PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

DECOMPOSITION OF TENSOR PRODUCTS OF IRREDUCIBLE UNITARY REPRESENTATIONS

DETLEV POGUNTKE

ABSTRACT. It is shown that the tensor product of an irreducible unitary representation of a (discrete) group G and an n-dimensional ($n < \infty$) unitary representation of G decomposes into at most n^2 irreducible subrepresentations; the multiplicity of each irreducible constituent is not greater than n. As an application it is shown that the restriction of an irreducible unitary representation to a subgroup of finite index is a finite sum of irreducible subrepresentations.

At the meeting "Harmonische Analyse und Darstellungstheorie lokalkompakter Gruppen" in Oberwolfach, R. Howe posed the following question: Let G be a (discrete) group and let π and ρ be irreducible unitary representations of G with $\dim \rho =: n < \infty$. Is $\rho \otimes \pi$ a sum of at most n^2 irreducible subrepresentations? He proved that the answer is yes if π is finite dimensional, too. In this paper it is shown that the answer is yes in the general case. Indeed, we will show a little bit more, namely:

Theorem. *Let π and π' be irreducible unitary representations of the group G in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{H}'$, respectively, and let ρ be a unitary representation of G in a Hilbert space of dimension $n < \infty$. Then the dimension of the space $\mathrm{Hom}_G(\pi', \rho \otimes \pi)$ of intertwining operators is not greater than n and is equal to n iff $\rho \otimes \pi$ is unitarily equivalent to $n\pi'$.*

Corollary 1. *Let $G, \rho, \pi, \mathfrak{H}$ be as in the Theorem. Then the dimension of the algebra $\mathrm{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ is not greater than n^2 and is equal to n^2 iff $\rho^* \otimes \rho \otimes \pi$ is unitarily equivalent to $n^2\pi$ (ρ^* denotes the contragradient representation). Especially, $\rho \otimes \pi$ is the direct sum of at most n^2 irreducible subrepresentations.*

Corollary 2. *Let $G, \pi, \mathfrak{H}$ be as in the Theorem and let N be a subgroup of G of finite index. Then the restriction of π to N is a finite sum of irreducible subrepresentations. If N is a normal subgroup the dimension of the algebra $\mathrm{Hom}_N(\pi, \pi)$ is not greater than the index $[G : N]$.*

Corollary 2 was used in the proof of Proposition 2.1 in [1] for locally compact groups of type I.

The basic idea in the proof of the Theorem is the introduction of an inner

Received by the editors August 8, 1974.
AMS (MOS) subject classifications (1970). Primary 22D10.
Key words and phrases. Irreducible unitary representations of groups, tensor products of representations, intertwining operators.

product $\langle\langle\ ,\ \rangle\rangle$ in $\mathrm{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ which is totally algebraic in nature. At the end of the paper, we will give an example of an n-dimensional (for all integers $n > 1$) irreducible representation ρ such that $\rho \otimes \rho^*$ is a sum of n^2 one-dimensional subrepresentations.

Let V be a finite dimensional Hilbert space and let $\mathfrak{H}$ be an arbitrary Hilbert space. The algebraic tensor product $V \otimes \mathfrak{H}$ becomes a Hilbert space if we define $\langle w \otimes k, v \otimes h\rangle = \langle w, v\rangle\langle k, h\rangle$. If $e_1, \ldots, e_n$ is an orthonormal basis of V, every element in $V \otimes \mathfrak{H}$ can be represented uniquely in the form $\Sigma_{i=1}^n e_i \otimes h_i$ and, by definition of the inner product, we get

$$\left\langle \sum_{i=1}^n e_i \otimes h_i, \sum_{i=1}^n e_i \otimes k_i \right\rangle = \sum_{i=1}^n \langle h_i, k_i\rangle.$$

Now, we indicate how Corollary 1 (the answer to Howe's question) follows from the Theorem.

Let $\mathfrak{H}'$ be another Hilbert space and let V^* denote the dual space of V. Then the spaces of bounded operators $\mathrm{Hom}(V \otimes \mathfrak{H}, \mathfrak{H}')$ and $\mathrm{Hom}(\mathfrak{H}, V^* \otimes \mathfrak{H}')$ are canonical isomorphic. The isomorphism $J : \mathrm{Hom}(V \otimes \mathfrak{H}, \mathfrak{H}') \to \mathrm{Hom}(\mathfrak{H}, V^* \otimes \mathfrak{H}')$ is given by $(JT)h = \Sigma_{i=1}^n e_i^* \otimes T(e_i \otimes h)$ if $e_1^*, \ldots, e_n^*$ is the dual basis of $e_1, \ldots, e_n$; JT is independent of the basis $e_1, \ldots, e_n$. J is not an isometry for the operator norms, but we have

Lemma 1. $n^{-1/2}\|T\| \le \|JT\| \le n^{1/2}\|T\|$ *for all* $T \in \mathrm{Hom}(V \otimes \mathfrak{H}, \mathfrak{H}')$. *Moreover, let G be a group, and let ρ, π and π' be unitary representations of G in V, $\mathfrak{H}$ and $\mathfrak{H}'$, respectively. Then J transforms the space of intertwining operators* $\mathrm{Hom}_G(\rho \otimes \pi, \pi')$ *onto* $\mathrm{Hom}_G(\pi, \pi^* \otimes \pi')$.

The simple proof of this lemma is omitted. For $\pi' = \rho \otimes \pi$ we get $\mathrm{Hom}_G(\rho \otimes \pi, \rho \otimes \pi) \cong \mathrm{Hom}_G(\pi, (\rho^* \otimes \rho) \otimes \pi)$ (Theorem $\Rightarrow$ Corollary 1).

Now, for a unitary representation ρ in a finite dimensional Hilbert space V and an irreducible unitary representation π in a Hilbert $\mathfrak{H}$, we introduce an inner product $\langle\langle\ ,\ \rangle\rangle$ in $\mathrm{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ which is crucial in the proof of the Theorem (we will use that inner product for different ρ's and π's and will denote it by the same symbol $\langle\langle\ ,\ \rangle\rangle$). We will compute $\langle\langle TT^*, TT^*\rangle\rangle$ for $T \in \mathrm{Hom}_G(\pi', \rho \otimes \pi)$ in two different ways. The inner product is defined as follows:

If $S, T \in \mathrm{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ then $JS, JT \in \mathrm{Hom}_G(\pi, \rho^* \otimes \rho \otimes \pi)$ and $(JS)^* JT \in \mathrm{Hom}_G(\pi, \pi)$. Since π is irreducible, that operator is a scalar multiple of the identity, and we define

$$\langle\langle T, S\rangle\rangle\, \mathrm{id}_{\mathfrak{H}} = (JS)^*(JT).$$

Lemma 2. (i) *$\langle\langle\ ,\ \rangle\rangle$ is linear in the first and conjugate-linear in the second variable.*

(ii) $\langle\langle S, S\rangle\rangle = \|J(S)\|^2$, *especially*, $\langle\langle S, S\rangle\rangle = 0$ *iff* $S = 0$.

(iii) $\langle\langle \mathrm{id}_{V\otimes\mathfrak{H}}, \mathrm{id}_{V\otimes\mathfrak{H}}\rangle\rangle = \dim V$.

(iv) $\langle\langle S, T\rangle\rangle = \overline{\langle\langle T, S\rangle\rangle}$.

(v) *Let* $e_1, \ldots, e_n$ *be an orthonormal basis of* V, *and let* $S, T \in \mathrm{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ *be given by*

$$S(e_i \otimes h) = \sum_{k=1}^{n} e_k \otimes S_{ki}(h) \quad \textit{and} \quad T(e_i \otimes h) = \sum_{k=1}^{n} e_k \otimes T_{ki}(h)$$

for $h \in \mathfrak{H}$ *and* $1 \leq i \leq n$. *Then the equation* $\langle\langle T, S\rangle\rangle \mathrm{id}_{\mathfrak{H}} = \sum_{i,k=1}^{n} S_{ik}^* T_{ik}$ *holds.*

Proof. (i) and (ii) are clear, (iii) and (iv) follow from (v). Let $e_1^*, \ldots, e_n^*$ be the dual basis of $e_1, \ldots, e_n$. By definition, we have $(JS)(h) = \sum_{i,k=1}^{n} e_i^* \otimes e_k \otimes S_{ki}(h)$. A simple computation shows that

$$(JS)^*\left(\sum_{i,k=1}^{n} e_i^* \otimes e_k \otimes h_{ki}\right) = \sum_{i,k=1}^{n} S_{ki}^* h_{ki}$$

and therefore,

$$(JS)^*(JT) = \sum_{i,k=1}^{n} S_{ik}^* T_{ik}.$$

Definition. Let ρ and π be as before. Let π' be another irreducible unitary representation of G in $\mathfrak{H}'$ and let $T \in \mathrm{Hom}_G(\pi', \rho \otimes \pi)$. The operator $T^* \in \mathrm{Hom}_G(\rho \otimes \pi, \pi')$ corresponds to an operator in $\mathrm{Hom}_G(\pi, \rho^* \otimes \pi')$ (by Lemma 1). This operator is denoted by T^a. Explicitly: if $e_1, \ldots, e_n$ is an orthonormal basis of V with dual basis $e_1^*, \ldots, e_n^*$ and T is represented as

$$Th' = \sum_{i=1}^{n} e_i \otimes T_i h', \quad \text{then} \quad T^a h = \sum_{i=1}^{n} e_i^* \otimes T_i^* h.$$

Lemma 3. *Let* $0 \neq T \in \mathrm{Hom}_G(\pi', \rho \otimes \pi)$. *Then* $T^*T \in \mathrm{Hom}_G(\pi', \pi')$, $(T^a)^* T^a \in \mathrm{Hom}_G(\pi, \pi)$, $TT^* \in \mathrm{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ *and* $T^a (T^a)^* \in \mathrm{Hom}_G(\rho^* \otimes \pi', \rho^* \otimes \pi')$. *Let the positive real numbers* α *and* β *be defined by* $T^*T = \alpha\, \mathrm{id}_{\mathfrak{H}'}$, *and* $(T^a)^* T^a = \beta\, \mathrm{id}_{\mathfrak{H}}$, *respectively* ($\pi'$ *and* π *are irreducible*). *Then one has*

$$\langle\langle TT^*, TT^*\rangle\rangle = \alpha\beta = \langle\langle T^a (T^a)^*, T^a (T^a)^*\rangle\rangle.$$

Moreover, let $S : \mathfrak{H}' \to V \otimes \mathfrak{H}$ *be another intertwining operator such that the ranges of* S *and* T *are orthogonal. Then one has* $\langle\langle SS^*, TT^*\rangle\rangle = 0 = \langle\langle S^a (S^a)^*, T^a (T^a)^*\rangle\rangle$.

Proof. The first statements are clear. Let $e_1, \ldots, e_n$ be an orthonormal basis of V and let T be given by $Th' = \sum_{i=1}^{n} e_i \otimes T_i h'$. Then

$$T^*T = \sum_{i=1}^{n} T_i^* T_i = \alpha\, \mathrm{id}_{\mathfrak{H}'}, \qquad (T^a)^* T^a = \sum_{i=1}^{n} T_i T_i^* = \beta\, \mathrm{id}_{\mathfrak{H}}$$

and

$$TT^*(e_i \otimes h) = \sum_{k=1}^{n} e_k \otimes T_k T_i^* h.$$

By Lemma 2(v), we get

$$\langle\langle TT^*, TT^*\rangle\rangle \mathrm{id}_{\mathfrak{H}} = \sum_{i,k=1}^{n} (T_k T_i^*)^* T_k T_i^* = \sum_{i,k=1}^{n} T_i T_k^* T_k T_i^* = \alpha\beta\, \mathrm{id}_{\mathfrak{H}}.$$

Since $(T^a)^a = T$ the same argument shows that $\alpha\beta = \langle\langle T^a(T^a)^*, T^a(T^a)^*\rangle\rangle$. Let S be as in the lemma. From $T(\mathfrak{H}') \perp S(\mathfrak{H}')$ we get $T^*S = 0$ and, therefore,

(1) $$\sum_{i=1}^{n} T_i^* S_i = 0.$$

Since $(T^a)^* S^a \in \mathrm{Hom}_G(\pi, \pi)$ there exists $\gamma \in \mathbb{C}$ such that

(2) $$\gamma\, \mathrm{id}_{\mathfrak{H}} = (T^a)^* S^a = \sum_{i=1}^{n} T_i S_i^*.$$

By Lemma 2(v):

$$\langle\langle SS^*, TT^*\rangle\rangle\, \mathrm{id}_{\mathfrak{H}} = \sum_{i,k=1}^{n} (T_k T_i^*)^* S_k S_i^* = \sum_{i,k=1}^{n} T_i T_k^* S_k S_i^* = 0 \quad (\text{by } (1)).$$

Now, $T^a(T^a)^*$ (and, analogously, $S^a(S^a)^*$) is given by $T^a(T^a)^*(e_i^* \otimes h') = \Sigma_{k=1}^{n} e_k^* \otimes T_k^* T_i h'$. Lemma 2(v) implies:

$$\langle\langle S^a(S^a)^*, T^a(T^a)^*\rangle\rangle \mathrm{id}_{\mathfrak{H}'} = \sum_{i,k=1}^{n} (T_k^* T_i)^* S_k^* S_i = \sum_{i,k=1}^{n} T_i^* T_k S_k^* S_i = 0$$

(by (2) and (1)).

Now we are able to give the

Proof of the Theorem. Let $T_1, \ldots, T_r$ be r linearly independent elements in $\mathrm{Hom}_G(\pi', \rho \otimes \pi)$. Without loss of generality we may assume that $T_i(\mathfrak{H}') \perp T_j(\mathfrak{H}')$ for $i \neq j$ (by orthogonalization with respect to the inner product $(S, T)\,\mathrm{id}_{\mathfrak{H}'} = T^*S$). We have to show that $r \leq n$. Let α_i, β_i, P_i and Q_i $(1 \leq i \leq r)$ be defined by $T_i^* T_i = \alpha_i \mathrm{id}_{\mathfrak{H}'}$, $(T_i^a)^* T_i^a = \beta_i \mathrm{id}_{\mathfrak{H}}$, $P_i = \alpha_i^{-1} T_i T_i^*$ and $Q_i = \beta_i^{-1} T_i^a (T_i^a)^*$. Then the P_i's and Q_i's are projections in $\mathrm{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ and $\mathrm{Hom}_G(\rho^* \otimes \pi', \rho^* \otimes \pi')$, respectively. By Lemma 3, one has

$$\langle\langle P_i, P_i\rangle\rangle = \beta_i \alpha_i^{-1}, \quad \langle\langle Q_i, Q_i\rangle\rangle = \alpha_i \beta_i^{-1},$$

and

$$\langle\langle Q_i, Q_j\rangle\rangle = 0 = \langle\langle P_i, P_j\rangle\rangle \quad \text{for } i \neq j.$$

Let P and Q be defined by the equations $\mathrm{id}_{V \otimes \mathfrak{H}} = P + \Sigma_{i=1}^{r} P_i$ and $\mathrm{id}_{V^* \otimes \mathfrak{H}'} = Q + \Sigma_{i=1}^{r} Q_i$. From Lemma 2, one can easily conclude that

$$\langle\langle \mathrm{id}_{V\otimes\mathfrak{H}}, P_i\rangle\rangle = \langle\langle P_i, P_i\rangle\rangle = \langle\langle P_i, \mathrm{id}_{V\otimes\mathfrak{H}}\rangle\rangle$$

and

$$\langle\langle \mathrm{id}_{V^*\otimes\mathfrak{H}'}, Q_i\rangle\rangle = \langle\langle Q_i, Q_i\rangle\rangle = \langle\langle Q_i, \mathrm{id}_{V^*\otimes\mathfrak{H}'}\rangle\rangle$$

(because the P_i's and Q_i's are projections). This shows that $\langle\langle P_j, P\rangle\rangle = 0 = \langle\langle Q_j, Q\rangle\rangle$ for $1 \le j \le r$. Lemma 2 and the above relations imply

$$\langle\langle P, P\rangle\rangle + \sum_{i=1}^{r} \beta_i \alpha_i^{-1} = n = \langle\langle Q, Q\rangle\rangle + \sum_{i=1}^{r} \alpha_i \beta_i^{-1}.$$

Since $\langle\langle\ ,\ \rangle\rangle$ is positive definite one has $2n \ge \Sigma_{i=1}^{r} (\beta_i \alpha_i^{-1} + \alpha_i \beta_i^{-1})$. From $x + x^{-1} \ge 2$ for positive real numbers x we get $r \le n$ as desired.

If the dimension of $\mathrm{Hom}_G(\pi', \rho \otimes \pi)$ is equal to n, choose n linearly independent elements $T_1, \dots, T_n$ in that space with $T_i(\mathfrak{H}') \perp T_j(\mathfrak{H}')$ for $i \ne j$ and form P_i and P as above. It is easy to see that $\langle\langle P, P\rangle\rangle = 0$ and, therefore, $P = 0$ or $\mathrm{id}_{V\otimes\mathfrak{H}} = \Sigma_{i=1}^{n} P_i = \Sigma_{i=1}^{n} \alpha_i^{-1} T_i T_i^*$. This shows that $V \otimes \mathfrak{H}$ is the (orthogonal) sum of the $T_i(\mathfrak{H}')$'s but the restriction of the representation $\rho \otimes \pi$ to $T_i(\mathfrak{H}')$ is unitarily equivalent to π'. The "if-part" is trivial. The Theorem is proved, we have already pointed out how Corollary 1 follows from the Theorem.

Remark. The fact that $\rho \otimes \pi$ is the direct sum of at most n^2 irreducible subrepresentations can be proved quicker if one uses Lemma 1 and a similar trick as in the proof of the Theorem. More precisely, let $P_1, \dots, P_r$ be orthogonal projections in $\mathrm{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$. As in the proof of the Theorem one gets $n \ge \Sigma_{i=1}^{r} \langle\langle P_i, P_i\rangle\rangle$. But $\langle\langle P_i, P_i\rangle\rangle = \|J(P_i)\|^2$ and $\|J(P_i)\| \ge n^{-1/2} \|P_i\| = n^{-1/2}$ (Lemma 1) and, therefore, $n \ge rn^{-1}$ or $r \le n^2$. Of course, the Theorem gives a more precise description.

Proof of Corollary 2. Since $\bigcap_{g\in G} gNg^{-1}$ is a normal subgroup of finite index, it suffices to prove the second statement. The group G, resp. $H := G/N$, acts linearly on the space $\mathrm{Hom}_N(\pi, \pi)$ by $g \cdot f = \pi(g) f \pi(g)^*$. Let ρ be any irreducible unitary representation of H in the finite dimensional Hilbert space E; we consider ρ as a representation of G, too. The space of intertwining operators $\mathrm{Hom}_H(\rho, \mathrm{Hom}_N(\pi, \pi)) = \mathrm{Hom}_G(\rho, \mathrm{Hom}_N(\pi, \pi))$ is isomorphic to $\mathrm{Hom}_G(\rho \otimes \pi, \pi)$. By the Theorem, the dimension of that space is not greater than $\dim E$. Since H is finite, every element in $\mathrm{Hom}_N(\pi, \pi)$ is contained in a finite dimensional H-invariant subspace of $\mathrm{Hom}_N(\pi, \pi)$. Therefore, the dimension of $\mathrm{Hom}_N(\pi, \pi)$ is not greater than $\Sigma_\rho (\dim \rho)^2$, ρ being an equivalence class of irreducible unitary representations of H. But, by a well-known theorem in the representation theory of finite groups, the value of this sum is exactly the order of H.

Example. For all integers $n > 1$ we will give an example of an n-dimensional irreducible representation ρ and another irreducible representation π such that $\rho \otimes \pi$ decomposes into exactly n^2 subrepresentations. To motivate our example let ρ and π be such representations, $\rho \otimes \pi = \bigoplus_{i=1}^{n^2} \pi_i$. Then the algebra $\mathrm{Hom}_G(\rho \otimes \pi, \rho \otimes \pi)$ is at least n^2 dimensional; from Corollary 1 we know that its dimension is exactly n^2. Therefore, the π_i's are unitarily nonequivalent. Moreover, again by Corollary 1, $\rho^* \otimes \rho \otimes \pi$ is unitarily equivalent to $n^2\pi$; but on the other hand $\rho^* \otimes \rho \otimes \pi$ is equal to $\rho^* \otimes (\bigoplus_{i=1}^{n^2} \pi_i)$ $= \bigoplus_{i=1}^{n^2} (\rho^* \otimes \pi_i)$. Hence π is unitarily equivalent to $\rho^* \otimes \pi_i$ for all i.

Let G be the Heisenberg group over $\mathbf{Z}/n\mathbf{Z}$,

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{Z}/n\mathbf{Z} \right\}.$$

The commutator subgroup G' is equal to the center

$$ZG = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbf{Z}/n\mathbf{Z} \right\}.$$

Let χ be a character of

$$N := \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : y, z \in \mathbf{Z}/n\mathbf{Z} \right\}$$

which is faithful on ZG and let ρ be the induced representation $\mathrm{ind}_{N\uparrow G}\chi$ in the n-dimensional space V. ρ is irreducible by the Frobenius reciprocity theorem: the restriction of ρ to N decomposes into n different characters since χ is faithful on ZG. Choose $\pi = \rho^*$. For $g \in ZG$ we have

$$\rho(g) = \chi(g)\,\mathrm{id}_V, \qquad \pi(g) = \rho^*(g) = \overline{\chi(g)}\,\mathrm{id}_{V^*}$$

and therefore,

$$(\rho \otimes \pi)(g) = \chi(g)\,\mathrm{id}_V \otimes \overline{\chi(g)}\,\mathrm{id}_{V^*} = \mathrm{id}_{V \otimes V^*}.$$

Hence the homomorphism $\rho \otimes \pi$ factors through G/ZG which is abelian, and $\rho \otimes \pi$ decomposes into one-dimensional subrepresentations; $\rho \otimes \pi$ is the sum of all n^2 nonequivalent (see above) one-dimensional representations of G.

Acknowledgement. Originally, I proved the Theorem for $\pi = \pi'$. The generalization was suggested by the referee. Moreover, the example is due to the referee; I gave an example only for $n = 2$.

REFERENCE

1. C. C. Moore, *Groups with finite dimensional irreducible representations*, Trans. Amer. Math. Soc. 166 (1972), 401–410. MR 46 #1960.

FAKULTÄT FÜR MATHEMATIK DER UNIVERSITÄT BIELEFELD, 48 BIELEFELD, KURT-SCHUMACHER-STRASSE 6, FEDERAL REPUBLIC OF GERMANY