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RESEARCH ARTICLE

WELL-BOUNDED SEMIGROUPS IN CONNECTED GROUPS

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In [1], Dobbins studied the so-called well-bounded semigroups in locally compact groups, i.e. open subsemigroups S such that the boundary ∂S is a subgroup (for instance a connected component of $V \setminus W$ where V is finite dimensional real vector space and W a subspace of codimension 1). He did not answer the question whether or not there exist such semigroups in the universal covering group $SL_2(\mathbb{R})^\sim$ of $SL_2(\mathbb{R})$. In this paper, it is shown that the answer is affirmative. One gets the existence from a general theorem about well-bounded semigroups in connected locally compact groups which also clarifies the nature of these semigroups and shows that the general situation is very similar to the vector space case. In the (short) proof of the theorem we make extensive use of the theory developed in Dobbins' paper. We also discuss the ideal theory in well-bounded semigroups. Especially, we will show that the closure of a well-bounded semigroup in $SL_2(\mathbb{R})^\sim$ has only a countable descending sequence of two-sided ideals.

I wish to express my thanks to K.H. Hofmann who told me the problem concerning $SL_2(\mathbb{R})^\sim$ at the meeting on "Categorical Topology" in Mannheim 1975.

Next, I will briefly recall the definitions and theorems of Dobbins which are crucial in the sequel.

1. DEFINITION. A well-bounded semigroup (in G) is a pair (G,S) consisting of a locally compact group G and an open subsemigroup

S of G such that ∂S is a subgroup of G.

2. DEFINITION. A well-bounded semigroup (G,S) is called *reduced* if there is no non-trivial (closed) normal subgroup N of G with $S \cap N = \emptyset$.

3. THEOREM. *Let (G,S) be a well-bounded semigroup. Then there exists a closed normal subgroup M in G such that $S \cap M = \emptyset$ and $(G/M, SM/M)$ is a reduced well-bounded semigroup.*

Moreover, $\partial(SM/M) = (M\partial S)^{-}/M$.

If G is connected, then M is unique, namely $M = \mathrm{Core}\,(\partial S) :=$
$:= \bigcap_{g \in G} g\partial S g^{-1}$.

4. THEOREM. *Let (G,S) be a reduced well-bounded semigroup, and let G be connected. Then G is isomorphic to one of the following groups.*

(1) $\mathbb{R}$

(2) $\{\begin{pmatrix} a & b \\ o & 1 \end{pmatrix} ;\ a,b \in \mathbb{R},\ a > o\} \subseteq GL_2(\mathbb{R})$

(3) $SL_2(\mathbb{R})^{\sim}$.

Moreover, up to conjugation, ∂S is $\{0\}$ in case (1), further $\{\begin{pmatrix} a & o \\ o & 1 \end{pmatrix} ,\ a > o\}$ in case (2), and finally the component of the unit element in $p^{-1}\{\begin{pmatrix} a & b \\ o & a^{-1} \end{pmatrix} ;\ a,b \in \mathbb{R},\ a > o\}$ in case (3), where $p : SL_2(\mathbb{R})^{\sim} \to SL_2(\mathbb{R})$ denotes the covering homomorphism.

Now we formulate the criterion for the existence of well-bounded semigroups.

5. THEOREM. *Let G be a connected locally compact group and let H be a closed subgroup of G. Then the following statements are equivalent:*

(1) There exists an open subsemigroup S of G with $\partial S = H$ such that (G,S) is a well-bounded semigroup.

(ii) There exists a closed subgroup E of G, isomorphic to $\mathbb{R}$, such that the multiplication $E \times H \to G$ is a homeomorphism.

<u>If</u> (i) (<u>or</u> (ii)) <u>holds then the sole open subsemigroups</u> S <u>with</u> $\partial S = H$ <u>are the two connected components of</u> $G \setminus H$, <u>and</u> $S = HC = CH$ <u>where</u> C <u>denotes one of the two connected components of</u> $E \setminus \{0\}$. <u>Clearly</u>, $G = S \,\dot\cup\, H \,\dot\cup\, S^{-1}$.

<u>REMARK</u>. If H is a closed subgroup of the connected locally compact group G then it can be shown that (i) and (ii) are equivalent to (iii) G/H is homeomorphic to $\mathbb{R}$.

(We prove the remark by using the reduction in [4], p. 236, applying Theorem 1 in [5] and using 14.3 of [3] as in the proof of 5.)

Before proving 5. we note that the group $G = SL_2(\mathbb{R})^\sim$ and the subgroup H described in 4. (3) satisfy condition (ii) of 5. (compare also [6] or [5]) . Indeed we observe:

<u>6</u>. <u>REMARK</u>. Let $p : SL_2(\mathbb{R})^\sim \to SL_2(\mathbb{R})$ be the covering homomorphism and let H be the component of the unit element in $p^{-1}\{\begin{pmatrix} a & b \\ o & a^{-1}\end{pmatrix} ;$ $a,b \in \mathbb{R},\ a > o\}$. There exists a unique isomorphism σ from $\mathbb{R}$ onto $p^{-1}\{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t\end{pmatrix} ;\ t \in \mathbb{R}\}$ with $p\sigma(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t\end{pmatrix}$. The center of $SL_2(\mathbb{R})^\sim$ is equal to $\sigma(\pi\mathbb{Z})$. The map $(t,h) \mapsto \sigma(t)h$ is a homeomorphism from $\mathbb{R} \times H$ onto $SL_2(\mathbb{R})^\sim$.

<u>Proof of 5</u>. (ii) $\Rightarrow$ (i) Using the inversion $G \to G$, $g \mapsto g^{-1}$, we see that the multiplication $H \times E \to G$ is also a homeomorphism. Let C be one of the connected components of $E \setminus \{0\}$. Define $S := CH$. Then S is one of the two components of $G \setminus H$. But HC is also a component of $G \setminus H$. Since HC and $S = CH$ intersect at least in C , they are equal. From $HC = CH$ and the fact that C and H are semigroups, one easily deduces that S is a semigroup. Of course, S is open and $\partial S = H$.

(i) $\Rightarrow$ (ii) Let $M :=$ Core (H) . By 3., $(G/M, SM/M)$ is a reduced well-bounded semigroup with $\partial(SM/M) = H/M$. From 4. and 6., it follows that there exists a closed subgroup E' of G/M , isomorphic to $\mathbb{R}$, such that the multiplication $E' \times H/M \to G/M$ is a homeomorphism. Then there exists (for instance by 14.3 in [3]) a closed subgroup E of G such that the quotient homomorphism $G \to G/M$ induces an isomorphism from E onto E' . Obviously, this E is the required group.

It remains to prove that every open subsemigroup S of G with $\partial S = H$ is of the form CH. Since $\overline{S}$ is a subsemigroup of G containing H we have $HS \cup SH \subset \overline{S} = S \cup H$, and, therefore, $S = HS = SH$ because $S \cap H = \emptyset$. Using this equality and $G = HE = EH$ we get $S = (E \cap S) \cdot H = H \cdot (E \cap S)$. Since S is an open subsemigroup of G with $\partial S = H$ the intersection $S \cap E$ is an open subsemigroup of $E \cong \mathbb{R}$ with $\partial(S \cap E) = \{0\}$ and is therefore one of the components of $E \setminus \{0\}$. The proof of 5. is complete.

In order to study the ideal theory in (well-bounded) semigroups it is useful to introduce (and to determine) the following equivalence relations.

7. DEFINITION. Let (G,S) be a well-bounded semigroup. Then equivalence relations L, R and J on G and, by restriction, on S are defined by $xLy \Leftrightarrow Sx = Sy$, $xRy \Leftrightarrow xS = yS$ and $xJy \Leftrightarrow SxS = SyS$. Let $D = L \cup R$ be the equivalence relation generated by L and R. Similarily, the semigroups $\overline{S}$ and $S^1 := S \cup \{1\}$ define equivalence relations L^*, R^*, D^* and J^* and L^1, R^1, D^1 and J^1, respectively. The restrictions of L^1, R^1, D^1 and J^1 resp. L^*, R^*, D^* and J^* on S resp. $\overline{S}$ are the Green's relations of S resp. $\overline{S}$.

8. PROPOSITION. Let (G,S) be a well-bounded semigroup with connected G, let $H = \partial S$ and $x,y \in G$. Then $L^1 = R^1 = D^1 =$ equality, $xLy \Leftrightarrow xL^*y \Leftrightarrow Hx = Hy$, $xRy \Leftrightarrow xR^*y \Leftrightarrow xH = yH$ and therefore $xDy \Leftrightarrow xD^*y \Leftrightarrow HxH = HyH$.

Proof. Suppose $x \neq y$ and xL^1y, i.e. $Sx \cup \{x\} = Sy \cup \{y\}$. Then there exist $s,t \in S$ such that $y = sx$ and $x = ty$. It follows $y = sty$ and $st = 1 \in S$, a contradiction. Similarily, $R^1 =$ equality and hence $D^1 =$ equality. Clearly, xLy implies $\overline{S}x = \overline{S}y$, i.e. xL^*y. From $\overline{S}x = \overline{S}y$ we get the existence of $s,t \in \overline{S}$ with $x = sy$ and $y = tx$, hence $st = 1$. Since H is the group of units in $\overline{S}$ (note $G = S \,\dot\cup\, H \,\dot\cup\, S^{-1}$) it follows that $s,t \in H$ and, consequently, $Hx = Hy$. From $Hx = Hy$, we get $Sx = SHx = SHy = Sy$, i.e. xLy. The rest of the proposition is clear.

The inclusions $D \subset J$ and $D^* \subset J^*$ are general phenomena. In order to study the precise relation between D and J we need some more information about D in the three example groups (1), (2), (3) of 4.

9. LEMMA. Let (G,S) be a reduced well-bounded semigroup with connected G, and let $H = \partial S$ be normalized as in 4. Then the equivalence relation D can be described in the following way:

In (1): D = equality.

In (2): D has 3 equivalence classes, namely, H, $S = HxH$ for any $x \in S$, and $S^{-1} = HxH$ for an arbitrary $x \in S^{-1}$.

In (3): Here we also use the notations of 6. There are two types of D-equivalence classes:

(a) $H\sigma(k\pi)H = \sigma(k\pi)H$, $k \in \mathbb{Z}$, and

(b) $H\sigma(\pi k + \frac{\pi}{2})H = \{\sigma(t)h \mid h \in H, \ \pi k < t < \pi k + \pi\} = \sigma((\pi k, \pi(k+1)))H$, $k \in \mathbb{Z}$. We write M_k for the D-class of type (b).

Proof. Cases (1) and (2) are trivial. Assume $G = SL_2(\mathbb{R})^{\sim}$. Let $B = p(H) = \{\begin{pmatrix} a & b \\ o & a^{-1}\end{pmatrix} \mid a,b \in \mathbb{R}, \ a > o\}$ and $w := p\sigma(\frac{\pi}{2}) = \begin{pmatrix} o & 1 \\ -1 & o\end{pmatrix}$. Recall that one has the "Bruhat decomposition" of $SL_2(\mathbb{R})$:

$$SL_2(\mathbb{R}) = B \ \dot{\cup}\ w^2B \ \dot{\cup}\ BwB \ \dot{\cup}\ Bw^3B \ .$$

More precisely, let $U := \{\begin{pmatrix} 1 & b \\ o & 1\end{pmatrix} \mid b \in \mathbb{R}\}$ and $m = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix} \in SL_2(\mathbb{R})$. Then

$\gamma = o$ and $\alpha > o$ iff $m \in B$,
$\gamma = o$ and $\alpha < o$ iff $m \in w^2B = -B$,
$\gamma < o$ iff $m \in BwB$ and m has a unique representation $m = hwu$ with $h \in B$ and $u \in U$,
$\gamma > o$ iff $m \in Bw^3B$ and m has a unique representation $m = hw^3u$ with $h \in B$ and $u \in U$. Another description of BwB and Bw^3B is given by

$$BwB = \{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t\end{pmatrix} \mid o < t < \pi\}\, B$$

and

$$Bw^3B = \{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t\end{pmatrix} \mid \pi < t < 2\pi\}\, B \ .$$

The rest of the proof consists of pulling these informations back to G . To this end, let $k \in \mathbb{Z}$ and $\pi k < s < \pi k + \pi$. The proof will be complete if we can show that $H\sigma(s)H = M_k$. First, we claim that $H\sigma(s)$ or $H\sigma(s)H$ is contained in M_k . Fix $h \in H$. Then a continuous function $f : \mathbb{R} \to \mathbb{R}$ is defined by $h\sigma(t) \in \sigma(f(t))H$. Since $\sigma(\pi\mathbb{Z})$ is the center of G it follows $f(n\pi) = n\pi$ for each $n \in \mathbb{Z}$. Moreover, f is injective and hence bijective; for if $h\sigma(t) = \sigma(f(t))x$ and $h\sigma(t') = \sigma(f(t'))x'$ with $x,x' \in H$ and $f(t) = f(t')$ then $x\sigma(-t) = x'\sigma(-t')$ and, therefore, $x = x'$ and $t = t'$. Especially, f maps the interval $\pi k < t < \pi k + \pi$ onto itself and, consequently, $H\sigma(s)H \subseteq M_k$. From this inequality and the above characterization of the B double cosets in $SL_2(\mathbb{R})$ one easily deduces $H\sigma(s)H = M_k$.

The following lemma is useful for determining the equivalence relations J, J^1 and J^* .

10. LEMMA. Let (G,S) be a well-bounded semigroup with connected G and let $H = \partial S$. Then $SxS = HxS = SxH$, $\overline{S}x\overline{S} = Hx\overline{S} = \overline{S}xH$ and $S^1xS^1 = SxS \cup \{x\}$ for each $x \in G$.

Proof. Let E and $S = CH = HC$ be as in 5. and $x = eh$ with $e \in E$ and $h \in H$. Then $SxS = SehHS = SeS = HCeS = HeCS = HeS = HehS = HxS$. The proof of the equalities $SxS = SxH$ and $\overline{S}x\overline{S} = Hx\overline{S} = \overline{S}xH$ is similar. The inclusion $SxS \cup \{x\} \subseteq S^1xS^1$ is obvious. But $S^1xS^1 = SxS^1 \cup xS^1 = SxS \cup Sx \cup xS \cup \{x\} \subseteq SxS \cup SxH \cup HxS \cup \{x\} = SxS \cup \{x\}$.

11. THEOREM. *Let* (G,S) *be a well-bounded semigroup with connected* G , *let* $H = \partial S$, *and let* $M = \mathrm{Core}\,(H)$. *The equivalence relations* $D^* = D$ *and* J^* *on* G *coincide. In general,* D *is coarser than* J^1 , *and* J *is coarser than* D , *even when restricted to* S , *namely in the case that* $G/M \cong SL_2(\mathbb{R})^\sim$. *If* $S = HC$ *in the usual notation then* $x \mapsto HxHC = SxS$ *induces a bijective map from* $\overline{S}/J$ *onto the set of all open two-sided ideals in* S . *The proper closed two-sided ideals in* S *are precisely the closures of the proper open two-sided ideals. If* $G/M \cong \mathbb{R}$, *then there is a bijective map from* $\overline{S}/J$ *onto* $[0,\infty)$. *If* $\dim G/M = 2$, *then* $\overline{S}/J$ *reduces to one point. If*

$G/M \cong SL_2(\mathbb{R})^{\sim}$ then $\overline{S}/J$ is countable and $HxH \mapsto Hx\overline{HC} = \overline{S}x\overline{S}$ defines a bijective map from $\overline{S}/D$ onto the set of all two-sided ideals in $\overline{S}$; in particular this set is countable.

Proof. Clearly, it suffices to prove the statements of the theorem in the reduced cases (1), (2), (3) (see 4.).

(1): Obviously, $D = J^1 = J = J^* =$ equality. The open ideals in $S = (o,\infty)$ are the intervals $\{(x,\infty) \mid x \geq o\}$ and the closed proper ideals are the intervals $\{[x,\infty) \mid x > o\}$.

(2): The D-equivalence classes are S, H, S^{-1} . Using 10. one sees that there are two J-equivalence classes $\overline{S}$ and S^{-1} (if $x \in \overline{S}$ then $SxS = S$, if $x \in S^{-1}$ then $SxS = G$) , that there are three J^*-equivalence classes S, H, S^{-1} (if $x \in S$ then $\overline{S}x\overline{S} = S$, if $x \in H$ then $\overline{S}x\overline{S} = \overline{S}$, and if $x \in S^{-1}$ then $\overline{S}x\overline{S} = G$) , and that there are infinitely many J^1-equivalence classes S, S^{-1} and $\{h\}$, $h \in H$, (if $x \in S$ then $S^1xS^1 = S$, if $x \in S^{-1}$ then $S^1xS^1 = G$, and if $x \in H$ then $S^1xS^1 = S \dot{\cup} \{x\}$) . The rest is clear because S is a simple semigroup $(SxS = S$ if $x \in \overline{S})$.

(3): Here, we use the notations of 6. By 9., the D-equivalence classes are $\sigma(k\pi)H$, $k \in \mathbb{Z}$, and $M_k = H\,\sigma(\pi k + \frac{\pi}{2})H =$ $= \sigma((\pi k, \pi k + \pi))H$, $k \in \mathbb{Z}$. Using 10., one gets

$$SxS = HxHC = \sigma((\pi k,\infty))H \quad \text{if} \quad x \in \sigma(k\pi)H \quad \text{or} \quad x \in M_k \ ,$$

and $$\overline{S}x\overline{S} = Hx\overline{HC} = \sigma([\pi k,\infty))H \quad \text{if} \quad x \in \sigma(k\pi)H \ ,$$

and $$\overline{S}x\overline{S} = Hx\overline{HC} = \sigma((\pi k,\infty))H \quad \text{if} \quad x \in M_k \ ,$$

and $$S^1xS^1 = SxS \cup \{x\} = \sigma((\pi k,\infty))H \quad \text{if} \quad x \in M_k \ ,$$

and $$S^1xS^1 = \sigma((\pi k,\infty))H \,\dot{\cup}\, \{x\} \quad \text{if} \quad x \in \sigma(k\pi)H \ .$$

In particular, $D = J^*$, and the D-equivalence classes $\sigma(k\pi)H$ and M_k are identified under J , and for a fixed $k \in \mathbb{Z}$ the J^1-equivalence classes $\{\sigma(\pi k)h\}$, $h \in H$, are identified under D .

To prove the rest of the theorem we compute the ideals in S and $\overline{S}$. Let I be a proper ideal in S , and let k be the smallest positive integer k such that $I \cap (M_k \cup \sigma(\pi k)H) \neq \emptyset$, and let $A := I \cap \sigma(\pi k)H$. The computation of the J-equivalence classes shows that $I = A \cup \sigma((\pi k,\infty))H$, and indeed for any positive integer k and any subset A of $\sigma(\pi k)H$ the union $A \cup \sigma((\pi k,\infty))H$ is a proper ideal in S. The closed (open) ideals correspond to the cases $A = \sigma(\pi k)H$ $(A = \emptyset)$, i.e. the $\sigma([\pi k,\infty))H$ are the proper closed ideals, and the $\sigma((\pi k,\infty))H$ are the proper open ideals. Now let K be an ideal in $\bar{S}$. If $K \cap H \neq \emptyset$ then $K = \bar{S}$. Thus we assume that K is a subset of S. Then $K = S$ or $K = A \cup \sigma((\pi k,\infty))H$ for some positive integer k and a subset A of $\sigma(\pi k)H$. But A has to be invariant under translations with elements of H, and hence $A = \emptyset$ or $A = \sigma(\pi k)H$. Therefore all ideals in $\bar{S}$ have the form $\sigma([\pi k,\infty))H$ or $\sigma((\pi k,\infty))H$ for some nonnegative integer k.

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