SYMMETRY (OR SIMPLE MODULES) OF SOME BANACH ALGEBRAS

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Recall that a Banach algebra A with isometric involution $a \to a^*$ is called symmetric if every element of the form a^*a, $a \in A$, has a real nonnegative spectrum. Several authors have investigated the question for which locally compact groups G the convolution algebra $L^1(G)$ is symmetric. Even for simply connected Lie groups G a necessary and sufficient criterion (e.g. in terms of the Lie algebra of G) is not known. Let $G = H \ltimes S$ be the Levi decomposition of the simply connected Lie group G with semisimple H and solvable S. Then the compactness of H is a necessary condition for the symmetry of $L^1(G)$ (non-compact semisimple Lie groups do never have symmetric group algebras, [1]). So, assyme that H is compact. Then the symmetry of $L^1(S)$ is sufficient for the symmetry of $L^1(G)$, [6]. $L^1(S)$ is symmetric if the Haar measure of S has polynomial growth, [7], which is equivalent, by [2], to the fact that all eigenvalues of all operators in the adjoint representation of S on the Lie algebra of S have absolute value 1. But there exist also other solvable Lie groups with symmetric group algebras, see e.g. [6]. On the other hand there is a lot of solvable Lie groups with nonsymmetric group algebras, [9]. Thus, for solvable Lie groups the question of symmetry seems to be very complicated.

In several cases, see [6] or [8], the question of symmetry can be reduced to the study of algebras of the following type which are the main theme in this article:

Let G be a locally compact group and let A be an involutive Banach algebra. Suppose that G acts strongly continuously on A, $(x,a) \to a^x$, by isometric $*$- isomorphisms. Then one can form the algebra $B = L^1(G,A)$ of left Haar integrable A-valued functions on G, see e.g. [3], with multiplication

$$(f * g)\ (x) = \int_G f(xy)^{y^{-1}}\ g(y^{-1})\ dy$$

and involution $f^*(x) = \Delta(x)^{-1} f(x^{-1})^{*x}$ where Δ denotes the modular function of G. Suppose further that U is a semisimple regular symmetric commutative Banach algebra on which G acts strongly continuously by isometric $*$-isomorphism, also denoted by $(x,u) \to u^x$. Then G acts also on the Gelfand space $\hat{U}$ of U. Fix any $\chi \in \hat{U}$. For $t \in G$ define $t\chi \in \hat{U}$ by $(t\chi)(u) = \chi(u^t) =: \hat{u}(t)$. Suppose that $t \to t\chi$ is an homeomorphism from G onto $\hat{U}$ (consequently, one can consider U as an algebra of functions on G via the Gelfand transform) and that the following two conditions hold:

(i) $U_o := \{u \in U | \hat{u}$ has compact support$\}$ is dense in U.

(ii) For every neighborhood W of e in G there exists $u \in U$, $u \neq 0$, and a continuous map $f: G \to U$ such that $\hat{u}$ is supported by W and $f(z)(x) = \hat{u}(xz)$ for all $x, z \in G$.

From these assumptions one can deduce, see [6], Theorem 4, that $L^1(G,U)$ is simple and symmetric and contains a lot of hermitian rank one projections (in fact, they span a dense two-sided ideal) which will be crucial in the sequel. Moreover, U and A are connected by the following assumptions: A has a U-module structure which is compatible with all the other operations,

i.e. $|u\,a| \leq |u|\;|a|$, $u(ab) = (ua)b = a(ub)$,

$(ua)^* = u^* a^*$, $(ua)^x = u^x a^x$

for all $a,b \in A$, $u \in U$, $x \in G$. In other words, when we form the Banach $*$-algebra $U \oplus A$ with the obvious operations then G acts strongly continuously by isometric $*$-isomorphisms on $U \oplus A$ and U is central in $U \oplus A$. Assume (and this is the last assumption) that UA is dense in A. What we want to do is to deduce properties of $B = L^1(G,A)$ from properties of A and vice versa. It is known that symmetry of A implies symmetry of B, [8]. In this paper I will give a different proof by a more general approach.

But before doing so, we should give an example of the situation described above in order to show that such a situation occurs "in nature".

Suppose that H is a Lie group with a normal subgroup $K = \mathbb{R}^2$ on which H acts (by inner automorphisms) via $\left\{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| \; x \; \mathbb{R}\right\}$. Then $Z = \{0\} \times \mathbb{R}$ is central in H. Let N be the centralizer of K. Then H is isomorphic to $\mathbb{R} \ltimes N$ and $L^1(H)$ is isomorphic to $L^1(\mathbb{R}, L^1(N))$ where the action of $\mathbb{R}$ is induced by the inner automorphisms. Choose a non-trivial character $\eta : Z \to T$ and form the algebra $L^1(H)_\eta$ of all measurable functions $f : H \to \mathbb{C}$ with $f(xz) = \eta(z)\, f(x)$ for (almost) all $z \in Z$, $x \in H$ and $\int_{H/Z} |f| < \infty$. Similarly, form $A := L^1(N)_\eta$ and $U := L^1(K)_\eta \simeq L^1(\mathbb{R})$. Then the triple $G = \mathbb{R}$, A, U satisfies the assumptions described above and $L^1(\mathbb{R}, A)$ is isomorphic to $L^1(H)_\eta$.

To attack the symmetry of $B = L^1(G,A)$ (in the general situation) we use the following characterization of symmetric algebras which is proved in Naimarks book "Normed rings" (in a different formulation), see also [5]:

An involutive Banach algebra C is symmetric iff for every algebraically irreducible representation ρ of C in a Banach space E (in the sequel, we will use the term "simple C-module" instead of "algebraically irreducible...") there exist a (topologically irreducible) $*$-representation π of C in the Hilbert space $\mathcal{H}$ and a non-zero interwining operator E in $\mathcal{H}$, i.e. $\mathrm{Hom}_C(E, \mathcal{H}) \neq 0$.

Thus, to decide whether a given involutive Banach algebra C is symmetric or not one may proceed in the following manner:

1° "Describe" all topologically irreducible $*$-representations of C.

2° "Describe" all simple C-modules.

3° Decide whether there exist interwining operators.

Of course, 1° and 2° are of independent interest, and for group algebras (or "related" algebras) a lot is known concerning 1°, but there seems to be more or less no information available on point 2° (except for group algebras of groups with "large" compact subgroups as semisimple Lie groups or semidirect products of compact groups with normal abelian subgroups, so-called motion-groups).

Another consequence of this characterization of symmetric algebras is the following: Let H be a Lie group as in the example. Then $L^1(H)$ is symmetric iff $L^1(H/Z)$ and $L^1(H)_\eta$ are symmetric for all non-trivial characters $\eta: Z \to T$. Therefore, one has to study the algebra $L^1(H)$ which are of the type discussed in this article.

From now on, we assume that G, U, A have the properties described above. For $B = L^1(G,A)$, we want to carry out the program $1^o - 3^o$. In fact, we can solve 1^o more or less completely (Theorem 1) and 2^o to such an extent (Theorem 2) that we can show that symmetry of A implies symmetry of B (Corollary to Theorem 2).

Fix $\chi \in \hat{U}$ once and for ever and define (for $t \in G$, $w \in U$)

$$\hat{w}(t) := \chi(w^t) .$$

Then we have the formula $w^s(t) = \hat{w}(st)$. Choose an $u \in U$, $0 \neq u = u^*$ such that $\hat{u}$ has a compact support. Form

$$v: = \int_G u^y u^y u \, dy \in U .$$

v has the property that $\hat{v}(x) = \Delta(x)^{-1} \hat{u}(x) \int_G \hat{u}(z)^2 \, dz$ for all $x \in G$. Assume that $\int_G \hat{u}(z)^2 \, dz = \int_G \hat{v}(z)^2 \, dz$

(if we start with an arbitrary u then a certain constant multiple satisfies this equation).

Define $p : G \to U$ by $p(x) = v^x u$. p has the following properties:

(1) p is a continuous function with compact support, especially $p \in L^1(G,U)$.

(2) $p(x) = \Delta(x)^{-1} u^x v$ for all $x \in G$.

(3) $p = p^*$

(4) $p * p = p$

(5) $p * L^1(G,U) * p = \mathbb{C}\, p \neq 0$.

Let I be the closure of $\operatorname{Kern}\chi A$ in A, I is an $*$-ideal in A. Let $A' : A/I$ be the quotient algebra and denote by

$$Q : A \to A'$$

the quotient morphism.

From the assumptions it follows that $B = L^1(G,A)$ is an $L^1(G,U)$-bimodule and $L^1(G,U) * B * L^1(G,U)$ is dense in B. Especially, we can form $p * B * p$ which is a closed subalgebra of B. The first step is to establish a dense $*$-morphism $T : p * B * p \to A'$.

Let f be an element of $p * B$ ($\supset p * B * p$), i.e. $f = p * f$. Then we have

$$f(x) = (p * f)(x) = \int_G p(xy)^{y^{-1}} f(y^{-1})\, dy =$$

$$= \int_G \{\Delta(xy)^{-1} u^{xy} v\}^{y^{-1}} f(y^{-1})\, dy =$$

$$= \Delta(x)^{-1} \int_G \Delta(y)^{-1} u^x v^{y^{-1}} f(y^{-1})\, dy =$$

$$= \Delta(x)^{-1} \int u^x v^y f(y)\, dy = \Delta(x)^{-1} u^x \varphi$$

with $\varphi = \int_G v^y f(y)\, dy$.

Now, let $f \in p * B * p$, let φ be as above and define

$$T : p * B * p \to A'$$

by

$$\text{If} := ||\hat{u}||_2^{-2} \int_G Q(\hat{v}(t)\ \phi^t)\, dt.$$

<u>Proposition</u> T is a dense $*$-morphism.

<u>Proof</u>. A straightforward computation shows that T is multiplicative

and involutive. To prove the density we observe the following two facts:

$$Q(wa) = \hat{w}(e)\, Q(a) \quad \text{for all} \quad w \in U, \quad a \in A.$$

$$\text{If} \quad g \in p{*}B, \quad g(x) = \Delta(x)^{-1}\, u^x \varphi,$$

$$\text{then} \quad T(g{*}p) = ||\hat{u}||_2^{-2} \int_G Q(\hat{v}(t)\varphi^t)\,dt.$$

Without loss of generality we may assume that $\hat{v}(e) \neq 0$. To approximate a given $Q(a) \in A'$, for every neighborhood W of e we choose $w \in U$ such that $\hat{w}$ is supported by W, $\hat{v}\,\hat{w}$ is nonnegative and the integral over $||\hat{u}||_2^{-2}\,\hat{v}\,\hat{w}$ is one. Define $g \in p{*}B$ by $g(x) = \Delta(x)^{-1}\, u^x\, wa$. Then

$$T(g\ p) = ||\hat{u}||_2^{-2} \int_G Q(\hat{v}(t)\ w^t\ a^t)\ dt =$$

$$= ||\hat{u}||_2^{-2} \int_G \hat{v}(t)\ \hat{w}(t)\ Q(a^t)\ dt.$$

This integral is arbitrarily close to $Q(a) =$

$$= ||\hat{u}||_2^{-2} \int_G \hat{v}(t)\ \hat{w}(t)\ Q(a)\ dt \quad \text{if} \quad W \text{ is small enough.}$$

For a Banach $*$-algebra C, we denote by $\mathrm{Rep}(C)$ the equivalence classes of non degenerated $*$-representations in Hilbert spaces. The morphism T induces a map $\overline{T} : \mathrm{Rep}(A') \to \mathrm{Rep}(p{*}B{*}p)$. By restriction we get a map $S : \mathrm{Rep}(B) \to \mathrm{Rep}\ (p{*}B{*}p)$; if $\pi \in \mathrm{Rep}(B)$ is a representation in $\mathcal{H}$ then $S(\pi)$ is a representation in $\pi(p)\,\mathcal{H}$; $\pi(p)$ is not zero because $B{*}p{*}B$ is dense in B. Moreover, we can construct a map $\mathrm{ind} : \mathrm{Rep}(A') \to \mathrm{Rep}(B)$ in the following way: Let ρ be a representation of A' in $\mathcal{K}$. Let the representation $\tilde{\rho}$ of A in $\mathcal{H} = L^2(G,\mathcal{K})$ be defined by $(\tilde{\rho}(a)\xi)(x) = \rho(Q(a^x))(\xi(x))$. For $t \in G$ and $\xi \in \mathcal{H}$ let $\xi^t \in \mathcal{H}$ be the function $\xi^t(x) = \xi(tx)$. Then we define the representation

$\pi = \mathrm{ind}(\rho)$ in H by

$$\pi(f)\xi = \int \tilde{\rho}(f(t)^{t^{-1}})\xi^{t^{-1}}\, dt.$$

Theorem 1 *The diagram*

$$\begin{array}{ccc} \mathrm{Rep}(B) & \xleftarrow{\ \mathrm{ind}\ } & \mathrm{Rep}(A') \\ & {\scriptstyle S}\searrow \quad \swarrow{\scriptstyle \bar{T}} & \\ & \mathrm{Rep}(p{*}B{*}p) & \end{array}$$

is commutative, and all three maps are bijections.

Remark *Since all three maps are compatible with direct sums, irreducible representations correspond to irreducible representations.*

Proof. For simplicity we assume that U is contained in A because the theorem is easily deduced from the corresponding theorem for $\tilde{A} = A \oplus U$, $\tilde{B} = L^1(G,\tilde{A})$, $\tilde{A}' = A' \oplus \mathbb{C}\,1$ and $p{*}\tilde{B}{*}p = p{*}B{*}p \oplus \mathbb{C}\,p$.

$\bar{T} = S \circ \mathrm{ind}$:

Let ρ be a $*$-representation of A' in K, let $H = L^2(G,K)$, $\tilde{\rho}$ and $\pi = \mathrm{ind}\,\rho$ be as above.

Then for $f \in B$, $\xi \in H$ and $s \in G$ we have

$$(\pi(f)\xi)(s) = \int_G dr\, \rho\,(Q(f(sr)^{r^{-1}}))\ (\xi(r^{-1})).$$

For $f \in p{*}B$, $f(x) = \Delta(x)^{-1}\Delta u^x\,\varphi$, we find

$$(\pi(f)\xi)\ (s) = \Delta(s)^{-1}\int_G dr\ \hat{u}(s)\ \rho(Q(\varphi^r))\xi(r) =$$

$$= ||\hat{u}||_2^{-2}\ \hat{v}(s)\int_G \rho(Q(\varphi^r))\xi(r)\ dr.$$

Especially, for $\varphi = v$, i.e. $f = p$, we get

$$(\pi(p)\xi)(s) = ||\hat{u}||_2^{-2}\, \hat{v}(s) \int_G \hat{v}(r)\xi(r)\, dr.$$

Therefore, $V : K \to H$ defined by

$(V\eta)(x) = ||\hat{v}||_2^{-1}\, \hat{v}(x)\eta$ is an isometry from K onto $\pi(p)\, H$.

A trivial computation shows that $\rho(Tf) = V^{-1}\, \pi(f) V$ for $f \in p*B*p$. Hence, $\rho \circ T$ and $S(\pi)$ are unitarily equivalent.

From the density of T it follows that $\bar{T}$ is injective. Therefore, it suffices to show that S and ind are onto.

<u>S is onto</u>: Let ρ be a $*$-representation of $p*B*p$. Since S is compatible with direct sums we may assume that ρ is cyclic, let ξ be a cyclic vector of norm 1 and let $f(x) := \langle\rho(x)\xi,\xi\rangle$ be the associated positive form. Define $F : B \to \mathbb{C}$ by $F(x) = f(p*x*p)$. As the computations in the proof of the Lemma in [8] show (approximate identities are not needed at this point), F is a positive form. Here we need that $\sum_{q,r\in P} q*B*r$ is dense in B (P = projections of rank one in $L^1(G,U)$) which follows from the fact that the linear span of P is dense in $L^1(G,U)$. Moreover, F can be extended as a positive form to $B \oplus \mathbb{C}\, 1$, $\tilde{F} : B \oplus \mathbb{C}\, 1 \to \mathbb{C}$ is defined by $\tilde{F}\,(x + \lambda) = F(x) + \lambda =$ $F(x) + \lambda = F(x) + \lambda f(p) = F(x + \lambda p)$. $\tilde{F}$ is positive because $\tilde{F}((x+\lambda)^*(x+\lambda)\quad \tilde{F}\,(p(x+\lambda)^*(x+\lambda)p) = F\,(|(x+\lambda)p|^*\,|(x+\lambda)p|) \geq 0$. From the GNS-construction it follows very easily that the representation π associated to F has the property that $S(\pi)$ is unitarily equivalent to ρ.

<u>ind is onto</u>: This fact can be deduced from a suitable version of the imprimitivity theorem, see e.g. [4]. But it can also be shown in the following simple way:

Let $C_\infty(G,A')$ be the Banach $*$-algebra of all continuous functions $G \to A'$ vanishing at infinity with the uniform norm and the pointwise operations. Let $\mathcal{F} : A \to C_\infty(G,A')$ be defined by $(\mathcal{F}a)(t) = Q(a^t)$, $\mathcal{F}$

is a dense $*$-morphism. F is G-equivariant if we let G act on $C_\infty(G, A')$ by left translations. Moreover, let $C^*(A')$ be the C^*-hull of A'. By composition, we find a G-equivariant dense $*$-morphism $\tilde{F} : A \to C_\infty(G, C^*(A')) =: \tilde{A}$. Since every irreducible $*$-representation of A (which is a character on U) factorizes through $\tilde{F}$, $\tilde{A}$ is just the C^*-hull of A.

Every $*$-representation of B factorizes through $B \to \tilde{B} = L^1(G, \tilde{A})$. Therefore, we may assume that $A = C_\infty(G,D)$ for some unital C^*-algebra D, $U = C_\infty(G)$ and G acts by left translations on A. In this case, A' is canonically isomorphic to D, and $T : p * B * p \to A'$ is an isometry onto A'. Now, let π be a $*$-representation of B, form the representation $S(\pi)$ of $p * B * p$ and define the representation ρ of A' by $\rho = S(\pi) \circ T^{-1}$. It is easy to see that $\mathrm{ind}(\rho)$ is unitarily equivalent to the given π.

Theorem 2: Let G, U, A and $B = L^1(G, A)$ be as always. Let p, A' and $T : p * B * p \to A'$ be as constructed above. If we associate to the simple B-module E the $p*B*p$ - module pE we get a bijection from the set of isomorphism classes of simple B-modules onto the set of isomorphism classes of simple $p * B * p$ - modules. Moreover, for every simple $p * B * p$ - module M there exists a simple A'-module M' such that $\mathrm{Hom}_T(M, M') \neq 0$, i.e. there is a non-zero linear map $R : M \to M'$ with $R(fm) = T(f)\, R(m)$ for $f \in p * B * P$, $m \in M$.

Corollary 1: Symmetry of A implies symmetry of B. To give at least one application, we formulate and prove (see also [6]).

Corollary 2: Let K be a compact group and let D be a symmetric Banach $*$-algebra on which K acts strongly continuously by isometric $*$-isomorphisms. Then $L = L^1(K, D)$ is symmetric.

Proof: As in the proof of Theorem 1 we may assume that U is contained in A.

If E is a simple B-module then $pE \neq 0$ because $B * p * B$ is dense in B, and it is easy to see that pE is a simple $p * B * p$ - module.

In order to show that $E \mapsto pE$ is a bijection we construct the inverse map. Let M be a simple $p*B*p$ - module. Then we form the induced B-module $M^i = B \otimes_{p*B*p} M$ and define $\tilde{M} := M^i/(\{\xi \in M^i \mid B\xi = 0\})$. M^i is not zero because the map $B \otimes M \to M$, $b \otimes \xi \to pbp\,\xi$, factorizes through M^i. pM^i (and hence $p\tilde{M}$) is isomorphic to the given M. Moreover, M^i (and hence $\tilde{M}$) is cyclic: for every ξ, $0 \neq \xi \in M$, $p \otimes \xi$ is a generator of M^i. Therefore, we can realize M^i as a quotient of B and we can introduce a Banach space structure on M^i and on $\tilde{M}$. So, $\tilde{M}$ is

(1) a cyclic Banach B-module, generated by every non-zero element in $p\tilde{M}$ with (2) $p\tilde{M} \cong M$.

From $(UA)^- = A$ it follows that B^2 is dense in B. But then $\tilde{M}$ has the property that

(3) $\xi \in \tilde{M}$, $B\xi = 0$ implies $\xi = 0$.

From (1), (2) and (3) it follows very easily that $\tilde{M}$ is a simple B-module.

The maps $E \mapsto pE$ and $M : \mapsto \tilde{M}$ are inverse to each other.

Now, let M be a simple $p*B*p$ - module, $M = pE$ for a (unique) simple B-module E. We want to construct a simple A'-module M' as in the theorem. The B-module structure on E is given by an A-module structure on E, $(a, \varepsilon) \mapsto a\varepsilon$, and a compatible G-action on E, $(t, \varepsilon) \mapsto \varepsilon^t$, compatible means that $(a\varepsilon)^t = a^t \varepsilon^t$. For $f \in B$ and $\varepsilon \in E$ we have the formula

$$f\varepsilon = \int_G (f(t)\varepsilon)^t \, dt = \int_G f(t)^{t^{-1}} \varepsilon^{t^{-1}} \, dt.$$

The functions $f \in B*p$ are of the form $f(x) = \psi^x u$ with some $\psi \in A$. If we define $A^\infty = \{\psi \in A \mid \int_G |\psi^x u| dx < \infty\}$ we get a surjective A-linear map $\Psi : A^\infty \to B*p$ (A^∞ and $B*p$ are considered as left A-modules); by the way, A^∞ is a two-sided, G-invariant, dense ideal in A.

For a fixed non-zero ξ in M we get by multiplication a B-linear map from $B * p$ onto E. Composition with Ψ gives a surjective A-linear map.

$$\Phi : A^{\infty} \to E$$

Especially, the kernel Ω of Φ is a left ideal in A.
For $\psi \in A^{\infty}$ we have the formula

$$\Phi(\psi) = \int_G \psi u^{t^{-1}} \xi^{t^{-1}} dt.$$

From $\xi = p\xi$ we obtain

$$\xi = \int_G p(t)^{t^{-1}} \xi^{t^{-1}} dt = \int_G vu^{t^{-1}} \xi^{t^{-1}} dt = \Phi(v).$$

Because Φ is G-equivariant, Ω is a G-invariant left ideal in A. Ω is contained in the annihilator $\mathrm{Ann}_A(\xi)$ of ξ in A because $\psi \in \mathrm{Kern}\Phi$ implies

$$0 = \int_G \psi u^{t^{-1}} \xi^{t^{-1}} dt, \text{ hence}$$

$$0 = \int_G v\psi u^{t^{-1}} \xi^{t^{-1}} dt = \int_G \psi vu^{t^{-1}} \xi^{t^{-1}} dt = \psi\xi.$$

Moreover, the integral representation of ξ shows that if we take $w \in U$ with $wv = v$ (clearly, such w's exist since $\hat{v}$ is compactly supported) then w is a right unit for the left ideal $\mathrm{Ann}_A(\xi)$. Therefore, $\mathrm{Ann}_A(\xi)$ (and hence Ω) is contained in a maximal modular left ideal Λ of A. From Schur's Lemma, it follows that U acts by a character on the simple A-module A/Λ, i.e. there exists $s \in G$ such that the kernel of $s\chi$ (recall $s\chi(w) = \chi(w^s)$ for $w \in U$) is contained in Λ. Λ^s is also a maximal modular left ideal in A. $M' := A/\Lambda^s$ is a simple A-module and in fact an A'-module because $\mathrm{Kern}\ \chi$ is contained in Λ^s. Since Ω is

G-invariant, Ω is contained in Λ^s.

To finish the proof of the theorem we have to construct a non-zero T-linear map $M \to M'$. Denote by $Q' : A \to M' = A/\Lambda^s$ the quotient map (Q' factorizes through $Q : A \to A'$) and define $\tilde{R} : A \to M'$ by

$$\tilde{R}(a) = \int_G \hat{v}(t)\, Q'(a^t)\, dt = \int_G Q'(v^t a^t)\, dt.$$

The restriction of $\tilde{R}$ to A^∞ factorizes through the quotient map $A^\infty \to A^\infty/\Omega$ and gives a linear map $A^\infty/\Omega \to M'$. Since Φ induces an isomorphism from A^∞/Ω onto E, we get a linear map $R : E \to M'$. We claim that the restriction of R to $M = pE$ is a dense T-linear map from M into M'.

Let $f, g \in p * B * p \subset p * B$, $f(x) = \Delta(x)^{-1} u^x \varphi$, $g(x) = \Delta(x)^{-1} u^x \psi$, and let $\varepsilon = g\xi \in M$. We have to show that $R(f\varepsilon) = Tf\, R\varepsilon$.

From $\xi = \int_G v u^{x^{-1}} \xi^{x^{-1}} dx$, it follows that

$$\varepsilon = g\xi = \int_G g(t)^{t^{-1}} \xi^{t^{-1}}\, dt = \int_G\int_G u\psi^t v^t u^{x^{-1}} \xi^{x^{-1}}\, dt\, dx = \Phi\Big(\int_G u\psi^t v^t\, dt\Big)$$

and, by definition, that $R\varepsilon = \tilde{R}\big(\int_G u v^t \psi^t\, dt\big) = \int_G \hat{v}(x) Q'\big(\int_G u^x v^{tx} \psi^{tx}\, dt\big)dx =$

$$= \int_G\int_G \hat{v}(x)\hat{u}(x)\Delta(x)^{-1} Q'(v^y \psi^y)\, dx\, dy = \int_G \hat{v}(y) Q'(\psi^y)\, dy, \text{ because}$$

$$\int_G \hat{v}(x)\hat{u}(x)\Delta(x)^{-1}\, dx = 1.$$

This is just the image of Tg under the quotient map $A' \to M'$.
Since T is dense, R is dense, too.
Now, we compute $R(f\varepsilon)$:

$$f\varepsilon = \int_G f(y)^{y^{-1}} \varepsilon^{y^{-1}}\, dy = \int_G \Delta(y)^{-1} u\varphi^{y^{-1}} \varepsilon^{y^{-1}}\, dy = \int_G u\varphi^y \varepsilon^y\, dy =$$

$$= \int_G\int_G\int_G u\varphi^y u^y \psi^{ty} v^{ty} u^{x^{-1}} \xi^{x^{-1}}\, dt\, dx\, dy = \Phi\Big(\int_G\int_G \Delta(y)^{-1} u\varphi^y u^y \psi^z v^z\, dy\, dz\Big),$$

hence $R(f\varepsilon) = \int\limits_G \hat{v}(t)Q'(\iint\limits_{GG}\Delta(y)^{-1}u^t\varphi^{yt}u^{yt}\psi^{zt}v^{zt}\,dy\,dz)dt =$

$$= \int\limits_G \hat{v}(t)Q'(\iint\limits_{GG} \Delta(y)^{-1}\Delta(t)u^t f^y u^y \psi^z v^z \Delta(t)^{-2}\,dy\,dz)\,dt =$$

$$= \iiint\limits_{GGG}\Delta(y)^{-1}\Delta(t)^{-1}\hat{v}(t)\hat{u}(t)\hat{v}(z)\hat{u}(y)Q'(\varphi^y\psi^z)\,dt\,dy\,dz =$$

$$= \iint\limits_{GG}\Delta(y)^{-1}\hat{u}(y)\hat{v}(z)Q'(\varphi^y\psi^z) = ||\hat{u}||_2^{-2}\iint\limits_{GG}\hat{v}(y)\hat{v}(z)Q'(\varphi^y\psi^z)dy\,dz\,.$$

From the definition of Tf and the formula for $R\varepsilon$ it follows that $R(f\varepsilon) = Tf\,R\varepsilon$.

The theorem is proved. For the proof of corollary 1 we use (of course) the characterization of symmetry by simple modules. Let E be a simple B-module, let $M = pE$. We may assume that $E = \tilde{M} = M^i/\{\gamma\in M^i|B\gamma = 0\}$. From the theorem it follows that there exists a non-zero T-linear map from M into a simple A'-module M'. By assumption we find a non-degenerated (irreducible)$*$-representation ρ of A' in the Hilbert space K and a non-zero A'-interwinning operator $M'\to K$. By composition we get a non-zero T-linear map $R : M \to K$. Let $\pi = \mathrm{ind}(\rho)$ be the induced representation of B in $H = L^2(G,K)$. We claim that we can embed E into H.

Define $\tilde{R} : M\to H$ by $(\tilde{R}\xi)(t) = \hat{v}(t)R\xi$ and $V : B\otimes M\to H$ by $V(g\otimes\xi) = \pi(g)(\tilde{R}\xi)$.

From the fact that $R : M \to K$ is T-linear one easily deduces the formula

$$\pi(f)(\tilde{R}\xi) = \tilde{R}(f\xi)$$

for $f\in p*B*p$ and $\xi\in M$.

Therefore, the map $V : B\otimes M \to H$ factorizes through $M^i = B\otimes_{p*b*p}M$ and gives a B-linear map $\tilde{V} : M^i \to H$. Since π is a non-degenerated representation the subspace $\{\gamma\in M^i|B\gamma = 0$ is contained in the kernel of $\tilde{V}$, and finally we find a non-zero B-linear map from $\tilde{M} = E$ into H.

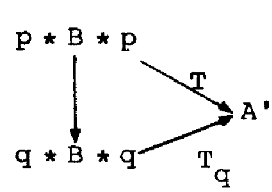
<u>Proof of Corollary 2</u>. Let $C = C(K,D)$ be the Banach $*$-algebra of all continuous functions from K into D with the pointwise operations and the uniform norm. We let K act on C by left translations, i.e. $f^t(x) = f(tx)$ for $t, x \in K$ and $f \in C$. The map $\alpha : D \to C$ defined by

$$\alpha(d)(x) = d^x$$

is a K-equivariant injective $*$-isomorphism from D onto a closed subalgebra of C and induces a $*$-isomorphism from $L = L^1(K,D)$ onto a closed subalgebra of $L^1(K, C)$. Since closed $*$-subalgebra of symmetric Banach algebras are symmetric it suffices to show that $L^1(K, C)$ is symmetric. But the triple $G = K$, $A = C$, $U = C(K, \mathbb{C})$ satisfies the usual assumptions of this paper and, consequently, the symmetry of C implies the symmetry of $L^1(K, C)$.

The selection of p was rather arbitrary. So, one could think that it might be useful to study different p's. That this is not the case is shown by the following remark which we state without proof.

<u>Remark</u> Let $q \in L^1(G, U)$ be another hermitean rank one projector. Then q has a similar structure as p even though the representing functions in U need not to have compactly supported Gelfand transforms. Nevertheless, in a similar way one can construct a $*$-morphism T_q from $q * B * q$ into A'. But there exists a unique (partial isometry) $k \in L^1(G, U)$ with $k^* * k = p$ and $k * k^* = q$ which gives rise to a $*$-isomorphism from $p * B * p$ onto $q * B * q$ such that the diagram



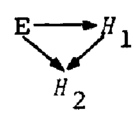
commutes.

We conclude this article with some open questions.

(1) Does the symmetry of B imply the symmetry of A ?

The answer is yes if A is commutative (since $T : p * B * p \to A'$ is dense and $p * B * p$ is symmetric every multiplicative functional on A' is hermitean) or if G is discrete because in this case the map $T : p * B * p \to A'$ is onto. In general, T is not onto, there is an example in [9]. I don't know the answer even for compact abelian G.

(2) Let C be a symmetric Banach $*$-algebra. The characterization of symmetry used in this paper tells us that for every simple C-module E there exists an irreducible $*$-representation in $\mathcal{H}$ and a non-zero C-linear map $E \to \mathcal{H}$. But is $E \to \mathcal{H}$ unique in the sense that if there are two embeddings $E \to \mathcal{H}_j$ $(j = 1,2)$ that there exists an interwinning operator $\mathcal{H}_1 \to \mathcal{H}_2$ such that



commutes ?

The answer is yes if C is the group algebra of a connected nilpotent Lie group. Suppose that A has this uniqueness property. Is it true that $B = L^1(G, A)$ has this uniqueness property ?

(3) Let [NeS] be the class of all Banach $*$-algebras C with the property that every topologically completely irreducible Banach C-module is Naimark-equivalent to an irreducible $*$-representation of C. Of course, every algebra in this class is symmetric. Moreover, the group algebras of connected two step nilpotent Lie groups and of motion groups, see [10], are in [NeS]. Is it true that B is in [NeS] provided that A is in [NeS]?

(4) It is known that in a certain sense it is impossible to classify the irreducible $*$-representations of a non-type I group but possibly it is simpler to classify the simple modules over a symmetric group algebra.

To test this one should treat the following example:

Let $C(T^2)$ be the C*-algebra of all continuous functions on the torus, let $\mathbb{R}$ act on $C(T^2)$ by

$$f^t(z, w) = f(e^{it}z, e^{i\alpha t}w)$$

for some irrational α, and form the algebra $L = L^1(\mathbb{R}, C(T^2))$; this algebra is closely related to the group algebra of the Mautner group. If we let act L on $E = C(T^2)$ by

$$f\xi = \int_{\infty}^{\infty} f(t)^{t^{-1}} \xi^{t^{-1}} dt$$

where the product of $f(t)^{t^{-1}} \in C(T^2)$ and $\xi^{t^{-1}} \in C(T^2)$ is the usual pointwise product in $C(\ ^2)$, we get a simple L-module. From this L-module we can construct further simple L-modules if we apply first the automorphisms $U_z : L \to L$ $(z \in \mathbb{R})$ defined by

$$(U_z f)(t) = e^{itz} f(t) \quad .$$

Are <u>all</u> simple L-modules obtained in this way ?

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