

Symmetry and Nonsymmetry for Locally Compact Groups

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The class [S] of locally compact groups G is considered, for which the algebra $L^1(G)$ is symmetric (=Hermitian). It is shown that [S] is stable under semidirect compact extensions, i.e., $H \in [S]$ and K compact implies $K \times H \in [S]$. For connected solvable Lie groups inductive conditions for symmetry are given. A construction for nonsymmetric Banach algebras is given which shows that there exists exactly one connected and simply connected solvable Lie group of dimension ≤ 4 which is not in [S]. This example shows that $G/Z \in [S]$, Z the center of G , in general does *not* imply $G \in [S]$. It is shown that nevertheless for discrete groups and a (possibly) stronger form of symmetry this implication holds, implying a new and shorter proof of the fact that [S] contains all discrete nilpotent groups.

Recall that a Banach algebra \mathcal{A} with isometric involution $a \rightarrow a^*$ is called *symmetric*, if the spectrum $Sp a^*a$ for every element $a \in \mathcal{A}$ is always contained in $\mathbb{R}^+ = [0, \infty[$. The theorem of Ford and Shirali [2] tells us that this is equivalent to \mathcal{A} being *Hermitian*, i.e., that $Sp b \subset \mathbb{R}$ for each $b \in \mathcal{A}$ with $b = b^*$. For other characterizations of symmetry for involutive Banach algebras see [16] or [10]. In this paper we are mainly concerned with group algebras $L^1(G)$ for locally compact groups G with left invariant Haar measure. Thus we define:

[S] is the class of all locally compact groups G for which the convolution algebra $L^1(G)$ is symmetric.

It is convenient to introduce also some other classes: [A] Abelian, [C] compact, [M] amenable, [Solv] solvable, [Nil] nilpotent, and [PG] polynomially growing groups. The subclass of all connected groups in a class [X] will be denoted by $[X]_0$. Detailed information about the relations among the various classes [X] can be found in the survey article [14], a short summary about known results on [S] is contained in [11].

The history of [S] is a line of destroyed hopes and wrong conjectures: After early results there was some hope that $[S] = [M]$, but 1969 Jenkins [5] showed that $[Solv] \not\subset [S]$, hence a fortiori $[M] \not\subset [S]$. Then there was an old conjecture that $[PG] = [S]$. This was shown to be false in [11]: Even $[S]_0 \not\subset [PG]$. On the

other hand in [3] a locally finite discrete group is constructed, which is *not* in [S], hence $[PG] \not\subset [S]$. As yet all known results indicated that $[S]_0 = [M]_0$, the inclusion $[S]_0 \subset [M]_0$, e.g., being true. But among others we will prove that the connected four-dimensional exponential group $G_{4,9}(0)$ with algebra $\mathfrak{g}_{4,9}(0)$ in the terminology of [1] has a nonsymmetric algebra. Thus $[S]_0 \neq [M]_0$. Actually this example is minimal: All other $G \in [\text{Solv}]_0$ with $\dim G \leq 4$ are in [S]. We also show that this group is not contained in the class [W] of groups with the Wiener property. This result fills the gap in [9, Corollary 2, p. 275]. It also implies that closed normal subgroups of Wiener groups need not be Wiener. The least example of this kind is the group of real upper triangular 3×3 matrices (a_{jk}) with $a_{11} > 0$, $a_{22} > 0$, $a_{33} = 1$. As a contracting extension of the Heisenberg group H_1 this group is in [W], see [13]. The subgroup $a_{11} = 1$ is exactly $G_{4,9}(0)$.

As in our previous papers we use also here extensively the machinery of generalized L^1 -algebras and consequently assume that the reader is familiar with the basic facts of this theory, see, e.g., [6].

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Throughout this paper let \mathcal{A} be a Banach algebra with an isometric involution $a \rightarrow a^*$. Let G be a locally compact group acting strongly continuously on \mathcal{A} as a group of isometric $*$ -automorphisms. As in our previous papers we form the generalized L^1 -algebra $\mathcal{L} = L^1(G, \mathcal{A})$ of left Haar integrable \mathcal{A} -valued functions on G . If $a \rightarrow a^x$ denotes the automorphism on \mathcal{A} defined by $x \in G$, convolution and involution on \mathcal{L} are given by

$$f \star g(x) = \int f(xy)^y{}^{-1} g(y^{-1}) dy, \quad f^*(x) = (f(x^{-1})^x)^* \Delta(x)^{-1}.$$

The standard example of such an \mathcal{L} comes from semidirect products: Let H be another locally compact group such that G acts continuously and automorphically on H . Then the semidirect product $\Gamma = G \times_{\cdot} H$ is well defined and one has $L^1(\Gamma) \cong L^1(G, L^1(H))$ in a canonical fashion.

THEOREM 1. *If G is compact and \mathcal{A} is symmetric, then $\mathcal{L} = L^1(G, \mathcal{A})$ is symmetric.*

Proof. Let $\mathcal{C} = C(G, \mathcal{A})$ be the involutive Banach algebra of all continuous functions from G into \mathcal{A} with pointwise operations and uniform norm. If G acts on \mathcal{C} by $f^z(x) = (f(zx))^z$, then \mathcal{C} is a G -algebra and \mathcal{A} can be identified with the G -subalgebra of all constant functions. But then \mathcal{L} can be considered as a closed subalgebra of $L^1(G, \mathcal{C})$ and hence it suffices to prove symmetry for the latter.

Now let \mathcal{A}_i be the G -algebra which coincides with \mathcal{A} as an involutive Banach

algebra but with trivial action of G . For $u \in C(G, \mathcal{A})$ let $u_t \in C(G, \mathcal{A}_t)$ be defined by $u_t(x) = u(x)^x$. Then

$$(u^z)_t(x) = (u^z(x))^x = u(zx)^{zx} = u_t(zx) = (u_t)^z(x).$$

Thus $u \rightarrow u_t$ is a G -isomorphism from \mathcal{C} onto $C(G, \mathcal{A}_t)$ and consequently we may assume from the beginning that $\mathcal{A} = \mathcal{A}_t$, i.e., that G acts trivially on \mathcal{A} .

Now we apply the first "lemma" in [15] to the algebra $\mathcal{B} = L^1(G, \mathcal{C})$. In order to do this we observe that the center of the adjoint algebra \mathcal{C}^b of \mathcal{C} contains $C(G)$ in such a way that all the assumptions on $\mathcal{C} = \mathcal{A}$ and $C(G) = \mathcal{Q}$ made in Theorem 1 of [9] are fulfilled. These assumptions imply that the algebra $\Gamma_*(G) = L^1(G, C(G))$ (see [7]) is contained in \mathcal{B}^b . By Corollary 3 of Theorem 1 in [9] or by Theorem 4 below we know that $\Gamma_*(G)$ is simple and contains Hermitian projectors e of rank 1. Thus we can proceed exactly as in the proof of Satz 1 in [15] and it remains to prove that the subalgebra $e \star L^1(G, \mathcal{C}) \star e$ is symmetric. To this end we take for e the constant function $e(x, y) \equiv 1$.

Then for $f \in L^1(G, \mathcal{C})$ we have

$$e \star f(x) = \int e(xy^{-1})^y f(y) dy = \int f(y) dy \in \mathcal{C},$$

i.e., $e \star f$ is a constant function on G . On the other hand for any constant function $g: x \rightarrow a \in \mathcal{C}$ one has $e \star g = g$, thus $e \star L^1(G, \mathcal{C})$ consists exactly of all constant functions from G into \mathcal{C} . Let $h \in e \star L^1(G, \mathcal{C})$ be constant. Then

$$h \star e(x) = \int h(xy^{-1})^y e(y) dy = \int h^y dy \in \mathcal{C}^G,$$

the invariant elements of \mathcal{C} . Again, if $g \in \mathcal{C}^G$, then also $g \star e = e$ and consequently

$$e \star L^1(G, \mathcal{C}) \star e \approx \mathcal{C}^G. \tag{1}$$

But $u \in \mathcal{C}^G$ means $u^z(x) = u(zx) = u(x)$ for all $x, z \in G$, hence \mathcal{C}^G is the subalgebra of constant functions from G into \mathcal{A} , which we had already identified with \mathcal{A} . It follows immediately from the definitions that (1) is a full isomorphism between $e \star L^1 \star e$ and \mathcal{A} . Thus $e \star L^1 \star e$ and consequently $L^1(G, \mathcal{C})$ and $L^1(G, \mathcal{A})$ are symmetric.

COROLLARY 1. *Let $G = K \times_s H$ be the semidirect product of a compact group K and a closed normal subgroup H . Then $H \in [S]$ implies $G \in [S]$.*

COROLLARY 2. *Let G be a Lie group with component G_0 , such that G/G_0 is finite. Then $G \in [S]$ if the radical R of G_0 is in $[S]$ and if G_0/R is compact.*

Proof. After [8, Theorem 3], we can assume that $G = G_0$ and moreover that G is simply connected. But then $G = K \times_s R$ with semisimple K .

The following example shows that the converse of Theorem 1 is false: Let $r > 1$ be a real number, and let \mathcal{A} be the $*$ -subalgebra of all $f \in l^1(\mathbb{Z})$ with

$$|f| = \sum_{-\infty}^{\infty} |f(n)| r^{|n|} < \infty.$$

Clearly \mathcal{A} is a commutative Banach $*$ -algebra, which is *not* symmetric, because it has non-Hermitian characters: The dual $\hat{\mathcal{A}}$ of \mathcal{A} consists of all characters

$$\chi_z: f \rightarrow \chi_z(f) = \sum_{-\infty}^{\infty} f(n) z^n$$

with $z \in \mathbb{C}$, $r^{-1} \leq |z| \leq r$. The Hermitian characters are exactly those with $|z| = 1$.

Now $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ acts continuously on \mathcal{A} by $f \rightarrow f^\zeta$, $f^\zeta(n) = \zeta^n f(n)$, $\zeta \in \mathbb{T}$. For $k \in \mathbb{Z} = \hat{\mathbb{T}}$ let $b_k \in \mathcal{A}$ be the function $n \rightarrow \delta_{k,n}$. Then, in the terminology of [10], b_k generates the subspace \mathcal{A}_k of $f \in \mathcal{A}$ with $f^\zeta = \zeta^k f$, in particular $\mathcal{A}_0 \cong \mathbb{C}$ is symmetric and the (only) normalized positive functional $f_0(\lambda b_0) = \lambda$ has, e.g., $F_0 = \chi_1$ as a positive extension on \mathcal{A} . But this was all that was needed in [10, Satz 4], to prove the symmetry of $L^1(\mathbb{T}, \mathcal{A})$.

Of course the converse of Corollary 1 may still be true. The only known connected solvable Lie group not in [S], i.e., $G_{4,9}(0)$, cannot be used for counterexamples, because it has no compact automorphismgroups with more than two elements.

Corollary 2 and our next theorem reduce the problem of symmetry for almost connected groups to the case of connected solvable Lie groups. The proof of the next theorem is the same as that for the corresponding theorem for the Wiener property, see [4].

THEOREM 2. *Let $G = \text{prolim } G_\alpha$ be the projective limit of the groups G_α . Then $G \in [S]$ if and only if all $G_\alpha \in [S]$.*

Proof. Let K_α be the normal compact subgroup in G with $G/K_\alpha = G_\alpha$. The normalized Haar measure m_α of K_α is a central idempotent in the measure algebra $M(G)$, thus $I_\alpha = m_\alpha \star L^1(G)$ is a closed $*$ -invariant ideal in $L^1(G)$ which is easily seen to be isomorphic with $L^1(G_\alpha)$. An easy application of the Stone-Weierstrass theorem shows that $\bigcup_\alpha I_\alpha$ is a dense ideal in $L^1(G)$. Now Theorem 2 follows from [11, Satz 1].

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Our first two theorems reduce the question whether an almost connected group G belongs to [S] or not more or less to the case of connected solvable Lie groups. In this case the obvious method to apply is induction. A powerful

tool in this context is Satz 1 in [15], which is the source of symmetry for nilpotent groups in that paper, but which actually gives much more. We will use it here to prove a criterion for symmetry which implies for instance that all connected solvable groups of dimension ≤ 4 and not locally isomorphic with $G_{4,9}(0)$ are in [S].

Now let G be a connected and simply connected solvable Lie group and let W be a minimal closed normal connected subgroup of positive dimension. Then $l = \dim W \leq 2$. Let \mathfrak{g} be the Lie algebra of G and let \mathfrak{w} be the l -dimensional ideal corresponding to W . Clearly W and \mathfrak{w} are commutative, hence $W \cong \mathbb{R}^l$, and the centralizer \mathfrak{h} of \mathfrak{w} in \mathfrak{g} has codimension at most 2. If $l = 1$ and W , resp. \mathfrak{w} is not central, then $\mathfrak{w} = (y)$ and $\mathfrak{g} = (x) \oplus \mathfrak{h}$ with $[x, y] = y$. It follows that $G = \mathbb{R} \times_s H$, $\mathbb{R} \cong W \triangleleft H$ and $\mathbb{R} \times_s W \cong S_2$, the connected affine group of \mathbb{R} .

If $l = 2$, then \mathfrak{w} has a basis $\{x, y\}$ and there exist linear forms φ and ψ on \mathfrak{g} with

$$[a, x] = \varphi(a)x - \psi(a)y, \quad [a, y] = \psi(a)x + \varphi(a)y.$$

Clearly φ and ψ are independent if and only if $\text{codim } \mathfrak{h} = 2$.

THEOREM 3. *Let G be a connected and simply connected solvable Lie group with algebra \mathfrak{g} , let \mathfrak{w} be a minimal nonzero ideal in \mathfrak{g} with centralizer $\mathfrak{h} \subset \mathfrak{g}$ and let W and H be the corresponding connected closed normal subgroups of G . Let $\varphi, \psi \in \mathfrak{g}^*$, the real dual of \mathfrak{g} , be defined as above, if $\dim \mathfrak{w} = l = 2$. Assume that $G/W \in [S]$. Then $G \in [S]$, provided that one of the following conditions is fulfilled:*

- (1) W is central and G/W has a central subgroup $U/W \cong \mathbb{R}$, so that the centralizer K of U in G is in [S].
- (2) $l = 1$, W is not central and $H \in [S]$.
- (3) $l = 2$, $\text{codim } H = 1$, $H \in [S]$ and $\varphi \neq 0$.
- (4) $l = 2$, $\text{codim } H = 2$, and the subgroup corresponding to the kernel of φ is in [S].

Remark. The cases which are not covered by Theorem 3 are where (i) G has one-dimensional center Z and G/Z has trivial center, and (ii) $l = 2$, $\text{codim } H = 1$ and G contains a one parameter subgroup acting via inner automorphisms as a rotation group on $W \cong \mathbb{R}^2$. This is of course equivalent with the existence of an element $a \in \mathfrak{g}$ with $[a, x] = y$, $[a, y] = -x$ for a suitable basis $\{x, y\}$ of \mathfrak{w} . If G is an exponential group, then this case (ii) can not occur, but clearly (i) is possible, and indeed $G_{4,9}(0)$ is the least possible example of this kind.

Proof of Theorem 3. Assume (1). There is nothing to show if $K = G$. Otherwise K must have codimension 1, because G acts trivially on W and U/W . Thus $G = \mathbb{R} \times_s K$ with $\mathbb{R} \cong U$, U is central in K , and $\mathbb{R} \times_s U \cong H_1$, the

three-dimensional Heisenberg group. The rest of the proof in this case is the same as for nilpotent Lie groups with one-dimensional center (see [15]).

In order to prove Theorem 3 for the remaining cases (2), (3), and (4) we need Satz 1 from [15], but in a more general form. In our present case (2) the subalgebras \mathcal{X}_{\pm} of $L^1(W) \cong L^1(\mathbb{R})$, defined below, unfortunately don't have bounded approximate units, thus a version of Satz 1 is required which avoids this assumption. For the benefit of the reader we will formulate this version and indicate how to prove it.

"SATZ 1." Let R be a locally compact Abelian group and let \mathcal{A} be an involutive Banach R -algebra. Assume that the center of the adjoint algebra \mathcal{A}^b of \mathcal{A} contains a closed involutive R -invariant subalgebra \mathcal{U} such that the action of R on \mathcal{U} is continuous, and that the dual $\hat{\mathcal{U}}$ of \mathcal{U} can be identified with R . Moreover let the Gelfand transform map \mathcal{U} injectively onto a subalgebra \mathcal{Q} of $C_{\infty}(R)$ and assume that \mathcal{Q} satisfies conditions (1) through (4) in [9, p. 262]. Let \mathcal{A}_0 be the closure of $\mathcal{U}\mathcal{A} = \mathcal{A}\mathcal{U}$ in \mathcal{A} and assume that \mathcal{A}_0 is symmetric, and \mathcal{A}_0^2 is dense in \mathcal{A}_0 . Then $\mathcal{L}_0 = L^1(R, \mathcal{A}_0)$ is symmetric.

Proof. The conditions on \mathcal{Q} imply that \mathcal{U}^2 is dense in \mathcal{U} , thus we have $\overline{\mathcal{U}\mathcal{A}_0} = \mathcal{A}_0$ and we may assume that $\mathcal{A} = \mathcal{A}_0$. Let $\mathcal{L} = L^1(R, \mathcal{A})$ and $\mathcal{D} = L^1(R, \mathcal{U})$. Then \mathcal{D} can be considered as part of \mathcal{L}^b and it is easy to see that $\overline{\mathcal{D}\mathcal{L}}$ contains $L^1(R, \mathcal{U}\mathcal{A})$, hence coincides with \mathcal{L} . Let \mathcal{P} be the system of all Hermitian projectors of rank 1 in \mathcal{D} . Since \mathcal{D} is simple we have $\overline{\mathcal{D}p\mathcal{D}} = \mathcal{D}$ for every $p \in \mathcal{P}$. Now an inspection of the proof of Satz 1 in [15] reveals that approximate units are used only twice: First one has to have $(p\mathcal{L}q)(q\mathcal{L}q)^{-} = p\mathcal{L}q$ for $p, q \in \mathcal{P}$ (we omit the " \star "). But this follows from

$$(p\mathcal{L}q)(q\mathcal{L}q)^{-} \supset (p\mathcal{L}\mathcal{D}q\mathcal{D}\mathcal{L}p)^{-} = p(\mathcal{L}\mathcal{D}\mathcal{L})^{-}q = p\overline{\mathcal{L}^2}q = p\mathcal{L}q.$$

Secondly one needs $\overline{\mathcal{D}\mathcal{L}} = \mathcal{L}$, which has been shown above to be true. Now the rest of the proofs from [15] applies unchanged.

Now assume that (2) holds. Then again $\text{codim } H = 1$ and $G = R \times_{\ast} H$ with $R \cong \mathbb{R}$, hence $L^1(G) \cong L^1(R, \mathcal{A}) = \mathcal{L}$ with $\mathcal{A} = L^1(H)$. Moreover the center of \mathcal{A}^b contains $\mathcal{Z} = L^1(W)$ which can be identified with $L^1(\mathbb{R})$. Let \hat{f} be the Fourier transform of $f \in \mathcal{Z}$. Then the ideals

$$\mathcal{Z}_0 = \{f \in \mathcal{Z}; \hat{f}(0) = 0\}, \quad \mathcal{Z}_{\pm} = \{f \in \mathcal{Z}; \hat{f}(x) = 0 \text{ for } \mp x \geq 0\},$$

considered already in [9, p. 272], are R -invariant and $\mathcal{Z}_+ + \mathcal{Z}_-$ is dense in \mathcal{Z}_0 . Moreover the closures \mathcal{A}_{\pm} of $\mathcal{Z}_{\pm}\mathcal{A}$ are R -invariant ideals in \mathcal{A} and so is \mathcal{A}_0 , the closure of $\mathcal{A}_+ + \mathcal{A}_-$. Now the dual of $\mathcal{U} = \mathcal{Z}_{\pm}$ can be identified with R , see [9] or [11], and also the other conditions in "Satz 1" are fulfilled, consequently the ideals $\mathcal{L}_{\pm} = L^1(R, \mathcal{A}_{\pm})$ in \mathcal{L} and also the closure $\mathcal{L}_0 =$

$L^1(R, \mathcal{A}_0)$ of their sum are symmetric. Because \mathcal{L}_0 is the kernel corresponding to the canonical epimorphism $L^1(G) \rightarrow L^1(G/W)$ we see that $\mathcal{L}/\mathcal{L}_0 \cong L^1(G/W)$, hence $\mathcal{L}/\mathcal{L}_0$ and consequently also \mathcal{L} is symmetric, see (5) in [9, p. 261].

Finally assume (3) or (4). Let K be the closed normal subgroup in G corresponding to the kernel of φ in \mathfrak{g} ; in case (3) we have $K = H$. In both cases $\text{codim } K = 1$, $G = R \times_s K$ with $R \cong \mathbb{R}$, and $L^1(G) \cong \mathcal{L} = L^1(R, \mathcal{A})$ with $\mathcal{A} := L^1(K)$. Let \mathcal{L}_0 denote the functions $f \in L^1(W)$ with $\int_W f(x) dx = 0$. Then the kernel \mathcal{A}_0 of the canonical epimorphism $L^1(K) \rightarrow L^1(K/W)$ is the closure of $\mathcal{A}\mathcal{L}_0\mathcal{A}$, and the kernel of $L^1(G) \rightarrow L^1(G/W)$ is $\mathcal{L}_0 := L^1(R, \mathcal{A}_0)$. As in case (2), it suffices to prove the symmetry of \mathcal{L}_0 . To this end, let \mathcal{U} denote the algebra of radial functions in \mathcal{L}_0 . K acts trivially on \mathcal{U} and therefore \mathcal{U} is in the center of \mathcal{A}^b . Moreover, it is easy to see that $\mathcal{U}\mathcal{L}_0$ is dense in \mathcal{L}_0 (actually, \mathcal{U} contains bounded approximate units of \mathcal{L}_0). We conclude that $\mathcal{U}\mathcal{A} = \mathcal{A}\mathcal{U}$ is dense in \mathcal{A}_0 .

Also the other assumptions of "Satz 1" are fulfilled: Similarly to case (2), resp. to [9] or [11], the dual of \mathcal{U} can be identified with \mathbb{R} such that the action of R becomes left translation. It follows that \mathcal{L}_0 and consequently $\mathcal{L} = L^1(G)$ is symmetric.

3

In our previous papers algebras of the form $L^1(G, \mathcal{Q})$ with \mathcal{Q} a G -invariant Banach subalgebra of $C_\infty(G)$ played an important role. Under certain fairly general conditions they turned out to be simple and symmetric [9, Corollary 4, p. 265; 11, Satz 3]. Among these conditions the most restrictive one is the fact that \mathcal{Q} was required to be *two-sided* translation invariant with both translations $z \rightarrow q^z$ and $z \rightarrow q_z$ being continuous from G into \mathcal{Q} . Here we will weaken the right-sided invariance, and we will simplify the proof of [9]. It is not known if one can omit all the hypotheses on right invariance.

THEOREM 4. *Let \mathcal{Q} be a $*$ -subalgebra of the algebra $C_\infty(G)$ of all continuous complex-valued functions on G vanishing at infinity. For $f \in C_\infty(G)$ and $z \in G$ let $f^z \in C_\infty(G)$ and $f_z \in C_\infty(G)$ be defined by $f^z(x) = f(zx)$ and $f_z(x) = f(xz)$. Assume that \mathcal{Q} satisfies the following conditions:*

- (1) \mathcal{Q} is a Banach algebra under a norm $\| \cdot \|$ with $\| q^* \| = \| q \| \geq \| q \|_\infty = \sup_{x \in G} | q(x) |$.
- (2) \mathcal{Q} is left invariant, i.e., $q \in \mathcal{Q}$ and $z \in G$ imply $q^z \in \mathcal{Q}$ and $\| q^z \| = \| q \|$.
- (3) The mapping $z \rightarrow q^z$ is continuous from G into \mathcal{Q} for every $q \in \mathcal{Q}$.
- (4) The compactly supported functions in \mathcal{Q} form a dense subalgebra \mathcal{Q}_0 .
- (5) For every neighborhood U of e in G there exists a function $u \in \mathcal{Q}$ with the following properties:

- (a) $u \neq 0$ and the support $\text{supp } u$ is contained in U ,
- (b) $u_z \in \mathcal{Q}$ for all $z \in G$,
- (c) the mapping $z \rightarrow u_z$ is continuous from G into \mathcal{Q} .

Then the algebra $L^1(G, \mathcal{Q})$ is simple and symmetric.

Remark. From Theorem 4 (5a,b,c) it follows easily that the subset of all $v \in \mathcal{Q}$ for which $z \rightarrow v_z$ is continuous from G into \mathcal{Q} is a two-sided translation and conjugation invariant subalgebra \mathcal{Q}_i of \mathcal{Q} and that for every nonempty open set $V \subset G$ there exists $v \in \mathcal{Q}_i$ with $\text{supp } v \subset V$, $v \geq 0$ and $\int_G v(x) dx = 1$. Moreover it follows that \mathcal{Q} is a regular function algebra.

Proof. We consider the regular representation ρ of $\mathcal{L} = L^1(G, \mathcal{Q})$ in $L^2(G)$. Recall that one can identify the elements $f \in \mathcal{L}$ with complex-valued functions on $G \times G$: For $x \in G$ it is $f(x) \in \mathcal{Q} \subset C_x(G)$, thus we may write $f(x)(y) = f(x, y)$, in particular $f(x)^z(y) = f(x, zy)$. Then ρ can be written (see, e.g., [7])

$$(\rho(f)\xi)(x) = \int_G f(xy, y^{-1}) \xi(y^{-1}) dy, \quad \xi \in L^2(G).$$

In our papers [9, 11] we had defined special elements $u \circ v$ in \mathcal{L} for functions u and v in \mathcal{Q}_0 . Our previous definition of $u \circ v$ is not quite appropriate in the present context, therefore we define now for arbitrary $u, v \in \mathcal{Q}_0$

$$(u \circ v)(x) = \Delta(x)^{-1/2} u^x \bar{v}.$$

Then clearly $u \circ v \in \mathcal{L}$ and $(u \circ v)(x, y) = \Delta(x)^{-1/2} u(xy) \overline{v(y)}$. Moreover, with $u'(x) = \Delta(x)^{-1/2} u(x)$ we have

$$(\rho(u \circ v)\xi)(x) = u'(x) \int_G \xi(y^{-1}) \overline{v'(y^{-1})} \Delta(y^{-1}) dy = (\xi | v') u'(x)$$

with the inner product $(\xi | \vartheta) = \int \xi(y) \overline{\vartheta(y)} dy$ in $L^2(G)$. Thus $\rho(u \circ v)$ is a rank 1 operator and $\rho(u \circ u)$ is a rank 1 Hermitian projector, if $|u'|_2 = 1$. Let \mathcal{E} be the two-sided ideal of all $f \in \mathcal{L}$ for which $\rho(f)$ has finite rank. The first three theorems in [11] show that symmetry for \mathcal{L} will follow, if we can prove that \mathcal{E} is dense in \mathcal{L} .

This will follow from (4) if we can show that every function $p' \otimes q: (x, y) \rightarrow p'(x) q(y)$ for any $q \in \mathcal{Q}_0$ and any compactly supported continuous function p' is contained in the closure of \mathcal{E} in \mathcal{L} .

Let $q \in \mathcal{Q}_0$ and $p' \in C_x(G)$ be given and assume that p' has compact support T . Then $p: x \rightarrow \Delta(x)^{1/2} p'(x)$ is also continuous and supported by T , hence for $\epsilon > 0$ there exists a symmetric compact neighborhood U of e in G with $|p - p^z|_\infty < \epsilon$, $|p - p_z|_\infty < \epsilon$ for $z \in U$. Now choose $u \in \mathcal{Q}$ with Theorem 4 (5a,b,c), $u \geq 0$ and $\int u(x) dx = 1$.

The integral

$$f := \int_G \int_G p(z^{-1}t)[u_z \circ (u_t q)] dt dz$$

exists because we have to integrate only over a compact set (note that the \mathcal{Q} -norms $|u_z|$ and $|u_t|$ are bounded on compact sets). Since $u_z \circ (u_t q) \in \mathcal{E}$ for all $t, z \in G$, the integral f is contained in the closure $\bar{\mathcal{E}}$. Let us compute $f(x, y)$:

$$\begin{aligned} f(x, y) &= \int_G \int_G p(z^{-1}t) \Delta(x)^{-1/2} u(xyz) u(yt) q(y) dt dz \\ &= \Delta(x)^{-1/2} \int_G \int_G p(z^{-1}xt) u(z) u(t) dt dz \cdot q(y). \end{aligned}$$

Let \tilde{p} be defined by $\tilde{p}(x) = \int_G \int_G p(z^{-1}xt) u(z) u(t) dt dz$. Then \tilde{p} is a continuous function supported by UTU . Moreover $f = \tilde{p} \otimes q$, and $\int u dx = 1$ implies

$$p(x) = \int_G \int_G p(x) u(z) u(t) dt dz,$$

and therefore

$$\begin{aligned} |\tilde{p}(x) - p(x)| &= \left| \int_G \int_G [p(z^{-1}xt) - p(x)] u(z) u(t) dz dt \right| \\ &\leq \int_G \int_G |p(z^{-1}xt) - p(x)| u(z) u(t) dz dt \leq 2\epsilon. \end{aligned}$$

Thus if $\Delta(x)^{-1/2}$ is bounded by c on UTU we obtain for the norm $|\cdot|_1$ in \mathcal{L} :

$$\begin{aligned} |f - p' \otimes q|_1 &= |\Delta^{-1/2}(\tilde{p} - p) \otimes q|_1 = |q| \int_{UTU} \Delta(x)^{-1/2} |\tilde{p}(x) - p(x)| dx \leq \\ &2 |q| c \epsilon |UTU|. \end{aligned}$$

Choosing U small enough we may assume that $|UTU| < 2|T|$ for the Haar measures of the compact sets UTU and T . Thus

$$|f - p' \otimes q|_1 < 4c |q| \cdot |T| \cdot \epsilon.$$

It follows that $\bar{\mathcal{E}} = \mathcal{L}$. Now the results of [11] imply that \mathcal{L} is symmetric. Finally, if \mathcal{J} is a closed proper two-sided ideal in \mathcal{L} , then $\mathcal{E} \not\subset \mathcal{J}$, and because \mathcal{E} is simple it follows that $\mathcal{E} \cap \mathcal{J} = 0$, thus $\mathcal{E}\mathcal{J} = 0$ and $\mathcal{L}\mathcal{J} = 0$. This implies $\mathcal{J} = 0$.

4

As a consequence of our last theorem we see that any algebra of the form $\mathcal{A} = L^1(H, \mathcal{Q})$ with $\mathcal{Q} \subset C_\infty(H)$ as before is the completion of a "matrix algebra," spanned by matrix units $e_{jk} = u_j \circ u_k$, $u_j \in \mathcal{Q}_0$. Given another locally compact group G which acts on H and hence on \mathcal{A} , it is natural to use these e_{jk} when studying $\mathcal{L} = L^1(G, \mathcal{A})$. Because \mathcal{A} is part of \mathcal{L}^b , the

algebra \mathcal{L} is in a way a matrix algebra over the algebra $e_{ii}^* \mathcal{L} e_{ii}^* = \mathcal{L}_i$ and the structure of these \mathcal{L}_i (which of course are all isomorphic) determines to a great extent the structure of \mathcal{L} .

We consider two locally compact groups G and H and assume that G acts continuously and automorphically on H , i.e., that there exists a continuous mapping $G \times H \rightarrow H, (g, x) \rightarrow x^g \in H$, with $(xy)^g = x^g y^g, (x^g)^h = x^{gh}, x^e = x$. Let Δ be the modular function of this action: If dx is the left Haar measure on H , then $dx^g = \Delta(g) dx$. Now let \mathcal{Q} be a subalgebra of $C_x(H)$ with all the properties listed in Theorem 4. For $u \in C_x(H)$ and $g \in G$ let $u \circ g$ be the function $u \circ g(x) = u(x^{g^{-1}})$. We assume that $q \rightarrow q \circ g$ defines an automorphism of \mathcal{Q} for every $g \in G$, in particular that $|q \circ g| = |q|$, and that $g \rightarrow q \circ g$ is continuous for every $q \in \mathcal{Q}$.

Under this assumption $\mathcal{A} = L^1(H, \mathcal{Q})$ is a G -algebra with

$$f^g(x) = \Delta(g)^{-1} f(x^{g^{-1}}) \circ g, \quad f \in \mathcal{A}, \quad g \in G.$$

More explicitly, this means $f^{g^{-1}}(x, y) = \Delta(g) f(x^g, y^g)$.

THEOREM 5. *If the modular function Δ is not trivial, then the algebra $\mathcal{L} = L^1(G, \mathcal{A})$ with $\mathcal{A} = L^1(H, \mathcal{Q})$ as above is not symmetric.*

Proof. Let $\mathfrak{H} = L^2(H)$ and let ρ be the faithful representation of $\mathcal{A} = L^1(H, \mathcal{Q})$ in \mathfrak{H} , as defined in Section 3. We fix a real element u in \mathcal{Q}_0 with $u(e) > 0, \|u\|_2 = 1$ and set $p = u \circ u$. Then $\rho(p)$ is a Hermitian projector of rank 1 in \mathfrak{H} and p is a minimal Hermitian idempotent in \mathcal{A} . As ρ is faithful and $\rho(p)$ has rank 1 it follows that $p\mathcal{A}p = \mathbb{C}p$. Moreover, we can consider \mathcal{A} as a subset, hence p as an element of \mathcal{L}^b . Thus Theorem 5 will be proved when we can show that the subalgebra $\mathcal{L}_p = p^* \mathcal{L} p^*$ of \mathcal{L} is not symmetric if $\Delta \not\equiv 1$. Note that $p^* \in \mathcal{L}^b$ is defined by $(p^* f)(g) = p^g f(g), (f p^*)(g) = f(g) p$, see [6].

Let \mathcal{K} be the C^* -algebra of all compact operators of \mathfrak{H} . We use the representation ρ to identify \mathcal{A} with a subset of \mathcal{K} . Next we define the unitary representation π of G in \mathfrak{H} by

$$(\pi(g)\xi)(x) = \Delta(g)^{1/2} \xi(x^g).$$

Then for any $g \in G, a \in \mathcal{A}$, and $\xi \in \mathfrak{H}$:

$$\begin{aligned} (\pi(g)^* a \pi(g)\xi)(x) &= \Delta(g)^{-1/2} (a \pi(g)\xi)(x^{g^{-1}}) \\ &= \Delta(g)^{-1/2} \int_H a(x^{g^{-1}}y, y^{-1}) (\pi(g)\xi)(y^{-1}) dy \\ &= \int_H a(x^{g^{-1}}y, y^{-1}) \xi((y^g)^{-1}) dy \\ &= \int_H \Delta(g)^{-1} a((xy)^{g^{-1}}, (y^{-1})^{g^{-1}}) \xi(y^{-1}) dy = (a^g \xi)(x), \end{aligned}$$

thus $\pi(g)^* a \pi(g) = a^g$, that is, π implements the action of G on \mathcal{A} . Now we consider \mathcal{X} as a trivial G -algebra and form the generalized algebra $L^1(G, \mathcal{X})$. We observe that for any compact operator $k \in \mathcal{X}$ the mapping $g \rightarrow \pi(g)k$ is measurable from G into the C^* -algebra \mathcal{X} , hence $\sigma: f \rightarrow f^{\natural}$ with

$$f^{\natural}(g) = \pi(g) f(g)$$

defines a norm decreasing injective linear mapping from \mathcal{L} into $L^1(G, \mathcal{X})$. A straightforward computation shows that σ is actually an injective $*$ -homomorphism. We will compute the image of $\mathcal{L}_p = p^* \mathcal{L} p^*$ in $L^1(G, \mathcal{X})$.

By definition for $f \in \mathcal{L}$ we have $(p^* f p^*)(g) = p^g f(g) p$, thus $f \in \mathcal{L}_p$ if and only if $f(g) = p^g f(g) p$ for (almost) all $g \in G$. Moreover

$$f^{\natural}(g) = \pi(g) f(g) = \pi(g) p^g f(g) p = p \pi(g) f(g) p = \varphi(g) p$$

with $\varphi(g) \in \mathbb{C}$ because $p \mathcal{X} p = \mathbb{C} p$. If $|\cdot|_*$ denotes the C^* -norm we have

$$|\varphi(g)| = |f^{\natural}(g)|_* = |f(g)|_* \leq |f(g)|.$$

Thus σ maps \mathcal{L}_p into $L^1(G, \mathbb{C} p) = L^1(G)$. We can consider \mathcal{Q}_0 as a subspace of $\mathfrak{H} = L^2(H)$, which clearly is invariant under π , in particular we have $(\pi(g)u)(x) = \Delta(g)^{-1/2} u(x^g)$ for $g \in G$ and $x \in H$. Let us write $\pi(g)u = gu$. Setting

$$w(g) = (g^{-1}u) \circ u \in \mathcal{A},$$

we get

$$p^g \mathcal{A} p = \mathbb{C} w(g), \quad |w(g)|_* = 1.$$

This is a consequence of the following remarks: If $u, v \in \mathcal{Q}_0$ with $\|u'\|_2 = \|v'\|_2 = 1$, then either by direct computation in $L^1(H, \mathcal{Q})$ or by geometric arguments one sees that $(v \circ v)(v \circ u)(u \circ u) = v \circ u$, specifically $p^g = (g^{-1}u) \circ (g^{-1}u)$ implies $p^g w(g) p = w(g) \in p^g \mathcal{A} p$. The rest follows because $p^g \mathcal{A} p$ has dimension 1.

Our result implies that any $f \in \mathcal{L}_p$ can be written as $f(g) = \varphi(g) w(g)$ with a complex-valued measurable function $\varphi: g \rightarrow \varphi(g)$. Moreover, as $\pi(g) w(g) = p$, it follows that

$$f^{\natural}(g) = \varphi(g) p.$$

Thus writing $\omega(g) = |w(g)|$ (norm in $\mathcal{A}!$) we obtain the following result: The image of \mathcal{L}_p under the homomorphism σ is exactly the Beurling subalgebra $L^1(G, \omega dx)$ of $L^1(G) = L^1(G, \mathbb{C} p)$ of $L^1(G, \mathcal{X})$ with respect to the weight function $\omega(g) = |g^{-1}u \circ u|$.

That ω is a weight function follows from the fact that σ is multiplicative. Furthermore, $\omega(g) \geq |\omega(g)|_* = 1$, thus $L^1(G, \omega)$ is a subalgebra of $L^1(G)$. For $\omega(g)$ we have the following estimate

$$\begin{aligned} \omega(g) &= |g^{-1}u \circ u| = \int_H |(g^{-1}u)^x u|_2 \Delta_H(x)^{-1/2} dx \\ &\geq \int_H |(g^{-1}u)^x u|_\infty \Delta_H(x)^{-1/2} dx \end{aligned}$$

with Δ_H the Haar module of H . Now

$$|(g^{-1}u)^x u|_\infty = \sup_{y \in H} |(g^{-1}u)(xy) u(y)| \geq |\Delta(g)^{-1/2} u(e) u(x^{-1})|,$$

consequently, with $\Omega = u(e) \int_H \Delta_H(x)^{-1/2} |u(x^{-1})| dx > 0$:

$$\omega(g) \geq \Delta(g)^{-1/2} \Omega.$$

This implies that

$$\delta(f) = \int_G f(g) \Delta(g)^{-1/2} dg \in \mathbb{C}.$$

for every $f \in L^1(G, \omega dx)$ and that $f \rightarrow \delta(f)$ defines a non-Hermitian character of $L^1(G, \omega dx)$ if Δ is not trivial. Hence this algebra and consequently also \mathcal{L}_p and \mathcal{L} cannot be symmetric in this case.

We will see that our last result also has consequences for the Wiener property for Banach $*$ -algebras. In [9, Definition 2], we called \mathcal{B} a Wiener algebra, if every proper closed two-sided ideal $\mathcal{I} \subset \mathcal{B}$ is annihilated by some nontrivial unitary representation (= $*$ -representation in a Hilbert space) of \mathcal{B} . The class [W] consists of all locally compact groups G for which $L^1(G)$ is Wiener. In [9, Corollary 2, p. 275], it was shown that the group $G_{4,9}(0)$ is the only connected solvable Lie group with dimension ≤ 4 which possibly does not belong to [W]. We are now able to prove that $G_{4,9}(0)$ indeed is *not* in [W] and also that it is the only of these groups *not* in [S]. First we show (same situation and notation as in Theorem 5):

THEOREM 6. *If the modular function Δ is not trivial, then $\mathcal{L} = L^1(G, \mathcal{A})$ is not a Wiener algebra.*

Proof. The subalgebra \mathcal{L}_p of \mathcal{L} is certainly not Wiener, because the kernel of the non-Hermitian character δ cannot be annihilated by nontrivial unitary representations of \mathcal{L}_p . While in general the Wiener property is not inherited by arbitrary closed $*$ -subalgebras, the following lemma shows that under certain additional assumptions subalgebras of Wiener algebras are again Wiener:

LEMMA. *Let \mathcal{B} be a Wiener algebra and let p be a Hermitian idempotent*

in the adjoint algebra \mathcal{B}^0 . If $\mathcal{B}p\mathcal{B}$ is dense in \mathcal{B} , then the closed subalgebra $\mathcal{B}_p = p\mathcal{B}p$ is Wiener.

Proof of the Lemma. Let \mathcal{I}_0 be a closed two-sided ideal in \mathcal{B}_p . If $\mathcal{I}_0 \neq \mathcal{B}_p$, then the two-sided closed ideal \mathcal{I} , generated by \mathcal{I}_0 , in \mathcal{B} is proper, because $\mathcal{I}_0 + \mathcal{B}\mathcal{I}_0 + \mathcal{I}_0\mathcal{B} + \mathcal{B}\mathcal{I}_0\mathcal{B}$ is dense in \mathcal{I} and consequently $p\mathcal{I}p$ is contained in \mathcal{I}_0 . Thus $\mathcal{I}_0 \neq \mathcal{B}_p$ implies the existence of a unitary representation π of \mathcal{B} with $\pi(\mathcal{I}) = 0$, hence $\pi(\mathcal{I}_0) = 0$. But $\pi(\mathcal{B}_p) \neq 0$, because otherwise $\pi((\mathcal{B}p\mathcal{B})^2) = 0$ and $\pi(\mathcal{B}p\mathcal{B}) = 0$, which contradicts the assumptions that $\mathcal{B}p\mathcal{B}$ is dense in \mathcal{B} and $\pi \neq 0$.

To apply the Lemma to \mathcal{L} and \mathcal{L}_p of Theorem 6 we only have to show that $\mathcal{L}p^*\mathcal{L}$ is dense in \mathcal{L} . This follows from the fact that $\mathcal{A}p\mathcal{A}$ is dense in \mathcal{A} by Theorem 4 and that $\mathcal{L}\mathcal{A}^*$, the span of all fa^* with $f \in \mathcal{L}$, $a \in \mathcal{A}$, contains $L^1(G, \mathcal{A}^2)$ and thus is dense in \mathcal{L} . Hence $\overline{\mathcal{L}p^*\mathcal{L}} = (\mathcal{L}(\mathcal{A}p\mathcal{A})^*\mathcal{L})^- = (\mathcal{L}\mathcal{A}^*\mathcal{L})^- = \overline{\mathcal{L}^2} = \mathcal{L}$.

As already mentioned Theorems 5 and 6 apply in particular to the group $G_{4,9}(0)$. This group can be written as $G = \mathbb{R} \times_s H_1$ with H_1 the three-dimensional Heisenberg group: $H_1 = \mathbb{R}^3$ as a manifold, with product $(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, x_1y_2 + z_1 + z_2)$. For $t \in \mathbb{R}$ the action on H_1 is given by $(x, y, z)^t = (e^{-t}x, e^ty, z)$, thus $Z = \{(0, 0, z); z \in \mathbb{R}\}$ is the center of G . We may write $L^1(G) = L^1(\mathbb{R}, L^1(H_1))$. The mapping $f \rightarrow f^t$ with

$$f^t(x, s) = \iint f(x, y, z) e^{-i(sv+zs)} dy dz$$

is a homomorphism of $L^1(H_1)$ onto $\Gamma(\mathbb{R}) = L^1(\mathbb{R}, A(\mathbb{R}))$, with

$$(f^t)(x, s) = \iint f(e^tx, e^{-t}y, z) e^{-i(sv+zs)} dy dz = e^tf(e^tx, e^ts).$$

It follows easily that $\mathcal{L} = L^1(\mathbb{R}, \Gamma(\mathbb{R}))$ is a quotient of $L^1(G)$. It follows that Theorems 5 and 6 apply with $G = H = \mathbb{R}$, $\mathcal{Q} = A(\mathbb{R})$, $\mathcal{A} = \Gamma(\mathbb{R})$, and $\Delta(t) = e^{-t}$. Hence \mathcal{L} and consequently $L^1(G)$ is neither symmetric, nor Wiener: $G_{4,9}(0) \notin [S]$, $G_{4,9}(0) \notin [W]$.

From the results of [11] it follows that all connected solvable groups G with $\dim G \leq 4$, which are semidirect products of Abelian groups are in [S]. The remaining groups are of the form $\mathbb{R} \times_s H_1$. In the classification in [1, pp. 180, 181] these are the cases $G_{4,9}(\alpha)$, $G_{4,10}$ and $G_{4,11}(\alpha)$. Here only $G_{4,9}(0)$ and $G_{4,11}(0)$ have one-dimensional center, the others contain the center of H_1 as one-dimensional noncentral subgroups, thus they are in [S], by Theorem 3(2). The "diamondgroup" $G_{4,11}(0)$ is in [S] by [9, Theorem 5] or by the general result $[PG]_0 \subset [S]$, proved recently by Ludwig [12]. So eventually we have found:

COROLLARY. *Among the solvable connected groups of dimension at most 4 the*

group $G_{4,9}(0)$ is the only one which is not in [S]. It is also the only among these groups which is not in [W].

This result is remarkable in so far as it shows that central compact subgroups are everything else but "harmless": There exist groups G with central subgroups $T \cong \mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ for which G/T is in [S], resp. [W], but G is not. This is equivalent with the existence of $G \in [S]$, resp. $G \in [W]$, for which there exist a factor system p with values in \mathbb{T} , i.e., a 2-cocycle in \mathbb{T} , such that the algebra $L^1(G, \mathbb{C}; p) = L^1(G; p)$ is not symmetric, resp. Wiener. In our last section we will show that with respect to symmetry this cannot occur for discrete groups.

As an interesting by-product of our results we find that a closed normal subgroup of a group in [S] need not be in [S]: Let G be the solvable group of all real matrices

$$g(s, t, x, y, z) = \begin{pmatrix} 1 & t & s & z \\ 0 & e^x & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The normal subgroup H of all g 's with $x = 0$ is isomorphic with $G_{4,9}(0)$, hence not in [S]. Let Z be the subgroup of all g 's with $s = t = x = y = 0$ and let W be the subgroup $s = t = y = 0$. Then Z is the center of G and W/Z is the center of G/Z . The centralizer of W is the nilpotent subgroup $s = 0$, hence Theorem 3(1) implies $G \in [S]$.

5

It seems still to be unknown whether arbitrary tensor products of symmetric algebras are again symmetric. It is true, if one factor is commutative. In [10, Satz 3], we proved that it is also true if one factor contains a dense "matrix algebra," in particular the product $\mathcal{A} \otimes \mathcal{K}$, complete with respect to some reasonable norm, of a symmetric algebra \mathcal{A} and the C^* -algebra $\mathcal{K} = K(\mathfrak{H})$ of compact operators of a Hilbert space \mathfrak{H} . But we do not know whether the projective tensor product $\mathcal{A} \widehat{\otimes} \mathcal{B}$ of a symmetric algebra \mathcal{A} and the C^* -algebra $\mathcal{B} = B(\mathfrak{H})$ of all bounded operators of an infinite-dimensional Hilbert space \mathfrak{H} is symmetric. Thus we define:

An involutive Banach algebra \mathcal{A} is called rigidly symmetric, if the projective tensor product $\mathcal{A} \widehat{\otimes} \mathcal{B}$ with a C^ -algebra \mathcal{B} is always symmetric. Let [RS] be the class of locally compact groups G for which $L^1(G)$ is rigidly symmetric.*

Clearly $[A] \cup [C] \subset [RS] \subset [S]$. Since for any Banach space E we have $L^1(G) \widehat{\otimes} E = L^1(G, E)$, it is clear that $G \in [RS]$ iff $L^1(G, A)$ (trivial action,

trivial factor system) is symmetric for every C^* -algebra A . Moreover, it is easy to see that [15, Satz 1] and our present Theorem 1, with obvious modifications, also apply to the class [RS]. Especially, [RS] contains every group G , for which the symmetry of $L^1(G)$ can be derived only from these two theorems and the fact that [A] and [C] are in [RS]. This is the case, e.g., for all connected nilpotent Lie groups. Our aim is to prove

THEOREM 7. *Let Z be a central open subgroup of the locally compact group G . Then $G/Z \in [RS]$ implies $G \in [RS]$.*

Proof. Let \mathcal{B} be a C^* -algebra. We have to show that $\mathcal{L} := L^1(G, \mathcal{B}) = L^1(G) \otimes \mathcal{B}$ is symmetric. After [11] this is equivalent to the fact that every bounded algebraically irreducible representation of \mathcal{L} in a Banach space is preunitary. Thus let τ be such a representation in the Banach space E . Then there exists a bounded (resp. bounded continuous) representation ρ of $L^1(G)$ (resp. of G) and a bounded representation σ of \mathcal{B} in E with $\tau(f \otimes b) = \rho(f) \sigma(b)$ for all $f \otimes b \in L^1(G) \otimes \mathcal{B}$ (resp. with $\tau(f) = \int \rho(x) \sigma(f(x)) dx$ for all $f \in \mathcal{L}$).

As τ is irreducible, the restriction of ρ to Z is a character $\chi \in \hat{Z}$. Because G/Z is discrete we can write $L^1(G) = l^1(G/Z, L^1(Z); P) \cong l^1(G/Z, A(\hat{Z}); \hat{P})$ with some factor system P , resp. $\hat{P} = \{\hat{P}_{x,y}\}$. Here the $\hat{P}_{x,y}$ are continuous functions on \hat{Z} with values in \mathbb{T} . As $\rho|_Z = \chi$, we can factor ρ over $l^1(G/Z, \mathbb{C}; \hat{P}(\chi)) = l^1(G/Z; p)$ with $p_{x,y}(\chi) = \hat{P}_{x,y}(\chi)$. Thus τ factors over $l^1(G/Z; p) \otimes \mathcal{B} = l^1(H, \mathcal{B}; p)$ with $H = G/Z$ acting trivially on \mathcal{B} . There exists a strongly continuous unitary projective representation π of H for the cocycle p in some Hilbert space \mathfrak{H} , i.e., $\pi(x) \pi(y) = p_{x,y} \pi(xy)$. Let \mathcal{A} be the algebra of bounded operators in \mathfrak{H} and $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ a C^* -tensor product of \mathcal{A} and \mathcal{B} . For $f \in l^1(H, \mathcal{B}; p)$ define $f^* \in l^1(H, \mathcal{C})$ (trivial action and trivial factor system) by

$$f^*(x) = \pi(x) \otimes f(x).$$

Then $|f^*(x)| = |\pi(x)| |f(x)| = |f(x)|$, and consequently $T: f \rightarrow f^*$ is a linear isometry from $l^1(H, \mathcal{B}; p)$ into $l^1(H, \mathcal{C})$. Now

$$\begin{aligned} (f * g)^*(x) &= \pi(x) \otimes \sum_y f(xy) g(y^{-1}) p_{xy, y^{-1}} \\ &= \sum_y (\pi(x) p_{xy, y^{-1}} \otimes f(xy) g(y^{-1})) \\ &= \sum_y \pi(xy) \pi(y^{-1}) \otimes f(xy) g(y^{-1}) \\ &= \sum_y (\pi(xy) \otimes f(xy)) (\pi(y^{-1}) g(y^{-1})) \\ &= \sum_y f^*(xy) g^*(y^{-1}) = (f^* * g^*)(x) \end{aligned}$$

Similarly $(f^*)^* = (f^*)^*$. It follows that T is an isometric isomorphism of

*-algebras, hence $L(H, \rho; \mathcal{B})$ is isomorphic with a closed subalgebra of $L(H, \mathcal{C})$. By hypothesis, the latter algebra is symmetric. Consequently also $L(H, \rho; \mathcal{B})$ is symmetric. Thus (τ, E) is preunitary and $G \in [RS]$.

As a corollary we obtain (a possibly stronger) version of Hulanickis result:

COROLLARY 3. *All finite extensions of discrete nilpotent groups are in [RS].*

It is not hard to see that the three-dimensional quotient $H = G_{4,9}(0)/Z$ is in [RS], while $G_{4,9}(0)$ is not even in [S]. This shows that the assumption that G/Z is discrete in Theorem 7 cannot be much weakened. The point is that the functions f^* are in general not measurable, if H is nondiscrete.

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