcal{G})_\lambda$ we are done if $\mathcal{A}$ is infinite–dimensional.

If $\mathcal{A}$ is finite–dimensional then $\mathfrak{X}$ is closed, and two cases are possible. Either the Mackey bicharacter $\alpha : G \times G \to \mathbb{T}$ associated with $(G,N,\mathcal{A},T,\tau)$ is trivial or not. This bicharacter is related to the structure of $(\mathcal{G},\lambda)$ by $\alpha(s_1,s_2) = \widetilde{\tau}([(s_1,v_1),(s_2,v_2)])$ for $(s_j,v_j) \in \widetilde{G}$, compare the proof of (3.4). If α is trivial the first alternative of the theorem occurs: Each irreducible unitary representation ρ of N with $\ker\rho = \mathcal{J}$ is extendible to G and $C^*(\mathcal{G})_\lambda$ is commutative because of the above equation for α. If α is not trivial then $C^*(\mathcal{G})_\lambda$ being the twisted convolution algebra on a vector space is isomorphic to the tensor product of $C^*(Z_\alpha/N)$ and the algebra of compact operators on an infinite–dimensional separable Hilbert space, compare also [31], and we are done as well.

If $\mathfrak{X}$ does not reduce to a singleton then choose a regularizing group $H = W \ltimes G$ as in front of (3.9). In particular, one has $\mathfrak{X} = H\mathcal{J}$ for any chosen point $\mathcal{J} \in \mathfrak{X}$. Take any non–trivial unitary character χ on the connected abelian

Lie group $H/H_{\mathcal{J}}$. The compositum $G \longrightarrow H \xrightarrow{\chi} \mathbb{T}$ is nontrivial (and hence surjective) as $H = (GH_{\mathcal{J}})^-$. This compositum turns $\mathbb{T}$ into a transitive G–space and the map $\mathfrak{X} = \text{Priv}\,(\mathcal{A}) \ni T_h(\mathcal{J}) \mapsto \chi(h) \in \mathbb{T}$ is G–equivariant. Hence by [17, Theorem 2.13] the algebra $C^*(G){*}k(\overline{\mathfrak{X}}\backslash\mathfrak{X})/C^*(G){*}k(\overline{\mathfrak{X}}) \cong C^*(G, N, \mathcal{A}, T, \tau)$ is isomorphic to the tensor product of the algebra of compact operators on $L^2(\mathbb{T})$ and another algebra. Here the other algebra does not matter because the only consequence we draw from this consideration is that $C^*(G, N, \mathcal{A}, T, \tau)$ is a stable algebra, and that therefore we are free to replace this algebra by a stably isomorphic copy.

Using $H = W \ltimes G$, as in (3.8) the algebra $C^*(G, N, \mathcal{A}, T, \tau)$ is isomorphic to a C^*–algebra of a certain regular system $(Q, N', \mathcal{A}, T', \tau')$. Applying one of Green's isomorphisms as in (3.7) one finds that $C^*(Q, N', \mathcal{A}, T', \tau')$ is stably isomorphic to $C^*(Q_{\mathcal{J}}, N', \mathcal{A}/\mathcal{J}, \dot{T}, \dot{\tau})$. As in the proof of (3.4) (and in the above case $\mathfrak{X} = \{\mathcal{J}\}$) the latter algebra is stably isomorphic to $C^*(\widetilde{Q}_{\mathcal{J}})_{\dot{\tau}^\sim}$ where $\dot{\tau}^\sim$ is a certain unitary character on a coabelian subgroup $(N')^\sim$ of $\widetilde{Q}_{\mathcal{J}}$. According to the case $\mathfrak{X} = \{\mathcal{J}\}$ we put $\mathcal{G} = \widetilde{Q}_{\mathcal{J}}/\ker \dot{\tau}^\sim$, $\mathcal{L} = (N')^\sim/\ker \dot{\tau}^\sim$, and let $\lambda \in \mathcal{L}^\wedge$ be the character induced by $\dot{\tau}^\sim$. Then $C^*(G, N, \mathcal{A}, T, \tau)$ is stably isomorphic to $C^*(\mathcal{G})_\lambda$. ∎

In the final part we shall consider simply connected solvable Lie groups G. In this case Pukanszky, [37], has given a parametrization of the primitive ideals $\mathcal{I}$ in $C^*(G)$. We shall compute $n_{\mathcal{I}}$ in terms of the parameters, in particular, we shall see why $n_{\mathcal{I}}$ is nonnegative, which is obvious from the definition of $n_{\mathcal{I}}$, but which is not evident from the formulas given in (3.9) or (3.11).

First we recall Pukanszky's parametrization. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{g}^*$ be its dual vector space, on which G acts by the coadjoint representation. For $f \in \mathfrak{g}^*$ let G_f be the stabilizer of f in G. On the connected component $(G_f)_0$ there is an associated unitary character η_f determined by

$$\eta_f(\exp X) = e^{if(X)}$$

for $X \in \mathfrak{g}_f$ where $\mathfrak{g}_f$ denotes the Lie algebra of G_f. The reduced stabilizer G_f^{red} is defined by requiring that $G_f^{\text{red}}/\ker \eta_f$ is the center of $G_f/\ker \eta_f$. In other words, on the group G_f there is a skew–symmetric bicharacter given by $(x, y) \longmapsto \eta_f(xyx^{-1}y^{-1})$, and G_f^{red} is the kernel of this bicharacter, hence G_f/G_f^{red} carries the structure of a (discrete) quasi-symplectic space.

Clearly, the character η_f can be extended to a unitary character of G_f^{red} (in several ways except for $G_f^{\text{red}} = (G_f)_0$). With each pair (f, χ), $f \in \mathfrak{g}^*$, $\chi \in (G_f^{\text{red}})^\wedge$, $\chi|_{(G_f)_0} = \eta_f$, Pukanszky has associated a primitive ideal $\mathcal{I} = \mathcal{I}_{f,\chi}$ in $C^*(G)$. The map $(f, \chi) \longmapsto \mathcal{I}_{f,\chi}$ is surjective, but of course not injective. The equation $\mathcal{I}_{f,\chi} = \mathcal{I}_{f',\chi'}$ defines an equivalence relation on the set of pairs (f, χ), which we shall describe next. Roughly speaking, the equivalence relation says that (f, χ) and (f', χ') are on the same G-quasi-orbit. In a strict sense to make the last statement precise would require to introduce a topology on the whole set of pairs (f, χ). This is possible, cf. [37], but for our purposes another way, also discussed in [37], seems to be more appropriate.

Let $H \leq \text{Aut}\,(\mathfrak{g})$ be the component of the Zariski closure of $\text{Ad}(G)$. One could also choose another auxiliary group H. The main points are that

H contains Ad (G), that H acts trivially on $\mathfrak{g}/\mathfrak{n}$ where again $\mathfrak{n}$ denotes the derived algebra, and that the H-orbits in $\mathfrak{g}^*$ and $\mathfrak{n}^*$ for the dual actions are locally closed. Let Λ be an H-orbit in $\mathfrak{n}^*$, let $\Omega = \{f \in \mathfrak{g}^* \mid f|_\mathfrak{n} \in \Lambda\}$, and let $\mathcal{L}(\Omega) = \{(f,\chi) \mid f \in \Omega, \chi \in (G_f^{\mathrm{red}})^\wedge, \chi|_{(G_f)_0} = \eta_f\}$. For $f, f' \in \Omega$ one has $G_f N = G_{f'} N$, $G_f^{\mathrm{red}} N = G_{f'}^{\mathrm{red}} N$ and $(G_f)_0 N = (G_{f'})_0 N$. In particular, $G_f^{\mathrm{red}} N/(G_f)_0 N$ and its dual $S := (G_f^{\mathrm{red}} N/(G_f)_0 N)^\wedge$ do not depend on the choice of a point $f \in \Omega$. The group $G_f^{\mathrm{red}} N/(G_f)_0 N$ is free abelian of finite rank and isomorphic to $G_f^{\mathrm{red}}/(G_f)_0$, hence S is a torus.

For $\sigma \in (\mathfrak{g}/\mathfrak{n})^*$ denote by $\widetilde{\sigma} \in (G/N)^\wedge$ the corresponding unitary character. The direct product $\Sigma = H \times (\mathfrak{g}/\mathfrak{n})^* \times S$ acts transitively on $\mathcal{L}(\Omega)$ by means of

$$(h,\sigma,s)(f,\chi) = (hf + \sigma, (h\chi)(\widetilde{\sigma}\ |_{G_{hf}^{\mathrm{red}}})(s\ |_{G_{hf}^{\mathrm{red}}}))$$

where $(hf)(X) = f(h^{-1}X)$ and $(h\chi)(x) = \chi(h^{-1}x)$ for $x \in G_{hf}^{\mathrm{red}}$; clearly by $h^{-1}x$ we mean the action of H on the simply connected group G. Further observe that $G_{hf+\sigma}^{\mathrm{red}} = G_{hf}^{\mathrm{red}}$. Via $\mathrm{Ad} : G \to H \subset \Sigma$, also G acts on $\mathcal{L}(\Omega)$. The set $\mathcal{L}(\Omega)$ is topologized by viewing it as a homogeneous Σ-space. The ideals $\mathcal{I}_{f,\chi}$ and $\mathcal{I}_{f',\chi'}$ coincide if and only if $f|_\mathfrak{n}$ and $f'|_\mathfrak{n}$ are in the same H-orbit Λ and if (f,χ) and (f',χ') lie in the same G-quasi-orbit in the corresponding $\mathcal{L}(\Omega)$. We just remark that already the subgroup $H \times (\mathfrak{g}/\mathfrak{n})^*$ of Σ acts transitively on $\mathcal{L}(\Omega)$. This follows easily from (a) and (c′) of (3.17) below. Clearly, viewing $\mathcal{L}(\Omega)$ as a homogeneous $H \times (\mathfrak{g}/\mathfrak{n})^*$ space leads to the same topology.

Now let $\mathcal{I}$ be a given primitive ideal in $C^*(G)$, $\mathcal{I} = \mathcal{I}_{f,\chi}$. The associated G-quasi-orbit $\mathfrak{X} \subset \mathrm{Priv}\ C^*(N)$ corresponds in the Kirillov picture to the relative closure $\mathfrak{X}'$ of $G(f|\mathfrak{n})$ in $H(f|_\mathfrak{n}) =: \Lambda$. The fundamental groups $\pi_1(\mathfrak{X})$ and $\pi_1(\mathfrak{X}')$ are canonically isomorphic, because the fibers of $\mathfrak{X}' \longrightarrow \mathfrak{X}$ are coadjoint orbits in $\mathfrak{n}^*$, which are simply connected as is well known. Let $\mathfrak{Y}'$ be the preimage of $\mathfrak{X}'$ under the canonical map $\mathcal{L}(\Omega) \longrightarrow \Lambda$. Then the set $\mathfrak{Y}$ of primitive ideals in $C^*(G)$ lying over $\mathfrak{X}$ is homeomorphic to the space of G-quasi-orbits in $\mathfrak{Y}'$.

Let $\mathcal{B}$ be the G-quasi-orbit in $\mathfrak{Y}'$ through (f,χ) (or any other G-quasi-orbit in $\mathfrak{Y}'$). The fibration

$$\mathcal{B} \longrightarrow \mathfrak{Y}' \longrightarrow \mathfrak{Y}$$

gives $\mathrm{rk}\,\pi_1(\mathfrak{Y}') = \mathrm{rk}\,\pi_1(\mathfrak{Y}) + \mathrm{rk}\,\pi_1(\mathcal{B})$.

On the other hand from the fibration $\mathfrak{Y}' \longrightarrow \mathfrak{X}'$, whose fibers are homeomorphic with $(\mathfrak{g}/\mathfrak{n})^* \times S$, we conclude that

$$\begin{aligned}\mathrm{rk}\,\pi_1(\mathfrak{Y}') &= \mathrm{rk}\,\pi_1(\mathfrak{X}') + \mathrm{rk}\,\pi_1(S)\\ &= \mathrm{rk}\,\pi_1(\mathfrak{X}) + \mathrm{rk}\,(G_f^{\mathrm{red}}/(G_f)_0).\end{aligned}$$

Our main result (3.9) implies that the invariant $n_\mathcal{I}$ is given by

$$\begin{aligned}n_\mathcal{I} &= \mathrm{rk}\,\pi_1(\mathfrak{X}) - \mathrm{rk}\,\pi_1(\mathfrak{Y}) = \mathrm{rk}\,\pi_1(\mathcal{B}) - \mathrm{rk}\,(G_f^{\mathrm{red}}/(G_f)_0)\\ &= \mathrm{rk}\,\pi_1(\mathcal{B}) - \mathrm{rk}\,(G_f/(G_f)_0) + \mathrm{rk}\,(G_f/G_f^{\mathrm{red}}).\end{aligned}$$

Finally, let $\mathcal{C}$ be the G-quasi-orbit through $f \in \mathfrak{g}^*$, which is the relative closure of Gf in Hf. Then by (1.34.ii, iii) the canonical map $\mathcal{B} \longrightarrow \mathcal{C}$ is surjective and its fibers are homeomorphic to the underlying space of a compact abelian Lie group F, say. Moreover, as $\pi_0(F)$ is finite, the exact sequence in (1.34.iii) together with (1.34.i) implies that $\mathrm{rk}\,\pi_1(\mathcal{B}) - \mathrm{rk}\,(G_f/(G_f)_0) = \mathrm{rk}\,\pi_1(\mathcal{B}) - \mathrm{rk}\,\pi_1(G/G_f) = \mathrm{rk}\,\pi_1(\mathcal{C}) - \mathrm{rk}\,\pi_1(G/G_f) + \mathrm{rk}\,\pi_1(F)$, hence

$$\begin{aligned} n_{\mathcal{I}} &= \mathrm{rk}\,\pi_1(F) + \mathrm{rk}\,(G_f/G_f^{\mathrm{red}}) + \mathrm{rk}\,\pi_1(\mathcal{C}) - \mathrm{rk}\,\pi_1(G/G_f) \\ &= \mathrm{rk}\,\pi_1(F) + \mathrm{rk}\,(G_f/G_f^{\mathrm{red}}) + \mathrm{rk}\,\pi_1(\mathcal{C})/\mathrm{im}\,\pi_1(\gamma), \end{aligned}$$

where $\gamma : G/G_f \longrightarrow \mathcal{C} \subset Hf \subset \mathfrak{g}^*$ denotes the canonical map.

We have proved the following theorem.

Theorem 3.13. *Let $\mathcal{I}$ be a primitive ideal in the group C^*-algebra of a simply connected solvable Lie group G corresponding to the G-quasi-orbit $\mathcal{B}$ through (f,χ) in the Pukanszky parametrization, let $\mathcal{C}$ be the G-quasi-orbit through $f \in \mathfrak{g}^*$, let F be the fiber of the canonical map from $\mathcal{B}$ onto $\mathcal{C}$, which is the underlying space of a compact abelian Lie group, and let $\gamma : G/G_f \to \mathcal{C}$ be the obvious map. Then the invariant $n_{\mathcal{I}}$ is given by the formula*

$$n_{\mathcal{I}} = rk\,\pi_1(F) + rk\,(G_f/G_f^{\mathrm{red}}) + rk\,\pi_1(\mathcal{C})/\mathrm{im}\,\pi_1(\gamma). \qquad \blacksquare$$

As a consequence one obtains the type I criterion, cf. also [35].

Corollary 3.14. *A primitive ideal $\mathcal{I} = \mathcal{I}_{f,\chi}$ is of type* I*, by which we mean here that the unique simple ideal $M(\mathcal{I})$ in $C^*(G)/\mathcal{I}$ is isomorphic to the algebra of compact operators, if and only if G_f/G_f^{red} is finite and the G-orbit through f is locally closed in $\mathfrak{g}^*$.*

If L is any line in $\mathfrak{g}^*$ through the origin and if G_f/G_f^{red} is finite for all f in L then G_f/G_f^{red} is necessarily trivial for all f in L. This is easy to see. Therefore, *all* primitive ideals are of type I, i.e., G is of type I, if and only if for all $f \in \mathfrak{g}^*$ the orbit Gf is locally closed and $G_f = G_f^{\mathrm{red}}$. This is the original Auslander–Kostant criterion.

Proof. A primitive ideal $\mathcal{I}$ is of type I if and only if $n_{\mathcal{I}} = 0$. If $n_{\mathcal{I}} = 0$ then $\mathrm{rk}\,(G_f/G_f^{\mathrm{red}})$ is zero, hence G_f/G_f^{red} is finite, and $\mathrm{rk}\,\pi_1(\mathcal{C})/\mathrm{im}\pi_1(\gamma)$ is zero, which implies by (1.34.i) that γ is bijective. Therefore, Gf is relatively closed in Hf, whence locally closed in $\mathfrak{g}^*$.

On the other hand, if Gf is locally closed then γ is bijective and by (1.34. iii) also $\mathcal{B} \to \mathcal{C}$ is bijective, whence F is trivial. One concludes $\mathrm{rk}\,\pi_1(F) + \mathrm{rk}\,\pi_1(\mathcal{C})/\mathrm{im}\pi_1(\gamma) = 0$ which implies $n_{\mathcal{I}} = 0$ in view of (3.13) and the finiteness of G_f/G_f^{red}. ■

There is also another parametrization of Priv $C^*(G)$ available, which we now discuss briefly; for a proof of this parametrization compare the remarks at the end of this article. Let again $\mathfrak{n}^*$ be the linear dual of the derived algebra $\mathfrak{n}$ in $\mathfrak{g}$. For each $g \in \mathfrak{n}^*$ there is a unitary character χ_g on the stabilizer N_g determined by

$$\chi_g(\exp X) = e^{\mathrm{i}g(X)}, X \in \mathfrak{n}_g.$$

The character χ_g defines a bicharacter on the stabilizer group G_g by $(x,y) \longmapsto \chi_g(xyx^{-1}y^{-1})$. Denote by $C(g)$ the kernel of this bicharacter, i.e., $C(g)/\ker\chi_g$

is the center of $G_g/\ker\chi_g$. With each pair (g,η), $g\in\mathfrak{n}^*, \eta\in C(g)^\wedge$, $\eta\mid_{N_g}=\chi_g$, one may associate a primitive ideal $\mathcal{I}_{g,\eta}$ in $C^*(G)$. Again one gets a surjective map from the set of all such pairs (g,η) onto Priv $C^*(G)$. If again $\Lambda\subset\mathfrak{n}^*$ is an H-orbit then the set Δ of all pairs (g,η) with $g\in\Lambda$ can be topologized, similar to the above approach, by viewing it as a homogeneous space with acting group $H\times(C(g)N/N)^\wedge$ or with acting group $H\times(G/N)^\wedge$: $(h,s)(g,\chi)=(hg,(h\chi)s\mid_{C(hg)})$. But here $C(g)N/N\cong C(g)/N_g$ is not discrete, hence the fibers of the canonical map $\Delta\to\Lambda$ are not compact. Two ideals $\mathcal{I}_{g,\eta}$ and $\mathcal{I}_{g',\eta'}$ coincide if and only if the functionals g and g' are in the same H-orbit Λ in $\mathfrak{n}^*$ and if (g,η) and (g',η') lie in the same G-quasi-orbit in the corresponding Δ.

Now let $\mathcal{I}=\mathcal{I}_{g,\eta}$ be a given primitive ideal in $C^*(G)$ and let $\mathfrak{X}\subset$ Priv $C^*(N)$ and $\mathfrak{X}'\subset\Lambda\subset\mathfrak{n}^*$ be as above. The preimage, say $\mathfrak{Y}''$, of $\mathfrak{X}'$ in Δ parametrizes the set $\mathfrak{Y}$ of primitive ideals in $C^*(G)$ lying over $\mathfrak{X}$. Hence $\mathfrak{Y}$ is homeomorphic to the space of G-quasi-orbits in $\mathfrak{Y}''$. If $\mathcal{B}'$ denotes the G-quasi-orbit through (g,η) in $\mathfrak{Y}''\subset\Delta$ then one gets reasoning as above

$$\begin{aligned}
&\operatorname{rk}\pi_1(\mathfrak{Y}'')=\operatorname{rk}\pi_1(\mathfrak{Y})+\operatorname{rk}\pi_1(\mathcal{B}')\\
\text{and}\quad &\operatorname{rk}\pi_1(\mathfrak{Y}'')=\operatorname{rk}\pi_1(\mathfrak{X}')+\operatorname{rk}\pi_1((C(g)/N_g)^\wedge)\\
&\qquad\qquad\;=\operatorname{rk}\pi_1(\mathfrak{X})+\operatorname{rk}(C(g)/C(g)_0)
\end{aligned}$$

because $C(g)_0/N_g$ is a vector group. Therefore, $n_{\mathcal{I}}=\operatorname{rk}\pi_1(\mathfrak{X})-\operatorname{rk}\pi_1(\mathfrak{Y})=\operatorname{rk}\pi_1(\mathcal{B}')-\operatorname{rk}C(g)/C(g)_0$. Since $\pi_1(G/G_g)$ is isomorphic to $G_g/(G_g)_0$ and since the difference

$$\operatorname{rk}G_g/(G_g)_0-\operatorname{rk}C(g)/C(g)_0=r(G_g/(G_g)_0)-r(C(g)/C(g)_0)$$

is equal to the "rank" $r(G_g/C(g))$ of the quasi-symplectic space $G_g/C(g)$, one gets $n_{\mathcal{I}}=\operatorname{rk}\pi_1(\mathcal{B}')-\operatorname{rk}\pi_1(G/G_g)+r(G_g/C(g))$. In view of (1.34.i) one finally obtains the following formula for $n_{\mathcal{I}}$.

Theorem 3.15. *The invariant $n_{\mathcal{I}}$ of $\mathcal{I}=\mathcal{I}_{g,\eta}$ is given by $n_{\mathcal{I}}=rk\,\pi_1(\mathcal{B}')/\mathrm{im}\pi_1(\alpha)+r(G_g/C(g))$, where $\alpha: G/G_g\longrightarrow\mathcal{B}'$ denotes the obvious map. The ideal $\mathcal{I}_{g,\eta}$ is of type* I *if and only if the G-quasi-orbit $\mathcal{B}'$ is an G-orbit and if the quasi-symplectic space $G_g/C(g)$ endowed with the obvious bicharacter satisfies the equivalent conditions of* (1.23), *i.e., $G_g/C(g)$ is so to speak strictly polarizable.*

In addition, we remark that, as one might guess in view of (3.14), the condition that $\mathcal{B}'$ is an G-orbit does in general not imply that the corresponding G-quasi-orbit in $\mathfrak{n}^*$ is an G-orbit. Only the opposite implication holds true. ■

Next we give a simple example for the discontinuity of the map $\mathcal{I}\longmapsto n_{\mathcal{I}}$.

Example 3.16. For two nonzero real numbers α_1,α_2 such that the quotient α_1/α_2 is irrational let $\mathfrak{g}$ be the real Lie algebra consisting of all 4×4 complex matrices of the form

$$A(\lambda,v_1,v_2,\mu)\overset{\text{def}}{=}\begin{pmatrix}0&v_1&v_2&i\mu\\0&i\lambda\alpha_1&0&\overline{v}_1\\0&0&i\lambda\alpha_2&\overline{v}_2\\0&0&0&0\end{pmatrix},$$

where $\lambda, \mu \in \mathbb{R}$ and $v_1, v_2 \in \mathbb{C}$. The subalgebra $\{A(0,0,0,\mu) \mid \mu \in \mathbb{R}\}$ is the center $\mathfrak{z}\mathfrak{g}$ of $\mathfrak{g}$. This algebra $\mathfrak{g}$ may be viewed as an extension of the Heisenberg ideal $\mathfrak{n} \stackrel{\text{def}}{=} \{A(0, v_1, v_2, \mu) \mid v_1, v_2 \in \mathbb{C}, \mu \in \mathbb{R}\}$ or, alternatively, as an extension of the quotient $\mathfrak{g}/\mathfrak{z}\mathfrak{g}$, which is isomorphic to the Mautner algebra.

Let G be the simply connected group with algebra $\mathfrak{g}$. Its center $Z(G)$ is equal to $\exp \mathfrak{z}\mathfrak{g}$. The primitive ideals $\mathcal{I}$ in $C^*(G)$, which are in general position, i.e., those which restrict to a non–constant unitary character on $Z(G)$, are of type I, because already $\pi(C^*(N))$ is equal to the algebra of compact operators if π denotes one of the irreducible representations with kernel $\mathcal{I}$. The primitive ideals $\mathcal{I}$ in Priv $C^*(G/Z(G)) \subset$ Priv $C^*(G)$, which are in general position relative to Priv $C^*(G/Z(G))$, i.e., those whose corresponding G-quasi-orbit in $(N/Z(G))^\wedge$ is a two-dimensional torus, are not of type I. Actually, their invariant $n_{\mathcal{I}}$ is equal to two. All the other primitive ideals in $C^*(G)$, i.e., the primitive ideals in $C^*(G/Z(G))$, whose corresponding G-quasi-orbit in $(N/Z(G))^\wedge$ is either a point or a one-dimensional torus, are again of type I. We conclude that the layers $\{\mathcal{I} \in \text{Priv } C^*(G) \mid n_{\mathcal{I}} = 0\}$ and $\{\mathcal{I} \in \text{Priv } C^*(G) \mid n_{\mathcal{I}} = 2\}$ are neither open nor closed. Hence $\mathcal{I} \longmapsto n_{\mathcal{I}}$ is not continuous for any non-trivial topology on $\{0, 2\}$. ■

In our organization of this section we have developed the theory of centrally regularizable systems so far to allow us the computation of the invariant $n = n_{\mathcal{I}}$. After that, in order to arrive on the shortest way at an interpretation of our formula for $n_{\mathcal{I}}$ in the case of solvable Lie groups, we have supposed as given the Pukanszky parametrization and another one, closely related. In fact, our results on centrally regularizable systems can be used to derive both parametrizations. In this last part of the paper this will be briefly indicated and will lead to an interpretation of some formerly introduced quantities in the context of solvable Lie groups.

In order to deduce and to relate both parametrizations we shall need some information on certain skew–symmetric bicharacters, which is collected in the following lemma.

Lemma 3.17. *Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{n}$, and let N be its derived group with Lie algebra $\mathfrak{n}$. Let $f \in \mathfrak{g}^*$, and put $g = f|_{\mathfrak{n}} \in \mathfrak{n}^*$. The functional g defines a unitary character χ_g on N_g by $\chi_g(\exp\ X) = e^{ig(X)}$ for $X \in \mathfrak{n}_g$. This character χ_g gives a skew–symmetric bicharacter $G_g \times G_g \to \mathbb{T}$ by $(x, y) \mapsto \chi_g([x, y])$. For a subset A of G_g the orthogonal set $A^\perp$ is formed w.r.t. this bicharacter, i.e., $A^\perp = \{x \in G_g | \chi_g([x, a]) = 1$ for all $a \in A\}$. The following assertions hold true.*

(a) $(G_g)_0 f = f + (\mathfrak{g}/(\mathfrak{n} + \mathfrak{g}_f))^*$,

(b) $N_g f = f + (\mathfrak{g}/(\mathfrak{n} + \mathfrak{g}_g))^*$,

(c) $((G_g)_0)^\perp = \{x \in G_g | xf \in f + (\mathfrak{g}/(\mathfrak{n} + \mathfrak{g}_g))^*\}$,

(c′) $((G_g)_0)^\perp = G_f N_g$,

(d) $((G_f)_0)^\perp = \{x \in G_g | xf \in f + (\mathfrak{g}/(\mathfrak{n} + \mathfrak{g}_f))^*\}$,

(d′) $((G_f)_0)^\perp = G_f (G_g)_0$,

(e) $(G_f^{\text{red}})^\perp = G_f (G_g)_0$ *where* $G_f^{\text{red}} = G_f \cap (G_f)^\perp$,

(f) $(G_f^{\text{red}})^{\perp\perp} = G_f^{\text{red}} N_g$,

(g) $C(g) = (G_g)^\perp \subset G_f^{\text{red}} N_g$,

(h) $G_f^{\text{red}} N_g = ((G_g)_0)^\perp \cap ((G_g)_0)^{\perp\perp}$.

Proof. Most of the assertions are well known. We just give some comments on their proofs. The inclusions $\mathfrak{g}_g f \subset (\mathfrak{g}/(\mathfrak{n}+\mathfrak{g}_f))^*$ and $\mathfrak{n}_g f \subset (\mathfrak{g}/(\mathfrak{n}+\mathfrak{g}_g))^*$ are evident. Actually, one has equalities, which are proved by computing the dimensions of the spaces in question. From these equalities one gets (a) and (b) by using that G, and hence $(G_g)_0$ and N_g, act trivially on $(\mathfrak{g}/\mathfrak{n})^*$.

Claims (c) and (d) follow quickly from the easily verified equation

$$\chi_g([x, \exp\ Y]) = e^{if(\mathrm{Ad}(x)Y - Y)}$$

for $x \in G_g$ and $Y \in \mathfrak{g}_g$. Their counterparts (c′) and (d′) are consequences of (b) and (c), resp. of (a) and (d).

Claim (e) follows from (c′) and (d′), claim (f) from (e) and (c′), and (g) is obvious in view of (f). Claim (h) follows from (f), (e) and (c′): $G_f^{\text{red}} N_g = ((G_f^{\text{red}} N_g)^\perp)^\perp = (G_f(G_g)_0)^\perp = G_f^\perp \cap ((G_g)_0)^\perp = ((G_g)_0)^{\perp\perp} \cap ((G_g)_0)^\perp$ as N_g is contained in the kernel $C(g)$ of the bicharacter. ∎

Let still (and for the rest of this article) $G, \mathfrak{g}, N$ and $\mathfrak{n}$ be as in the lemma. Furthermore we now choose a connected Lie group H containing G such that N is normal and coabelian in H and that the restriction of the adjoint group of H to $\mathfrak{g}$ coincides with the connected component of the Zariski closure of the adjoint group of G. Each $\mathcal{I} \in \text{Priv}\ C^*(G)$ defines as a first invariant an H–orbit in $\widehat{N} = \text{Priv}\ C^*(N)$, say Ψ, which is determined by requiring that the hull $h(\mathcal{I}|_N)$ is contained in the closure Ψ^-, but not contained in the boundary $\Psi^- \setminus \Psi$. The set of primitive ideals in $C^*(G)$ with the same invariant Ψ is homeomorphic with the set of primitive ideals in $C^*(G, N, \mathcal{A}, T, \tau)$ where $\mathcal{A} \overset{\text{def}}{=} k(\Psi^- \setminus \Psi)/k(\Psi^-)$ has the obvious G–action T, and the twist τ is deduced from translations with elements in N, compare also (3.9) and the comments in front of this theorem.

To parametrize Priv $C^*(G)$ it suffices to parametrize each individual Priv $C^*(G, N, \mathcal{A}, T, \tau)$. To this end we apply (3.4), of course using the regularization given by H. We know that Priv $C^*(G, N, \mathcal{A}, T, \tau)$ is a transitive $H \times (G/N)^\wedge$–space or, as we now prefer, a transitive $H \times (\mathfrak{g}/\mathfrak{n})^*$–space in view of the canonical identification of $(G/N)^\wedge$ with $(\mathfrak{g}/\mathfrak{n})^*$, $(\mathfrak{g}/\mathfrak{n})^* \ni \sigma \mapsto \widetilde{\sigma} \in (G/N)^\wedge$. We also know the stabilizers in $H \times (\mathfrak{g}/\mathfrak{n})^*$ once we know the map $\zeta_\mathcal{J}$, $\mathcal{J} \in \Psi$. Observe that here all $H_\mathcal{J}$ and all $\zeta_\mathcal{J}$ are independent of $\mathcal{J} \in \Psi$ as H/N is abelian. Again we denote by $\Lambda \subset \mathfrak{n}^*$ the H–orbit of functionals corresponding to Ψ in the Kirillov picture. The stabilizers $H_\mathcal{J}$ are equal to $H_g N$, $g \in \Lambda$.

For $g \in \Lambda$ the attached $\mathcal{J} \in \Psi$ may be realized as the kernel of a representation ρ of N as follows. Choose an H_g–invariant positive polarization $\mathfrak{p}$ in $\mathfrak{n}_\mathbb{C}$, cf. [1], let $D = \exp\ (\mathfrak{p} \cap \mathfrak{n})$, define $\sigma_g : D \to \mathbb{T}$ by $\sigma_g(\exp X) = e^{ig(X)}$ for $X \in \mathfrak{p} \cap \mathfrak{n}$, and let ρ be the subrepresentation of $\text{ind}_D^N \sigma_g$ acting in the closure $\mathfrak{H}$ of the subspace consisting of all smooth functions ξ (in the space of $\text{ind}_D^N \sigma_g$) such that

$$\frac{d}{dt}|_{t=0}\{\xi(x \exp\ tY_1) + i\xi(x \exp\ tY_2)\} = \xi(x)(g(Y_2) - ig(Y_1))$$

for all $x \in N$ and all $Y_1 + iY_2 \in \mathfrak{p} \subset \mathfrak{n} + i\mathfrak{n}$. Of course, here one of the most delicate points is the result due to Auslander and Kostant that this construction leads, independent of the choice of $\mathfrak{p}$, to the same equivalence class of unitary representations as the traditional Kirillov method based on real polarizations.

Next for $h \in H_g$ define the unitary operator $u(h)$ on $\mathfrak{H}$ by

$$(u(h)\xi)(x) = \delta(h)\xi(h^{-1}xh) \tag{3.18}$$

where $\delta(h)$ is the square root of the appropriate modular function. It is easy to check that $\rho(hyh^{-1}) = u(h)\rho(y)u(h)^{-1}$ for $y \in N$, whence $\rho(T_h f) = u(h)\rho(f)u(h)^{-1}$ for $f \in C^*(N)$.

Since $\zeta_{\mathcal{J}} : H_{\mathcal{J}} = H_g N \to (G_{\mathcal{J}}/N)^\wedge = (G_g N/N)^\wedge$ is trivial on N this homomorphism is completely determined by the values $\zeta_{\mathcal{J}}(h)(s)$, $h \in H_g$, $s \in G_g$. By definition one has

$$\rho([h,s]) = \zeta_{\mathcal{J}}(h)(s)[u(h), u(s)].$$

The commutator $[h,s]$ is contained in $N_g \subset D = \exp(\mathfrak{p} \cap \mathfrak{n})$. Using this one readily checks that

$$\zeta_{\mathcal{J}}(h)(s) = \sigma_g([h,s]) = \chi_g([h,s])$$

where as above χ_g denotes the obvious character on N_g. In particular the associated Mackey bicharacter α on $G_{\mathcal{J}} \times G_{\mathcal{J}}$ is given

$$\alpha(s_1, s_2) = \chi_g([s_1, s_2])$$

for $s_1, s_2 \in G_g$.

By (3.5) the primitive ideals in $C^*(G, N, \mathcal{A}, T, \tau)$ are induced from primitive ideals in $C^*(C(g)N, N, \mathcal{A}, T, \tau)$. For (g, η) in the above considered set $\Delta = \{(g,\eta) | g \in \Lambda, \eta \in C(g)^\wedge, \eta|_{N_g} = \chi_g\}$ define an extension $\widetilde{\rho}$ of the representation ρ of N to $C(g)N$ by letting $\widetilde{\rho}(s) = \eta(s)u(s)$ for $s \in C(g)$, u as in (3.18). This irreducible representation of $C(g)N$ yields a primitive ideal in $C^*(C(g)N, N, \mathcal{A}, T, \tau)$, and $\mathrm{ind}_{C(g)N}^G \widetilde{\rho}$ yields a primitive ideal $\mathcal{I}_{g,\eta}$ in $C^*(G)$ or in $C^*(G, N, \mathcal{A}, T, \tau)$.

It is easy to check that the map $\Delta \ni (g,\eta) \mapsto \mathcal{I}_{g,\eta} \in \mathrm{Priv}\ (G, N, \mathcal{A}, T, \tau)$ is $H \times (\mathfrak{g}/\mathfrak{n})^*$–equivariant and hence onto. We claim:

3.19. *The space of G-quasi-orbits in Δ is homeomorphic with* Priv $C^*(G, N, \mathcal{A}, T, \tau)$.

Clearly, it is enough to show that $(G\ \mathrm{stab}(g,\eta))^-$, where $\mathrm{stab}(g,\eta)$ denotes the stabilizer of $(g,\eta) \in \Delta$ in $H \times (\mathfrak{g}/\mathfrak{n})^*$, coincides with the stabilizer of $\mathcal{I}_{g,\eta}$, which by (3.4) and our computation of $\zeta_{\mathcal{J}}$ is equal to $(G\Gamma_g)^-$ where $\Gamma_g \overset{\mathrm{def}}{=} \{(x,\sigma) \in H_g \times (\mathfrak{g}/\mathfrak{n})^* | \widetilde{\sigma}(w) = \chi_g([x,w])$ for all $w \in G_g\}$. From the equivariance of $(g,\eta) \mapsto \mathcal{I}_{g,\eta}$ follows immediately that $\mathrm{stab}(g,\eta)$ is contained in the stabilizer of $\mathcal{I}_{g,\eta}$; hence $(G\ \mathrm{stab}(g,\eta))^-$ is contained in $(G\Gamma_g)^-$. For the reverse inclusion one just observes that Γ_g is contained in $\mathrm{stab}(g,\eta)$. ∎

Putting together the descriptions of the various Priv $C^*(G,N,\mathcal{A},T,\tau)$, $\mathcal{A}=\mathcal{A}_\Psi$, $\Psi\in\widehat{N}/H$, one obtains a bijective correspondence between Priv $C^*(G)$ and the set of G–quasi–orbits in $\{(g,\eta)|g\in\mathfrak{n}^*,\eta\in C(g)^\wedge,\eta|_{N_g}=\chi_g\}$.

Finally we discuss the Pukanszky parametrization. Again we consider first a portion Priv $C^*(G,N,\mathcal{A},T,\tau)$ of Priv $C^*(G)$ where $\mathcal{A}=\mathcal{A}_\Psi$, Ψ and Λ are as above. According to former notation let $\Omega=\{f\in\mathfrak{g}^*|f|_\mathfrak{n}\in\Lambda\}$ and $\mathcal{L}(\Omega)=\{(f,\chi)|f\in\Omega,\chi\in(G_f^{\rm red})^\wedge,\chi|_{(G_f)_0}=\eta_f\}$. The groups $G_f^{\rm red}N$ are independent of $f\in\Omega$ and by (3.17.g) they contain the kernel $C(f|_\mathfrak{n})N$ of the Mackey bicharacter. Therefore, by (3.5) the primitive ideals in $C^*(G,N,\mathcal{A},T,\tau)$ are induced from primitive ideals in $C^*(G_f^{\rm red}N,N,\mathcal{A},T,\tau)$. For $(f,\chi)\in\mathcal{L}(\Omega)$ let $g=f|_\mathfrak{n}$ and extend the above representation $\rho=\rho_g$ of N to a representation ρ' of $G_f^{\rm red}N$ by letting $\rho'(s)=\chi(s)u(s)$ for $s\in G_f^{\rm red}$ where u is as in (3.18). The kernel of $\mathrm{ind}_{G_f^{\rm red}N}^G\rho'$ is the primitive ideal $\mathcal{I}_{f,\chi}$ in $C^*(G)$ or in $C^*(G,N,\mathcal{A},T,\tau)$. Again the map $\mathcal{L}(\Omega)\ni(f,\chi)\mapsto\mathcal{I}_{f,\chi}$ is $H\times(\mathfrak{g}/\mathfrak{n})^*$–equivariant and hence onto Priv $C^*(G,N,\mathcal{A},T,\tau)$.

Not surprisingly both parametrizations are closely related. There is a canonical $H\times(\mathfrak{g}/\mathfrak{n})^*$–equivariant map $\nu:\mathcal{L}(\Omega)\to\Delta$ such that $\mathcal{I}_{f,\chi}=\mathcal{I}_{g,\eta}$ if $(g,\eta)=\nu(f,\chi)$. If $(f,\chi)\in\mathcal{L}(\Omega)$ then put $g=f|_\mathfrak{n}$ and let $\nu(f,\chi)=(g,\eta)$ where $\eta\in C(g)^\wedge$ is constructed as follows. The characters $\chi\in(G_f^{\rm red})^\wedge$ and $\chi_g\in(N_g)^\wedge$ yield a character on $G_f^{\rm red}N_g$ by $ab\mapsto\chi(a)\chi_g(b)$ for $a\in G_g^{\rm red}$, $b\in N_g$; and η is the restriction of this character to the subgroup $C(g)$ of $G_g^{\rm red}N_g$. Further details are omitted.

Using this relation we shall now show the following claim.

3.20. *The map* $(f,\chi)\mapsto\mathcal{I}_{f,\chi}$ *yields a homeomorphism from the space of* G*–quasi–orbits in* $\mathcal{L}(\Omega)$ *onto* Priv $C^*(G,N,\mathcal{A},T,\tau)$.

As we remarked earlier, from (a) and (c′) of (3.17) it follows that $\mathcal{L}(\Omega)$ is a transitive $H\times(\mathfrak{g}/\mathfrak{n})^*$–space. Therefore, we are left to show that for $(f,\chi)\in\mathcal{L}(\Omega)$ the stabilizer $\mathrm{stab}(\mathcal{I}_{f,\chi})$ in $H\times(\mathfrak{g}/\mathfrak{n})^*$ coincides with $(\mathrm{stab}(f,\chi)G)^-$. We know already $\mathrm{stab}(\mathcal{I}_{f,\chi})=(\mathrm{stab}(g,\eta)G)^-=(\Gamma_gG)^-$ where $(g,\eta)\overset{\rm def}{=}\nu(f,\chi)$ and Γ_g is the formerly introduced group. From the equivariance of ν follows $\mathrm{stab}(f,\chi)\subset\mathrm{stab}(g,\eta)$, hence it remains to prove that Γ_g is contained in $(\mathrm{stab}(f,\chi)G)^-$. Let $(x,\sigma)\in H_g\times(\mathfrak{g}/\mathfrak{n})^*$ with $\chi_g([x,-])=\widetilde{\sigma}|_{G_g}$ be given. Define $\gamma\in(\mathfrak{g}/\mathfrak{n})^*$ by the equation $xf+\gamma=f$. It is easy to see that $\widetilde{\sigma}\widetilde{\gamma}^{-1}$ is trivial on $(G_f)_0$, hence $\sigma-\gamma$ is contained in $(\mathfrak{g}/(\mathfrak{g}_f+\mathfrak{n}))^*$. ¿From (a) of (3.17) we obtain an $y\in(G_g)_0$ such that $yf=f+(\sigma-\gamma)$. Using that (c′) of (3.17) gives in particular $\chi_g([y,w])=1$ for all $w\in G_f^{\rm red}$ one concludes that (xy^{-1},σ) is in $\mathrm{stab}(f,\chi)$, hence $(x,\sigma)=(xy^{-1},\sigma)y$ belongs to $\mathrm{stab}(f,\chi)G$. ■

Remark 3.21. In the above discussion we have applied (3.4) only for regularizations where H/N is abelian. The full force of (3.4) can be used to derive somewhat stronger results along the same lines. Let K be the algebraic group of all Lie automorphisms of $\mathfrak{g}$, which induce the identity on $\mathfrak{g}/\mathfrak{n}$. Let Λ' be a K–orbit in $\mathfrak{n}^*$, let $\Omega'=\{f\in\mathfrak{g}^*|f|_\mathfrak{n}\in\Lambda'\}$, and let $\mathcal{L}(\Omega')=\{(f,\chi))|f\in\Omega',\chi\in(G_f^{\rm red})^\wedge,\chi|_{(G_f)_0}=\eta_f\}$. Then the portion of Priv $C^*(G)$ parametrized by $\mathcal{L}(\Omega')$ is homeomorphic to the space of G–quasi–orbits in $\mathcal{L}(\Omega')$. This can be seen by taking $K\ltimes G$ (with the obvious action of K on G) as regularizing group. – Clearly, a similar result can be formulated and proved for the other parametrization. ■

As a conclusive remark we repeat a question, which was already posed in [31] and which still seems to be unsettled, namely the question if similar results, in particular the description of simple subquotients of $C^*(G)$, hold true for finite extensions of connected Lie groups.

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