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Discrete nilpotent groups have a T_1 primitive ideal space

by

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Abstract. Let G be a discrete nilpotent group, let $C^*(G)$ be the C^*-hull of $L^1(G)$, and let Prim (G) be the space of all primitive ideals in $C^*(G)$, equipped with the Jacobson topology. It is shown that Prim (G) is a T_1 space, i.e. the primitive ideals are maximal. As a consequence, the set of maximal two-sided ideals in $L^1(G)$ coincides with the set of primitive ideals and with the set of kernels of irreducible *-representations of $L^1(G)$.

For a locally compact group G let $C^*(G)$ be the C^*-hull of $L^1(G)$ and let $\operatorname{Prim} G = \operatorname{Prim} C^*(G)$ denote the space of kernels of irreducible *-representation of $C^*(G)$, equipped with the Jacobson topology. Using R. Howe's results on representations of a certain type of discrete nilpotent groups, C.C. Moore and J. Rosenberg have shown that Prim(G) is a T_1 space (i.e. the primitive ideals in $C^*(G)$ are maximal) for all *finitely generated* discrete nilpotent groups. In this paper, I want to give a short direct proof for the T_1 property of Prim(G) for *all* discrete nilpotent groups.

The heart of the proof is the following lemma.

LEMMA 1. *Let G be a locally compact group, let N be an open normal subgroup of G, and let W be a subgroup of G with $N \subset W$ and such that W/N is central in G/N. Let λ be the left regular representation of G in $L^2(G/N)$, and let σ and τ be unitary representations of G. Suppose that σ is irreducible and weakly contained in $\lambda \otimes \tau$ (symbolically: $\sigma \ll \lambda \otimes \tau$). Then there exists a unitary character χ of W, $\chi \equiv 1$ on N, such that*

$$\ker_{L^1}(\chi \otimes \tau|_W) \subset \ker_{L^1} \sigma|_W,$$

where $\ker_{L^1}$ means that we take the kernel of the corresponding representation of $L^1(W)$.

Remark. In [1], the so-called class $[\psi]$ of locally compact groups was introduced. For a locally compact group G, denote by $\operatorname{Priv}_*(L^1(G))$ the space of kernels of irreducible $*$-representations of $L^1(G)$, equipped with the Jacobson topology. G belongs, by definition, to $[\psi]$ if the canonical map $\operatorname{Prim}(G) \to \operatorname{Priv}_*(L^1(G))$ is an homeomorphism. It was shown that every locally compact group with polynomially growing Haar measure (e.g. every nilpotent locally compact (discrete) group) is in $[\psi]$. Moreover, it was shown that a locally compact group G belongs to $[\psi]$ iff for every pair π, ϱ of unitary representations of G the inclusion $\ker_{L^1}\pi \subset \ker_{L^1}\varrho$ implies $\ker_{C^*}\pi \subset \ker_{C^*}\varrho$. Thus, if we assume in Lemma 1 additionally that W belongs to $[\psi]$, then the assertion means that $\sigma|_W$ is weakly contained in $\chi\otimes\tau|_W$.

Proof of Lemma 1. Here, I use an idea from [7] where we proved the symmetry of the group algebra of a discrete nilpotent group. Let $\mathscr{C} := \tau(C^*(G))$, and let $\mathscr{A} = \mathscr{L}^1(G/N, \mathscr{C})$ be the Leptin algebra with trivial factor system and trivial action. In other words, $\mathscr{A}$ is just the (projective) tensor product of the two algebras $\mathscr{L}^1(G/N)$ and $\mathscr{C}$. We define

$$E\colon \mathscr{L}^1(G)\to\mathscr{A}$$

by

$$(Ef)(\dot{x}) = \int_N f(xn)\tau(xn)\,dn = \tau(x)\int_N f(xn)\tau(n)\,dn$$

where $\dot{x}$ denotes the image of x under the quotient morphism $G\to G/N$. Since N is open in G there are no measurability problems. Trivial computations show that E is an *-morphism.

Let $\mathscr{H}$ be the representation space of τ. Then $\tau' := \lambda\otimes\tau$ acts in $\mathscr{H}' := L^2(G/N, \mathscr{H})$, and we have

$$\big(\tau'(f)\xi\big)(x) = \int_G f(y)\tau(y)\xi(\dot{y}^{-1}x)\,dy,$$

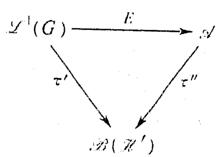
where $f\in\mathscr{L}^1(G)$, $x\in G/N$, $\xi\in\mathscr{H}'$.

Moreover, we define $\tau'' := \mathscr{A}\to\mathscr{B}(\mathscr{H}')$ by

$$\big(\tau''(f)\xi\big)(x) = \int_{G/N} f(y)\xi(y^{-1}x)\,dy$$

where $x\in G/N$, $\xi\in\mathscr{H}'$, and $f\in\mathscr{A} = \mathscr{L}^1(G/N, \mathscr{C})$, especially note that $f(y)\in\mathscr{B}(\mathscr{H})$.

One verifies very easily that τ'' is an *-representation of $\mathscr{A}$ and that the diagram



commutes. Let $\tau'\big(\mathscr{L}^1(G)\big)^-$ and $\tau''(\mathscr{A})^-$ be the closures in the norm (of $\mathscr{B}(\mathscr{H}')$). From $\sigma\preccurlyeq\tau'$ it follows that $|\sigma(f)|\leqslant|\tau'(f)|$ for all $f\in L^1(G)$. Therefore, there exists a unique irreducible *-representation $\bar{\sigma}$ (in the representation space of σ) of $\tau'\big(L^1(G)\big)^-$ with $\bar{\sigma}\tau' = \sigma$. From 2.10.2 in [2] it follows that there exists an irreducible *-representation σ' of $\tau''(\mathscr{A})^-$ such that the restriction of σ' to $\tau'\big(L^1(G)\big)^-$ contains $\bar{\sigma}$ as a subrepresentation. Let $\tilde{\sigma} = \sigma'\tau''$, $\tilde{\sigma}$ is an irreducible *-representation of $\mathscr{A}$. By construction, we have

$$(*)\qquad f\in L^1(G),\quad Ef\in\ker\tilde{\sigma}\Rightarrow f\in\ker\sigma.$$

The representation $\tilde{\sigma}$ of $\mathscr{A}$ can be extended to an *-representation of the adjoint algebra $\mathscr{A}^b$, see [6], which contains $L^1(G/N)$ and $\mathscr{C}$, and we find a unitary representation σ_1 of G/N and an *-representation σ_2 of $\mathscr{C}$ such that

$$\tilde{\sigma}(f) = \int_{G/N}\sigma_1(x)\sigma_2\big(f(x)\big)\,dx$$

for $f\in\mathscr{A} = \mathscr{L}^1(G/N, \mathscr{C})$.

Since $L^1(W/N) = L^1(W/N, C)$ is central in $\mathscr{A}^b$, the restriction of σ_1 to W/N is the multiple of a unitary character χ of W/N. We consider χ also as a character of W and assert that $\ker_{L^1(W)}(\chi\otimes\tau|_W)\subset\ker_{L^1(W)}\sigma$. Using $(*)$ we see that it is enough to show:

$$f\in L^1(W),\quad (\chi\otimes\tau|_W)(f) = 0\ \Rightarrow\tilde{\sigma}(Ef) = 0.$$

The assumption says that $\int_W f(\omega)\chi(\omega)\tau(\omega)\,d\omega = 0$. But

$$\begin{aligned}\tilde{\sigma}(Ef) &= \int_{G/N}\sigma_1(x)\sigma_2\big(Ef(x)\big)\,dx = \int_{G/N}\sigma_1(\dot{x})\sigma_2\Big(\int_N f(xn)\tau(xn)\,dn\Big)\,d\dot{x}\\ &= \int_{W/N}\chi(\dot{x})\sigma_2\Big(\int_N f(xn)\tau(xn)\,dn\Big)\,d\dot{x} = \sigma_2\Big(\int_{W/N}\chi(\dot{x})\Big(\int_N f(xn)\tau(xn)\,dn\Big)\,d\dot{x}\Big)\\ &= \sigma_2\Big(\int_W\chi(\omega)f(\omega)\tau(\omega)\,d\omega\Big) = 0,\end{aligned}$$

and the proof of Lemma 1 is finished.

LEMMA 2. *Let G be a discrete nilpotent group, let $G_0 = \{e\}\subset G_1\subset G_2\subset\ldots\subsetneq G_n = G$ be the ascending central series of G, i.e. G_k/G_{k-1} = center of G/G_{k-1} for $k\geqslant 1$. Let π be an irreducible unitary representation of G such that $\ker_{C^*}\pi$ is a maximal ideal in $C^*(G)$. Let τ be another irreducible unitary representation of G, let $1\leqslant k\leqslant n$, and let γ be a unitary character of G_k/G_{k-1} such that $\gamma\otimes\tau|G_k\preccurlyeq\pi|G_k$.*

Then $\pi|G_k$ and $\gamma\otimes\tau|G_k$ are weakly equivalent.

Proof. We consider π as fixed. Suppose that the Lemma is false. Then we take the largest k, $1\leqslant k\leqslant n$, such that there exist an irreducible unitary representation τ of G and a unitary character γ of G_k/G_{k-1} with the property that $\gamma\otimes\tau|G_k$ is properly weakly contained in $\pi|G_k$. Then we have $k<n$ because $\gamma\otimes\tau\preccurlyeq\pi$ implies $\pi\preccurlyeq\gamma\otimes\tau$ since $\ker_{C^*}\pi$ is maximal. From $\gamma\otimes\tau|G_k\preccurlyeq\pi|G_k$ it follows that

$$\operatorname*{ind}_{G_k\uparrow G}\gamma\otimes\tau|G_k=(\operatorname*{ind}_{G_k\uparrow G}\gamma)\otimes\tau\preccurlyeq\operatorname*{ind}_{G_k\uparrow G}\pi|G_k=\pi\otimes\lambda$$

where λ denotes the left regular representation in $L^2(G/G_k)$. The reader should note that the "continuity of the induction" was shown in [3] only in the separable case. But here it is also true because one can compute the L^1-kernels of the induced representations explicitly in terms of the kernels of the original representations. And it is enough to know the L^1-kernels since G belongs to $[\psi]$. Now, we choose an irreducible unitary representation σ of G which is weakly contained in $(\operatorname*{ind}_{G_k\uparrow G}\gamma)\otimes\tau$. From Lemma 1 (together with the Remark), applied to $N=G_k$, $W=G_{k+1}$, it follows that there exists a unitary character χ of G_{k+1}/G_k with $\sigma|G_{k+1}\preccurlyeq\chi\otimes\pi|G_{k+1}$ or $\bar\chi\otimes\sigma|G_{k+1}\preccurlyeq\pi|G_{k+1}$. Since k was maximal, $\bar\chi\otimes\sigma|G_{k+1}$ and $\pi|G_{k+1}$ are weakly equivalent. Then $\sigma|G_k$ and $\pi|G_k$ are weakly equivalent, too. But from $\sigma\preccurlyeq(\operatorname*{ind}_{G_k\uparrow G}\gamma)\otimes\tau\preccurlyeq\pi\otimes\lambda$ it follows that

$$\sigma|G_k\preccurlyeq\gamma\otimes\tau|G_k\preccurlyeq\pi|G_k(\preccurlyeq\sigma|G_k).$$

Hence, $\gamma\otimes\tau|G_k$ and $\pi|G_k$ are weakly equivalent which is a contradiction

THEOREM. *Let G be a discrete group which is a finite extension of a nilpotent group. Then* $\operatorname{Prim}(G)$ *is a T_1 space.*

Proof. Since the T_1 property is stable under finite extensions, see [10], we may assume that G is nilpotent. Let ϱ be an irreducible unitary representation of G. We have to show that $\ker_{C^*}\varrho$ is maximal. Since $C^*(G)$ has a unit, $\ker_{C^*}\varrho$ is contained in a maximal two-sided ideal which is the kernel of an irreducible *-representation, say π. Let $G_0=\{e\}\subset G_1\subset\ldots\subsetneq G_n=G$ be the ascending central series of G. Since π and ϱ are irreducible, their restrictions to G_1 are weakly equivalent to characters. These characters are identical because $\pi|G_1$ is weakly contained in $\varrho|G_1$, hence $\varrho|G_1$ is weakly contained in $\pi|G_1$. Next, we show that $1\leqslant k<n$, $\varrho|G_k\preccurlyeq\pi|G_k$ implies $\varrho|G_{k+1}\preccurlyeq\pi|G_{k+1}$. From $\varrho|G_k\preccurlyeq\pi|G_k$ it follows that $\varrho\otimes\lambda\preccurlyeq\pi\otimes\lambda$ where λ denotes the left regular representation of G in $L^2(G/G_k)$. Since G (or better: G/G_k) is amenable the trivial representation is weakly contained in λ. Hence ϱ is weakly contained in $\pi\otimes\lambda$. From Lemma 1 it follows that there exists a unitary character χ of G_{k+1}/G_k with $\varrho|G_{k+1}\preccurlyeq\chi\otimes\pi|G_{k+1}$. Moreover, we have $\pi|G_{k+1}\preccurlyeq\varrho|G_{k+1}$. By Lemma 2, $\pi|G_{k+1}$ and $\chi\otimes\pi|G_{k+1}$ are weakly equivalent. Hence $\pi|G_{k+1}$ and $\varrho|G_{k+1}$ are weakly equivalent, too. By finite induction, we get that π and ϱ are weakly equivalent. Therefore, $\ker_{C^*}\varrho=\ker_{C^*}\pi$ is maximal, and the Theorem is proved.

The Theorem has also some consequences for the ideal theory in the group algebra $L^1(G)$.

COROLLARY. *Let G be a discrete group which is a finite extension of a nilpotent group. Let* $\operatorname{Max}(L^1(G))$ *denote the set of all maximal two-sided ideals, let* $\operatorname{Priv}(L^1(G))$ *denote the set of primitive ideals (in the algebraic sense), and let* $\operatorname{Priv}_*(L^1(G))$ *denote the set of all kernels of irreducible *-representations of $L^1(G)$. Then* $\operatorname{Max}(L^1(G))=\operatorname{Priv}(L^1(G))=\operatorname{Priv}_*(L^1(G))$.

Proof. It is known from [8] or [7] and [5] that $L^1(G)$ is a symmetric Banach*-algebra which implies the inclusion $\operatorname{Priv}(L^1(G))\subset\operatorname{Priv}_*(L^1(G))$. The inclusion $\operatorname{Max}(L^1(G))\subset\operatorname{Priv}(L^1(G))$ is obvious. Now, let $P=\ker_{L^1}\pi$, π irreducible, be in $\operatorname{Priv}_*(L^1(G))$. Since $L^1(G)$ has a unit, P is contained in a maximal two-sided ideal M. There exists an irreducible *-representation μ of $L^1(G)$ with $\ker_{L^1}\mu=M$. Since G as a group with polynomially growing Haar measure belongs to $[\psi]$, the inclusion $\ker_{L^1}\pi\subset\ker_{L^1}\mu$ implies $\ker_{C^*}\pi\subset\ker_{C^*}\mu$. The Theorem gives $\ker_{C^*}\pi=\ker_{C^*}\mu$ and, consequently, $P=\ker_{L^1}\pi=\ker_{L^1}\mu=M$.

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