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THE COPRODUCT OF TWO CIRCLES

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As is well known, the category of compact groups is complete and cocomplete. Whereas the structure of limits is well known, it is much harder to get some insight into the structure of colimits. In this paper we described the internal structure of the coproduct of two circle groups.

Introduction. The purpose of this paper is to characterize internally the structure of the coproduct of two circle groups in the category whose objects are all compact (Hausdorff) groups and whose morphisms are the continuous homomorphisms, thus solving a problem posed by K.H. Hofmann.

Let K be the compact abelian group of complex numbers of modulus 1 and let C together with morphisms $\varphi, \psi: K \to C$ be the coproduct of K with itself in the category of compact groups. C is connected because K is connected. According to the structure theorem for compact connected groups (see e.g. [4, 6.59, p. 75]), the following hold:

(1) The commutator subgroup C' of C is a closed connected subgroup of C.

(2) If $Z_0(C)$ denotes the component of the identity in the centre ZC of C, then the multiplication $m: Z_0(C) \times C' \to C$ is a surjective morphism with totally disconnected central kernel.

(3) There exist a family $(C_i)_{i\in I}$ of simply connected compact connected simple Lie groups and a surjective morphism $\mu': \Pi_{i\in I} C_i \to C'$ with totally disconnected central kernel.

Let $\mu: Z_0(C) \times \Pi_{i\in I} C_i \to C$ be the composite morphism $m\cdot(1_{Z_0(C)} \times \mu')$.

In this paper we will show that μ' is an isomorphism (isomorphism is always meant in the categorial sense), that the Pontryagin character group of the compact abelian group $Z_0(C)$ is isomorphic to $\mathbf{Q}^2$ ($\mathbf{Q}$ denotes the additive group of rational numbers with the discrete topology), and that for each simply connected compact connected simple Lie group G the set of indices

$$\{i \in I \mid C_i \text{ is isomorphic to } G\}$$

has the cardinality of the continuum. Moreover, it will be described how the kernel of μ is embedded in $Z_0(C) \times \Pi_{i \in I} C_i$.

We begin with two definitions.

Definition 1. Let G be a compact group, and let G_i, $i = 1, 2$, be subgroups of G. G is said to be *generated* by G_1 and G_2 if G is the smallest closed subgroup of G which contains G_1 and G_2.

Definition 2. Let G be a compact group, and let (f, g) be a pair of morphisms from K into G. The pair (f, g) is called *admissible* if G is generated by $f(K)$ and $g(K)$.

Remark 3. Let C together with the morphisms $\varphi, \psi : K \to C$ be – as always – the coproduct of K with itself in the category of compact groups. Then (φ, ψ) is an admissible pair.

In the following proposition we show that for each semisimple, compact, connected Lie group G there exists an admissible pair of morphisms from K into G. In particular, every such group is a quotient of C.

Proposition 4. *Let G be a semisimple, compact, connected Lie group, and let u and v be elements of the centre ZG of G. Then there exist subgroups K_1 and K_2 of G with the following properties:*

(i) *K_i is isomorphic to K for $i = 1, 2$;*

(ii) *$u \in K_1$ and $v \in K_2$;*

(iii) *G is generated by K_1 and K_2.*

Proof. The standard theorems from Lie theory used in the proof are found in every book on this subject, e.g. in [1], [8] or [9]. Let T be a maximal torus in G, i.e., a closed subgroup of G which is maximal with respect to the property of being isomorphic to K^n for a suitable n. Let $\mathrm{L}G$ (resp. $\mathrm{L}T$) be the Lie algebra of G (resp. T), and let $\exp : \mathrm{L}G \to G$ be the exponential map. We consider $\mathrm{L}T$ as a subalgebra of $\mathrm{L}G$:

$$\mathrm{L}T = \{X \in \mathrm{L}G \mid \exp(tX) \in T \text{ for all } t \in \mathbf{R}\}.$$

We proceed with the proof by stating two lemmas.

Lemma 5. *The set of all* $X \in \mathrm{L}T$ (*resp.* $X \in \mathrm{L}G$) *for which there exists* $t \in \mathbf{R}\setminus\{0\}$ *with* $\exp(tX) = u$ *is a dense subset of* $\mathrm{L}T$ (*resp.* $\mathrm{L}G$).

Proof. Since T is abelian and connected, exp induces a covering homomorphism e from the additive group $\mathrm{L}T$ onto T (see e.g. [9, pp. 94, 95]); $\ker e$ is a free abelian group whose rank equals the dimension of $\mathrm{L}T$. ZG is contained in T because the conjugates of the maximal torus T cover the group G (cf. [1, 4.21, p. 89]). In particular, there exists an $A \in \mathrm{L}T$ with $\exp(A) = u$. Therefore the set of all $X \in \mathrm{L}T$ for which there exists $t \in \mathbf{R}\setminus\{0\}$ with $\exp(tX) = u$ is $(\mathbf{R}\setminus\{0\}) \cdot (A + \ker e)$. The latter set is dense in $\mathrm{L}T$ because the rank of $\ker e$ coincides with the dimension of $\mathrm{L}T$. To prove the other part, let Y be an arbitrary element in $\mathrm{L}G$. Let H be the closure of the one-parameter subgroup $\{\exp tY \mid t \in \mathbf{R}\}$ in G. Since H is a compact connected abelian Lie group, it has to be a torus in G. Hence H is contained in a maximal torus S of G. From the above it follows that Y can be approximated by elements $X \in \mathrm{L}S$ for which there exist $t \in \mathbf{R}\setminus\{0\}$ with $\exp(tX) = u$. This proves the lemma. □

For statement (and later use) of Lemma 6, we collect some elementary facts about semisimple Lie algebras. Proofs can be found, e.g., in [7, §§ 62, 63]. Let $\mathfrak{L} = \mathrm{L}G \oplus \mathrm{i}\mathrm{L}G$ be the complexification of $\mathrm{L}G$. $\mathfrak{L}$ is a complex semisimple Lie algebra, and $\mathfrak{H} := \mathrm{L}T \oplus \mathrm{i}\mathrm{L}T$ is a (Cartan) subalgebra of $\mathfrak{L}$. Let $\mathrm{ad}: \mathfrak{L} \to \mathrm{End}(\mathfrak{L}, \mathfrak{L})$ be the adjoint representation. If we define $-(X, Y)$ to be the trace of the linear map $\mathrm{ad}(X) \cdot \mathrm{ad}(Y)$, then $(-,-)$ is a positive definite scalar product on $\mathrm{L}T$. There exists a finite set Δ in $\mathrm{L}T$ and linearly independent vectors X_α, $\alpha \in \Delta$, in $\mathfrak{L}$ with the following properties:

(1) a real multiple $r\alpha$ of an element α in Δ is contained in Δ if and only if r equals ± 1;

(2) Δ generates $\mathrm{L}T$ as a real vector space;

(3) $\mathfrak{L} = \mathfrak{H} \oplus \bigoplus_{\alpha \in \Delta} \mathbf{C} X_\alpha$;

(4) $H + \Sigma_{\alpha \in \Delta} z_\alpha X_\alpha$, where $H \in \mathfrak{H}$ and $z_\alpha \in \mathbf{C}$ for all $\alpha \in \Delta$, is contained in $\mathrm{L}G$ if and only if $H \in \mathrm{L}T$ and $z_{-\alpha} = \overline{z_\alpha}$ for all $\alpha \in \Delta$;

(5) $[H, X_\alpha] = \mathrm{ad}(H)(X_\alpha) = \mathrm{i}(\alpha, H) X_\alpha$ for $H \in \mathrm{L}T$ and $\alpha \in \Delta$;

(6) $[X_\alpha, X_{-\alpha}] = \mathrm{i}\alpha$ for $\alpha \in \Delta$.

Lemma 6. *Let X be in* $\mathrm{L}T$ *such that* $(\alpha, X) \neq 0$ *for all* $\alpha \in \Delta$, *and* $(\alpha, X) \neq (\beta, X)$ *for all* $\alpha, \beta \in \Delta$ *with* $\alpha \neq \beta$. *Let* $Y = H + \Sigma_{\alpha \in \Delta} z_\alpha X_\alpha$ *be in* $\mathrm{L}G$ *with* $H \in \mathfrak{H}$ *and* $\overline{z_{-\alpha}} = z_\alpha \neq 0$ *for all* $\alpha \in \Delta$. *Then* $\mathrm{L}G$ *is the smallest real subalgebra of* $\mathrm{L}G$ *containing X and Y.*

The proof of Lemma 6 is omitted because it follows from the proof of [5, Theorem 1].

We continue with the proof of the proposition. Let Δ^* denote the set

$$\Delta^* := \{\alpha - \beta \mid \alpha, \beta \in \Delta, \alpha \neq \beta\}.$$

Then

$$D := \bigcup_{\gamma \in \Delta^*} \{X \in LT \mid (\gamma, X) = 0\} = \bigcup_{\gamma \in \Delta \cup \Delta^*} \{X \in LT \mid (\gamma, X) = 0\}$$

is a finite union of affine hyperspaces (of codimension 1) and hence a closed proper subset of LT. By Lemma 5, there exists A in the open non-void set $LT \setminus D$ and $t \in \mathbf{R}$, $t \neq 0$, with $\exp(tA) = u$. Since

$$E := \{H + \Sigma_{\alpha \in \Delta} z_\alpha X_\alpha \mid H \in LT,\ z_{-\alpha} = \overline{z_\alpha} \neq 0 \text{ for all } \alpha \in \Delta\}.$$

is an open non-void subset of LG, there exists (by Lemma 5) $B \in E$ and $t \in \mathbf{R}$, $t \neq 0$, with $\exp(tB) = v$. By Lemma 6, the smallest subalgebra of LG containing A and B is LG itself. Now we define

$$K_1 := \{\exp(sA) \mid s \in \mathbf{R}\}, \qquad K_2 := \{\exp(sB) \mid s \in \mathbf{R}\};$$

we have to show that K_1 and K_2 have the properties (i)–(iii) of the proposition.

(ii). Trivial.

(iii). Let M be the smallest closed subgroup of G containing K_1 and K_2. As a closed subgroup of G, M is an analytic subgroup. Let LM be the Lie algebra of M (considered as a subalgebra of LG). Clearly, A and B are contained in LM, hence $LM = LG$; therefore $M = G$.

(i). Let f denote the continuous homomorphism $s \to \exp(sA)$ from $\mathbf{R}$ onto K_1. Since ZG, as the centre of a compact, semisimple Lie group, is finite, the order of u is finite, say n. If t is a non-zero real number with $\exp(tA) = u$, then nt is contained in the kernel of f. Hence K_1 is either isomorphic to K or the zero-subgroup of G. The second case is impossible because G is generated by K_1 and K_2 and G is semisimple. Analogously, K_2 is isomorphic to K. □

Definition 7. Let G_1 and G_2 be compact connected groups, and let (f_1, g_1) (resp. (f_2, g_2)) be a pair of morphisms from K into G_1 (resp. G_2). Then (f_1, g_1) and (f_2, g_2) are called *equivalent* if there exists an isomorphism $h: G_1 \to G_2$ satisfying $hf_1 = f_2$ and $hg_1 = g_2$.

Remark 8. A pair of morphisms (f_1, g_1) from K into G_1 which is equivalent to an admissible pair (f_2, g_2) from K into G_2 is also admissible.

Remark 9. In the case $G_1 = G_2$ the relation defined above is an equivalence relation on the set of (admissible) pairs of morphisms from K into G_1.

Proposition 10. *Let G be a compact, connected, semisimple Lie group. Then the set of equivalence classes of admissible pairs of morphisms from K into G has the cardinality of the continuum.*

Proof. Clearly, this set has at most the cardinality of the continuum because the set of all continuous homomorphisms from K into G has the cardinality of the continuum.

Let Aut(G) be the group of all isomorphisms from G onto itself, let Inn(G) be the subgroup of all inner automorphisms, let $I: G \to \text{Inn}(G)$ be the canonical homomorphism, and let $\text{Ad}: G \to \text{Aut}(\text{L}G)$ be the adjoint representation. It is a well-known fact that one can introduce a unique topology on Aut(G) such that Aut(G) is a compact Lie group, Inn(G) is the component of the identity, and Inn(G) carries the quotient topology with respect to I.

We use the notation introduced in the proof of Proposition 4. Let $u = v$ be the unit element in G, let A be in LT with $(\alpha, A) \neq (\beta, A)$ for $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, and let $B = H + \Sigma_{\alpha \in \Delta} z_\alpha X_\alpha$; with $H \in \text{L}T$ and $0 \neq z_\alpha = \overline{z_{-\alpha}} \in \mathbf{C}$ for all $\alpha \in \Delta$. Assume that

$$K_1 = \{\exp(tA) \mid t \in \mathbf{R}\}, \qquad K_2 = \{\exp(tB) \mid t \in \mathbf{R}\}$$

are isomorphic to K, and let f_1 and f_2 be injective morphisms from K into G with $f_i(K) = K_i$ for $i = 1, 2$. As we have seen, (f_1, f_2) is an admissible pair.

Now we consider pairs of the form $(f_1, \rho f_2)$, $\rho \in \text{Aut}(G)$. If $\rho = I(g)$ is an inner automorphism, then the Lie algebra of $\rho f_2(K)$ equals $\mathbf{R} \cdot \text{Ad}(g)(B)$. From the proof of Proposition 4 it is clear that $(f_1, I(g) f_2)$ is an admissible pair if $\text{Ad}(g)(B)$ is contained in

$$\{H' + \Sigma_{\alpha \in \Delta} c_\alpha X_\alpha \mid H' \in \text{L}T \text{ and } 0 \neq c_\alpha = \overline{c_{-\alpha}} \text{ for all } \alpha \in \Delta\}.$$

Hence there exists an open connected nieghborhood U of the identity in G such that $(f_1, I(g) f_2)$ is an admissible pair for all $g \in U$.

Two automorphisms ρ and τ in Aut(G) are called equivalent if the pairs $(f_1, \rho f_2)$ and $(f_1, \tau f_2)$ are equivalent in the sense of Definition 7.

Let F denote the quotient space with respect to this equivalence relation, and let $\nu: \mathrm{Aut}(G) \to F$ denote the associated natural map. Let C_i, $i = 1, 2$, be the closed (and hence compact) subgroups $\{\rho \in \mathrm{Aut}(G) \mid \rho f_i = f_i\}$ of $\mathrm{Aut}(G)$. A simple computation shows that ρ and τ are equivalent if and only if $C_1 \rho C_2 = C_1 \tau C_2$. Hence F is a Hausdorff space, and ν is an open map. Consequently, $\nu(I(U))$ is a connected, locally compact Hausdorff space. Therefore it remains to show that $\nu(I(U))$ has at least two points.

Assume, to the contrary, that $\nu(I(U))$ consists of a single point or, equivalently, that $C_1 I(U) C_2 = C_1 C_2$. Especially, $C_1 C_2$ is an open and closed neighborhood of the identity in $\mathrm{Aut}(G)$. Let $D_i = C_i \cap \mathrm{Inn}(G)$ for $i = 1, 2$. $D_1 D_2$ is an open and closed subset of $\mathrm{Inn}(G)$, because D_i is of finite index in C_i. Since G and hence $\mathrm{Inn}(G)$ are connected, we obtain $\mathrm{Inn}(G) = D_1 D_2$ or $G = I^{-1}(D_1) I^{-1}(D_2) = Z(K_1) \cdot Z(K_2)$ where $Z(K_i)$ denotes the centralizer of K_i in G. The subgroups K_i are contained in maximal tori of G. Since maximal tori are conjugate (see [1, 4.23, p. 92]), there exist a maximal torus T in G and $g \in G$ such that $K_1 \subseteq T$ and $K_2 \subseteq g^{-1} T g$. Write g as $g = xy$ with $x \in Z(K_1)$ and $y \in Z(K_2)$. It is easy to see that K_1 and K_2 are contained in $x^{-1} T x$. Since G is generated by K_1 and K_2, it follows that $G = x^{-1} T x$, a contradiction to the assumption that G is semisimple. □

The next two lemmas are needed in the sequel.

Lemma 11 (see [6, p. 36]). *Let G and H be compact connected groups, and let $f: G \to H$ be a surjective morphism. If $Z_0(G)$ (resp. $Z_0(H)$) denotes the component of the identity in the centre of G (resp. H) then $Z_0(H) = f(Z_0(G))$.*

Proof. Clearly, $f(Z_0(G))$ is contained in $Z_0(H)$. By the structure theorem for compact connected groups, cited in the introduction, $G = G' Z_0(G)$ and $H = H' Z_0(H)$. Moreover, $H' \cap Z_0(H)$ is totally disconnected. Since f is surjective, we have $f(G') = H'$ and

$$H = f(G) = f(G') f(Z_0(G)) = H' f(Z_0(G)).$$

The embedding from $Z_0(H)$ into H induces a monomorphism from $Z_0(H)/f(Z_0(G))$ into $H/f(Z_0(G)) = H'/H' \cap f(Z_0(G))$. The image of this monomorphism is contained in $H' \cap Z_0(H)/H' \cap f(Z_0(G))$, which is totally disconnected because it is a quotient of a totally disconnected compact group (see e.g. [3, 1.19, p. 18]). This shows that $Z_0(H)/f(Z_0(G))$ is the trivial group and the lemma is proved. □

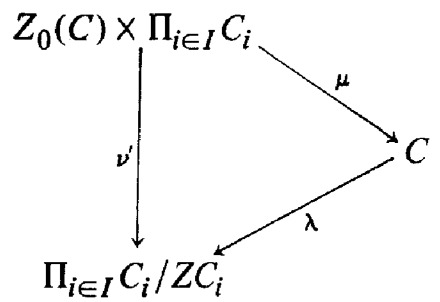
Lemma 12. *Let G and H be compact connected groups, and let $f: G \to H$ be a surjective morphism. Moreover, let H_1 and H_2 be closed connected subgroups of H, and let G_i be the component of the identity in $f^{-1}(H_i)$ for $i = 1, 2$. G is generated by G_1 and G_2 provided H is generated by H_1 and H_2.*

Proof. Let G^* denote the smallest closed subgroup of G containing G_1 and G_2. A surjective morphism $M \to N$ between compact groups maps the component of the identity of M onto the component of the identity of N. Hence $f(G_i) = H_i$ for $i = 1, 2$. Therefore $f(G^*) = H$ or $G = G^* \ker f$. For $i = 1, 2$ and $x \in \ker f$ we have $x f^{-1}(H_i) x^{-1} \subseteq f^{-1}(H_i)$ and therefore $x G_i x^{-1} \subseteq G_i$ and $x G^* x^{-1} \subseteq G^*$. Hence G^* is a normal subgroup of G. Since $\ker f$ is contained in $f^{-1}(H_1)$, the component of the identity in $\ker f$ is contained in $G_1 \subseteq G^*$, and, as a consequence, the group $\ker f / G^* \cap \ker f$ is totally disconnected. On the other hand, the connected space $G/G^* = G^* \ker f / G^*$ is homeomorphic to $\ker f / \ker f \cap G^*$. This shows that $G = G^*$ as desired. □

Recall (see the introduction) that $(\varphi, \psi: K \to C)$ denotes the coproduct of K with itself in the category of compact groups, $m: Z_0(G) \times C' \to C$ is the multiplication, $(C_i)_{i \in I}$ is a family of simply connected compact connected simple Lie groups, $\mu': \Pi_{i \in I} C_i \to C'$ is a surjective morphism with totally disconnected central kernel, and $\mu = m \cdot (1_{Z_0(C)} \times \mu'$. Moreover, let ZC_i denote the (finite) centre of C_i, let $\nu_i: C_i \to C_i / ZC_i$ denote the quotient morphism (the C_i / ZC_i are simple adjoint Lie groups), let ν be the product $\Pi_{i \in I} \nu_i$, and let

$$\nu': Z_0(C) \times \prod_{i \in I} C_i \to \prod_{i \in I} C_i / ZC_i$$

be defined by $\nu'(z, x) = \nu(x)$ for $z \in Z_0(C)$ and $x \in \Pi_{i \in I} C_i$. Since the kernel of μ is totally disconnected and hence central, there exists a unique morphism λ such that the diagram



commutes. Moreover, let

$$\tilde{\pi}_i : \prod_{i \in I} C_i \to C_i, \qquad \pi_i : \prod_{i \in I} C_i/ZC_i \to C_i/ZC_i$$

denote the projections, and let

$$\varphi_i := \pi_i \lambda \varphi, \qquad \psi_i := \pi_i \lambda \psi.$$

We fix these notations for the rest of the paper.

Proposition 13. *For all $i \in I$, (φ_i, ψ_i) is an admissible pair of morphisms from K into C_i/ZC_i. Moreover, for any compact connected simple adjoint Lie group H and any admissible pair (σ, τ) of morphisms from K into H there exists a unique $i \in I$ such that (φ_i, ψ_i) and (σ, τ) are equivalent.*

Corollary 14. *Let $(G_j)_{j \in J}$ be a family of compact connected simple adjoint Lie groups, and let (f_j, g_j), $j \in J$, be admissible pairs of morphisms from K into G_j. Let G be the product of the G_j, let $p_j : G \to G_j$ denote the projections, and let $f, g : K \to G$ be the unique morphisms satisfying $p_j f = f_j$ and $p_j g = g_j$ for all j. Then*

(a) *(f, g) is an admissible pair*

if and only if

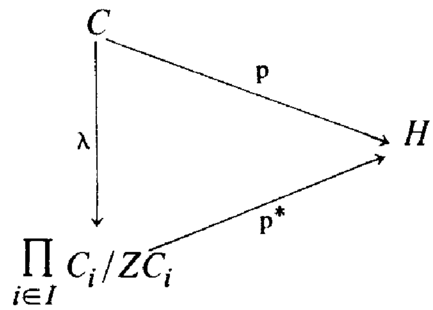
(b) *the (f_j, g_j) are pairwise nonequivalent.*

Proof. First we show (a) ⇒ (b) of the corollary. Assume that (b) is false. Then there exist $i, j \in J$, $i \neq j$, and an isomorphism $h : G_i \to G_j$ with $hf_i = f_j$ and $hg_i = g_j$. Obviously,

$$\{(x, h(x)) \mid x \in G_i\} \times \prod_{\substack{k \in I \\ k \neq i, j}} G_k$$

is a proper closed subgroup of G containing $f(K)$ and $g(K)$, a contradiction to (a).

Since $(\lambda\varphi, \lambda\psi)$ and the (φ_i, ψ_i) are admissible pairs, the implication (a) ⇒ (b) implies that the (φ_i, ψ_i), $i \in I$, are pairwise non-equivalent. It remains to show that there exists an index i such that (σ, τ) and (φ_i, ψ_i) are equivalent. By definition of the coproduct there exists a unique surjective morphism $p : C \to H$ with $p\varphi = \sigma$ and $p\psi = \tau$. Moreover, there exists a unique surjective morphism p^* such that the diagram



commutes (kerλ is central and the centre of H is trivial). Let C_i^*, $i \in I$, denote the connected normal subgroup

$$\{x \in \Pi_{i\in I} C_i/ZC_i \mid \pi_k(x) = 1 \text{ for all } k \in I\setminus\{i\}\}$$

of $\Pi_{i\in I} C_i/ZC_i$. Then $p^*(C_i^*)$ is a connected normal subgroup of the simple Lie group H. Moreover, $p^*(C_i^*)$ centralizes $p^*(C_k^*)$ for $i, k \in I$, $i \neq k$. This shows that there is a unique index $i \in I$ such that $p^*(C_i^*) = H$ and $p^*(C_k^*) = \{1\}$ for $k \in I\setminus\{i\}$. Since C_i^* is a simple adjoint group, p^* induces an isomorphism from C_i^* onto H, say h. It is clear that $h\varphi_i = \sigma$ and $h\varphi_i = \tau$. Thus (φ_i, ψ_i) and (σ, τ) are equivalent and the proposition is proved.

The implication (b) ⇒ (a) of the corollary follows from the proposition. □

In the sequel, the (discrete) Pontryagin character group of a compact abelian group A is denoted by $A^\wedge$.

Proposition 15. *$Z_0(C)^\wedge$ is isomorphic to $\mathbf{Q}^2$. C/C' is isomorphic to K^2.*

Proof. For $n \geqslant 2$ the group SU($n!$) is a simply connected compact connected simple Lie group. The centre ZSU($n!$) of SU($n!$) is a cyclic group of order $n!$; let a_n be a fixed generator of ZSU($n!$), and let e_n be the unit element in SU($n!$). According to Proposition 4 there is an admissible pair $(\check{f}_n, \check{g}_n)$ of injective morphisms from K into SU($n!$) × SU($n!$) with $(a_n, e_n) \in \check{f}_n(K)$ and $(e_n, a_n) \in \check{g}_n(K)$. Let

$$\tilde{G}_n := \mathrm{SU}(n!)^2 \times \ldots \times \mathrm{SU}(2!)^2 \times K^2,$$

and let $\tilde{f}_n, \tilde{g}_n : K \to \tilde{G}_n$ be defined by

$$\tilde{f}_n(x) = (\check{f}_n(x), \ldots, \check{f}_k(x)^{n!/k!}, \ldots, \check{f}_2(x)^{n\cdot\ldots\cdot 3}, x, 1)$$

$$\tilde{g}_n(x) = (\check{g}_n(x), \ldots, \check{g}_k(x)^{n!/k!}, \ldots, \check{g}_2(x)^{n\cdot\ldots\cdot 3}, 1, x)\,.$$

$(\tilde{f}_n, \tilde{g}_n)$ is an admissible pair, for: Let H be the smallest closed subgroup of $\tilde{G}_n$ containing $\tilde{f}_n(K)$ and $\tilde{g}_n(K)$. Moreover, let

$$p:\tilde{G}_n \to \prod_{k=2}^{n} \mathrm{SU}(k!)^2$$

be the projection, and let

$$q:\prod_{k=2}^{n} \mathrm{SU}(k!)^2 \to \prod_{k=2}^{n} (\mathrm{SU}(k!)/Z\,\mathrm{SU}(k!))^2$$

be the quotient morphism. By Corollary 14, $(qp\tilde{f}_n, qp\tilde{g}_n)$ is an admissible pair. By Lemma 12, $\Pi_{k=2}^{n}\mathrm{SU}(k!)^2$ is generated by $[q^{-1}(qp\tilde{f}_n(K))]_0 = p\tilde{f}_n(K)$ and $p\tilde{g}_n(K)$. Consequently, $p(H) = \Pi_{k=2}^{n}\mathrm{SU}(k!)^2$. Since the latter group is semisimple (hence perfect), we obtain $p(H') = \Pi_{k=2}^{n}\mathrm{SU}(k!)^2$. Now it is easy to see that $H = \tilde{G}_n$ as desired.

Since $\check{f}_n$ is injective, $\check{f}_n^{-1}(a_n, e_n)$ is an element of order $n!$ in K and therefore a generator of the group $\mathrm{Tor}_{n!}(K)$ of roots of unity of order $n!$. Hence

$$\check{f}_n(\mathrm{Tor}_{n!}(K)) = \{(y, e_n) \mid y \in Z\mathrm{SU}(n!)\}$$

and analogously

$$\check{g}_n(\mathrm{Tor}_{n!}(K)) = \{(e_n, y) \mid y \in Z\mathrm{SU}(n!)\}.$$

Define

$$D_n := \tilde{f}_n(\mathrm{Tor}_{n!}(K))\tilde{g}_n(\mathrm{Tor}_{n!}(K)).$$

D_n is a finite central subgroup of $\tilde{G}_n$. Let

$$G_n := \tilde{G}_n/D_n$$

and let $\rho_n: \tilde{G}_n \to G_n$ be the quotient morphism. For $n \geqslant 3$, let $\tilde{p}_n: \tilde{G}_n \to \tilde{G}_{n-1}$ be defined by

$$\tilde{p}_n(x_n, \ldots, x_2, x, y) = (x_{n-1}, \ldots, x_2, x^n, y^n)$$

for $x, y \in K$ and $x_j \in \mathrm{SU}(j!)^2$. Because $\tilde{p}_n(D_n) = D_{n-1}$ (as is easy to see), there exists a unique surjective morphism p_n such that the diagram

$$\begin{array}{ccc} \tilde{G}_n & \xrightarrow{\tilde{p}_n} & \tilde{G}_{n-1} \\ \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho_{n-1}} \\ G_n & \xrightarrow{p_n} & G_{n-1} \end{array}$$

commutes. One verfies easily that $\tilde{p}_n(\tilde{f}_n(K)) = \tilde{f}_{n-1}(K)$ and analogously $\tilde{p}_n(\tilde{g}_n(K)) = \tilde{g}_{n-1}(K)$. This implies

$$\rho_{n-1}\tilde{f}_{n-1}(K) = \rho_{n-1}\tilde{p}_n\tilde{f}_n(K) = p_n\,\rho_n\tilde{f}_n(K),$$

$$\rho_{n-1}\tilde{g}_{n-1}(K) = p_n\,\rho_n\tilde{g}_n(K).$$

Therefore p_n induces a surjective morphism from $\rho_n\tilde{g}_n(K)$ (resp. $\rho_n\tilde{f}_n(K)$) onto $\rho_{n-1}\tilde{g}_{n-1}(K)$ (resp. $\rho_{n-1}\tilde{f}_{n-1}(K)$). A simple computation shows that these morphisms are injective. Choose injective morphisms f_2 and g_2 from K into G_2 with $f_2(K) = \rho_2\tilde{f}_2(K)$ and $g_2(K) = \rho_2\tilde{g}_2(K)$, and then for $n \geqslant 3$ choose injective morphisms f_n, g_n from K into G_n satisfying

$$f_n(K) = \rho_n\tilde{f}_n(K), \qquad g_n(K) = \rho_n\tilde{g}_n(K),$$

$$p_3 \ldots p_n f_n = f_2, \qquad p_3 \ldots p_n g_n = g_2.$$

For $n \geqslant 3$, the f_n and g_n are uniquely defined by these properties. Moreover, $p_n f_n = f_{n-1}$ and $p_n g_n = g_{n-1}$ hold for $n \geqslant 3$. (f_n, g_n) is admissible because $(\tilde{f}_n, \tilde{g}_n)$ is admissible. Therefore, for each $n \geqslant 2$, there exists a unique surjective morphism $q_n : C \to G_n$ with $q_n\varphi = f_n$ and $q_n\psi = g_n$. The equations

$$(p_n q_n)\varphi = p_n f_n = f_{n-1} = q_{n-1}\varphi$$

and

$$(p_n q_n)\psi = g_{n-1} = q_{n-1}\psi$$

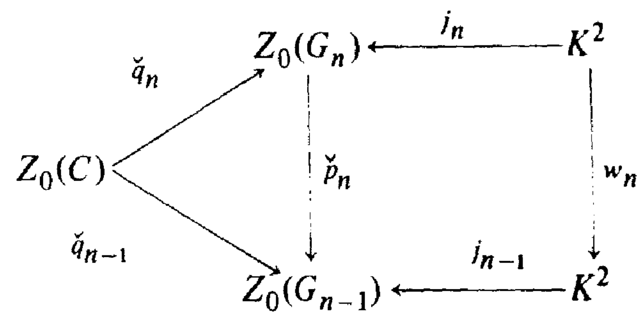
imply $p_n q_n = q_{n-1}$ for $n \geqslant 3$. By Lemma 11, the p_n and q_n induce surjective morphisms $\check{p}_n : Z_0(G_n) \to Z_0(G_{n-1})$ and $\check{q}_n : Z_0(C) \to Z_0(G_n)$. For $n \geqslant 3$, $\check{p}_n\check{q}_n = \check{q}_{n-1}$ holds. Moreover,

$$Z_0(G_n) = \rho_n(Z_0(\tilde{G}_n)) = \rho_n(\{(e_n, e_n, \ldots, e_2, e_2, x, y) \mid x, y \in K\}).$$

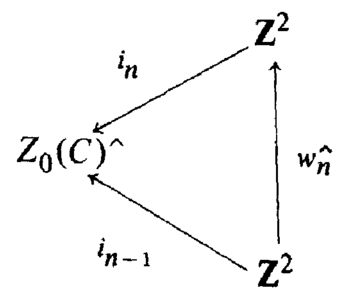
Therefore $j_n : K^2 \to Z_0(G_n)$ defined by

$$j_n(x, y) = \rho_n(e_n, e_n, \ldots, e_2, e_2, x, y)$$

is a surjective morphism from K^2 onto $Z_0(G_n)$. A simple computation shows that j_n is an isomorphism. Let $w_n : K^2 \to K^2$ be defined by $w_n(x, y) = (x^n, y^n)$. The diagrams



commute for $n \geqslant 3$. Apply the Pontryagin duality functor. The character group of K^2 is $\mathbf{Z}^2$. The dual map $w_n^{\wedge} : \mathbf{Z}^2 \to \mathbf{Z}^2$ of w_n is given by $w_n^{\wedge}(x, y) = (nx, ny)$. Since $j_n^{-1} \check{q}_n$ is an epimorphism, the dual map i_n of $j_n^{-1} \check{q}_n$ is a monomorphism. We obtain commutative diagrams



for $n \geqslant 3$. Let

$$A := \bigcup_{n=2}^{\infty} i_n(\mathbf{Z}^2) .$$

A is a subgroup of $Z_0(C)^{\wedge}$ since $i_{n-1}(\mathbf{Z}^2)$ is contained in $i_n(\mathbf{Z}^2)$ for $n \geqslant 3$. Let $\delta : A \to \mathbf{Q}^2$ be defined in the following way: if $a \in A$ and $a = i_n(x, y)$ for some $n \geqslant 2$, then $\delta(a) = (x/n!, y/n!)$. It is easy to see that δ is well-defined, and that δ is an isomorphism. It remains to show $A = Z_0(C)^{\wedge}$. This will follow from:

Lemma 16. *Let B be a compact connected abelian group, and let T be a totally disconnected closed subgroup of B such that B/T is isomorphic to K^n. Then $B^{\wedge}$ is isomorphic to a subgroup of $\mathbf{Q}^n$.*

Proof. Dualizing the exact sequence

$$0 \to T \to B \to B/T \to 0,$$

we get the exact sequence of discrete abelian groups

$$0 \to \mathbf{Z}^n \to B^\wedge \to T^\wedge \to 0.$$

Since $\mathbf{Q}$ is a flat $\mathbf{Z}$-module, the sequence

$$0 \to \mathbf{Z}^n \otimes_{\mathbf{Z}} \mathbf{Q} \to B^\wedge \otimes_{\mathbf{Z}} \mathbf{Q} \to T^\wedge \otimes_{\mathbf{Z}} \mathbf{Q} \to 0$$

is also exact. $T^\wedge \otimes_{\mathbf{Z}} \mathbf{Q}$ is zero because $T^\wedge$ is the character group of a totally disconnected, compact group and hence a torsion group. Hence $B^\wedge \otimes_{\mathbf{Z}} \mathbf{Q}$ is isomorphic to $\mathbf{Z}^n \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}^n$. Since $B^\wedge$ is torsion free, there exists a monomorphism from $B^\wedge$ into $B^\wedge \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}^n$, and the lemma is proved. □

Let $d: C \to C/C'$ denote the quotient morphism. Obviously, $(d\varphi, d\psi: K \to C/C')$ is the coproduct of K with itself in the category of compact abelian groups. Therefore C/C' is isomorphic to K^2 (and the second part of the proposition is proved). By Lemma 11, d induces a surjective morphism from $Z_0(C)$ onto C/C', say d^*. Since the kernel of d^* is totally disconnected (by the structure theorem for compact connected groups), $Z_0(C)^\wedge$ is isomorphic to a subgroup of $\mathbf{Q}^2$ (Lemma 16). As we have already seen, $Z_0(C')^\wedge$ contains a subgroup (namely A) which is isomorphic to $\mathbf{Q}^2$. The above facts imply $A = Z_0(C)^\wedge$. □

For further analysis we need the following lemma.

Lemma 17. *Assume that*

(a) *H and A are compact connected groups,*

(b) *A is abelian and $A^\wedge$ is isomorphic to $\mathbf{Q}^2$,*

(c) *L and M are closed abelian subgroups of $A \times H$ generating $A \times H$,*

(d) *$M^\wedge$ and $L^\wedge$ are isomorphic to subgroups of $\mathbf{Q}$.*

If p_1 (resp. p_2) denotes the projection $A \times H \to A$ (resp. $A \times H \to H$), then the following assertions are valid:

(i) *$p_1(M) \cap p_1(L) = \{0\}$, $A = p_1(M)p_1(L)$.*

(ii) *$L^\wedge$ and $M^\wedge$ are isomorphic to $\mathbf{Q}$.*

(iii) *p_1 induces isomorphisms from M onto $p_1(M)$ and from L onto $p_1(L)$.*

(iv) *Let α (resp. β) denote the inverse of the isomorphism from M onto $p_1(M)$ (resp. from L onto $p_1(L)$) induced by p_1. Let i_M (resp. i_L) denote the embedding from M (resp. L) into $A \times H$, and let $u := p_2 i_M \alpha$ and $v := p_2 i_L \beta$. Then*

$$M = \{(x, u(x)) \mid x \in p_1(M)\},$$

$$L = \{(x, v(x)) \mid x \in p_1(L)\}.$$

The simple proof of this lemma is omitted. We apply the lemma by choosing $A = Z_0(C)$, $H = \Pi_{i \in I} C_i$, $M = [\mu^{-1}(\varphi(K))]_0$, and $L = [\mu^{-1}(\psi(K))]_0$. By Lemma 12, $A \times H$ is generated by M and L. Moreover, M and L are abelian since the commutator subgroups M' and L' are connected and contained in the totally disconnected group $\ker\mu$. From the exact sequence

$$0 \to M \cap \ker\mu \to M \to \varphi(K) \to 0$$

we get by Lemma 16 that $M^\wedge$ is isomorphic to a subgroup of $\mathbf{Q}$; the analogous statement holds for $L^\wedge$. Hence in this case all assumptions of Lemma 17 are satisfied.

In the sequel we use the assertions (i)–(iv) and the notations introduced in Lemma 17.

Proposition 18. (i) $\ker\mu = (\ker\mu \cap M)(\ker\mu \cap L)$ $= \{(xy, u(x) \cdot v(y)) \mid x \in p_1(\ker\mu \cap M), y \in p_1(\ker\mu \cap L)\}$.

(ii) $\mu' : \Pi_{i \in I} C_i \to C'$ *is an isomorphism.*

Proof. (i) $F := (\ker\mu \cap M)(\ker\mu \cap L)$ is a closed central subgroup of $Z_0(C) \times \Pi_{i \in I} C_i$. Let $P := (Z_0(C) \times \Pi_{i \in I} C_i)/F$, and let $r := Z_0(C) \times \Pi_{i \in I} C_i \to P$ denote the quotient morphism. There exists a unique surjective morphism $s : P \to C$ with $sr = \mu$. We claim that s is an isomorphism. Since $\varphi(K) = \mu(M) = s(r(M))$, s induces a surjective morphism from $r(M)$ onto $\varphi(K)$. Since $\ker\mu \cap M$ is contained in F, this morphism is also injective. Hence there exist morphisms $\varphi', \psi' : K \to P$ satisfying $\varphi'(K) = r(M)$, $\psi'(K) = r(L)$, $s\varphi' = \varphi$ and $s\psi' = \psi$. The universal property of the coproduct guarantees the existence of a morphism $s' : C \to P$ with $s'\varphi = \varphi'$ and $s'\psi = \psi'$. s' is surjective because P is generated by $r(M) = \varphi'(K)$ and $\psi'(K)$. The equations $(ss')\varphi = s\varphi' = \varphi$ and $(ss')\psi = \psi$ imply $ss' = 1_C$. Hence the surjectivity of s' implies that s is an isomorphism. This proves the first assertion.

By Lemma 17,

$$\ker\mu \cap M = \{(x, u(x)) \mid x \in p_1(M \cap \ker\mu)\}$$

$$\ker\mu \cap L = \{(x, v(x)) \mid x \in p_1(L \cap \ker\mu)\},$$

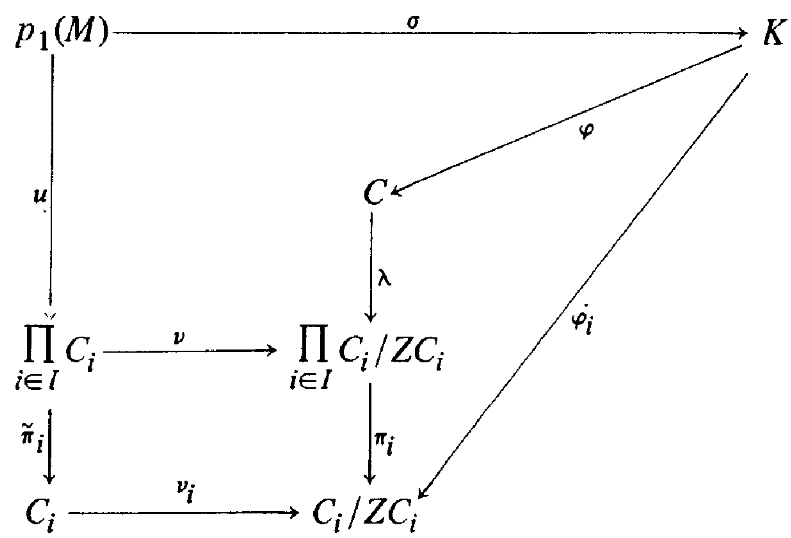
and hence the second equation holds.

(ii) Since $p_1(M) \cap p_1(L) = \{0\}$ and, in particular,

$$p_1(M \cap \ker\mu) \cap p_1(L \cap \ker\mu) = \{0\},$$

this description of $\ker\mu$ shows that the kernel of μ' is trivial. □

Proposition 19. *There exist unique surjective morphisms $\sigma : p_1(M) \to K$ and $\tau : p_1(M) \to K$ such that the diagram*

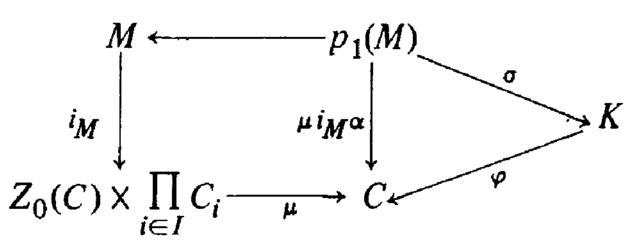


and the analogous diagram with τ, υ and ψ commute. Moreover,

$$\ker\sigma = p_1(\ker\mu \cap M),$$

$$\ker\tau = p_1(\ker\mu \cap L).$$

Proof. The uniqueness is obvious. Since φ is injective and $\mu i_M \alpha(p_1(M)) = \mu(M) = \varphi(K)$, there exists a surjective morphism σ such that



commutes. It is easy to see that σ has the property asserted in the proposition. Moreover,

$$\ker\sigma = \ker(\mu i_M \alpha) = (i_M \alpha)^{-1}(\ker\mu) = \alpha^{-1}(M \cap \ker\mu)$$
$$= p_1(M \cap \ker\mu)$$

by the definition of α. □

Summarizing our results, we see that the coproduct of K with itself in the category of compact groups can be described as follows: Let

$$\left\{ K \underset{\psi_i}{\overset{\varphi_i}{\rightrightarrows}} G_i \mid i \in I \right\}$$

be a family of admissible pairs (φ_i, ψ_i) of morphisms from K into compact connected simple adjoint Lie groups G_i such that for each admissible pair (γ, δ) of morphisms from K into some compact connected simple adjoint Lie group there exists precisely one $i \in I$ such that (φ_i, ψ_i) is equivalent to (γ, δ) in the sense of Definition 7. Then for every compact connected simple adjoint Lie group H the set of indices $i \in I$ such that H is isomorphic to G_i has the cardinality of the continuum. Moreover, I has the cardinality of the continuum (since there are only countably many non-isomorphic compact connected simple adjoint Lie groups).

For $i \in I$, let $\nu_i : \tilde{G}_i \to G_i$ denote the universal covering. Moreover, let $\sigma : \mathbf{Q}^\wedge \to K$ be an arbitrary surjective morphism (with totally disconnected kernel S). For each $i \in I$ there exist unique morphisms u_i and v_i (use Pontryagin duality) such that the diagrams

$$\begin{array}{ccc} \tilde{G}_i & \xrightarrow{\nu_i} & G_i \\ {\scriptstyle u_i}\uparrow & & \uparrow{\scriptstyle \varphi_i} \\ \mathbf{Q}^\wedge & \xrightarrow{\sigma} & K \end{array} \qquad \begin{array}{ccc} \tilde{G}_i & \xrightarrow{\nu_i} & G_i \\ {\scriptstyle v_i}\uparrow & & \uparrow{\scriptstyle \psi_i} \\ \mathbf{Q}^\wedge & \xrightarrow{\sigma} & K \end{array}$$

commute. Let

$$u = \prod_{i \in I} u_i : \mathbf{Q}^\wedge \to \prod_{i \in I} \tilde{G}_i,$$

$$v = \prod_{i \in I} v_i : \mathbf{Q}^\wedge \to \prod_{i \in I} \tilde{G}_i,$$

$$G = \mathbf{Q}^\wedge \times \mathbf{Q}^\wedge \times \prod_{i \in I} \tilde{G}_i,$$

$$D = \{(x, y, u(x)v(y)) \mid x, y \in S\} \subseteq G.$$

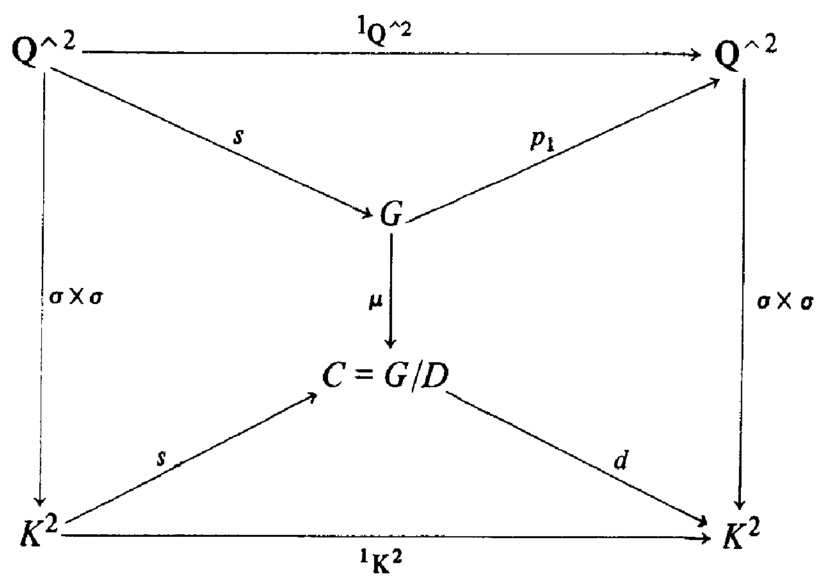
Then D is a closed totally disconnected central subgroup of G. Let $\mu : G \to G/D$ denote the quotient morphism. Moreover, let

$\epsilon, \eta : \mathbf{Q}^\wedge \to G/D$ be defined by

$$\epsilon(x) = \mu(x, 0, u(x)), \qquad \eta(x) = \mu(0, x, v(x)).$$

There exist unique morphisms $\varphi, \psi : K \to G/D$ with $\varphi\sigma = \epsilon$ and $\psi\sigma = \eta$. Then G/D together with the morphisms φ, ψ is the coproduct of K with itself in the category of compact groups. Moreover, the map $(x, y) \to u(x)v(y)$ from S^2 into $\Pi_{i\in I}\tilde{G}_i$ is injective and μ induces an isomorphism from $\mathbf{Q}^{\wedge 2}$ onto $Z_0(G/D)$.

Recently, K.H. Hofmann has shown that every compact connected group H splits over its commutator subgroup H'. In the following, we shall briefly describe this splitting in our special situation that $H = C$ is the coproduct of K with itself. The homomorphism $(x, y) \to u(x)v(y)$ from S^2 into $\Pi_{i\in I}\tilde{G}_i$ takes values in the centre of $\Pi_{i\in I}\tilde{G}_i$ and hence in $\Pi_{i\in I}T_i$ if T_i denotes a maximal torus in $\tilde{G}_i$. Since the character group of $\Pi_{i\in I}T_i$ is free, there exists a morphism $t : \mathbf{Q}^{\wedge 2} \to \Pi_{i\in I}\tilde{G}_i$ with $t(x, y) = u(x)v(y)$ for $x, y \in S$. Let $p_1 : G \to \mathbf{Q}^{\wedge 2}$ denote the first projection, and let $\tilde{s}(x, y) = (x, y, t(x, y))$. Then there exist unique morphisms s and d such that the diagram



commutes. C is the semidirect product of $s(K^2) \cong K^2$ and $\ker d = C' \cong \Pi_{i\in I}\tilde{G}_i$. The space C is homeomorphic to $K^2 \times \Pi_{i\in I}\tilde{G}_i$.

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