Symmetry and nonsymmetry for a class of exponential Lie groups

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Abstract. This paper is a contribution to the question for which simply connected Lie groups G the group algebra $L^1(G)$ is symmetric (= hermitean). For groups G in a certain subclass of the class of exponential Lie groups a necessary and sufficient condition for the symmetry of $L^1(G)$ is given in terms of the Lie algebra of G. This subclass contains all groups with Lie algebra $\mathfrak{g}$ such that the (additive) Jordan decomposition is possible in $\mathrm{ad}(\mathfrak{g})$. The condition was introduced by Boidol in exploring the $*$-primitive ideal space, and so the main result of the paper implies that for some exponential Lie groups G the symmetry of $L^1(G)$ is equivalent to a certain property of the $*$-primitive ideal space. Moreover, an example of a seven-dimensional exponential Lie group G with symmetric group algebra is given where the existing general methods are not applicable to get the symmetry.

Recall that a Banach algebra A with isometric involution $a \to a^*$ is called symmetric if every element of the form a^*a, $a \in A$, has a real nonnegative spectrum. Several authors have investigated the question for which locally compact groups G the convolution algebra $L^1(G)$ is symmetric. Even for simply connected Lie groups G a necessary and sufficient condition (e.g. in terms of the Lie algebra of G) is not known. Let $G = H \ltimes S$ be the Levi decomposition of the simply connected Lie group G with semisimple H and solvable S. Then the compactness of H is a necessary condition for the symmetry of $L^1(G)$ (non-compact semisimple Lie groups do never have symmetric group algebras, [7]). So, assume that H is compact. Then the symmetry of $L^1(S)$ is sufficient for the symmetry of $L^1(G)$, [13]. Therefore, before treating the general case one should first decide for which simply connected solvable Lie groups S the algebra $L^1(S)$ is symmetric. But at the moment, also this question seems to be too general. While for solvable Lie groups with polynomially growing Haar measure the answer is known (they always have symmetric group algebras, [14]) the problem is not solved for exponential Lie groups. In this paper, I will give a partial solution, namely for the following class of groups.

Definition 1. An exponential real Lie algebra $\mathfrak{g}$ belongs to the class $[EA]$ provided that there exists a semidirect decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ with a nilpotent ideal $\mathfrak{n}$ and a commutative subalgebra $\mathfrak{s}$ which acts by semisimple (i.e. diagonalizable in the complexification $\mathfrak{n}_{\mathbb{C}}$ of $\mathfrak{n}$) derivations on $\mathfrak{n}$. Exponential means that the weights of $\mathfrak{s}$ in $\mathfrak{n}_{\mathbb{C}}$ are of

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the form $(1+it)\alpha$ with real t and a real linear form α on $\mathfrak{s}$. The class of simply connected Lie groups whose Lie algebra belongs to $[EA]$ is also denoted by $[EA]$.

Remarks. If $\mathfrak{g}$ is in $[EA]$ then in the above decomposition one may choose $\mathfrak{n}$ to be the nilradical. — Suppose that the exponential Lie algebra $\mathfrak{g}$ has the property that the (additive) Jordan decomposition is possible in the space $\mathrm{ad}(\mathfrak{g})$ of linear transformations on $\mathfrak{g}$. Then $\mathfrak{g}$ belongs to $[EA]$. For $G \in [EA]$, I will give a necessary and sufficient condition for the symmetry of $L^1(G)$. This condition was created by Boidol in [2] to study the class "$[\psi]$", see § 1 and [3]. Besides simple algebraic considerations the proof that this condition implies symmetry requires only Satz 1 in [15] (see also Satz 1 in [13] or Corollary 1 to Theorem 2 in [17]) which contains a method to deduce symmetry from the symmetry of group algebras of lower dimensional groups. But in § 5, I will give an example of a symmetric seven-dimensional exponential Lie group where the method of Satz 1 in [15] is not applicable to get symmetry. In fact, this group is the first example of a symmetric exponential Lie group where the symmetry-proof is not based on Satz 1 in [15].

§ 1. For the convenience of the reader, I want to collect some known facts on symmetry and to introduce Boidols condition.

(A) (see [11]) *An involutive Banach algebra A is symmetric iff for every algebraically irreducible representation ρ of A in a Banach space E there exist a (topologically irreducible) $*$-representation π of A in the Hilbert space $\mathfrak{H}$ and a non-zero intertwining operator from E in $\mathfrak{H}$.*

This characterization has especially consequences for group algebras of groups with central subgroups. Before stating the result, we first introduce the following notation.

Definition 2. Let G be a Lie group with closed connected central subgroup Z, and let χ be a unitary character of Z. Then we denote by $L^1(G)_\chi$ the algebra of all measurable functions $f\colon G \to \mathbb{C}$ with $f(xz)=\chi(z)f(x)$ for (almost) all $z \in Z$, $x \in G$, and $\int_{G/Z} |f| < \infty$. $L^1(G)_\chi$ is a quotient algebra of $L^1(G)$.

With this notation we have

(B) (see [11] or [15]) *$L^1(G)$ is symmetric iff $L^1(G)_\chi$ is symmetric for all unitary characters χ of Z.*

A slight modification of the proof of Theorem 3 in [13] together with (B) gives the following Theorem which is based on Satz 1 in [15], for a more adequate formulation see Theorem 2 in [17].

(C) *Let G be a simply connected solvable Lie group, let Z be a closed connected central subgroup and let χ be a unitary character of Z. Let A be an abelian non-central closed connected normal subgroup with $A \cap Z = \{e\}$, and let C be the centralizer of A. Then $L^1(G)_\chi$ is symmetric if one of the following two conditions hold:*

(i) $\dim A = 1$, *$L^1(G/A)_\chi$ is symmetric and $L^1(C)_{\tilde\chi}$ is symmetric for every unitary character $\tilde\chi$ of AZ with $\tilde\chi|_Z=\chi$ and $\tilde\chi|_A \neq 1$.*

(ii) $\dim A = 2$, *G acts via inner automorphisms on $A \cong \mathbb{C}$ as multiplication by $e^{\alpha(g)(1+it)}$ with real $t \neq 0$ and an homomorphism $\alpha\colon G \to \mathbb{R}$, $L^1(G/A)_\chi$ is symmetric and $L^1(C)_{\tilde\chi}$ ($C=\ker\alpha$) is symmetric for every unitary character $\tilde\chi$ of AZ with $\tilde\chi|_Z=\chi$ and $\tilde\chi|_A \neq 1$.*

In [3], the class $[\psi]$ of involutive Banach algebras was introduced (in [2], these algebras are called $*$-regular): Let $\mathscr{A}$ be such an algebra, and let $\mathscr{A} \to C^*(\mathscr{A})$ be the C^*-hull of $\mathscr{A}$. We denote by $\mathrm{Priv}_*(\mathscr{A})$, resp. $\mathrm{Priv}_*(C^*(\mathscr{A}))$, the space of kernels of irreducible $*$-representations of $\mathscr{A}$, resp. $C^*(\mathscr{A})$, with the Jacobson topology. $\mathscr{A}$ is $*$-regular if the induced map $\mathrm{Priv}_*(C^*(\mathscr{A})) \to \mathrm{Priv}_*(\mathscr{A})$ is an homeomorphism. In [2], the class of exponential Lie groups G with $*$-regular group algebra is determined. Let f be a real linear functional on the Lie algebra $\mathfrak{g}$ of G, let $\mathfrak{m}(f) = \mathfrak{g}(f) + [\mathfrak{g}, \mathfrak{g}]$ with

$$\mathfrak{g}(f) = \{X \in \mathfrak{g} \mid f([X, \mathfrak{g}]) = 0\}$$

and let $\mathfrak{m}(f)^\infty$ be the smallest ideal $\mathfrak{a}$ in $\mathfrak{m}(f)$ such that $\mathfrak{m}(f)/\mathfrak{a}$ is nilpotent. Boidol defines

Definition 3. f is called *inductive* if $\mathfrak{m}(f)^\infty$ is contained in the kernel of f,

and shows: $L^1(G)$ is $*$-regular (G = exponential (solvable) Lie group) iff every functional is inductive.

Remark. For our purposes the following (trivial) characterization of inductive functionals is useful: Let $\mathfrak{n}$ be the nilradical of the (exponential) solvable Lie algebra $\mathfrak{g}$, let $\mathfrak{k}(f) = \mathfrak{g}(f) + \mathfrak{n}$, and let $\mathfrak{k}(f)^\infty$ be the smallest ideal $\mathfrak{a}$ in $\mathfrak{k}(f)$ such that $\mathfrak{k}(f)/\mathfrak{a}$ is nilpotent. Then f is inductive iff $\mathfrak{k}(f)^\infty$ is contained in the kernel of f. The reason is the obvious fact that $\mathfrak{k}(f)^\infty = \mathfrak{m}(f)^\infty$.

§ 2. Here we show that a group G in $[EA]$ has a symmetric group algebra if every functional on the Lie algebra $\mathfrak{g}$ of G is inductive (i.e. $L^1(G)$ is $*$-regular). In the proof we use (C) of § 1 and the following proposition.

Proposition 1. *Let $\mathfrak{g}$ be a solvable Lie algebra, let $\mathfrak{a}$ be an abelian ideal in $\mathfrak{g}$, let $\mathfrak{c}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, and let f be a real linear functional on $\mathfrak{g}$ with $f|_\mathfrak{a} \neq 0$. Then we have:*

(i) *Suppose that $\mathfrak{a}$ is onedimensional and non-central. f is inductive iff $f|_\mathfrak{c}$ is inductive.*

(ii) *Suppose that $\mathfrak{a}$ is twodimensional, $\mathfrak{a} \cong C$, and $\mathfrak{g}$ acts on $\mathfrak{a}$ by multiplication with $\alpha(X)(1+it)$, $X \in \mathfrak{g}$, for some real $t \neq 0$ and some non-zero real functional α on $\mathfrak{g}$. f is inductive iff $f|_\mathfrak{c}$ is inductive.*

Proof. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, $\mathfrak{n}$ is also the nilradical of $\mathfrak{c}$. Let g be the restriction of f to $\mathfrak{c}$. We claim that $\mathfrak{c}(g) = \mathfrak{a} + \mathfrak{g}(f)$. The inclusion $\mathfrak{a} + \mathfrak{g}(f) \subseteq \mathfrak{c}(g)$ is obvious since $\mathfrak{g}(f)$ is contained in $\mathfrak{c}$. For the other inclusion we select an $Y \in \mathfrak{g} \setminus \mathfrak{c}$. Let X be an element of $\mathfrak{c}(g)$, i.e. $f([X, \mathfrak{c}]) = 0$. Let $\lambda = f([X, Y])$. From $f|_\mathfrak{a} \neq 0$ it follows that $f([\mathfrak{a}, Y]) = \mathbb{R}$, especially there exists an $A \in \mathfrak{a}$ with $f([A, Y]) = \lambda$. We get $f([X-A, Y]) = 0$. Since X and A are in $\mathfrak{c}(g)$ we have also $f([X-A, \mathfrak{c}]) = 0$ and, consequently, $X - A \in \mathfrak{g}(f)$ or $X \in \mathfrak{g}(f) + A \subseteq \mathfrak{g}(f) + \mathfrak{a}$. Since $\mathfrak{a}$ is contained in $\mathfrak{n}$ we find

$$\mathfrak{k}(g) = \mathfrak{c}(g) + \mathfrak{n} = \mathfrak{g}(f) + \mathfrak{a} + \mathfrak{n} = \mathfrak{g}(f) + \mathfrak{n} = \mathfrak{k}(f),$$

and hence $\mathfrak{k}(g)^\infty = \mathfrak{k}(f)^\infty$. f vanishes on $\mathfrak{k}(f)^\infty$ iff g vanishes on $\mathfrak{k}(g)^\infty$.

Moreover, we will need the following obvious lemma.

Lemma 1. *Let $\mathfrak{g}$ be a solvable Lie algebra, and let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. Let f be a real linear functional on $\mathfrak{g}$ with $f(\mathfrak{a}) = 0$, and denote by f' the induced functional on $\mathfrak{g}/\mathfrak{a}$. Then f is inductive iff f' is inductive.*

(Only) for the formulation and the proof of Theorem 1 we introduce

Definition 4. Let $\mathfrak{g}$ be in $[EA]$, let $\mathfrak{z}$ be a central ideal in $\mathfrak{g}$, and let c be a real linear functional on $\mathfrak{z}$. Then the triple $(\mathfrak{g}, \mathfrak{z}, c)$ is called *admissible* if every real linear functional f on $\mathfrak{g}$ with $f|_{\mathfrak{z}} = c$ is inductive.

Theorem 1. *Let $(\mathfrak{g}, \mathfrak{z}, c)$ be an admissible triple. Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$, let Z be the central subgroup corresponding to $\mathfrak{z}$, and denote by χ the unitary character on Z corresponding to c. Then $L^1(G)_\chi$ is symmetric.*

Corollary. *Let G be a group in $[EA]$ with the property that every real functional on the Lie algebra of G is inductive. Then $L^1(G)$ is symmetric.*

Proof. The Corollary is an immediate consequence of the Theorem. We prove the Theorem by induction on $\dim \mathfrak{g}/\mathfrak{z}$. For $\dim \mathfrak{g}/\mathfrak{z} = 0$ there is nothing to show. Suppose $\dim \mathfrak{g}/\mathfrak{z} > 0$. W.l.o.g. we may assume that the kernel of c is zero: if not, we substitute the admissible triple $(\mathfrak{g}, \mathfrak{z}, c)$ by the admissible triple $(\mathfrak{g}/\ker(c), \mathfrak{z}/\ker(c), c')$; the algebra $L^1(G)_\chi$ corresponding to the triple $(\mathfrak{g}, \mathfrak{z}, c)$ is isomorphic to the algebra which corresponds to the new triple. — We distinguish two cases.

Case 1. There exists a non-zero minimal ideal $\mathfrak{a}$ with $\mathfrak{a} \cap \mathfrak{z} = (0)$. Three subcases 1. 1, 1. 2, 1. 3 are possible.

1. 1. $\mathfrak{a}$ is central. We apply (A) of § 1. Let E be an algebraically irreducible $L^1(G)_\chi$-module. Then E is also a G-module, and the central subgroup AZ corresponding to $\mathfrak{a} + \mathfrak{z}$ operates by a unitary character η on E with $\eta|_Z = \bar{\chi}$. This means that we may consider E as an $L^1(G)_{\bar{\eta}}$-module. But, by induction, $L^1(G)_{\bar{\eta}}$ is symmetric because $L^1(G)_{\bar{\eta}}$ corresponds to the admissible triple $(\mathfrak{g}, \mathfrak{a} + \mathfrak{z}, \tilde{c})$ where $\tilde{c}$ denotes the differential of $\bar{\eta}$. Therefore, one can find an irreducible $*$-representation of $L^1(G)_{\bar{\eta}}$ (of $L^1(G)_\chi$) in a Hilbert space $\mathfrak{H}$ and an intertwining operator from E into $\mathfrak{H}$.

1. 2. $\mathfrak{a}$ is onedimensional and non-central. Of course, we apply (C), (i) of § 1. The triple $(\mathfrak{g}/\mathfrak{a}, \mathfrak{a} + \mathfrak{z}/\mathfrak{a}, c)$ (c is considered as a linear functional on $\mathfrak{a} + \mathfrak{z}/\mathfrak{a} \cong \mathfrak{z}$) is admissible by Lemma 1. By induction, $L^1(G/A)_\chi$ is symmetric. Let $\mathfrak{c}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. From Proposition 1, it follows that $(\mathfrak{c}, \mathfrak{a} + \mathfrak{z}, \tilde{c})$ is admissible for every $\tilde{c}$ with $\tilde{c}|_{\mathfrak{z}} = c$ and $\tilde{c}|_{\mathfrak{a}} \neq 0$. By induction, the corresponding Banach algebra is symmetric and the assumptions of (C), (i) are fulfilled.

1. 3. $\mathfrak{a}$ is twodimensional and non-central. Since $\mathfrak{a}$ is minimal and $\mathfrak{g}$ is an exponential Lie algebra, G acts on $\mathfrak{a}$ as described in (C), (ii). As in the case 1. 2 above, one can show the symmetry of $L^1(G)_\chi$ using Lemma 1, Proposition 1, and (C), (ii).

Case 2. $\dim \mathfrak{z} = 1$ and $\mathfrak{z}$ is contained in every non-zero ideal of $\mathfrak{g}$. Here, we really use the assumption that $\mathfrak{g}$ is in $[EA]$. Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ be a decomposition as in Definition 1. Select a functional f on $\mathfrak{g}$ with $f|_{\mathfrak{z}} = c \neq 0$ and with the property that $\ker f \oplus \mathfrak{z}$ is an $\mathfrak{s}$-invariant decomposition of $\mathfrak{g}$. Then $\mathfrak{g}(f)$ contains $\mathfrak{s}$, and we find $\mathfrak{k}(f) = \mathfrak{g}$ (see the Remark after Definition 3). If $\mathfrak{k}(f)^\infty = \mathfrak{g}^\infty = 0$ then $\mathfrak{g}$ is nilpotent and $L^1(G)_\chi$ is symmetric by [14] of [15]. If $\mathfrak{k}(f)^\infty \neq 0$ then $\mathfrak{z}$ is contained in $\mathfrak{k}(f)^\infty$ and f is not inductive which is a contradiction to the assumptions.

§ 3. In this paragraph, we prepare the proof of the opposite direction of the Corollary to Theorem 1. What we will really show in § 4 is the following: If there exist non-inductive functionals then the group algebra is nonsymmetric. Here we give a more general (representation theoretic) criterion for nonsymmetry.

Theorem 2. *Let G be a locally compact group. Suppose that there exist a closed subgroup H, an (irreducible) unitary representation ρ of H, and a quasi-invariant measure μ on the homogeneous space $X = G/H$ with the following properties:*

1^0 $\mathrm{d}\mu(gx) = \chi(g)\,\mathrm{d}\mu(x)$ $(g \in G,\ x \in X)$ *for a non-trivial character* $\chi\colon G \to \mathbb{R}_+$. *Denote by G_0 the kernel of χ.*

2^0 *If* $\pi = \operatorname{ind}_{H \uparrow G} \rho$ *denotes the induced representation then the restriction σ of π to G_0 is irreducible.*

3^0 $\sigma(L^1(G_0))$ *contains non-zero operators of finite rank.*

Then $L^1(G)$ is not symmetric.

Proof. We construct some not necessarily unitary representations of G. Let $\mathfrak{K}$ be the representation space of ρ, and let α be a complex number with $0 < \operatorname{Re}\alpha < 1$. Then we denote by E_α the space of all continuous functions $\varphi\colon G \to \mathfrak{K}$ with $\varphi(gh) = \chi(h)^\alpha\, \rho(h)^{-1}\, \varphi(g)$ $(g \in G,\ h \in H)$ and with the property that there exists a compact set $L = L\varphi$ such that the support of φ is contained in LH. We define a norm $|\ \ |_\alpha$ on E_α by

$$|\varphi|_\alpha = \Big[\int_X |\varphi(g)|^p\, \chi(g)^{-1}\, \mathrm{d}\mu(\dot g)\Big]^{\frac{1}{p}}$$

with $\frac{1}{p} = \operatorname{Re}\alpha$ and $\dot g = gH \in X$. *Let* $\bar E_\alpha$ be the completion of E_α in this norm. G acts on E_α via $(x\varphi)(g) = \varphi(x^{-1}g)$ by isometries, and we get a (strongly continuous) representation π_α of G in the Banach space $\bar E_\alpha$. For $\alpha = \frac{1}{2}$, we just obtain the induced representation $\operatorname{ind}_{H \uparrow G} \rho$, i.e. $\pi = \pi_{\frac{1}{2}}$; we denote $\bar E_{\frac{1}{2}}$ also by $\mathfrak{H}$. The restriction of π_α to G_0 is denoted by σ_α. Define

$$U_{\alpha,\beta}\colon E_\alpha \to E_\beta \quad \text{by} \quad (U_{\alpha,\beta}\varphi)(g) = \chi(g)^{\beta-\alpha}\varphi(g).$$

$U_{\alpha,\beta}$ is an (in general, unbounded) operator which admits a closure, the closure is also denoted by $U_{\alpha,\beta}$; $U_{\alpha,\beta}$ is bounded if $\operatorname{Re}\alpha = \operatorname{Re}\beta$. Moreover, $U_{\alpha,\beta}$ is a G_0-intertwining operator. In the following, the operators $U_\alpha := U_{\alpha,\frac{1}{2}}\colon E_\alpha \to \mathfrak{H}$ are important. From the fact that U_α is a closed G_0-intertwining operator, we get

(*) If $f \in L^1(G_0)$ and η is in the domain $D(U_\alpha)$ of U_α then $\sigma_\alpha(f)\eta$ is in $D(U_\alpha)$ and $U_\alpha\sigma_\alpha(f)\eta = \sigma(f)\,U_\alpha\eta$.

Since σ is an irreducible representation it follows from 3° that there exists a $q = q^* \in L^1(G_0)$ such that $\sigma(q)$ is a projector of rank one, let $\sigma(q)\mathfrak{H} = \mathbb{C}\xi$ with $|\xi| = 1$. Since $U_\alpha(E_\alpha)$ is dense in $\mathfrak{H}$ we get

$$\mathbb{C}\xi = \sigma(q)\,U_\alpha(E_\alpha) = U_\alpha\sigma_\alpha(q)E_\alpha \quad \text{by } (*).$$

Especially, there exists a $\xi_\alpha \in \sigma_\alpha(q)E_\alpha \subseteq D(U_\alpha)$ with

$$U_\alpha\xi_\alpha = \xi.$$

By the way, ξ_α is uniquely determined by this equation. Moreover, we have

$$\sigma_\alpha(q)E_\alpha = \mathbb{C}\xi_\alpha = \sigma_\alpha(q)(\bar{E}_\alpha) \quad \text{and} \quad \sigma_\alpha(q)\xi_\alpha = \xi_\alpha.$$

Let $\mathscr{A} \subseteq L^1(G)$ be the closure of the involutive subalgebra $q * L^1(G) * q$ of $L^1(G)$. If f is in $\mathscr{A}$ then $\pi_\alpha(f)$ maps the onedimensional subspace $\mathbb{C}\xi_\alpha \subseteq \bar{E}_\alpha$ into itself, $\pi_\alpha(f)\xi_\alpha = \tau_\alpha(f)\xi_\alpha$. Obviously, τ_α is a multiplicative linear functional on $\mathscr{A}$.

Let $g \in C_c(G)$, i.e. g is a continuous function on G with compact support, and let $f := q * g * q$. Then $\alpha \to \tau_\alpha(f)$ is an holomorphic function in $0 < \mathrm{Re}\,\alpha < 1$ because of the following reason: Define f_α by $f_\alpha(x) = f(x)\,\chi(x)^{\frac{1}{2}-\alpha}$. Then $\alpha \to f_\alpha = q * g_\alpha * q$ is a holomorphic function from $\mathbb{C}$ into $L^1(G)$; the equality $\pi(f_\alpha)U_\alpha = U_\alpha \pi_\alpha(f)$ holds (on suitable domains) for $0 < \mathrm{Re}\,\alpha < 1$ and, consequently, we get $\tau_\alpha(f) = \tau_{\frac{1}{2}}(f_\alpha)$.

We claim that $\alpha \to \tau_\alpha(q * g * q)$ is not constant for some $g = g^* \in C_c(G)$. Suppose to the contrary that it is constant for all these functions. Then it follows that

$$\tau_\alpha(q * g * q) = \tau_{\frac{1}{2}}(q * g * q)$$

for all $g \in L^1(G)$ and all α, $0 < \mathrm{Re}\,\alpha < 1$, especially for $\mathrm{Re}\,\alpha = \frac{1}{2}$. In this case $\bar{E}_\alpha$ is a Hilbert space, and we have

$$\tau_\alpha(q * g * q) = \langle \pi_\alpha(q * g * q)\,\xi_\alpha, \xi_\alpha \rangle = \langle \pi_\alpha(g)\,\xi_\alpha, \xi_\alpha \rangle = \langle \pi(g)\,\xi, \xi \rangle.$$

The last equality implies that all the irreducible unitary representations π_α are unitarily equivalent to π. Since U_α is the unique intertwining operator between the irreducible representations σ_α and σ, it has to intertwine also π_α and π. But it is easy to see that this is not the case because χ is not trivial.

We have shown that there exists a $g = g^* \in C_c(G)$ such that $\alpha \to \tau_\alpha(q * g * q)$ is a non-constant holomorphic function in $0 < \mathrm{Re}\,\alpha < 1$. This function assumes also non-real values, we find a β, $0 < \mathrm{Re}\,\beta < 1$, such that $\tau_\beta(q * g * q) \notin \mathbb{R}$. Therefore, τ_β is a non-hermitean multiplicative functional on $\mathscr{A}$. Hence $\mathscr{A}$ and, consequently, $L^1(G)$ is not symmetric.

§ 4. Here, we want to establish the assumptions of Theorem 2 in the case that $G \in [EA]$ and that there exist non-inductive functionals on the Lie algebra of G. We start with the following purely algebraic Lemma.

Lemma 2. *Let $\mathfrak{g}$ be a Lie algebra in $[EA]$, let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, and let $\mathfrak{z}\mathfrak{n}$ be the center of $\mathfrak{n}$. Suppose that there exist non-inductive real functionals on $\mathfrak{g}$ and that all real functionals on all proper quotients of $\mathfrak{g}$ which belong to $[EA]$ are inductive. Then $\mathfrak{z}\mathfrak{n}$ is an irreducible $\mathfrak{g}$-module, and the centralizer of $\mathfrak{z}\mathfrak{n}$ is not nilpotent.*

Proof. $\mathfrak{z}\mathfrak{n}$ is a semisimple $\mathfrak{g}$-module, let

$$\mathfrak{z}\mathfrak{n} = \bigoplus_{\lambda \in \Lambda} V_\lambda$$

be the decomposition of $\mathfrak{z}\mathfrak{n}$ into root spaces, i.e. Λ is a set of complex-valued linear functionals on $\mathfrak{g}$, and an $X \in \mathfrak{g}$ acts on V_λ by multiplication with $\lambda(X)$ (if λ is not real one can introduce on V_λ the structure of a complex vector space). From the minimality of $\mathfrak{g}$ (together with Lemma 1) it follows:

(i) Each V_λ is irreducible.

(ii) If f is a non-inductive functional on $\mathfrak{g}$ then $f(V_\lambda) \neq 0$ for all $\lambda \in \Lambda$.

Let $\mathfrak{c} (= \bigcap_{\lambda \in \Lambda} \ker \lambda)$ be the centralizer of $\mathfrak{z}\mathfrak{n}$ in $\mathfrak{g}$. By (i), (ii) and repeated use of Proposition 1 we get

(iii) If f is a non-inductive functional on $\mathfrak{g}$ then $f|_\mathfrak{c}$ is not inductive. Especially, $\mathfrak{c}$ is not nilpotent. Let $\mathfrak{c}^\infty$ be the smallest ideal in $\mathfrak{c}$ such that $\mathfrak{c}/\mathfrak{c}^\infty$ is nilpotent. $\mathfrak{c}^\infty$ is a non-zero ideal in $\mathfrak{g}$, hence $\mathfrak{z}\mathfrak{n} \cap \mathfrak{c}^\infty \neq 0$, and we find a $\lambda \in \Lambda$ with $V_\lambda \subset \mathfrak{z}\mathfrak{n} \cap \mathfrak{c}^\infty$.

Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ be a decomposition as in Definition 1, and choose an $\mathfrak{s}$-invariant (vector space) complement $\mathfrak{w}$ to $\mathfrak{z}\mathfrak{n}$ in $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{z}\mathfrak{n}$. Select a real functional f on $\mathfrak{g}$ with $f(V_\lambda) \neq 0$, $f(\mathfrak{w}) = 0$, and $f(V_\mu) = 0$ for $\mu \in \Lambda$, $\mu \neq \lambda$. Denote by g the restriction of f to $\mathfrak{d} := \ker \lambda$. We have $\mathfrak{k}(g) = \mathfrak{d}$ (cf. Remark to Definition 3) and $\mathfrak{k}(g)^\infty = \mathfrak{d}^\infty \supset \mathfrak{c}^\infty \supset V_\lambda$. Hence g is not inductive. But then also f is not inductive (if $\lambda = 0$ this is obvious; if $\lambda \neq 0$ we use Proposition 1). From (ii) it follows that $\Lambda = \{\lambda\}$, and the proof of Lemma 2 is finished.

Theorem 3. *Let G be a group in $[EA]$ with Lie algebra $\mathfrak{g}$. If there exist non-inductive real functionals on $\mathfrak{g}$ then $L^1(G)$ is not symmetric.*

Proof. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, let $\mathfrak{z}\mathfrak{n}$ be the center of $\mathfrak{n}$, and let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$ be a decomposition as in Definition 1. By Lemma 2, we may assume that $\mathfrak{z}\mathfrak{n}$ is an irreducible $\mathfrak{g}$-module. Moreover, let $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{z}\mathfrak{n}$ be an $\mathfrak{s}$-invariant decomposition of $\mathfrak{g}$. The proof of Lemma 2 shows that if f is a functional with $f(\mathfrak{w}) = 0$ and $f(\mathfrak{z}\mathfrak{n}) \neq 0$ then f is not inductive. We fix such an f. Let $\mathfrak{c}$ be the centralizer of $\mathfrak{z}\mathfrak{n}$, and let $g = f|_\mathfrak{c}$. g is also not inductive, in fact we have $\mathfrak{c}(g) + \mathfrak{n} = \mathfrak{c}$.

It was shown in [2] (for arbitrary non-inductive functionals on exponential Lie algebras) that there exist a Vergne polarization $\mathfrak{p}$ for g and a subalgebra $\mathfrak{h}$, $\mathfrak{p} \subset \mathfrak{h} \subset \mathfrak{c}$, such that $Y \to \mathrm{Trace}_{\mathfrak{c}/\mathfrak{h}} \operatorname{ad}(Y) = \mathrm{Trace}_{\mathfrak{g}/\mathfrak{h}} \operatorname{ad}(Y)$ is a non-zero functional on $\mathfrak{h}$. Since this functional vanishes on $\mathfrak{n} \cap \mathfrak{h}$ there exists an extension δ of this functional to $\mathfrak{g}$ vanishing on $\mathfrak{n}$. Let $\chi: G \to \mathbb{R}_+$ be the corresponding real character, i.e. $\chi(\exp Y) = e^{\delta(Y)}$. Moreover, let ZN, P, H, C and N denote the subgroups corresponding to $\mathfrak{z}\mathfrak{n}$, $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{c}$ and $\mathfrak{n}$, respectively, let $\mathfrak{g}_0 = \ker \delta$ and $G_0 = \ker \chi$. Let the unitary character η on P be defined by $\eta(\exp Y) = e^{ig(Y)}$, and let $\rho = \operatorname{ind}_{P \uparrow H} \eta$ be the induced representation. On $X = G/H$ there exists a quasi-invariant measure μ as in Theorem 2. We claim that ρ and H satisfy also the other assumptions of Theorem 2, i.e. the restriction σ of $\pi = \operatorname{ind}_{H \uparrow G} \rho = \operatorname{ind}_{P \uparrow G} \eta$ to G_0 is irreducible, and $\sigma(L^1(G_0))$ contains non-zero operators of finite rank. First we note that π is irreducible because $\mathfrak{p}$ is a Vergne polarization for f, too, which follows from the fact that $\mathfrak{g}(f)$ is contained in $\mathfrak{c}$. Now, we distinguish the following three possibilities for the $\mathfrak{g}$-module $\mathfrak{z}\mathfrak{n}$.

Case 1. $\dim \mathfrak{z}\mathfrak{n}=1$, and $\mathfrak{z}\mathfrak{n}$ is central in $\mathfrak{g}$. In this case, we have $\mathfrak{c}=\mathfrak{g}$, $f=g$, and $\mathfrak{g}(f)+\mathfrak{n}=\mathfrak{g}$. The last equality implies that even the restriction τ of π to $N\subset G_0$ is irreducible (compare [2], in fact this is more or less obvious from the construction of the irreducible representations of exponential Lie groups). Since $\tau(L^1(N))$ contains operators of finite rank, see e.g. [18], the same is true for $\sigma(L^1(G_0))$.

Case 2. $\dim \mathfrak{z}\mathfrak{n}=1$, and $\mathfrak{z}\mathfrak{n}$ is not central in $\mathfrak{g}$. Let W be any element in $\mathfrak{g}_0\setminus\mathfrak{c}$ ($\delta\neq 0$ on $\mathfrak{c}$), let $\mathfrak{d}=\mathbb{R}W+\mathfrak{n}$, and let D be the corresponding subgroup of G. From $\mathfrak{c}(g)=\mathfrak{g}(f)+\mathfrak{z}\mathfrak{n}$ (see the proof of Proposition 1) it follows that $\mathfrak{c}=\mathfrak{c}(g)+\mathfrak{n}=\mathfrak{g}(f)+\mathfrak{n}$ and hence

$$\mathfrak{g}=\mathfrak{c}+\mathfrak{d}=\mathfrak{g}(f)+\mathfrak{d}.$$

This equality shows that the restriction τ of π to D is irreducible, and, in fact, τ corresponds to the functional $f|_{\mathfrak{d}}$ in the description of the irreducible unitary representations of the exponential Lie group D. We claim that $\tau(L^1(D))$ contains operators of finite rank.

First, we show that $\tau(L^1(D))$ contains compact operators. D is isomorphic to $R\ltimes N$ with $R=\exp(\mathbb{R}W)$ and hence $L^1(D)$ is isomorphic to the Leptin algebra $L^1(R, L^1(N))$ (see [9] or [10], where these algebras are called generalized L^1-algebras). We identify $J=L^1(ZN)$ with $L^1(\mathbb{R})$ such that $r=\exp(tW)\in R$ acts on $h\in J$ via $h^r(x)=e^t h(e^t x)$. Denote by $\hat{h}$ the Fourier transform of $h\in J$, and let

$$J_0=\{h\in J;\ \hat{h}(0)=0\},$$
$$J_+=\{h\in J;\ \hat{h}(x)=0 \quad \text{for all } x\leqq 0\},$$
$$\text{and } J_-=\{h\in J;\ \hat{h}(x)=0 \quad \text{for all } x\geqq 0\}.$$

J_0, J_+ and J_- are R-invariant ideals in J. Since $f(\mathfrak{z}\mathfrak{n})\neq 0$, the representation τ does not vanish on the ideal

$$L^1(R, \{J_0 * L^1(N)\}^-)=[L^1(R, \{J_+ * L^1(N)\}^-)+L^1(R, \{J_- * L^1(N)\}^-)]^-.$$

τ vanishes on precisely one of the latter two ideals; on which one this depends on $\eta|_{ZN}$ (and on the identification of $L^1(ZN)$ with $L^1(\mathbb{R})$). Suppose that τ does not vanish on $L^1(R, \{J_+ * L^1(N)\}^-)$. Let $A=\{J_+ * L^1(N)\}^-$, let $B=L^1(R, A)$, and let $U=J_0/J_-$. U is an R-algebra, A is an U-module, and the triple R, A, U satisfies the assumptions of [17]. Let $p\in L^1(R, U)$ be an hermitean projector of rank one. Since the image of every irreducible $*$-representation of A consists of compact operators it follows from [17] that $\tau(p * B * p)$ is contained in the compact operators. But then the same is true for $\tau(B)$ because $B * p * B * p * B$ is dense in B.

Now, I want to establish the existence of finite rank operators in the image of τ. To this end, I will use ideas which I learnt when studying [12]. By [5], $f|_{\mathfrak{d}}+\mathfrak{n}^\perp$ is contained in $D(f|_{\mathfrak{d}})$ because the stabilizer of $f|_{\mathfrak{d}}$ in D is contained in N. Using this fact it follows from [2]: If γ is another irreducible $*$-representation of $L^1(D)$ with $\ker\tau\subset\ker\gamma$ then $\ker\tau_*\subset\ker\gamma_*$ where γ_* (resp. τ_*) denotes the extension of γ (resp. τ) to the C^*-hull of $L^1(D)$. From this fact we get very easily the following: Let $B\to C^*(B)$ be the C^*-hull of B, and let τ' be the restriction of τ to B. If γ is another irreducible $*$-representation of B with $\ker \tau' \subset \ker \gamma$ then $\ker \tau'_* \subset \ker \gamma_*$ where $\gamma_*(\tau'_*)$ denotes the extension of $\gamma(\tau')$ to $C^*(B)$.

But $C^*(B)/\ker \tau'_*$ which is isomorphic to the algebra of compact operators has precisely one irreducible representation. Therefore, also $B' = B/\ker \tau'$ has precisely one irreducible $*$-representation. Moreover, B (see [17], A is symmetric by [15]) and, consequently, B' are symmetric algebras. Then it follows from Raikovs Theorem, see e.g. [20], that

$$\nu_{B'}(x) = |\tau'(x)| \quad \text{for all} \quad x = x^* \in B'$$

where $\nu_{B'}$ denotes the spectral radius. This fact implies by Proposition 2.5 in [6] that the spectrum $Sp_{B'}(x)$ of $x = x^* \in B'$ coincides with the spectrum of the operator $\tau'(x)$. By the way, this equality of spectra can also be proved more or less directly by using (A) of § 1 instead of Raikovs and Hulanickis Theorem. Now, choose an $h = h^* \in B'$ such that $\lambda > 0$ is the largest eigenvalue of the compact operator $\tau'(h)$, and let Γ be a small positively oriented circle in the complex plane around λ such that there is no eigenvalue of $\tau'(h)$ on Γ or in the interior of Γ except λ. For $z \in \mathbb{C} \setminus Sp_{B'}(h)$ let $R(z)$ be the inverse of $h - z$ in $B' \oplus \mathbb{C}1$, and let

$$h_\lambda = -\frac{1}{2\pi i} \int_\Gamma R(z)\,dz.$$

Then $h_\lambda \in B'$, and $\tau'(h_\lambda)$ is the projection on the λ-eigenspace of $\tau'(h)$. Especially, $\tau'(h_\lambda)$ is a non-zero finite rank operator, q.e.d.

Case 3. $\dim \mathfrak{z}\mathfrak{n} = 2$, $\mathfrak{z}\mathfrak{n} \cong \mathbb{C}$, and an $Y \in \mathfrak{g}$ acts on $\mathfrak{z}\mathfrak{n}$ by multiplication with $(1 + is)\,\mu(Y)$ for some real $s \neq 0$ and some non-zero real functional μ on $\mathfrak{g}$.

We proceed in more or less the same way as in case 2. Let $W \in \mathfrak{g}_0 \setminus \mathfrak{c}$, say $\mu(W) = 1$, let $R = \exp(\mathbb{R}W)$, let $\mathfrak{d} = \mathbb{R}W + \mathfrak{n}$, and let $D = \exp(\mathfrak{d}) = R \ltimes N$. Again, the restriction τ of π to D is irreducible, and we claim that $\tau(L^1(D))$ contains operators of finite rank.

$L^1(D)$ is isomorphic to the Leptin algebra $L^1(R, L^1(N))$. We identify $J = L^1(ZN)$ with $L^1(\mathbb{C})$ such that $r = \exp(tW) \in R$ acts on $h \in J$ via $h^r(z) = e^{2t} h(e^{t+ist}z)$. Denote by $\hat{h}$ the Fourier transform of $h \in J$, and let

$$U = \{h \in J;\ \hat{h}(0) = 0, \quad \text{and} \quad \hat{h}(u) = \hat{h}(v) \quad \text{if} \quad |u| = |v|\}.$$

U is R-invariant, and the Gelfand space of U can be identified with R such that R acts by translation on this Gelfand space (compare also [13]). Let $A = [U * L^1(N)]^-$, and let $B = L^1(R, A)$. B is an ideal in $L^1(D) = L^1(R, L^1(N))$, namely the kernel of the quotient morphism $L^1(D) \to L^1(D/ZN)$. Since $f(\mathfrak{z}\mathfrak{n}) \neq 0$, τ does not vanish on B, and hence the restriction τ' of τ to B is an irreducible representation of B.

Moreover, the triple R, A, U satisfies the assumptions of [17]. As in case 2, we see that $\tau'(B)$ consists of compact operators, that $B' = B/\ker \tau'$ has only one irreducible $*$-represensation, and that B' is a symmetric algebra. From these data it follows, as in case 2, that $\tau'(B)$ contains non-zero finite rank operators.

§ 5. In the preceding sections, I have shown that for a group G in $[EA]$ symmetry and $*$-regularity of $L^1(G)$ are equivalent. It is very natural to ask whether one can extend this result (at least) to all exponential Lie groups. By Theorem 2 (and Boidols results) the implication: $L^1(G)$ symmetric $\Rightarrow L^1(G)$ $*$-regular, is proved if one can show that the image of every irreducible $*$-representation of the group algebra of an exponential Lie group contains non-zero operators of finite rank. But I don't know whether this is true or not. — I believe that the opposite implication : G exponential, $L^1(G)$ $*$-regular $\Rightarrow L^1(G)$ symmetric is much harder. I want to "prove" this believe by an example. While one could prove: $G \in [EA]$, $L^1(G)$ $*$-regular $\Rightarrow L^1(G)$ symmetric, only by using Satz 1 in [15] (and derived results), in this paragraph I will give an example of a seven-dimensional exponential Lie group G with symmetric and $*$-regular group algebra where the symmetry cannot be deduced from Satz 1 in [15]. So, the example shows that it requires new methods if one wants to prove: G exponential, $L^1(G)$ $*$-regular $\Rightarrow L^1(G)$ symmetric. In the example, we get the symmetry by using the fact that the group contains the Heisenberg group as a normal subgroup; in this case we can apply Satz 1 in [16].

The example is the following: Let the real seven-dimensional Lie algebra $\mathfrak{g}$ with basis $A_1, A_2, X_1, X_2, Y_1, Y_2, Z$ be defined by

$$[X_1, Y_1]=Z,\ [X_2, Y_2]=Z,\ [A_2, Y_2]=Y_2,\ [A_2, X_2]=-X_2,$$
$$[A_1, Y_1]=Y_1,\ [A_1, X_1]=-X_1, \quad \text{and} \quad [A_1, A_2]=Z.$$

Let G be the corresponding simply connected Lie group. Using Boidols criterion one can show very easily that $L^1(G)$ is $*$-regular: if, for instance, f is a functional on $\mathfrak{g}$ with $f(Z) \neq 0$ then $\mathfrak{g}(f)=\mathbb{R}Z$ and, consequently, $\mathfrak{m}(f)^{\chi}=0$.

Now we want to show that $L^1(G)$ is symmetric. By (B) of § 1, the symmetry of $L^1(G)$ is equivalent to the fact that $L^1(G)_\chi$ is symmetric for every unitary character χ of the center $C=\exp(\mathbb{R}Z)$ of G.

For trivial χ, $L^1(G)_\chi$ is just the group algebra of G/C which is symmetric because the commutator subgroup of G/C is abelian (see (D) in [16]).

So, let us suppose that χ is non-trivial. Let M be the normal subgroup of G whose Lie algebra is generated by X_2, Y_2 and Z. The adjoint algebra $L^1(G)_\chi^b$ (see [9] or [10]) contains $L^1(M)_\chi$ (which is defined analogously). In $L^1(M)_\chi$, there is a lot of hermitean projections of rank one (in fact, they span a dense ideal), see e.g. [13]. Let p be one of these projections. By the Lemma in [15], $L^1(G)_\chi$ is symmetric iff $p * L^1(G)_\chi * p$ is symmetric. In [16], Satz 1, the latter algebra is computed: Let the real five-dimensional Lie algebra $\mathfrak{k}$ with basis A_1, A_2, X, Y, Z be defined by $[X, Y]=Z$, $[A_1, X]=-X$, $[A_1, Y]=Y$, $[A_1, A_2]=Z$, and let K be the corresponding simply connected Lie group. Then $p * L^1(G)_\chi * p$ is isomorphic to the Beurling subalgebra $L^1(K, w_2)_\chi$ of $L^1(K)_\chi$ where the weight w_2 is given by

$$w_2(\exp B)=\cosh^{\frac{1}{2}} f_2(B) \qquad (B \in \mathfrak{k})$$

for some non-zero real functional f_2 on $\mathfrak{k}$ vanishing on A_1, X, Y and Z.

We apply Satz 1 in [16] again. Let N be the normal subgroup of K whose Lie algebra is generated by X, Y and Z. $L^1(N)_\chi$ is contained in $L^1(K)^b_\chi$ and in $L^1(K, w_2)^b_\chi$. $L^1(K, w_2)_\chi$ is symmetric iff $q * L^1(K, w_2)_\chi * q$ is symmetric for one of the hermitean rank one projections q in $L^1(N)_\chi$. From Satz 1 in [16] we get the following: Let $\mathfrak{h}$ be the three dimensional Heisenberg algebra generated by A_1, A_2 and Z, and let H be the Heisenberg group. Then $q * L^1(K, w_2)_\chi * q$ is isomorphic to the Beurling subalgebra $L^1(H, w)_\chi$ of $L^1(H)_\chi$ where the weight w is given by

$$w(\exp B) = \cosh^{\frac{1}{2}} f_1(B) \cosh^{\frac{1}{2}} f_2(B) \qquad (B \in \mathfrak{h})$$

for some non-zero real functionals f_1 and f_2 on $\mathfrak{h}$ with

$$f_1(Z) = f_2(Z) = f_1(A_2) = f_2(A_1) = 0.$$

Now, it remains to show (after certain normalizations) the following Proposition.

Proposition 2. *Let H be the three dimensional Heisenberg group,* i.e.

$$H = \{[x, y, z] \mid x, y, z \in \mathbb{R}\}$$

with multiplication $[x, y, z]\,[x', y', z'] = [x + x', y + y', z + z' - x'y]$, *let χ be a non-trivial unitary character on the center C of H, and let the weight $w : H \to \mathbb{R}$ be defined by* $w(x, y, z) = \cosh^{\frac{1}{2}} x \cosh^{\frac{1}{2}} y$. *Then the Beurling algebra $L^1(H, w)_\chi$ is symmetric.*

Proof. Define $\varphi : H \to H$ by

$$\varphi(x, y, z) = \left[\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}, z + \tfrac{1}{2}xy + \tfrac{1}{4}(y^2 - x^2)\right].$$

φ is a group automorphism leaving the center C pointwise fixed. Moreover, φ leaves the Haar measure of H invariant. Therefore, φ induces a $*$-isomorphism V from $L^1(H)_\chi$ onto itself,

$$Vf(x, y, z) = f(\varphi(x, y, z)).$$

It is enough to show that the image of $L^1(H, w)_\chi$ under this map V is a symmetric algebra. One easily computes that this image is just $L^1(H, \tilde{w})_\chi$ where the weight $\tilde{w}$ is given by

$$\tilde{w}(x, y, z) = \cosh^{\frac{1}{2}} \frac{x-y}{\sqrt{2}} \cosh^{\frac{1}{2}} \frac{x+y}{\sqrt{2}} = \frac{1}{\sqrt{2}} \{\cosh\sqrt{2}x + \cosh\sqrt{2}y\}^{\frac{1}{2}}.$$

Now, define the weights $\sigma_1, \sigma_2 : H \to \mathbb{R}$ by

$$\sigma_1(x, y, z) = (\cosh\sqrt{2}x)^{\frac{1}{2}} \quad \text{and} \quad \sigma_2(s, y, z) = (\cosh\sqrt{2}y)^{\frac{1}{2}},$$

and let $\sigma = \sigma_1 + \sigma_2$.

One shows very easily that there exist positive constants C, D with

$$C\sigma(x, y, z) \leqq \tilde{w}(x, y, z) \leqq D\sigma(x, y, z) \quad \text{for all } [x, y, z] \in H.$$

From this we get $L^1(H, \tilde{w})_\chi = L^1(H, \sigma_1)_\chi \cap L^1(H, \sigma_2)_\chi$. But the algebras $L^1(H, \sigma_j)_\chi$ are symmetric by [8], p. 30, because $L^1(H, \sigma_j)_\chi$ is isomorphic to the Leptin algebra $L^1(\mathbb{R}, A(\mathbb{R}), (\cosh\sqrt{2}-)^{\frac{1}{2}})$ where $\mathbb{R}$ acts on $A(\mathbb{R})$ by translations. Obviously, the intersection of two symmetric algebras is again symmetric.

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