Arch. Math., Vol. 59, 38–41 (1992)

0003-889X/92/5901-0038 $ 2.30/0


# An addendum to K. H. Hofmann's article "A memo on the exponential function and regular points"

By

Detlev Poguntke

In this short note we solve two problems posed in the above mentioned article, [2]. We will use freely the notations introduced in [2]. In particular, for a regular element $x$ in a connected Lie group $G$ let $\mathfrak{a}(x)$ and $\mathfrak{h}(x)$, respectively, be the Fitting one- and Fitting null-component of $\mathrm{Ad}(x) - 1$ in the Lie algebra $\mathfrak{g}$ of $G$, and let $N(x)$ denote the normalizer of $H(x) := \exp \mathfrak{h}(x)$. Moreover, let $W(x) = \{h \in H(x);\, xh \text{ is regular}\}$. In the notations of [2] the open subset $W(x)$ of $H(x)$ is nothing but $\exp V(x)$, and $W(x)$ is, via exp, analytically equivalent to $V(x)/\Gamma(x)$ where $\Gamma(x) = \exp^{-1}(1) \cap \mathfrak{h}(x)$ is a discrete (additive) subgroup in the center of $\mathfrak{h}(x)$ and $V(x) = \exp^{-1}(W(x)) \cap \mathfrak{h}(x)$.

**Proposition 1.** *Let $g$ be a regular element in a connected Lie group $G$. Then there exists an open neighborhood $U(g)$ of zero in $\mathfrak{a}(g)$ such that the map $\beta\colon U(g) \times W(g) \to G$ defined by $\beta(X, h) = \exp X\, gh \exp(-X)$ has an open image and that $\beta$ induces an isomorphism of analytic manifolds from $U(g) \times W(g)$ onto the image of $\beta$.*

Remark 1. The map $\beta$ is essentially the map $\alpha$ studied in the Foliation Theorem (Proposition 13) of [2]. Essentially means that $\alpha$ and $\beta$ are transformed into each other by the canonical identification of $W(g)$ with $V(g)/\Gamma(g)$, see above. Proposition 1 allows us to replace in the Foliation Theorem the words "analytic homeomorphism" by "isomorphism of analytic manifolds" in part (i) and to dispose of part (iii).

Remark 2. The map $\alpha$ was also investigated in [1], p. 35, Lemma 3, even in the case of non-regular elements. It was shown that $\alpha$ is regular in a neighborhood of $(0, 0)$.

Proof. Obviously, independent of the choice of $U(g)$ the map $\beta$ will be analytic. The only two things we have to arrange for are that $\beta$ is injective and that $\beta$ is regular everywhere. As $g$ is fixed we let $\mathfrak{a} = \mathfrak{a}(g)$, $\mathfrak{h} = \mathfrak{h}(g)$, $W = W(g)$ and $N = N(g)$. From the results in [2] we quote that $\mathfrak{a}(gh) = \mathfrak{a}$ and $\mathfrak{h}(gh) = \mathfrak{h}$ for all $h \in W$. For $X \in \mathfrak{a}$ define $\psi_X\colon \mathfrak{a} \to G$ by $\psi_X(Z) = \exp(-X)\exp(Z + X)$, and denote by $d\psi_X\colon \mathfrak{a} \to \mathfrak{g}$ the differential of $\psi_X$ at 0. Clearly, the map $X \mapsto d\psi_X$ from $\mathfrak{a}$ into $\mathrm{End}(\mathfrak{a}, \mathfrak{g})$ is analytic. Now we choose an open neighborhood $U(g)$ of zero in $\mathfrak{a}$ with the following two properties.

(i) For $X \in U(g)$ the composition $\mathrm{pr}_{\mathfrak{a}} \circ d\psi_X$ of $d\psi_X$ with the projection $\mathrm{pr}_{\mathfrak{a}}$ from $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$ onto $\mathfrak{a}$ is an invertible linear transformation on $\mathfrak{a}$.

(ii) The map $(X, n) \mapsto \exp(X)n$ from $U(g) \times N$ in $G$ is injective.

To see that such an $U(g)$ exists one notes concerning (i) that $\mathrm{pr}_{\mathfrak{a}} \circ d\psi_X$ depends continuously on $X \in \mathfrak{a}$ and that $d\psi_O$ is the inclusion of $\mathfrak{a}$ in $\mathfrak{g}$. Concerning (ii) one observes that $N$ is a closed (not necessarily connected) Lie subgroup of $G$, that the Lie algebra of $N$ equals $\mathfrak{h}$ as $\mathfrak{h}$ is a Cartan subalgebra and that $\mathfrak{a}$ is complementary to $\mathfrak{h}$ in $\mathfrak{g}$. Moreover, one sees that the required properties of $U(g)$ do not explicitly depend on $g$ but rather on $\mathfrak{a}$ and on $\mathfrak{h}$ (and on $N$ which is derived from $\mathfrak{h}$).

With this choice of $U(g)$ we claim first that $\beta: U(g) \times W \to G$ is injective. Let $X_1, X_2 \in U(g)$ and $h_1, h_2 \in W$ and suppose that

$$\exp(X_1) g h_1 \exp(-X_1) = \exp(X_2) g h_2 \exp(-X_2).$$

Then $gh_2 = \exp(-X_2)\exp(X_1) g h_1 \exp(-X_1)\exp(X_2)$ and, therefore,

$$\begin{aligned}\mathfrak{h} = \mathfrak{h}(gh_2) &= \mathfrak{h}(\exp(-X_2)\exp(X_1) g h_1 \exp(-X_1)\exp(X_2)) \\ &= \mathrm{Ad}(\exp(-X_2)\exp(X_1))\mathfrak{h}(gh_1) = \mathrm{Ad}(\exp(-X_2)\exp(X_1))(\mathfrak{h});\end{aligned}$$

here we used that the assignment $x \mapsto \mathfrak{h}(x)$ transforms appropriately under (inner) automorphisms, see [2]. It follows that $\exp(-X_2)\exp(X_1)$ is contained in the normalizer $N$ of $H := \exp \mathfrak{h}$. From (ii) we deduce that $X_1 = X_2$ which immediately implies that also $h_1 = h_2$.

It remains to show that $\beta$ is regular at each given point $(X_0, h_0) \in U(g) \times W$. Evidently, this is equivalent to the fact that the map

$$(Z, Y) \mapsto \exp(Z + X_0) g h_0 \exp(Y) \exp(-Z - X_0)$$

from $\mathfrak{a} \times \mathfrak{h}$ into $G$ is regular at $(0, 0)$. Modifying this map once more by translations with fixed group elements our claim is equivalent to the regularity at $(0, 0)$ of the map $\varrho: \mathfrak{a} \times \mathfrak{h} \to G$ defined by

$$\varrho(Z, Y) = \exp(-X_0)\exp(Z + X_0) g h_0 \exp(Y) \exp(-Z - X_0)\exp(X_0) h_0^{-1} g^{-1}.$$

We have to compute the differential $d\varrho: \mathfrak{a} \oplus \mathfrak{h} \to \mathfrak{g}$ of $\varrho$ at $(0, 0)$ and to verify that it is an invertible linear transformation. Using the above introduced family $\psi_X$ of maps we obtain

$$\varrho(Z, Y) = \psi_{X_0}(Z) I_{gh_0}(\exp(Y) \psi_{X_0}(Z)^{-1})$$

where $I$ denotes the canonical homomorphism from $G$ into the group of inner automorphisms of $G$. To compute $d\varrho$ we write $\varrho$ as a composition of four maps, namely

$$\gamma: \mathfrak{a} \times \mathfrak{h} \to G \times H \times G,$$

where $\gamma(Z, Y) = (\psi_{X_0}(Z), \exp Y, \psi_{X_0}(Z)^{-1})$,

$$\mu: G \times H \times G \to G \times G,$$

where $\mu(a, b, c) = (a, bc)$,

$$\mathrm{id}_G \times I_{gh_0}: G \times G \to G \times G$$

and

$$m: G \times G \to G,$$

where $m(a, b) = ab$.

Using the chain rule one gets

$$d\varrho(A \oplus B) = d\psi_{X_0}(A) + \mathrm{Ad}(gh_0)(B - d\psi_{X_0}(A))$$
$$= \mathrm{Ad}(gh_0)(B) + (1 - \mathrm{Ad}(gh_0))(d\psi_{X_0}(A))$$

for $A \in \mathfrak{a}$, $B \in \mathfrak{h}$.

To see that $d\varrho$ is invertible, for dimensional reasons it is sufficient to show that $d\varrho(A \oplus B) = 0$ implies $A = B = 0$. To this end, apply $\mathrm{pr}_{\mathfrak{a}}$ to the equation $d\varrho(A \oplus B) = 0$. As $\mathfrak{h} = \mathfrak{h}(g) = \mathfrak{h}(gh_0)$ and $\mathfrak{a} = \mathfrak{a}(g) = \mathfrak{a}(gh_0)$ these two spaces are $\mathrm{Ad}(gh_0)$-invariant. Hence $\mathrm{pr}_{\mathfrak{a}}\mathrm{Ad}(gh_0)(B)) = 0$ and

$$0 = \mathrm{pr}_{\mathfrak{a}}(1 - \mathrm{Ad}(gh_0))(d\psi_{X_0}(A)) = (1 - \mathrm{Ad}(gh_0))\,\mathrm{pr}_{\mathfrak{a}}(d\psi_{X_0}(A)).$$

As $1 - \mathrm{Ad}(gh_0)$ is invertible on $\mathfrak{a}$ it follows $\mathrm{pr}_{\mathfrak{a}}(d\psi_{X_0}(A)) = 0$ whence $A = 0$ by (i). Then $B$ is zero, too. Proposition 1 is proved.

In Proposition 19 of [2] it was shown that in the solvable case each regular element $x$ is contained in $H(x)$. The proof used the theory developed in [2] and some results of [3] as well. Below we will give a short direct proof of this fact.

**Proposition 2.** *If $G$ is a connected solvable Lie group and if $x$ is a regular element of $G$ then $x$ is contained in $H(x)$.*

Proof. Evidently, it is sufficient to show the assertion in the simply connected case. Let's suppose that (the simply connected version of) the proposition is wrong, and let $G$ be a simply connected solvable Lie group of minimal dimension which contains a regular element $x$ such that $x \notin H(x)$. Of course, $G$ can't be nilpotent. Let $\mathfrak{m}$ be an ideal in the Lie algebra $\mathfrak{g}$ of $G$ of minimal positive dimension. Then $\mathfrak{m}$ is necessarily abelian, and as $G$ is simply connected $M := \exp \mathfrak{m}$ is a closed abelian normal subgroup of $G$. Denote by $\pi: G \to G/M$ the quotient map. Since $\pi(x)$ is regular, see Proposition 2 on p. 35 in [1], and since $d\pi$ maps $\mathfrak{h}(x)$ onto the Fitting null-component of $\mathrm{Ad}(\pi(x)) - 1$ in $\mathfrak{g}/\mathfrak{m}$ the proposition applied to $G/M$ gives that $x$ is contained in $H(x)M$. Hence we may write $x = hm$ with $h \in H(x)$ and $m \in M$. Moreover, the minimality of $G$ implies that $G = H(x)M$ and $\mathfrak{g} = \mathfrak{h}(x) + \mathfrak{m}$. Hence $\mathfrak{h}(x) \cap \mathfrak{m}$ is an ideal in $\mathfrak{g}$. By the minimality of $\mathfrak{m}$ either $\mathfrak{h}(x) \cap \mathfrak{m} = \mathfrak{m}$ or $\mathfrak{h}(x) \cap \mathfrak{m} = 0$. In the first case one concludes $G = H(x)$, a contradiction. So we are left with $\mathfrak{h}(x) \cap \mathfrak{m} = 0$. As $\mathfrak{h}(x)$ is invariant under $\mathrm{Ad}(x)$ and $\mathrm{Ad}(h)$, it is invariant under $\mathrm{Ad}(m)$ as well. Let $Z$ be the unique element in $\mathfrak{m}$ with $\exp Z = m$, and let $Y$ be an arbitrary element in $\mathfrak{h}(x)$. We know that $\mathrm{Ad}(m)(Y) - Y$ is in $\mathfrak{h}(x)$. On the other hand,

$$\mathrm{Ad}(m)(Y) - Y = \mathrm{Ad}(\exp Z)(Y) - Y = \exp(\mathrm{ad}\, Z)(Y) - Y = [Z, Y]$$

is contained in $\mathfrak{m}$. Hence $[Z, \mathfrak{h}(x)] = 0$ and, therefore, $Z$ is contained in the center of $\mathfrak{g}$ and in $\mathfrak{h}(x)$. Using $\mathfrak{h}(x) \cap \mathfrak{m} = 0$ we conclude that $Z = 0$, $m = 1$, $x = h \in H(x)$, a contradiction to $x \notin H(x)$.

## References

[1] N. Bourbaki, Groupes et algèbres de Lie, Chap. 7 et 8. Paris 1975.
[2] K. H. Hofmann, A memo on the exponential function and regular points. Arch. Math. **59**, 24–37 (1992).
[3] K. H. Hofmann and A. Mukherjea, On the density of the image of the exponential function. Math. Ann. **234**, 263–273 (1978).

Eingegangen am 3. 2. 1992

Anschrift des Autors:

Detlev Poguntke
Fakultät für Mathematik
Universität Bielefeld
Postfach 8640
DW-4800 Bielefeld 1