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A UNIVERSAL PROPERTY OF THE TAKAHASHI QUASI-DUAL

DETLEV POGUNTKE

Introduction. Topological group always means Hausdorff topological group, homomorphism (isomorphism) between topological groups always means continuous homomorphism (homeomorphic isomorphism). For a topological group G, the topological commutator subgroup (the closure of the algebraic commutator subgroup) is denoted by G'. For each locally compact group G, Takahashi has constructed a locally compact group G_T (called the Takahashi quasi-dual) and a homomorphism $G \to G_T$ such that G_T is maximally almost periodic, and G_T' is compact. The category of all locally compact groups with these two properties is denoted by [TAK]. Takahashi's duality theorem states that $G \to G_T$ is an isomorphism if $G \in$ [TAK]. In this paper we show that for each locally compact group G the homomorphism $G \to G_T$ has a universal property, namely that for each homomorphism $G \to H$, H being in [TAK], there is exactly one homomorphism $G_T \to H$ such that the diagram

$$\begin{array}{ccc} G & \longrightarrow & G_T \\ & \searrow \quad \swarrow & \\ & H & \end{array}$$

commutes. In the language of category theory this means that [TAK] is reflective in the category of all locally compact groups. Takahashi's duality theorem is a simple consequence of this result. Moreover, we give another description of the group G_T and show that $G \in$ [TAK] if and only if G can be embedded as a closed subgroup of a product of a compact group and a locally compact abelian group.

1. In this section we describe Takahashi's construction $G \to G_T$ and show that $G \to G_T$ is dense and induces an isomorphism $G/G' \to G_T/G_T'$.

Let G be a locally compact group. The set $\mathrm{Hom}(G, U(n))$ of all homomorphisms from G into the unitary group $U(n)$ in n dimensions is topologized as follows:

(i) $\mathrm{Hom}(G, U(1))$ is equipped with the compact-open topology.

(ii) If $n > 1$ and $D \in \mathrm{Hom}(G, U(n))$ then the sets $\{D \otimes \chi \mid \chi \in U\}$, U any neighborhood of the identity in the group $\mathrm{Hom}(G, U(1))$, form a fundamental system of neighborhoods of D in $\mathrm{Hom}(G, U(n))$.

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1. *Remark.* $\operatorname{Hom}(G, U(1))$ is isomorphic to $\operatorname{Hom}(G/G', U(1))$, the Pontryagin character group of G/G', because every compact subset of G/G' is the image of a compact subset in G under the natural homomorphism $G \to G/G'$.

Let $G^{\vee}$ and $\mathfrak{A}$ be the topological sums (over the positive integers) of the spaces $\operatorname{Hom}(G, U(n))$ and $U(n)$, respectively.

2. *Definition.* A map $Q: G^{\vee} \to \mathfrak{A}$, satisfying the conditions:
(1) if $D \in \operatorname{Hom}(G, U(n))$ then $Q(D) \in U(n)$;
(2) if $D, D' \in G^{\vee}$ then
(a) $Q(D \oplus D') = Q(D) \oplus Q(D')$ and
(b) $Q(D \otimes D') = Q(D) \otimes Q(D')$;
(3) if $D \in \operatorname{Hom}(G, U(n))$ and $U \in U(n)$ then

$$Q(UDU^{-1}) = UQ(D)U^{-1};$$

is called a unitary mapping.

If x is any element of G then $r(x): G^{\vee} \to \mathfrak{A}$, defined by $r(x)(D) = D(x)$, is a unitary mapping. Moreover, the set of unitary mappings forms a group under pointwise multiplication, and r is a homomorphism from G into this group.

3. *Remark.* If $s: G \to bG$ is the Bohr compactification of G then s induces a bijective map $s^{\vee}: (bG)^{\vee} \to G^{\vee}$, and $s^{\vee}$ induces an isomorphism $s^{\vee\vee}$ from the group $G^{\vee\vee}$ of unitary mappings on $G^{\vee}$ onto the group $(bG)^{\vee\vee}$ of unitary mappings on $(bG)^{\vee}$. If both groups are endowed with the finite-open topology (as we will always assume in the sequel), this isomorphism is an isomorphism of topological groups. Tannaka's duality theorem states that there is an isomorphism i from bG onto $(bG)^{\vee\vee}$ with $r = (s^{\vee\vee})^{-1}is$. Therefore $r: G \to G^{\vee\vee}$ is a Bohr compactification, too.

G_T is defined (i) algebraically, as the subgroup of $G^{\vee\vee}$ consisting of those elements which are continuous with respect to the topology defined above, and (ii) topologically, as being equipped with the compact-open topology. Then it can be shown (see [**2**], e.g.):

4. PROPOSITION. (1) *G_T is a locally compact group.*
(2) *G_T' is compact.*

This and the following (trivial) proposition are the only parts of Takahashi's paper needed here.

5. PROPOSITION. *If $u: G_T \to G^{\vee\vee}$ denotes the inclusion map then:*
(1) *u is a homomorphism of topological groups (hence G_T is maximally almost periodic);*
(2) *r factors through u, i.e. there exists a homomorphism $w: G \to G_T$ with $uw = r$.*

In order to establish the universal property of $w\colon G \to G_T$ we need some more information on u and w, given in the following lemmas.

6. LEMMA. (i) $u(G_T') = (G^{\vee\vee})' = \overline{r(G')}$, *and*
(ii) $u^{-1}((G^{\vee\vee})') = G_T' = \overline{w(G')}$.

Proof. Of course, $r(G') \subset u(G_T') \subset (G^{\vee\vee})'$. Since G_T' is compact, $u(G_T')$ is closed and therefore, $\overline{r(G')} \subset \overline{u(G_T')} = u(G_T') \subset (G^{\vee\vee})'$. For (i) it remains to prove that $(G^{\vee\vee})'$ is contained in $\overline{r(G')}$. Since r is dense ($r\colon G \to G^{\vee\vee}$ is a Bohr compactification of G), $\overline{r(G')}$ is a normal subgroup of $G^{\vee\vee}$, and r induces a dense homomorphism $G/G' \to G^{\vee\vee}/\overline{r(G')}$. Therefore, $G^{\vee\vee}/\overline{r(G')}$ is an abelian topological group, and $(G^{\vee\vee})'$ is contained in $\overline{r(G')}$. The equality $u^{-1}((G^{\vee\vee})') = G_T'$ follows from (i) because u is injective. Moreover, $w(G')$ and, therefore, $\overline{w(G')}$ is contained in G_T'. Since G_T' and hence $\overline{w(G')}$ are compact, $u(\overline{w(G')})$ is closed, and we get

$$u(\overline{w(G')}) \supset \overline{uw(G')} = \overline{r(G')} = u(G_T')$$

which implies $\overline{w(G')} \supset G_T'$ because u is injective.

7. LEMMA. *If* Ch(Hom$(G, U(1))$) *denotes the Pontryagin character group of the locally compact abelian* (*see* 1) *group* Hom$(G, U(1))$ *then* $v\colon G_T \to$ Ch(Hom$(G, U(1))$), *defined by*

$$v(Q) = Q \Big|_{\mathrm{Hom}(G,\, U(1))}^{U(1)},$$

is a homomorphism of topological groups. The kernel of v is G_T'.

Proof. Since any element Q of G_T is continuous and Q satisfies conditions (1) and (2)(b) in 2, Q induces a homomorphism

$$v(Q)\colon \mathrm{Hom}(G, U(1)) \to U(1)$$

of topological groups.

The proof that v is a homomorphism of topological groups is immediate and omitted.

Since v is a homomorphism into an abelian topological group, G_T' is contained in the kernel of v. Let Q be any element in G_T. $r\colon G \to G^{\vee\vee}$ induces a bijective (because r is a Bohr compactification of G) homomorphism $\tilde{r}$ from Hom$(G^{\vee\vee}, U(1))$ onto Hom$(G, U(1))$.

$$\begin{array}{ccc} \mathrm{Hom}(G^{\vee\vee}, U(1)) & \xrightarrow{\tilde{r}} & \mathrm{Hom}(G, U(1)) \\ & {\scriptstyle v(Q)\tilde{r}} \searrow \quad \swarrow {\scriptstyle v(Q)} & \\ & U(1) & \end{array}$$

Explicitly: $(v(Q)\tilde{r})(\chi) = v(Q)(\chi r) = Q(\chi r)$ for $\chi \in \mathrm{Hom}(G^{\vee\vee}, U(1))$, and it can be shown easily that $Q(\chi r) = \chi(u(Q))$. (Fix χ and prove that the continuous homomorphisms χ and $P \mapsto P(\chi r)$ from $G^{\vee\vee}$ to $U(1)$ coincide on $r(G)$ and hence on $G^{\vee\vee}$.) Thus, if Q is in the kernel of v then $\chi(u(Q)) = 1$ for all $\chi \in \operatorname{Hom}(G^{\vee\vee}, U(1))$ and hence $u(Q) \in (G^{\vee\vee})'$; by 6 (ii) we get $Q \in G_T'$.

8. Lemma. *w: $G \to G_T$ induces an isomorphism w': $G/G' \to G_T/G_T'$ of topological groups.*

Proof. Of course, the induced map w' is a homomorphism. By 7 there is an injective homomorphism v' such that

$$\begin{array}{ccc} G_T & \xrightarrow{v} & \operatorname{Ch}(\operatorname{Hom}(G, U(1))) \\ \searrow & & \nearrow_{v'} \\ & G_T/G_T' & \end{array}$$

commutes. As we remarked in 1, $\operatorname{Hom}(G, U(1))$ is isomorphic to $\operatorname{Hom}(G/G', U(1))$. Therefore, by the Pontryagin duality theorem, there exists an isomorphism d: $G/G' \to \operatorname{Ch}(\operatorname{Hom}(G, U(1)))$ given by $d([y])(\chi) = \chi(y)$ where $y \in G$, $[y]$ denotes the image of y under the natural homomorphism $G \to G/G'$, and $\chi \in \operatorname{Hom}(G, U(1))$. A simple computation shows $d = v'w'$ or, equivalently, $(d^{-1}v')w' = \mathrm{id}_{G/G'}$. On the other hand, from $(d^{-1}v')w'(d^{-1}v') = \mathrm{id}_{G/G'}(d^{-1}v') = d^{-1}v' = (d^{-1}v')\mathrm{id}_{G_T/G_T'}$ we obtain $w'(d^{-1}v') = \mathrm{id}_{G_T/G_T'}$ because $d^{-1}v'$ is injective. Thus, w' is an isomorphism with inverse $d^{-1}v'$.

9. Lemma. *w: $G \to G_T$ is a dense mapping.*

Proof. By 8 and 6 we get

$$G_T = w(G)\cdot G_T' = w(G)\cdot \overline{w(G')} = \overline{w(G)}.$$

10. *Remark.* Especially, 9 implies that w is an epimorphism in the category of locally compact groups.

2. In order to be able to give another description of the groups in [TAK] we need the following lemma whose simple proof is omitted.

11. Lemma. *Let G, G_1, G_2 be topological groups, and let f_i:$G \to G_i$ ($i = 1, 2$) be homomorphisms. Then the following conditions are equivalent:*

(1) *the homomorphism $G \to G_1 \times G_2$ induced by f_1 and f_2 is a homeomorphism onto a subgroup of $G_1 \times G_2$;*

(2) *for each neighborhood U of the identity in G there exist neighborhoods V_i of the identity in G_i such that $f_1^{-1}(V_1) \cap f_2^{-1}(V_2)$ is contained in U.*

12. Theorem. *Let G be a locally compact group, let s:$G \to bG$ be the Bohr compactification of G, and let q: $G \to G/G'$ be the natural homomorphism. Then the following conditions are equivalent:*

(a) $G \in$ [TAK];

(b) *the homomorphism $G \to bG \times G/G'$, induced by s and q, is a homeomorphism onto a closed subgroup;*

(c) *G can be embedded as a closed subgroup of a product of a compact group and a locally compact abelian group.*

Proof. (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are trivial.

(a) $\Rightarrow$ (b). Let U be a compact neighborhood of the identity in G. Because of 11 we need only to construct neighborhoods V and W in bG and G/G', respectively, such that

$$(*)\ q^{-1}(W) \cap s^{-1}(V) \text{ is contained in } U.$$

Choose $W = q(U)$. Since UG' is compact and G is maximally almost periodic, s induces a homeomorphism from UG' onto $s(UG')$; especially, $s(U)$ is a neighborhood of the identity in the space $s(UG')$. Therefore, there exists a neighborhood V of the identity in bG such that $s(UG') \cap V$ is contained in $s(U)$; (*) is easily verified.

In order to show that the image of G is closed in $bG \times G/G'$, take a net $(x_\alpha)_{\alpha\in I}$ in G such that

$$s(x_\alpha) \underset{\alpha\in I}{\longrightarrow} x \text{ and } q(x_\alpha) \underset{\alpha\in I}{\longrightarrow} q(y).$$

We have to construct $z \in G$ with $s(z) = x$ and $q(z) = q(y)$. Without loss of generality, we may assume that $y \in G'$ (if this is not the case consider the net $(x_\alpha y^{-1})_{\alpha\in I}$). Let U be a compact neighborhood of the identity in G. Since UG' is compact and $q(U)$ is a neighborhood of $q(y)$ in G/G' there exists a subnet $(x_\alpha)_{\alpha\in J}$ such that (i) $\alpha \in J \Rightarrow q(x_\alpha) \in q(U)$ or, equivalently, $x_\alpha \in UG'$, and (ii) $\lim_{\alpha\in J} x_\alpha$ exists. $z = \lim_{\alpha\in J} x_\alpha$ is the desired element of G.

In order to prove the main theorem we need the following lemma whose simple proof is omitted.

13. LEMMA. *Let G, L, H_1, H_2 be topological groups, let H be a closed subgroup of $H_1 \times H_2$, and let w be a dense homomorphism from G to L with the property that for homomorphisms f_i from G to H_i $(i = 1, 2)$ there exist homomorphisms $\hat{f}_i$ from L to H_i such that $f_i = \hat{f}_i w$. Then for each homomorphism f from G to H there exists a unique homomorphism $\tilde{f}$ from L to H with $\tilde{f}w = f$.*

14. THEOREM. *Let G be a locally compact group, let G_T and w: $G \to G_T$ be as in §1. Then $G_T \in$ [TAK], w is a dense homomorphism, and for each $H \in$ [TAK] and each homomorphism f: $G \to H$ there exists exactly one homomorphism $\hat{f}$: $G_T \to H$ with $f = \hat{f}w$.*

Moreover, this property determines (w, G_T) uniquely up to isomorphism. More precisely, if $G^* \in$ [TAK] and w^*: $G \to G^*$ is a homomorphism such that for each $H \in$ [TAK] and each homomorphism f: $G \to H$ there is a unique homomorphism $\hat{f}$: $G^* \to H$ with $f = \hat{f}w^*$ then there exists an isomorphism i: $G_T \to G^*$ with $iw = w^*$.

15. *Corollary. If $G \in$ [TAK] then w: $G \to G_T$ is an isomorphism.*

16. *Remark.* In the language of category theory the theorem states that the full subcategory [TAK] is an epireflective subcategory of the category

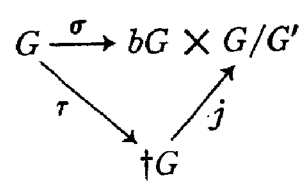
of all locally compact groups, and that for each locally compact group G its [TAK]-epireflection is given by w: $G \to G_T$. (For the definition of epireflective subcategories see [**1**], e.g.)

Proof of the Theorem. Because of 9, 12, and 13 it suffices to show that for each compact group K, each locally compact abelian group A, each homomorphism g: $G \to K$, and each homomorphism h: $G \to A$ there exist homomorphisms $\hat{g}$ and $\hat{h}$ from G_T to K and A, respectively, such that $g = \hat{g}w$ and $h = \hat{h}w$. Since $uw = r$: $G \to G^{\vee\vee}$ is the Bohr compactification of G (see §1) there exists a homomorphism $\bar{g}$: $G^{\vee\vee} \to K$ with $g = \bar{g}r$. Then $\hat{g} = \bar{g}u$ solves the problem. The fact that w induces an isomorphism w' from G/G' onto G_T/G_T' (see 8) implies the existence of $\hat{h}$.

The last assertion of the theorem is a standard computation. The proof of the corollary is trivial because (G, id_G) has the universal property if $G \in$ [TAK].

Now, we will give another description of the group G_T.

17. Proposition. *Let G be a locally compact group, s: $G \to bG$ the Bohr compactification of G, and q: $G \to G/G'$ the natural homomorphism. The homomorphisms s and q induce a homomorphism σ: $G \to bG \times G/G'$. Define $\dagger G := \overline{\sigma(G)}$, denote by j: $\dagger G \to bG \times G/G'$ the inclusion homomorphism, and let τ be the unique homomorphism such that*



commutes. Then $\dagger G \in$ [TAK], and for each homomorphism f: $G \to H$, H being in [TAK], *there exists one and only one homomorphism $\hat{f}$: $\dagger G \to H$ with $\hat{f}\tau = f$. Especially, G_T is isomorphic to $\dagger G$.*

Proof. Use 13 and the characterization of the groups in [TAK] from 12 as in the proof of 14, and then use the universal properties of $G \to bG$ and $G \to G/G'$.

18. *Remark.* The last proposition can be considered as a special case of the following categorical proposition (for terminology compare [**1**]): Let $\mathscr{C}$ be an $(\mathfrak{E}, \mathfrak{M})$-(bi)category with finite products, and let $\mathscr{A}$ and $\mathscr{B}$ be full $\mathfrak{E}$-reflective subcategories of $\mathscr{C}$. Then the full subcategory $\mathscr{D}$ of $\mathscr{C}$ whose objects are the $\mathfrak{M}$-subobjects of products $A \times B$ of an $\mathscr{A}$-object A and a $\mathscr{B}$-object B is an $\mathfrak{E}$-reflective subcategory of $\mathscr{C}$. Moreover, for any $\mathscr{C}$-object C, if r: $C \to A$ (respectively s: $C \to B$) is the reflection of C in $\mathscr{A}$ (respectively in $\mathscr{B}$), f: $C \to A \times B$ the morphism induced by r and s, and

$$C \xrightarrow{f} A \times B = C \xrightarrow{e} D \xrightarrow{m} A \times B$$

the $(\mathfrak{E}, \mathfrak{M})$-factorization of f, then e: $C \to D$ is the reflection of C in $\mathscr{D}$.

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Universität Bielefeld,
Bielefeld, West Germany