

ALGEBRAICALLY IRREDUCIBLE REPRESENTATIONS OF L^1 -ALGEBRAS OF EXPONENTIAL LIE GROUPS

DETLEV POGUNTKE

Introduction. Let G be a locally compact group. The classical task of the representation theory of G is the determination of the strongly continuous irreducible unitary representations of G . These representations give, for example, in the unimodular type I case the ingredients in the Plancherel theorem. But for other purposes as the computation of spectra of L^1 -functions considered as convolution operators on $L^1(G)$ the knowledge of the irreducible unitary representations does not suffice. This is like in the case of an involutive commutative Banach algebra \mathcal{A} where in order to compute the spectrum of an element in \mathcal{A} one has to use the full Gelfand transformation, i.e., one has to deal with all multiplicative linear functionals and not only with the hermitian ones. It turns out that for the determination of spectra of functions in $L^1(G)$ the proper class of representations to consider are the algebraically irreducible representations or, as I prefer to say, the simple $L^1(G)$ -modules. Of course, in the case of commutative L^1 -group algebras there is no difference between simple modules and irreducible unitary representations: every multiplicative linear functional is hermitian. But for noncommutative groups, even for solvable Lie groups, there are big differences. There exist (a lot of) solvable Lie groups G and simple $L^1(G)$ -modules E such that the annihilator of E in $L^1(G)$ is not the kernel of a unitary representation. The main result of this paper is a parametrization of the set of isomorphism classes of simple $L^1(G)$ -modules for exponential Lie groups G . While simple modules play some role in the general theory of Banach algebras (for instance in the proof of Johnson's theorem on the uniqueness of the norm on a semisimple Banach algebra) the L^1 -group algebras of exponential Lie groups are, up to my knowledge, the first example of a sufficiently wide class of noncommutative Banach algebras where all the simple modules can be determined explicitly. Of course, the simple modules were already known for L^1 -group algebras of nilpotent Lie groups, but this case is less interesting in the sense that there is always a bijective correspondence between the simple modules and the irreducible unitary representations which means that one does not find new phenomena and a proper extension of the unitary representation theory. As a consequence of the parametrization theorem one obtains a characterization of the exponential Lie groups with symmetric L^1 -group algebras.

This article is divided into six sections. In the first paragraph we summarize

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some general facts on simple modules, in particular we introduce the Fell topology on the set of isomorphism classes of simple modules. Next, for weighted twisted convolution algebras on vector groups the set of isomorphism classes of simple modules is determined. In the third paragraph it is shown that orbits of nilpotent Lie groups in the unitary dual of nilpotent Lie groups are "almost" sets of synthesis which can be deduced from Ludwig's results on points, [20]. Using this fact and the imprimitivity theorem of [25] one obtains in §4 that $L^1(N)/\text{Ann}_{L^1(N)}(E)$ is a semisimple algebra if G is an exponential Lie group, if N is its nilradical, and if E is a simple $L^1(G)$ -module. This result is the main tool to give in §5 the desired parametrization. In this parametrization enters a certain weight function on a vector group which is estimated in §6. Also in §6 the consequences of the parametrization theorem to questions of symmetry and $*$ -regularity of L^1 -group algebras are discussed.

§1. General properties of simple modules and the primitive ideal space. As one may expect from the title I do not claim originality for the results of this section. In fact, except for theorem 2 all ideas and results described below can be found in Naimark's book, [21], and in two articles of Fell's, [9] and [10].

Let \mathcal{A} be a complex algebra, let $a \in \mathcal{A}$ and let z be a nonzero complex number in the left spectrum of \mathcal{A} , i.e., $\mathcal{A}(a - z) = \{xa - zx; x \in \mathcal{A}\}$ is a proper left ideal. Then $u := (1/z)a$ is a right modular unit for the left ideal $\mathcal{A}(a - z)$, i.e., $xu \equiv x \pmod{\mathcal{A}(a - z)}$ for all $x \in \mathcal{A}$. By Zorn's lemma, $\mathcal{A}(a - z)$ is contained in a maximal left ideal Λ with right modular unit u . Then $E := \mathcal{A}/\Lambda$ is a simple \mathcal{A} -module. In fact, it is a left \mathcal{A} -module, but I will always omit the word "left": all modules in this paper are left modules. Moreover, the image ξ of u in E is different from zero, and one computes easily that $a\xi = z\xi$. Hence to every nonzero element z in the left spectrum of $a \in \mathcal{A}$ there exist a simple \mathcal{A} -module E and a nonzero $\xi \in E$ with $a\xi = z\xi$. In this way, from the knowledge of the simple \mathcal{A} -modules one may compute the left spectra.

Let's suppose for a moment that \mathcal{A} is an involutive Banach algebra. Then clearly the knowledge of all the left spectra gives the knowledge of all the (two-sided) spectra. It is particularly easy to compute the spectra if for every simple \mathcal{A} -module E it is possible to find an (irreducible) involutive representation π of \mathcal{A} in an Hilbert space \mathcal{H} and an \mathcal{A} -linear embedding of E in \mathcal{H} . In this case, one only has to consider the (point) spectra of the operators $\pi(a)$, $a \in \mathcal{A}$, in \mathcal{H} . In particular, it follows that for all $a \in \mathcal{A}$ the spectra of a^*a in \mathcal{A} are contained in the positive real axis, i.e., \mathcal{A} is a so-called symmetric Banach algebra. On the other hand, it is known, [21], that in the case of a symmetric Banach algebra \mathcal{A} for every simple \mathcal{A} -module E there exist \mathcal{H} and π as above.

The following theorem will be an important tool in the construction and classification of simple modules.

THEOREM 1. *Let \mathcal{A} be a complex algebra, and let \mathcal{J} be a two-sided ideal in \mathcal{A} .*

(i) If E is a simple \mathcal{A} -module with $\mathcal{J}E \neq 0$, then E is a simple \mathcal{J} -module. If F is a simple \mathcal{J} -module then there is a unique \mathcal{A} -module structure on F extending the \mathcal{J} -module structure. Hence there is a canonical bijection between the set of isomorphism classes of simple \mathcal{J} -modules and the set of isomorphism classes of simple \mathcal{A} -modules E with $\mathcal{J}E \neq 0$.

(ii) Let $p \in \mathcal{A}$ be an idempotent. If E is a simple \mathcal{A} -module with $pE \neq 0$ then pE is a simple $p\mathcal{A}p$ -module. If F is a simple $p\mathcal{A}p$ -module there exists a unique (up to isomorphism) \mathcal{A} -module E such that pE is isomorphic to F . Hence there is a canonical bijection between the set of isomorphism classes of simple $p\mathcal{A}p$ -modules and the set of isomorphism classes of simple \mathcal{A} -modules E with $pE \neq 0$.

(iii) Let $p \in \mathcal{A}$ be an idempotent. Then there is a canonical bijection between the set of isomorphism classes of simple $p\mathcal{J}p$ -modules and the set of isomorphism classes of simple \mathcal{J} -modules E with $(p\mathcal{J})E \neq 0$.

Sketch of the proof.

(i) Let E be a simple \mathcal{A} -module with $\mathcal{J}E \neq 0$. Then $E_0 := \{\xi \in E; \mathcal{J}\xi = 0\}$ is a proper \mathcal{A} -submodule, hence $E_0 = 0$. Therefore $\mathcal{J}\xi$ is a nonzero \mathcal{A} -submodule for every nonzero $\xi \in E$, hence $\mathcal{J}\xi = E$. If F is a simple \mathcal{J} -module, an \mathcal{A} -module structure is given by $a(f\xi) := (af)\xi$ for $a \in \mathcal{A}$, $f \in \mathcal{J}$, $\xi \in E$.

(ii) Let E be a simple \mathcal{A} -module with $pE \neq 0$. It is obvious that pE is a simple $p\mathcal{A}p$ -module. If E' is another simple \mathcal{A} -module then it is easy to see that an $p\mathcal{A}p$ -linear isomorphism between pE and pE' extends to an \mathcal{A} -linear isomorphism between E and E' .—If F is a simple $p\mathcal{A}p$ -module we form the induced \mathcal{A} -module $F_i := \mathcal{A} \otimes_{p\mathcal{A}p} F$. pF_i is isomorphic to F and every nonzero vector in pF_i is cyclic for F_i . Hence F_i has at least one simple quotient module E and pE is isomorphic to F .

(iii) is a consequence of (i) and (ii).

Now we assume that \mathcal{A} is a Banach algebra. On a simple \mathcal{A} -module E one may introduce a complete norm such that E becomes a Banach \mathcal{A} -module: Choose $\xi \in E$, $\xi \neq 0$, and put $\|\eta\| := \inf\{\|a\|; a \in \mathcal{A}, a\xi = \eta\}$ for $\eta \in E$. Of course, this norm depends on ξ , but different ξ 's give equivalent norms. If \mathcal{A} happens to be the L^1 -algebra of a locally compact group G then there exists a strongly continuous representation $G \times E \rightarrow E$ by isometric operators such that the $L^1(G)$ -module structure on E is given by

$$f\eta = \int_G f(x)(x\eta) dx$$

for $f \in L^1(G)$ and $\eta \in E$. If G is a closed subgroup of another locally compact group H such that H/G has an H -invariant measure ν (this is the only case needed in the sequel) one may form the induced module $\text{ind}_G^H E = L_G^2(H, E)$ consisting of all measurable functions $\varphi: H \rightarrow E$ with $\varphi(hx) = x^{-1}(\varphi(h))$, $x \in G$

and $h \in H$, such that

$$\|\varphi\|^2 := \int_{H/G} \|\varphi(h)\|^2 d\nu(\dot{h}) \text{ is finite.}$$

The group H acts strongly continuously in $L_G^2(H, E)$ by $(h\varphi)(k) := \varphi(h^{-1}k)$ for $h, k \in H$, and that gives rise to an $L^1(H)$ -module structure on $\text{ind}_G^H E$.

Now let \mathcal{A} again be an arbitrary Banach algebra. We want to introduce a topology on the set $\mathcal{S}(\mathcal{A})$ of isomorphism classes of simple \mathcal{A} -modules and on the set $\text{Priv}(\mathcal{A})$ of primitive ideals in \mathcal{A} . An ideal in \mathcal{A} is called primitive if it is the annihilator of a simple \mathcal{A} -module. By definition, there is a surjective map $\mathcal{S}(\mathcal{A}) \rightarrow \text{Priv}(\mathcal{A})$. In general, this map is not a bijection; but in the case that for every simple \mathcal{A} -module E there exists an $a \in \mathcal{A}$ such that aE is one-dimensional, the map is bijective.

Definition (Fell). For every Banach space X the space of bounded linear functionals is denoted by X' . For a simple \mathcal{A} -module E let $\Phi(E)$ be the linear span (in \mathcal{A}') of the functionals $a \rightarrow \varphi(a\xi)$, $\xi \in E$, $\varphi \in E'$. Of course, $\Phi(E)$ depends only on the isomorphism class $[E]$ of E . Let \mathcal{M} be a subset of $\mathcal{S}(\mathcal{A})$. $[E] \in \mathcal{S}(\mathcal{A})$ is contained in the closure of \mathcal{M} if $\Phi(E)$ is contained in the weak $*$ -closure of $\bigcup_{M \in \mathcal{M}} \Phi(M)$.

This defines a topology which we will call the Fell topology. Since the weak $*$ -closure of $\Phi(E)$ is nothing else but $\text{Ann}_{\mathcal{A}}(E)^\perp$ one may rephrase: $[E] \in \mathcal{M}$ if and only if $\text{Ann}_{\mathcal{A}}(E)^\perp$ is contained in the weak $*$ -closure of $\bigcup_{M \in \mathcal{M}} \text{Ann}(M)^\perp$. This gives a topology on $\text{Priv}(\mathcal{A})$: $P \in \text{Priv}(\mathcal{A})$ is contained in the closure of $\mathcal{T} \subseteq \text{Priv}(\mathcal{A})$ if P^\perp is contained in the weak $*$ -closure of $\bigcup_{Q \in \mathcal{T}} Q^\perp$. The open sets in $\mathcal{S}(\mathcal{A})$ are precisely the preimages of the open sets in $\text{Priv}(\mathcal{A})$ under $\mathcal{S}(\mathcal{A}) \rightarrow \text{Priv}(\mathcal{A})$, i.e. the topology does not distinguish \mathcal{A} -modules with the same annihilator.

Properties of the Fell topology:

(1) The Fell topology on $\text{Priv}(\mathcal{A})$ is stronger than the Jacobson topology.

(2) The correspondences of theorem 1 are homeomorphisms: Let \mathcal{J} be a closed two-sided ideal in \mathcal{A} and p an idempotent in \mathcal{A} . Then $\mathcal{S}(\mathcal{J})$ and $\mathcal{S}(p\mathcal{A}p)$ are homeomorphic to open parts of $\mathcal{S}(\mathcal{A})$, $\mathcal{S}(p\mathcal{J}p)$ is homeomorphic to an open part of $\mathcal{S}(\mathcal{J})$. Moreover, $\mathcal{S}(\mathcal{A}/\mathcal{J})$ is homeomorphic to a closed subset of $\mathcal{S}(\mathcal{A})$.

(3) In the case of commutative algebras the Fell topology coincides with the Gelfand topology. Hence the space $\text{Priv}(\mathcal{A})$ may be considered as a generalization of the Gelfand dual. Recall that a commutative Banach algebra is completely regular if the Gelfand topology coincides with the Jacobson topology. In an obvious manner, one may extend this notion to the noncommutative case.

(4) In the following, a special type of algebras will be of interest. Let G be a locally compact abelian group, and let w be a continuous weight on G . Then one may form the weighted (or Beurling) L^1 -algebra $\mathcal{A} = L^1(G, w)$. $\mathcal{S}(\mathcal{A})$ is in bijective correspondence to the set of continuous characters $\chi: G \rightarrow \mathbb{C}^\times$ with

$|\chi(x)| \leq w(x)$. This correspondence is an homeomorphism if the set of continuous characters is equipped with the compact open topology.

The following theorem does not fit well in the frame of this section. It is taken in this article because we will need it in §4. The proper place would have been the article [25], because it follows easily from the results obtained there. But when writing the latter paper I did not know that this result will be useful for applications.

THEOREM 2. *Let \mathcal{A} be an involutive Banach \mathbb{R} -algebra, i.e., \mathcal{A} is an involutive Banach algebra and \mathbb{R} acts strongly continuously by isometric $*$ -isomorphisms on \mathcal{A} . Let E be a simple $L^1(\mathbb{R}, \mathcal{A})$ -module, and suppose that there exist p, q in the adjoint algebra \mathcal{A}^b (for the definition and properties of the adjoint algebra, see [15]) with $pq = p = qp$, with $pE \neq 0$ and with $q^x \mathcal{A} q = 0$ for $x \in \mathbb{Z}, x \neq 0$. Then there exist a simple \mathcal{A} -module F and an $L^1(\mathbb{R}, \mathcal{A})$ -linear embedding from E into the induced module $L^2(\mathbb{R}, F)$. Moreover, $\text{Ann}_{\mathcal{A}}(E) = \bigcap_{x \in \mathbb{R}} \text{Ann}_{\mathcal{A}}(F)^x$.*

Proof. From Satz 2 in [25] it follows that there exists a simple $L^1(\mathbb{Z}, \mathcal{A})$ -module F_1 such that E can be embedded into the induced module \tilde{F}_1 . Conjugating p and q by an element $r \in \mathbb{R}$ (if necessary) we may assume that $pF_1 \neq 0$. By the theorem on induction in stages it suffices to show that there exist a simple \mathcal{A} -module F and an $L^1(\mathbb{Z}, \mathcal{A})$ -linear embedding from F_1 in the induced module $L^2(\mathbb{Z}, F)$. The assumption $q^x \mathcal{A} q = 0$ for $x \in \mathbb{Z}, x \neq 0$, implies that $q * L^1(\mathbb{Z}, \mathcal{A}) * q$ is contained in \mathcal{A} , i.e., in the functions supported by 0. For every $\xi \in pF_1$ we have $q\xi = \xi$. From the latter two facts and the irreducibility of F_1 one deduces very easily that $\mathcal{A}pF_1$ is a cyclic \mathcal{A} -module, in fact every nonzero $\xi \in pF_1$ is a cyclic vector. The cyclic \mathcal{A} -module $\mathcal{A}pF_1$ has a simple quotient F . Denote by $T: \mathcal{A}pF_1 \rightarrow F$ the quotient map. Then we define

$$S: F_1 \rightarrow L^2(\mathbb{Z}, F)$$

by

$$(S\xi)(z) = \delta_{z,0} T\xi \quad \text{for } \xi \in pF_1,$$

and

$$S(f\xi) = fS\xi \quad \text{for } f \in L^1(\mathbb{Z}, \mathcal{A}) \text{ and } \xi \in pF_1.$$

Of course, we have to show that S is well defined. Let $\xi \in pF_1$ and $f \in L^1(\mathbb{Z}, \mathcal{A})$ with $f\xi = 0$. We have to show that $fS\xi = 0$. Put $g = f * q$, i.e., $g(x) = f(x)q$. From $q\xi = \xi$ it follows that $g\xi = 0$. Since $qS\xi = S\xi$ it suffices to show that $gS\xi = 0$. By the definition of the induced representation we have $gS\xi = \sum_{x \in \mathbb{Z}} g(x)^{-x} (S\xi)^{-x}$ where η^x is defined by $\eta^x(y) = \eta(x + y)$. Hence $(gS\xi)(y) = \sum_{x \in \mathbb{Z}} g(x)^{-x} (S\xi)^{-x}(y - x) = \sum_{x \in \mathbb{Z}} g(y - x)^{-y} T\xi$, and we have to show that $g(y)^{-y} T\xi = 0$ for all $y \in \mathbb{Z}$. For $y \neq 0$, $g(y)^{-y} T\xi = (f(y)q)^{-y} T\xi = f(y)^{-y} q^{-y} q T\xi$ because $T\xi = qT\xi$. And $q^{-y} \mathcal{A} q = 0$ implies $q^{-y} q F = 0$. From $g\xi = 0$ it follows that $0 = \sum_{x \in \mathbb{Z}} g(x)^{-x} (x\xi)$, hence $0 = \sum_{x \in \mathbb{Z}} q \mathcal{A} g(x)^{-x} (x\xi) = \sum_{x \in \mathbb{Z}} q \mathcal{A} f(x)^{-x} q^{-x} (x\xi)$.

Since $q\mathcal{A}q^{-x} = 0$ for $x \neq 0$ we obtain $0 = q\mathcal{A}f(0)q\xi = q\mathcal{A}g(0)\xi$. Hence $g(0)\xi = 0$ because $g(0)\xi \neq 0$ would imply that $\mathcal{A}g(0)\xi = F_1$ and then $qF_1 = 0$, a contradiction to $pF_1 \neq 0$. From $g(0)\xi = 0$ and the \mathcal{A} -linearity of T it follows that $g(0)T\xi = 0$.

So, we know that E can be embedded into $L^2(\mathbb{R}, F)$. The equation on the annihilators is obvious.

§2. Simple modules over weighted twisted convolution algebras on vector groups.

Let W be a finite-dimensional real vector space, let $w : W \rightarrow \mathbb{R}$ be a continuous weight, i.e., $w(x) \geq 1$ and $w(x + y) \leq w(x)w(y)$ for all $x, y \in W$, and let $m : W \times W \rightarrow \mathbb{T}$ be a continuous cocycle on W . Then one may form the weighted twisted convolution algebra $L^1(W, m, w)$ consisting of all measurable functions $g : W \rightarrow \mathbb{C}$ such that $\|g\| := \int_W |g(x)|w(x)dx$ is finite where the multiplication is given by

$$(g * h)(x) = \int_W dy \bar{m}(x + y, -y)g(x + y)h(-y).$$

In this section we will determine the space $\mathcal{S}(L^1(W, m, w))$.

For the following discussion, see also [26].

The 2-cocycles on vector spaces are known, one may assume that m is of the form $m(x, y) = e^{\mu(x,y)}$ with an antisymmetric bilinear real form μ on W . Let Z be the kernel of the form μ . We choose a vector space complement X to Z in W and a direct decomposition $X = X_1 \oplus X_2$ such that $\mu(X_j, X_j) = 0$. The restriction of μ gives a nondegenerate pairing $\mu : X_1 \times X_2 \rightarrow \mathbb{R}$. Moreover we choose a positive definite inner product $\langle \cdot, \cdot \rangle$ on X_2 . μ and $\langle \cdot, \cdot \rangle$ give an isomorphism $T : X_1 \rightarrow X_2$ defined by $\langle Tx_1, x_2 \rangle = \mu(x_1, x_2)$. And one gets an inner product on X_1 , also denoted by $\langle \cdot, \cdot \rangle$, by $\langle x, y \rangle := \langle Tx, Ty \rangle = \mu(x, Ty)$. Then we define the function p from $X = X_1 + X_2$ into \mathbb{C} by $p(x_1 + x_2) = e^{-1/2\langle x_1, x_1 \rangle - 1/2\langle x_2, x_2 \rangle}$. In the case of a nondegenerate μ , i.e., $Z = 0$, one has the following theorem.

THEOREM 3. *Let $X = X_1 + X_2$, m, μ and p as above, and let $\mathcal{A} := L^1(X, m, w)$. Then p is contained in \mathcal{A} , a suitable positive multiple of p is an idempotent in \mathcal{A} , $p * \mathcal{A} * p = \mathbb{C}p$, and $\mathcal{A} * p * \mathcal{A}$ is a dense ideal. In particular, \mathcal{A} is a simple Banach algebra, and \mathcal{A} has (up to isomorphism) precisely one simple module.*

Proof. It was already shown in [26] that $p \in \mathcal{A}$, that $p * \mathcal{A} * p = \mathbb{C}p$ and that p is a multiple of an idempotent. By convolution with point measures one sees that $\mathcal{A} * p * \mathcal{A}$ contains $p\chi$ for every unitary character $\chi \in \hat{X}$. But $\{p\chi; \chi \in \hat{X}\}$ is total in \mathcal{A} : Let φ be in the dual \mathcal{A}' with $\varphi(p\chi) = 0$ for all $\chi \in \hat{X}$. Then φ can be identified with a measurable function on X such that $\|\varphi w^{-1}\|_\infty < \infty$, and we have

$$\int_X \chi(x)p(x)\varphi(x)dx = 0 \quad \text{for all } \chi \in \hat{X}.$$

Thus, $p\varphi$ is an L^1 -function on X whose Fourier transform is zero. Hence $p\varphi$ is zero almost everywhere, and as p has no zeros, φ has to be zero almost

everywhere. One concludes that $\mathcal{A} * p * \mathcal{A}$ is dense in \mathcal{A} . The latter statements in the theorem are easy consequences of the former.

Now, let E be a simple $L^1(W, m, w)$ -module. There exists a strongly continuous projective representation ρ of W in E with $\rho(x)\rho(y) = m(x, y)\rho(x + y)$ for $x, y \in W$, and $\rho(f)\xi := f\xi = \int_W f(x)\rho(x)\xi dx$ for $f \in L^1(W, m, w)$ and $\xi \in E$. As Z is the kernel of μ (and m) every $\rho(z)$, $z \in Z$, commutes with every $\rho(f)$. From Schur's lemma it follows that $\rho(z)$ has to be a scalar multiple of the identity, $\rho(z) = \eta(z)id_E$ with a continuous character $\eta = \eta_E$ from Z in \mathbb{C}^x . Moreover, η is bounded by w , i.e., $|\eta(z)| \leq w(z)$ for all $z \in Z$; denote the set of these continuous characters by \hat{Z}_w .

THEOREM 4. *The space $\mathcal{S}(L^1(W, m, w))$ of isomorphism classes of simple modules is homeomorphic to \hat{Z}_w via $E \rightarrow \eta_E$. Moreover, for every simple $L^1(W, m, w)$ -module E there exists an $h \in L^1(W, m, w)$ such that hE is one-dimensional.*

Proof. Let $\mathcal{J} := L^1(W, m, w)$. The algebra $\mathcal{A} = L^1(X, m, w)$ operates (by convolution) from the left and from the right on \mathcal{J} , and we may form the algebra $\mathcal{B} := \mathcal{J} \oplus \mathcal{A}$ which contains \mathcal{J} as an ideal. For every simple \mathcal{J} -module E we have $(p * \mathcal{J})E \neq 0$, p as above, because \mathcal{A} is a simple Banach algebra. From theorem 1 (iii) it follows that $\mathcal{S}(\mathcal{J})$ is homeomorphic to $\mathcal{S}(p * \mathcal{J} * p)$. An easy computation shows that $p * \mathcal{J} * p$ is isometrically isomorphic to $L^1(Z, w_0)$ where the weight w_0 is given by $w_0(z) = \int_X p(x)w(x + z)$, and the isomorphism $L^1(Z, w_0) \rightarrow p * \mathcal{J} * p$, $\varphi \rightarrow \tilde{\varphi}$, is given by $\tilde{\varphi}(x + z) = p(x)\varphi(z)$ for $x \in X$, $z \in Z$. Hence $\mathcal{S}(\mathcal{J})$ is homeomorphic to $\mathcal{S}(L^1(Z, w_0))$ and by §1 (4) homeomorphic to \hat{Z}_{w_0} . But the weight w_0 is equivalent to the weight $w|_Z$, because

$$w_0(z) = \int_X p(x)w(x + z) \leq \int_X p(x)w(x)w(z) = w(z) \int_X p(x)w(x),$$

and, as $w(x + z) \geq w(z)w(-x)^{-1}$, one has

$$w_0(z) \geq w(z) \int_X p(x)w(-x)^{-1}.$$

Therefore, $\hat{Z}_{w_0} = \hat{Z}_w$ and $\mathcal{S}(\mathcal{J})$ is homeomorphic to \hat{Z}_w . If one goes through the various identifications, one sees that the final homeomorphism is given by $[E] \rightarrow \eta_E$.—Moreover, $(p * \mathcal{J})E$ is one-dimensional for every simple \mathcal{J} -module E , and $hE = (p * \mathcal{J})E$ for a suitable $h \in p * \mathcal{J} * p$.

§3. On the synthesis problem of orbits of nilpotent Lie groups. Let N be a connected nilpotent Lie group. If one wants to classify the closed two-sided ideals \mathcal{I} in the L^1 -group algebra $L^1(N)$ of N one may proceed as follows. First one considers the hull $h(\mathcal{I})$ of \mathcal{I} , i.e., the set of all kernels of irreducible involutive representations π of $L^1(N)$ with $\mathcal{I} \subseteq \ker \pi$. $h(\mathcal{I})$ corresponds to a closed subset of the unitary dual \hat{N} of N , and evidently, it is an invariant of the

ideal. By a theorem of Leptin, [16], $h(\mathcal{I})$ is not void if \mathcal{I} is a proper ideal. By definition, \mathcal{I} is contained in the kernel $k(h(\mathcal{I}))$ of $h(\mathcal{I})$, and from the symmetry of the involutive algebra $L^1(N)$ it follows that $k(h(\mathcal{I}))/\mathcal{I}$ is always a radical Banach algebra. Unfortunately, $h(\mathcal{I})$ is not a complete variant of \mathcal{I} . Even in the case of abelian groups N there exist closed ideals \mathcal{I} such that \mathcal{I} is strictly contained in $k(h(\mathcal{I}))$. Closed subsets \mathfrak{X} of \hat{N} such that there is precisely one closed ideal \mathcal{I} with $h(\mathcal{I}) = \mathfrak{X}$, namely $\mathcal{I} = k(\mathfrak{X})$, are called Wiener sets or sets of synthesis. It was shown by Ludwig, [18], that for every closed subset \mathfrak{X} of \hat{N} there exists a unique smallest closed two-sided ideal $j(\mathfrak{X})$ with $h(j(\mathfrak{X})) = \mathfrak{X}$. In particular, any closed two-sided ideal \mathcal{I} in $L^1(N)$ lies between $j(h(\mathcal{I}))$ and $k(h(\mathcal{I}))$. In order to classify the ideals in $L^1(N)$, secondly one has to determine the ideals in the radical Banach algebras $k(\mathfrak{X})/j(\mathfrak{X})$ for closed subsets \mathfrak{X} of \hat{N} . Of course, one can hardly imagine a solution of the latter problem in full generality. For many purposes it is sufficient to know that $k(\mathfrak{X})/j(\mathfrak{X})$ is finitely nilpotent, i.e., there exists $n \in \mathbb{N}$ with $k(\mathfrak{X})^n \subset j(\mathfrak{X})$. It was shown by Ludwig, [20], that this is true if \mathfrak{X} consists of a single point π . What he actually proved is the following: Denote by $\mathcal{S}(N)$ and $\mathcal{D}(N)$ the space of Schwartz functions on N and the space of test functions on N , respectively. Let $\ker d\pi$ be the kernel of the associated infinitesimal representation of the universal enveloping algebra $\mathbb{U}\mathfrak{n}$ of \mathfrak{n} . Then $(\ker d\pi) * \mathcal{S}(N)$ is dense in $\ker \pi$. Consequently, also $(\ker d\pi) * \mathcal{D}(N)$ is dense in $\ker \pi$. Since $(\ker d\pi) * \mathcal{D}(N)$ is contained in $\mathcal{D}(N)$ it follows that $\ker \pi \cap \mathcal{D}(N)$ is dense in $\ker \pi$. Dixmier's symbolic calculus yields that $f^m \in j(\{\pi\})$ for all $f^* = f \in \ker \pi \cap \mathcal{D}(N)$ where $m = d + 4$ and $d = d_N$ denotes the degree of the growth of the Haar measure of N , i.e., for every compact set A in N there exists a positive constant C such that the measure of A^k is less or equal to Ck^d for all $k \in \mathbb{N}$. A little algebra shows that the same is true for all $f \in \mathcal{D}(N) \cap \ker \pi$: Put $f = f_1 + if_2$ with hermitian elements f_1, f_2 and form the polynomial $Q(z) = (f_1 + zf_2)^m$, $z \in \mathbb{C}$. Q is zero modulo $j(\{\pi\})$ for real z , hence for all $z \in \mathbb{C}$, in particular for $z = i$. Since $\ker \pi \cap \mathcal{D}(N)$ is dense in $\ker \pi$ one obtains that $f^m \in j(\{\pi\})$ for all $f \in \ker \pi$. The Nagata-Higman theorem, see [12], Appendix C, implies that $f_1 * \dots * f_n \in j(\{\pi\})$ for all $f_1, \dots, f_n \in \ker \pi$ where $n = 2^m - 1$. Using Ludwig's results we are going to prove in this section a generalization of his theorem, namely to the case where \mathfrak{X} is an orbit of another nilpotent Lie group acting on N . This generalization will be useful in the determination of the annihilators of topologically irreducible modules, in particular for simple modules.

THEOREM 5. *Let M and N be connected nilpotent Lie groups. Suppose that M acts continuously by automorphisms on N such that the associated semidirect product $M \rtimes N$ is nilpotent, too. Let \mathfrak{X} be an M -orbit in the unitary dual \hat{N} . Then $\mathcal{D}(N) \cap k(\mathfrak{X})$ is dense in $k(\mathfrak{X})$, and $\{k(\mathfrak{X})/j(\mathfrak{X})\}^n = 0$ where $n = 2^{d+4} - 1$ and d denotes the degree of the growth of the Haar measure of N .*

Remark. The fact that $M \rtimes N$ is nilpotent implies that M and $M \rtimes N$ act on the Lie algebra \mathfrak{n} and its dual \mathfrak{n}^* by unipotent automorphisms. Since \mathfrak{X} corresponds in the Kirillov picture to an $M \rtimes N$ -orbit in \mathfrak{n}^* and since orbits

under unipotent actions are closed, it follows from Brown's theorem, [6], that \mathfrak{X} is closed in \hat{N} .

Proof of the Theorem. We write $(m, x) \rightarrow x^m$, $m \in M$, $x \in N$, for the action of M on N , and form the auxiliary group H . As a set, H is the cartesian product $M \times N \times N$, and the multiplication is given by

$$(m', b', c')(m, b, c) = (m'm, b'mb, b^{-1}c'mbc).$$

Obviously, H is a connected nilpotent Lie group, too. We define a strongly continuous representation ρ of H on the Banach space $L^1(N)$ by

$$\{\rho(m, b, c)f\}(x) = f(c^{-1}b^{-1}x^mb).$$

$\rho(m, b, c)$ is a linear isometry on $L^1(N)$. Hence ρ can be integrated to a representation of $L^1(H)$ on $L^1(N)$, also denoted by ρ . The ideals $k(\mathfrak{X})$ and $\mathcal{I} := [k(\mathfrak{X}) \cap \mathcal{D}(N)]^-$ in $L^1(N)$ are invariant under ρ , hence one obtains representations ρ_k and $\rho_{\mathcal{I}}$ of H (and $L^1(H)$) in $L^1(N)/k(\mathfrak{X})$ and $L^1(N)/\mathcal{I}$, respectively. Obviously, $\ker_{L^1(H)} \rho_{\mathcal{I}}$ is contained $\ker_{L^1(H)} \rho_k$. We want to show that they are equal. From the theorem of Leptin mentioned above it follows easily that ρ_k is a topologically irreducible representation of H because a proper closed ρ_k -invariant subspace of $L^1(N)/k(\mathfrak{X})$ defines an ρ -invariant subspace Y between $k(\mathfrak{X})$ and $L^1(N)$. ρ -invariance means that Y is an M -invariant two-sided closed ideal in $L^1(N)$. Since $h(Y)$ is an M -invariant non-void set it has to coincide with \mathfrak{X} , hence $Y = k(\mathfrak{X})$. As ρ_k is topologically irreducible, $\ker_{L^1(H)} \rho_k$ is a prime ideal in $L^1(H)$ and hence, by a theorem of Ludwig, [20], the kernel of an irreducible involutive representation γ of $L^1(H)$. By the way, it is not necessary to use this theorem. One can construct such an γ directly in the following manner: Let τ be an irreducible unitary representation of N such that the equivalence class $[\tau]$ of τ in \hat{N} belongs to \mathfrak{X} , and let M_τ be the stabilizer of $[\tau]$. Let \mathfrak{H} denote the representation space of τ . We choose a strongly continuous projective representation U of M_τ in \mathfrak{H} with $\tau(x^m) = U(m)^{-1}\tau(x)U(m)$ for $x \in N$, $m \in M_\tau$. Then we define the (ordinary) unitary representation σ of the subgroup $H_\tau = M_\tau \times N \times N$ of H in the space $\mathcal{H}\mathcal{S}(\mathfrak{H}, \mathfrak{H})$ of Hilbert-Schmidt operators on \mathfrak{H} by

$$\sigma(m, b, c)(T) = U(m)\tau(b)\tau(c)T\tau(b)^{-1}U(m)^{-1}.$$

Obviously, σ is irreducible (even the restriction of σ to $\{e\} \times N \times N$ is irreducible), and the induced representation $\gamma = \text{ind}_{H_\tau}^H \sigma$ is irreducible. Some computations which are omitted show that $\ker_{L^1(H)} \gamma = \ker_{L^1(H)} \rho_k$. $\ker_{L^1(H)} \rho_{\mathcal{I}}$ is contained in $\ker_{L^1(H)} \rho_k$ but intersected with the space $\mathcal{D}(H)$ of test functions they coincide: If $f \in \mathcal{D}(H) \cap \ker_{L^1(H)} \rho_k$ then $\rho(f)(g) \in k(\mathfrak{X})$ for all $g \in \mathcal{D}(N)$. But $\rho(f)(g)$ is a test function, hence $\rho(f)(g) \in \mathcal{I} = [k(\mathfrak{X}) \cap \mathcal{D}(N)]^-$ for all $g \in \mathcal{D}(N)$. By continuity, $\rho(f)(g) \in \mathcal{I}$ for $g \in L^1(N)$, hence $f \in \ker_{L^1(H)} \rho_{\mathcal{I}}$. Therefore, $\ker_{L^1(H)} \rho_{\mathcal{I}} \supseteq [\ker_{\mathcal{D}(H)} \rho_{\mathcal{I}}]^- = [\ker_{\mathcal{D}(H)} \rho_k]^- = [\ker_{\mathcal{D}(H)} \gamma]^-$. By Lud-

wigs's theorem, $[\ker_{\mathcal{O}(H)}\gamma]^-$ is equal to $\ker_{L^1(H)}\gamma = \ker_{L^1(H)}\rho_k$. Hence $\ker_{L^1(H)}\rho_{\mathcal{I}} = \ker_{L^1(H)}\rho_k$ which implies that $\mathcal{I} = k(\mathfrak{X})$, as desired. The arguments explained above (Dixmier's symbolic calculus and the Nagata-Higman theorem) give that $k(\mathfrak{X})^n \subset j(\mathfrak{X})$ where $n = 2^{d+4} - 1$.

To apply this theorem to the annihilators of simple modules we start with a simple lemma.

LEMMA. *Let G be a locally compact group, let B be a closed normal subgroup of G , and let \mathcal{I} be a G -invariant closed two-sided ideal in $L^1(B)$. Let π be a uniformly bounded strongly continuous topologically irreducible representation of G in the Banach space E . If there exists a natural number n with $\pi(\mathcal{I}^n)E = 0$ then $\pi(\mathcal{I})E = 0$.*

Proof. Suppose that $\pi(\mathcal{I})E \neq 0$, and let m be the smallest natural number m with $\pi(\mathcal{I}^m)E = 0$. Then $m > 1$, and from the G -invariance of \mathcal{I} and the irreducibility of E it follows that $[\pi(\mathcal{I}^{m-1})E]^- = E$. Therefore, $\pi(\mathcal{I})E$ is contained in the closure of $\pi(\mathcal{I})\pi(\mathcal{I}^{m-1})(E) = \pi(\mathcal{I}^m)(E)$ which is zero.

For the statement of the last theorem of this section let's first give a definition which will be used in the following section. As we pointed out above the L^1 -group algebra of a connected nilpotent Lie group B contains "nice ideals", namely the kernels of closed subsets of \hat{B} , and besides those a number of worse ideals lying between the kernel of a closed set \mathfrak{X} in \hat{B} and $j(\mathfrak{X})$. We introduce a name for the "nice ideals".

Definition. Let B be a connected nilpotent Lie group. An ideal \mathcal{I} in $L^1(B)$ is called *pithy* if \mathcal{I} is the kernel of a (closed) subset of \hat{B} or, equivalently, if $\mathcal{I} = k(h(\mathcal{I}))$.

Combining theorem 5 and the lemma one obtains the following theorem which will be applied several times in the next section.

THEOREM 6. *Let G be a locally compact group, and let B and N be closed normal subgroups of G with $B \subset N$. Assume that B and N are connected nilpotent Lie groups. Let E be a simple $L^1(G)$ -module. Suppose that the hull of $\text{Ann}_{L^1(B)}(E)$ in \hat{B} is an N -orbit. Then $\text{Ann}_{L^1(B)}(E)$ is pithy, i.e., $\text{Ann}_{L^1(B)}(E) = k(h(\text{Ann}_{L^1(B)}(E)))$.*

Proof. Let $\mathcal{I} = \text{Ann}_{L^1(B)}(E)$, and let \mathfrak{X} be the hull of \mathcal{I} in \hat{B} . Since \mathcal{I} is G -invariant, \mathfrak{X} is G -invariant, too, and hence $k(\mathfrak{X})$ is G -invariant. From theorem 5 it follows that there is a natural number n with $k(\mathfrak{X})^n \subset \mathcal{I}$. By the lemma, $k(\mathfrak{X})$ annihilates E , hence $k(\mathfrak{X}) = \mathcal{I}$.

To conclude this section I would like to mention two open problems. Let N be a connected nilpotent Lie group, and let \mathfrak{X} be a closed subset of \hat{N} . Then \mathfrak{X} may be considered as a subset of the primitive ideal space Priv Un of the universal enveloping algebra Un . And one may form the "infinitesimal" kernel $\mathfrak{p} = k_{\text{Un}}(\mathfrak{X})$ of \mathfrak{X} in Un .

Problem 1. Find sufficient conditions for \mathfrak{X} such that $p^* \mathcal{S}(N)$ is dense in $k(\mathfrak{X})$.

Of course, \mathfrak{X} can't be arbitrary. For instance one has to have that the intersection of the closure of \mathfrak{X} in Priv Un (in the Jacobson topology) and the hermitian part of Priv Un is just \mathfrak{X} . I have in mind the case where \mathfrak{X} is the orbit of a "nice" group. A solution of the problem in this case would give another method to obtain information on group representations in Banach spaces from the associated infinitesimal representation. Note that for abelian groups N a lot is known on the synthesis problems for orbits of Lie groups in \hat{N} , see [13].—For the determination of all primitive ideals in L^1 -groups algebras of arbitrary solvable connected Lie groups it would be very helpful to solve the special case where \mathfrak{X} is an orbit of a torus (acting by automorphisms on N). So, let me state this particular case as

Problem 2. Let \mathfrak{X} be an orbit of a torus. Is $p^* \mathcal{S}(n)$ dense in $k(\mathfrak{X})$? Is $k(\mathfrak{X})/j(\mathfrak{X})$ nilpotent?

§4. The annihilators in $L^1(N)$ of simple $L^1(G)$ -modules. After the first three preparatory sections we now come to the proper theme of this article. Let G be an exponential Lie group, i.e., the exponential map \exp from the Lie algebra \mathfrak{g} of G into G is a diffeomorphism. An exponential Lie group is always solvable and, obviously, simply connected. The Lie algebra \mathfrak{g} has the property that $\text{ad } X$ has no nonzero purely imaginary eigenvalue for all $X \in \mathfrak{g}$. And this property characterizes the Lie algebras of exponential groups. Let N be the nilradical of G . In this section we will study the $L^1(N)$ -annihilators of simple $L^1(G)$ -modules. In particular, we will show that these annihilators are pithy. We will follow the usual conventions: a small German letter denotes the Lie algebra of a Lie group denoted by the corresponding large Latin letter; if H is a group acting on a set, and if x is a point in this set then H_x denotes the stabilizer of x .

THEOREM 7. *Let G be an exponential Lie group with nilradical N , and let E be a simple $L^1(G)$ -module. Then the annihilator $\text{Ann}_{L^1(N)}(E)$ of E in $L^1(N)$ is pithy. More precisely, there exists a unique G -orbit in \hat{N} , say $G\tau$, such that $k(G\tau) = \text{Ann}_{L^1(N)}(E)$. And there exists a simple $L^1(G_\tau)$ -module F with $\text{Ann}_{L^1(N)}(F) = \ker \tau$ such that E can be embedded into the induced module $\text{ind}_G^G F$.*

Proof. Let B be maximal in the set of nilpotent closed connected normal subgroups of G which have the property that the hull of $\text{Ann}_{L^1(B)}(E)$ is an N -orbit. By theorem 6, $\text{Ann}_{L^1(B)}(E)$ is pithy. If $B = N$ then $\text{Ann}_{L^1(N)}(E)$ is the kernel of an irreducible unitary representation of N , say τ . Moreover, G_τ coincides with G , and the theorem is proved in the case $B = N$. So, we may suppose that B is a proper subgroup of N . Then we choose a minimal closed connected normal subgroup C of G with $B \subsetneq C \subset N$. The dimension of C/B is one or two. Let \mathfrak{X} be the hull of $\text{Ann}_{L^1(C)}(E)$ in \hat{C} , and let \mathfrak{Y} be the hull of

$\text{Ann}_{L^1(B)}(E)$ in \hat{B} . By the Kirillov picture we may consider \mathfrak{X} resp. \mathfrak{Y} as a (closed) C -invariant subset of \mathfrak{c}^* resp. as a B -invariant subset of \mathfrak{b}^* . Let g be any element of \mathfrak{Y} . Then, by assumption, $\mathfrak{Y} = Ng$. Let $\tilde{\mathfrak{Y}} := \{f \in \mathfrak{c}^*; f|_{\mathfrak{b}} \in \mathfrak{Y}\}$. Of course we have $\tilde{\mathfrak{Y}} = Nf + \mathfrak{b}^\perp$ for all $f \in \tilde{\mathfrak{Y}}$ where \mathfrak{b}^\perp denotes the orthogonal space to \mathfrak{b} in \mathfrak{c}^* . Moreover, it is easy to see that $k(\tilde{\mathfrak{Y}})$ (if $\tilde{\mathfrak{Y}}$ is considered as a subset of \hat{C}) is equal to the closure of $L^1(C) * k(\mathfrak{Y}) * L^1(C)$. Hence, $k(\tilde{\mathfrak{Y}})$ is contained in $\text{Ann}_{L^1(C)}(E)$ and \mathfrak{X} is contained in $\tilde{\mathfrak{Y}}$.

Now, we have to distinguish several cases. Let's first assume that the dimension of C/B is one. Recall that $g \in \mathfrak{b}^*$ is an element of \mathfrak{Y} .

(1.1) N_f is strictly contained in N_g for one (or for all) $f \in \mathfrak{c}^*$ with $f|_{\mathfrak{b}} = g$.

Then $N_g f = f + \mathfrak{b}^\perp$, hence $\tilde{\mathfrak{Y}} = Nf$, and the G -invariant subset \mathfrak{X} of $\tilde{\mathfrak{Y}}$ has to coincide with $\tilde{\mathfrak{Y}}$. Hence \mathfrak{X} is an N -orbit, in contradiction to the maximality of B .

So, we may assume that $N_f = N_g$ for all extensions f of g to \mathfrak{c} . Next, suppose that even

(1.0.0) $G_f = G_g$ for all $f \in \mathfrak{c}^*$ with $f|_{\mathfrak{b}} = g$.

Since $G = NG_g = NG_f$, every G -orbit in $\tilde{\mathfrak{Y}}$ is an N -orbit. We claim that \mathfrak{X} is equal to an N -orbit (which will contradict the maximality of B). Suppose that \mathfrak{X} contains two different N -orbits Nf_1 and Nf_2 . Since the orbit space $\tilde{\mathfrak{Y}}/G = \tilde{\mathfrak{Y}}/N$ is homeomorphic to \mathfrak{b}^\perp and hence Hausdorff there exist closed N -invariant subsets A_1 and A_2 of $\tilde{\mathfrak{Y}}$ with $A_1 \cup A_2 = \tilde{\mathfrak{Y}}$ and $Nf_1 \cap A_1 = \emptyset = Nf_2 \cap A_2$. $k(\tilde{\mathfrak{Y}}) = k(A_1) \cap k(A_2)$ annihilates E , hence $L^1(G) * k(A_1) * k(A_2) * L^1(G)$ annihilates E . But the closures \mathcal{I}_1 resp. \mathcal{I}_2 of $L^1(G) * k(A_1)$ resp. $k(A_2) * L^1(G)$ are two-sided ideals in $L^1(G)$ because the $k(A_j)$ are invariant under conjugation by elements of G , and $\mathcal{I}_1 \mathcal{I}_2$ annihilates E . Since $\text{Ann}_{L^1(G)}(E)$ is a primitive ideal and especially a prime ideal, we conclude that \mathcal{I}_1 or \mathcal{I}_2 is contained in $\text{Ann}_{L^1(G)}(E)$. It follows that $k(A_1)$ or $k(A_2)$ is contained in $\text{Ann}_{L^1(C)}(E)$ and then that \mathfrak{X} is contained in A_1 or in A_2 , which is impossible by construction. Therefore \mathfrak{X} has to be an N -orbit which contradicts the maximality of B . It remains to consider the case

(1.0.1) There exists $f \in \mathfrak{c}^*$ with $f|_{\mathfrak{b}} = g$, $G_f \subsetneq G_g$ and $N_f = N_g$.

We distinguish between

- (A) G_g acts trivially on $\mathfrak{c}/\mathfrak{b}$ (or on \mathfrak{b}^\perp), and
- (B) G_g acts on \mathfrak{b}^\perp in a nontrivial way.

ad (A) From the assumption that $G_f \subsetneq G_g$ and that G_g acts trivially one deduces very easily that $G_g f = f + \mathfrak{b}^\perp$. From $G = G_g N$ it follows that $Gf = NG_g f = N(f + \mathfrak{b}^\perp) = Nf + \mathfrak{b}^\perp = \tilde{\mathfrak{Y}}$. Since \mathfrak{X} is an G -invariant subset of $\tilde{\mathfrak{Y}}$ it coincides with $\tilde{\mathfrak{Y}}$. Moreover, $k(\tilde{\mathfrak{Y}}) = k(\mathfrak{X})$ annihilates E , hence $\text{Ann}_{L^1(C)}(E) = k(\mathfrak{X})$. Put $M := G_f N = NG_f$. Of course, M depends only on \mathfrak{X} , and G is isomorphic to a semidirect product $\mathbb{R} \ltimes M$. Using theorem 2 we will show that there exist a

simple $L^1(M)$ -module D and an $L^1(G)$ -linear embedding from E into the induced module $L^2(\mathbb{R}, D)$. To this end, let $\mathcal{U} := L^1(C)/k(\mathfrak{X})$, and let $\mathcal{A} := L^1(M)/(L^1(M) * k(\mathfrak{X}))^-$. \mathcal{U} is contained in \mathcal{A}^b , and E may be considered as a simple $L^1(\mathbb{R}, \mathcal{A})$ -module. Since the unitary dual $\hat{\mathcal{U}}$ of \mathcal{U} is a transitive G -space, by lemma 2 in [26] (also all the other assumptions are satisfied) there exist $p, q \in \mathcal{U}$ with $pq = p = qp \neq 0$ such that the "Fourier transform" of q has an arbitrary small compact support. Especially, we may assume that $q^x * \mathcal{U} * q = 0$ for all $x = (z, m) \in \mathbb{Z} \times M$ with $z \neq 0$, where q^x is defined by conjugation. Then also $q^x * \mathcal{A} * q = 0$ for $x = (z, m) \in \mathbb{Z} \times M$ with $z \neq 0$. Since $pE \neq 0$ (as \mathcal{U} acts faithfully on E), all assumptions of theorem 2 are now established and we conclude that there exist a simple \mathcal{A} -module D (which may be considered as an $L^1(M)$ -module) and an $L^1(G)$ -linear embedding $E \rightarrow L^2(\mathbb{R}, D) = \text{ind}_M^G D$ such that $\text{Ann}_{L^1(M)}(E) = \bigcap_{x \in \mathbb{R}} \text{Ann}_{L^1(M)}(D)^x$ and, consequently, $\text{Ann}_{L^1(N)}(E) = \bigcap_{x \in \mathbb{R}} \text{Ann}_{L^1(N)}(D)^x$. Since M is an exponential Lie group of smaller dimension than G we know (by induction) that $\text{Ann}_{L^1(N)}(D) = k(M\tau)$ for some $\tau \in \hat{N}$, and that there exist a simple $L^1(M_\tau)$ -module F with $\text{Ann}_{L^1(N)}(F) = \ker \tau$ and an $L^1(M)$ -linear embedding $D \rightarrow \text{ind}_{M_\tau}^M F$. It follows that $\text{Ann}_{L^1(N)}(E) = \bigcap_{x \in \mathbb{R}} k(M\tau)^x = k(G\tau)$. The uniqueness of the orbit $G\tau$ follows from the fact that G -orbits in \hat{N} are locally closed (see [1]) which implies that different orbits have different closures. Considering $\text{Ann}_{L^1(C)}(E)$ one sees easily that the stabilizer G_τ is contained in M , hence $M_\tau = G_\tau$. From the theorem on induction in stages we obtain that E can be embedded into $\text{ind}_G^G F$.

ad (B) In this case, $x\lambda = \alpha(x)\lambda$ for $x \in G_g, \lambda \in \mathfrak{b}^\perp$ with a character $\alpha : G \rightarrow \mathbb{R}^\times$, nontrivial on G_g . Let f be any extension of g to \mathfrak{c} with $G_f \subsetneq G_g$. N_g is normal in G_g and G_g/N_g is abelian. Since G_f contains $N_f = N_g$, also G_f is normal in G_g and G_g/G_f is abelian. For $x \in G_g$ we write

$$xf = f + \varphi(x) \quad \text{with } \varphi(x) \in \mathfrak{b}^\perp.$$

Then $\varphi(xy) = \varphi(yx)$ and $\varphi(xy) = \varphi(x) + \alpha(x)\varphi(y)$ for all $x, y \in G_g$. These two equations imply that $\varphi(x)(1 - \alpha(y)) = \varphi(y)(1 - \alpha(x))$ for all $x, y \in G_g$. It follows that $G_f = \ker \alpha \cap G_g$ and that

$$\varphi(x) = (1 - \alpha(x))\rho \quad \text{for some non-zero } \rho \in \mathfrak{b}^\perp.$$

If \tilde{f} is another extension of g to \mathfrak{c} , $\tilde{f} = f + \mu$ with $\mu \in \mathfrak{b}^\perp$, then $G_{\tilde{f}} = G_g$ if and only if $\mu = \rho$, i.e., except for the extension $f_0 := f + \rho$ all other extensions \tilde{f} have the property that $G_{\tilde{f}} \subsetneq G_g$. Moreover, if we put $f_\pm := f_0 \pm \rho, \mathfrak{X}_\pm := Gf_\pm$ and $\mathfrak{S} = Gf_0 = Nf_0$ then $\mathfrak{X}_+ \cup \mathfrak{X}_- \cup \mathfrak{S}$ is a decomposition of \mathfrak{Y} into G -orbits, and \mathfrak{S} is contained in the closure of \mathfrak{X}_+ and \mathfrak{X}_- , respectively. From the facts that $k(\mathfrak{Y}) = k(\mathfrak{X}_+) \cap k(\mathfrak{X}_-)$ annihilates E and that $k(\mathfrak{X}_\pm)$ are G -invariant ideals one deduces that $k(\mathfrak{X}_+)$ or $k(\mathfrak{X}_-)$ annihilates E . W.l.o.g. we assume that $k(\mathfrak{X}_+)$ annihilates E . Then \mathfrak{X} is contained in $\mathfrak{X}_+ = \mathfrak{X}_+ \cup \mathfrak{S}$. Since B is maximal, \mathfrak{X} is not contained in \mathfrak{S} , hence $\mathfrak{X} \cap \mathfrak{X}_+ \neq \emptyset$ and, consequently, $\mathfrak{X} = \mathfrak{X}_+$. Put $M := G_{f_+} N = \ker \alpha$. G is isomorphic to a semidirect product $\mathbb{R} \times M$. Again we will use

theorem 2 in order to show that there exist a simple $L^1(M)$ -module D and an $L^1(G)$ -linear embedding from E into the induced module $L^2(\mathbb{R}, D) = \text{ind}_M^G(D)$. To this end, let $\mathcal{U} := k(\mathbb{S})/k(\mathbb{X})$, and let $\mathcal{A} := L^1(M)/(L^1(M) * k(\mathbb{X}))^-$. \mathcal{U} is contained in \mathcal{A}^b , and E may be considered as a simple $L^1(\mathbb{R}, \mathcal{A})$ -module. The unitary dual $\hat{\mathcal{U}}$ of \mathcal{U} is homeomorphic to $(\mathbb{X} \setminus \mathbb{S})/C = \mathbb{X}_+/C$. In particular, it is a transitive G -space. Also all the other assumptions of lemma 2 in [26] are satisfied, and we obtain that there exist $p, q \in U$ with $pq = qp = p \neq 0$ such that the "Fourier transform" of q has an arbitrary small support. Especially, we can arrange that $q^x * \mathcal{U} * q = 0$ for all $x = (z, m) \in Z \times M$ with $z \neq 0$. For those x we also have $q^x * \mathcal{A} * q = 0$. Now, we apply theorem 2 and argue as in the case (A).

Next we consider the case that $\dim C/B = 2$. From $G = G_g N$ it follows that $(\mathfrak{c}/\mathfrak{b})^* \cong \mathfrak{b}^\perp \subset \mathfrak{c}^*$ is an irreducible G_g -module because otherwise C wouldn't be minimal. \mathfrak{b}^\perp can be identified with the space \mathbb{C} of complex numbers such that the action of G on \mathfrak{b}^\perp can be written in the form $x\rho = \alpha(x)\rho$ where α is an homomorphism from G into \mathbb{C}^* . Since G is exponential α is of the form $\alpha(x) = e^{z\lambda(x)}$ where $\lambda: G \rightarrow \mathbb{R}$ is an homomorphism, and where z is a complex number with $\text{Re } z \neq 0 \neq \text{Im } z$.

We distinguish after the possible dimensions of N_g/N_f where f is an extension of g to \mathfrak{c} .

$$(2.2) \quad \dim N_g/N_f = 2$$

Then $\tilde{\mathfrak{Y}} = Nf$ and hence $\tilde{\mathfrak{X}} = \tilde{\mathfrak{Y}}$ is an N -orbit, contradicting the maximality of B .

$$(2.1) \quad \dim N_g/N_f = 1$$

We show that this case is impossible. For $x \in G_g$ we put $xf = f + \varphi(x)$ with $\varphi(x) \in \mathfrak{b}^\perp$. Then φ satisfies the equation

$$\varphi(xy) = \varphi(x) + \alpha(x)\varphi(y).$$

Putting $y = x^{-1}$ this equation implies

$$0 = \varphi(x) + \alpha(x)\varphi(x^{-1}) \quad \text{for all } x \in G_g.$$

For $x \in G_g, y \in N_g$ we get

$$\begin{aligned} \varphi(xyx^{-1}) &= \varphi(x(yx^{-1})) = \varphi(x) + \alpha(x)\varphi(yx^{-1}) \\ &= \varphi(x) + \alpha(x)(\varphi(y) + \varphi(x^{-1})) = \alpha(x)\varphi(y). \end{aligned}$$

Since N_g is normal in G_g the set $\varphi(N_g)$ is invariant under $\alpha(G_g)$. On the other hand, $\varphi(N_g)$ is an one-dimensional real subspace of \mathfrak{b}^\perp , contradicting the irreducibility of \mathfrak{b}^\perp .

$$(2.0) \quad N_f = N_g$$

We distinguish between

$$(2.0.0) \quad G_f = G_g \quad \text{for all extensions } f \text{ of } g.$$

and

$$(2.0.1) \quad \text{there exists an extension } f \text{ with } G_f \subsetneq G_g.$$

ad (2.0.0) This case is analogous to (1.0.0). Again every G -orbit in $\tilde{\mathfrak{Y}}$ is an N -orbit, and $\tilde{\mathfrak{Y}}/G = \tilde{\mathfrak{Y}}/N$ is homeomorphic to \mathfrak{b}^\perp and hence Hausdorff. A similar argument as in (1.0.0) shows that \mathfrak{X} has to be an N -orbit, in contradiction to the maximality of B .

ad (2.0.1) Let f be any extension of g to \mathfrak{c} with $G_f \subsetneq G_g$. N_g is normal in G_g and G_g/N_g is abelian. Since G_f contains $N_f = N_g$, also G_f is normal in G_g and G_g/G_f is abelian. Again we write

$$xf = f + \varphi(x) \quad \text{for } x \in G_g,$$

and find that

$$\varphi(xy) = \varphi(x) + \alpha(x)\varphi(y),$$

and

$$\varphi(xy) = \varphi(yx) \quad \text{for all } x, y \in G_g.$$

Therefore, $\varphi(x)(1 - \alpha(y)) = \varphi(y)(1 - \alpha(x))$ for all $x, y \in G_g$. From the last equation one deduces that $G_f = \ker \alpha \cap G_g$, and that

$$\varphi(x) = (1 - \alpha(x))\rho \quad \text{for some non-zero } \rho \in \mathfrak{b}^\perp.$$

If \tilde{f} is another extension of g to \mathfrak{c} , $\tilde{f} = f + \delta$ with $\delta \in \mathfrak{b}^\perp$, then $G_{\tilde{f}} = G_g$ if and only if $\delta = \rho$, i.e., except for the extension $f_0 := f + \rho$ all other extensions \tilde{f} have the property that $G_{\tilde{f}} \subsetneq G_g$, in which case $G_{\tilde{f}} = \ker \alpha \cap G_g$. Put $\mathfrak{S} = Gf_0 = Nf_0$, and $M = \ker \alpha (= NG_{\tilde{f}}$ for all extensions $\tilde{f} \neq f_0$ of g). G is isomorphic to a semidirect product $\mathbb{R} \ltimes M$. The closure of every G -orbit in $\tilde{\mathfrak{Y}}$ contains \mathfrak{S} ; in particular \mathfrak{X} contains \mathfrak{S} . Since B is maximal, \mathfrak{S} is a proper subset of \mathfrak{X} ; we choose a point π in $\mathfrak{X} \setminus \mathfrak{S}$. Again we apply theorem 2. Let $\mathcal{U} := k(\mathfrak{S})/k(\tilde{\mathfrak{Y}})$, and let $\mathcal{A} := L^1(M)/(L^1(M) * k(\tilde{\mathfrak{Y}}))^-$. \mathcal{U} is contained in \mathcal{A}^b , and E may be considered as a simple $L^1(\mathbb{R}, \mathcal{A})$ -module. The unitary dual of \mathcal{U} is homeomorphic to the space of \mathbb{C} -orbits in $\tilde{\mathfrak{Y}} \setminus \mathfrak{S}$. This space is homogeneous for the action of the (real) algebraic hull of the adjoint group $\text{Ad}(G)$. Hence we may again apply lemma 2 of [26] to the algebra \mathcal{U} , and we find $p, q \in U$ with $pq = qp = p \neq 0$, even $\pi(p) \neq 0$ where π is the above chosen element in $\mathfrak{X} \setminus \mathfrak{S}$, such that the "Fourier transform \hat{q} " of q has a small support. We choose the support of \hat{q} so small that $q^x * \mathcal{U} * q = 0 = q^x * \mathcal{A} * q$ for all $x = (z, m) \in \mathbb{Z} \ltimes M$ with $z \neq 0$ and argue as in the case (1.0.1). Note that $\pi(p) \neq 0$ implies that $pE \neq 0$.

§5. The parametrization theorem. Since the annihilators of simple $L^1(G)$ -modules in $L^1(N)$ are now known we are going to classify all the simple $L^1(G)$ -modules with a given annihilator in $L^1(N)$. The idea for doing this is quite simple. One uses the existence of rank one operators in the image of L^1 -group algebras under irreducible unitary representations γ (which was proved in [26]) for suitable exponential groups and suitable γ 's and applies first theorem 1 (iii) to suitable quotients of L^1 -group algebras and then theorem 4. The rest consists of straightforward calculations. Let's first introduce and fix some notations.

Notations. Let G be an exponential Lie group with nilradical N . Let τ be a fixed irreducible unitary representation of N in the Hilbert space \mathfrak{R} . Denote by K the stabilizer of the unitary equivalence class of τ in \hat{N} . It is known that K is connected and that $K = G_g N$ if $g \in \mathfrak{n}^*$ is a functional corresponding to τ . Then τ admits a projective extension $\tilde{\tau}$ to K , and we may assume that the cocycle m , corresponding to $\tilde{\tau}$, is given by a skew symmetric bilinear form μ on the vector group K/N , i.e.,

$$\tilde{\tau}(x)\tilde{\tau}(y) = m(x, y)\tilde{\tau}(xy),$$

and $m(x, y) = e^{i\mu(\dot{x}, \dot{y})}$ for $x, y \in K$, where $x \rightarrow \dot{x}$ denotes the canonical homomorphism $G \rightarrow G/N$. In the sequel, the kernel $Z := \{x \in K; m(x, K) = 1\}$ of the bicharacter m will play an important role. To the quotient homomorphism $G \rightarrow G/N$ we choose a continuous cross section $s: G/N \rightarrow G$ with $s(0) = e$ and $s(-y) = s(y)^{-1}$. Moreover, we choose a vector space complement V to K/N in G/N , i.e., $G/N = V \oplus K/N$, and put $H := Ns(V)$. Next, we form the induced representation $\gamma = \text{ind}_N^H \tau$ in the space $\mathfrak{S} = L^2_N(H, \mathfrak{R})$. γ is irreducible because the stabilizer in H of the class of τ in \hat{N} is equal to N . Moreover, γ admits a projective extension $\tilde{\gamma}$ to G , given by

$$(\tilde{\gamma}(kh)\xi)(x) = \tilde{\tau}(k)\xi(h^{-1}k^{-1}xk)$$

for $h, x \in H$ and $k \in K$. $\tilde{\gamma}$ is an m -projective representation where, via the isomorphism $G/H \cong K/N$, m is considered as cocycle on G .

The cross section $s: G/N \rightarrow G$ defines a cross section $\sigma: G/H \rightarrow G$ by combining the isomorphism $G/H \cong K/N$ with the restriction of s to K/N . Using σ the Banach space $L^1(G)$ can be identified with $L^1(G/H, L^1(H))$. In [14], it was computed how the convolution and the involution in $L^1(G)$ are transferred into $L^1(G/H, L^1(H))$: The group G acts on $L^1(H)$ by

$$a^x(h) = \Delta(x)^{-1}a(xhx^{-1}) \quad \text{for } x \in G,$$

$a \in L^1(H)$, $h \in H$, where Δ denotes the modular function of the action of G on H (or on N), i.e. $d(xhx^{-1}) = \Delta(x)^{-1}dh$. For $x, y \in G/H$ the "multiplier" $P_{x,y}: L^1(H) \rightarrow L^1(H)$ is defined by

$$(P_{x,y}a)(h) = a(\sigma(y)^{-1}\sigma(x)^{-1}\sigma(x+y)h).$$

Then the convolution and the involution in $L^1(G/H, L^1(H))$ are given by

$$(f * g)(x) = \int_{G/H} dy P_{x+y, -y} (f(x+y)^{\sigma(-y)}) g(-y)$$

and

$$f^*(x) = f(-x)^{\sigma(x)}.$$

Since the ideal $\ker \gamma$ in $L^1(H)$ is invariant under the action of G we may form the algebra $L^1(G/H, L^1(H)/\ker \gamma)$ with induced action and multiplier. This algebra is isomorphic to the quotient algebra $L^1(G)/(L^1(G) * \ker \gamma)^-$. Let q be an element in $L^1(H)/\ker \gamma$ such that $\gamma(q)$ is a projection of rank one in \mathfrak{S} , say $\gamma(q) = \langle -, \lambda \rangle \lambda$ (by the theorem in [26], such a q exists). As was shown in [26], it is possible to "compute" the algebra $q * L^1(G/H, L^1(H)/\ker \gamma) * q$. Here, I will only describe the result:

(1) There exists a unique continuous function $v: G \rightarrow L^1(H)/\ker \gamma$ with $\gamma(v(x)) = \langle -, \lambda \rangle \tilde{\gamma}(x)^{-1} \lambda$. $v(x)$ is contained in $q^x * (L^1(H)/\ker \gamma) * q$. The function $\tilde{w}: G \rightarrow \mathbb{R}$, defined by $\tilde{w}(x) = \|v(x)\|$, is constant on cosets modulo H and defines a symmetric weight function on G/H . Then one may form the algebra $L^1(G/H, m, \tilde{w})$ as in §2. Here it is even an involutive Banach algebra, the involution being given by $g^*(x) = \overline{g(-x)}$. The map $h \rightarrow h'$, $h'(x) := h(x)v(\sigma(x))$, is an isometric $*$ -isomorphism from $L^1(G/H, m, \tilde{w})$ onto the subalgebra $q * L^1(G/H, L^1(H)/\ker \gamma) * q$ of $L^1(G/H, L^1(H)/\ker \gamma)$.

In a similar way one may treat the algebra $L^1(K)/(L^1(K) * \ker \tau)^-$. Using the cross section s this algebra turns out to be isomorphic to $L^1(K/N, L^1(N)/\ker \tau)$. Let p be an element in $L^1(N)/\ker \tau$ such that $\tau(p)$ is a projection of rank one, say $\tau(p) = \langle -, \xi \rangle \xi$. Then one finds:

(2) There exists a unique continuous function $u: K \rightarrow L^1(N)/\ker \tau$ with $\gamma(u(x)) = \langle -, \xi \rangle \tilde{\tau}(x)^{-1} \xi$. $u(x)$ is contained in $p^x * (L^1(N)/\ker \tau) * p$. The function $w: K \rightarrow \mathbb{R}$, defined by $w(x) = \|u(x)\|$ is constant on cosets modulo N and defines a symmetric weight function on K/N . Form the algebra $L^1(K/N, m, w)$ as above. The map $h \rightarrow h'$, $h'(x) = h(x)u(s(x))$, is an isometric $*$ -isomorphism from $L^1(K/N, m, w)$ onto $p * L^1(K/N, L^1(N)/\ker \tau) * p$.

Let E be a simple $L^1(G)$ -module with $\text{Ann}_{L^1(N)}(E) = k(G\tau)$. Since $\ker \gamma = \{L^1(H) * k(G\tau)\}^-$, E may be considered as an $L^1(H)/\ker \gamma$ -module. I claim that $qE \neq 0$. To this end, let \mathcal{I}_0 be the ideal of all $f \in L^1(H)$ such that $\gamma(f)$ is an operator of finite rank, and let \mathcal{I} be the closure of \mathcal{I}_0 in $L^1(H)$. Since $\gamma \otimes \chi$ is unitarily equivalent to γ for all unitary characters χ of H/N the ideals \mathcal{I}_0 and \mathcal{I} are invariant under multiplications with χ . It follows that \mathcal{I} is generated by an ideal \mathcal{I}' in $L^1(N)$, i.e., $\mathcal{I} = (L^1(H) * \mathcal{I}')^-$. Since $\ker \gamma$ is strictly contained in \mathcal{I} , $k(G\tau) = \text{Ann}_{L^1(N)}(E)$ has to be strictly contained in \mathcal{I}' . Hence \mathcal{I}' does not annihilate E from which one deduces that $\mathcal{I}_0 E \neq 0$. From $(\mathcal{I}_0/\ker \gamma) * q * (\mathcal{I}_0/\ker \gamma) = \mathcal{I}_0/\ker \gamma$ (compare [7]) it follows that $qE \neq 0$.

qE is a simple module over $q * L^1(G/H, L^1(H)/\ker \gamma) * q$, and theorem 1 (iii) shows that E is uniquely determined by this module qE .

qE may be considered as a simple $L^1(G/H, m, \tilde{w})$ -module via the isomorphism in (1). By theorem 4, to this simple module there corresponds a character η of the kernel of m , i.e., of $ZH/H \cong Z/N$, with $|\eta| \leq \tilde{w}$. Denote all these characters by $\hat{Z}_{\tilde{w}}$. On the other hand, every $\eta \in \hat{Z}_{\tilde{w}}$ gives rise to a simple module over $q * L^1(G/H, L^1(H)/\ker \gamma) * q$ and then to a simple $L^1(G)$ -module E_η with $\text{Ann}_{L^1(N)}(E_\eta) = k(G\tau)$. Let's summarize:

(3) There is a bijection between the set of isomorphism classes of simple $L^1(G)$ -modules E with $\text{Ann}_{L^1(N)}(E) = k(G\tau)$ and the set $\hat{Z}_{\tilde{w}}$ of characters η on $ZH/H \cong Z/N$ with $|\eta| \leq \tilde{w}$. Moreover, there exists $h \in L^1(G)$ such that hE is one-dimensional.

The last assertion follows from the corresponding statement in theorem 4. Similarly, one gets

(4) There is bijection between the set of isomorphism classes of simple $L^1(K)$ -modules F with $\text{Ann}_{L^1(N)}(F) = \ker \tau$ and the set \hat{Z}_w of characters χ on Z/N with $|\chi| \leq w$.

The following connection between the modules in (4) and certain $L^1(G)$ -modules will be used in the proof of theorem 8.

(5) Let F be the simple $L^1(K)$ -module corresponding to $\chi \in \hat{Z}_w$, and denote by ρ the isometric action of K in F . Form the induced module $\text{ind}_K^G F = L_K^2(G, F)$. Then $q * L^1(HZ/H, L^1(H)/\ker \gamma) * q$ acts in $qL_K^2(G, F)$ by the character χ if $q * L^1(HZ/H, L^1(H)/\ker \gamma) * q$ is identified with $L^1(HZ/H, \tilde{w}) = L^1(Z/N, \tilde{w})$ as above.

In order to prove (5) we need some preparations. First, we claim the following formula which reflects the fact that F is the simple module corresponding to χ .

(6) $\rho(z)\rho(u(z)) = \chi(z)\rho(p)$ for all $z \in Z$.

Let $h \in L^1(Z/N, w)$, and let h' be the corresponding element in $p * L^1(Z/N, L^1(N)/\ker \tau) * q$, i.e., $h'(t) = h(t)u(s(t))$. For all $\beta \in \rho(p)F$, we get

$$\int_{Z/N} dt h(t)\chi(t)\beta = \rho(h')\beta = \int_{Z/N} dt h(t)\rho(s(t))\rho(u(s(t)))\beta.$$

Since this equation holds true for all $h \in L^1(Z/N, w)$, one obtains $\chi(t)\beta = \rho(s(t))\rho(u(s(t)))\beta$ for all $\beta \in \rho(p)F$ and all $t \in Z/N$, and hence $\chi(t)\rho(p) = \rho(s(t))\rho(u(s(t)))$ for all $t \in Z/N$. From this equation one induces (6) by using the fact that $u(zn) = \delta_{n^{-1}} * u(z)$ ($n \in N, z \in Z$; $\delta_{n^{-1}}$ denotes the Dirac measure at n^{-1}) and, therefore, $\rho(u(zn)) = \rho(n)^{-1}\rho(u(z))$.

Next, let \mathfrak{R}_0 be the span of $\{\tau(f)\xi; f \in L^1(N), \tau(f) \text{ is of finite rank}\}$. It is known that \mathfrak{R}_0 is a simple $L^1(N)$ -module (see [7]), in fact it is the unique simple $L^1(N)$ -module whose annihilator is $\ker \tau$. Especially, we have $\mathfrak{R}_0 = \tau(L^1(N))\xi$; recall that $\tau(p) = \langle -, \xi \rangle \xi$. Using ξ , one may introduce a norm on \mathfrak{R}_0 by $\|\alpha\| = \inf\{\|f\|; f \in L^1(N), \tau(f)\xi = \alpha\}$. Then \mathfrak{R}_0 becomes a Banach space and N acts, via τ , isometrically on \mathfrak{R}_0 . It is easy to see that \mathfrak{R}_0 is invariant under $\tilde{\tau}(K)$. In particular, one gets a representation τ' of Z in \mathfrak{R}_0 . For every $z \in Z$, $\tau'(z)$ is a

bounded operator on \mathfrak{R}_0 , and τ' is a strongly continuous representation of Z in \mathfrak{R}_0 , but not necessarily uniformly bounded. In fact, it is not hard to see that the weight w' on Z/N defined by $w'(z) = \|\tau'(z)\|$ is equivalent to w . Let $L_N^2(H, \mathfrak{R}_0)$ be the induced H -module, and denote by δ the isometric representation of H in $L_N^2(H, \mathfrak{R}_0)$. The representation ϵ of Z in $L_N^2(H, \mathfrak{R}_0)$ is defined by

$$(\epsilon(z)\alpha)(x) = \chi(z)\tau'(x^{-1}zx)\alpha(x) = \chi(z)\tau'(z)\alpha(z^{-1}xz).$$

δ can be integrated to a representation of $L^1(H)$, also denoted by δ , and it may be considered as a representation of $L^1(H)/\ker \gamma$. We claim that

$$(7) \quad \epsilon(z)\delta(v(z)) = \chi(z)\delta(q) \text{ for all } z \in Z.$$

The injection of \mathfrak{R}_0 into \mathfrak{R} induces an injection of $L_N^2(H, \mathfrak{R}_0)$ into $\mathfrak{S} = L_N^2(H, \mathfrak{R})$. This injection intertwines δ and γ , and it intertwines ϵ and $\tilde{\gamma}|_Z \otimes \chi$. Hence it suffices to show that $\tilde{\gamma}(z)\chi(z)\gamma(v(z)) = \chi(z)\gamma(q)$ for all $z \in Z$. But this equation is obvious.

(8) Denote by π the representation of G in the induced module $L_K^2(G, F)$. Then $\pi(z)\pi(v(z)) = \chi(z)\pi(q)$ for all $z \in Z$.

The space $L_K^2(G, F)$ may be identified with $L_N^2(H, F) = \text{ind}_N^H F|_N$, and the representation $\pi|_{HZ}$ in this space is given by

$$(\pi(h)\alpha)(x) = \alpha(h^{-1}x),$$

$$(\pi(z)\alpha)(x) = \rho(x^{-1}zx)\alpha(x) = \rho(z)\alpha(z^{-1}xz)$$

for $z \in Z$ and $h, x \in H$. Let β be a nonzero element in $\rho(p)F$. Then we define the operator $S = S_\beta$ from \mathfrak{R}_0 into F by $S(\tau'(g)\xi) = \rho(g)\beta$ for $g \in L^1(N)$. It is easy to see that S is a well-defined bounded operator. S induces a bounded operator $W = W_\beta$ from $L_N^2(H, \mathfrak{R}_0)$ into $L_N^2(H, F)$ by putting $(W\alpha)(x) = S(\alpha(x))$. S intertwines $\tau' \otimes \chi$ and $\rho|_Z$:

Let $z \in Z$ and $\vartheta \in \mathfrak{R}_0$. Put $\vartheta = \tau'(g)\xi = \tau(g)\xi$ with $g \in L^1(N)$. Then

$$\begin{aligned} S((\tau \otimes \chi)(z)\vartheta) &= \chi(z)S(\tau'(z)\tau'(g)\xi) \\ &= \chi(z)S(\tau'(z)\tau'(g)\tau'(z)^{-1}\tau'(z)\xi) \\ &= \chi(z)S(\tau'(g^{z^{-1}})\tau'(u(z^{-1}))\xi) \end{aligned}$$

because $\tau(z)\xi = \tau(u(z^{-1}))\xi$. It follows that $S((\tau \otimes \chi)(z)\vartheta) = \chi(z)\rho(g^{z^{-1}})\rho(u(z^{-1}))\beta = \chi(z)\rho(g^{z^{-1}})\rho(z)\chi(z^{-1})\rho(p)\beta$ because of (6), hence

$$S((\tau \otimes \chi)(z)\vartheta) = \rho(g^{z^{-1}})\rho(z)\beta.$$

But $\rho(z)S\vartheta = \rho(z)\rho(g)\beta = \rho(z)\rho(g)\rho(z)^{-1}\rho(z)\beta = \rho(g^{z^{-1}})\rho(z)\beta$. From this intertwining property of S one deduces very easily that $W: L_N^2(H, \mathfrak{R}_0) \rightarrow L_N^2(H, F)$ intertwines ϵ and $\pi|_Z$. Moreover, it is obvious that W intertwines δ and $\pi|_H$. From these facts and (7) it follows that for $z \in Z$ the operators $\pi(z)\pi(v(z))$ and

$\chi(z)\pi(q)$ coincide on the range of $W = W_\beta$. But the union of the ranges of W_β , $\beta \in \rho(p)F$, is total in $L_N^2(H, F)$, and (8) is proved.

Now, (5) is an easy consequence of (8), compare the computations in the proof of (6). After these preparations it is not hard to prove the parametrization theorem.

THEOREM 8. *Let G be an exponential Lie group with nilradical N , and let τ be an irreducible unitary representation of N . For every simple $L^1(G)$ -module E with $\text{Ann}_{L^1(N)}(E) = k(G\tau)$ there exists a unique simple $L^1(G_\tau)$ -module F with $\text{Ann}_{L^1(N)}(F) = \ker \tau$ such that E can be embedded into the induced module $\text{ind}_{G_\tau}^G F = L_{G_\tau}^2(G, F)$. For every simple $L^1(G_\tau)$ -module F with $\text{Ann}_{L^1(N)}(F) = \ker \tau$ the induced module $L_{G_\tau}^2(G, F)$ contains a unique simple $L^1(G)$ -module E with $\text{Ann}_{L^1(N)}(E) = k(G\tau)$. Hence there is a bijection between the set of isomorphism classes of simple $L^1(G)$ -modules E with $\text{Ann}_{L^1(N)}(E) = k(G\tau)$ and the set of isomorphism classes of simple $L^1(G_\tau)$ -modules F with $\text{Ann}_{L^1(N)}(F) = \ker \tau$. The latter set can be parametrized by the set of characters \hat{Z}_w on Z/N which depends, of course, on τ . Moreover, for every simple $L^1(G)$ -module E there exists an $h \in L^1(G)$ such that hE is one-dimensional.*

Proof. The existence of an $L^1(G_\tau)$ -module F such that E can be embedded into $\text{ind}_{G_\tau}^G F$ was already established in theorem 7. The uniqueness is an immediate consequence of the following fact, which is easily deduced from (5):

(9) Suppose that the simple $L^1(G)$ -module E corresponds to $\eta \in \hat{Z}_w$ in the sense of (3) and that the simple $L^1(G_\tau)$ -module F corresponds to $\chi \in \hat{Z}_w$ in the sense of (4). If E can be embedded into $\text{ind}_{G_\tau}^G F$ then $\chi = \eta$.

Now, let F be a simple $L^1(G_\tau)$ -module with $\text{Ann}_{L^1(N)}(F) = \ker \tau$. Suppose that F corresponds to $\chi \in \hat{Z}_w$. By (5), $q * L^1(HZ/H, L^1(H)/\ker \gamma) * q \cong L^1(Z/N, \tilde{w})$ acts in $q(\text{ind}_{G_\tau}^G F)$ by the character χ . Since this is a bounded representation it follows that $|\chi(z)| \leq \tilde{w}(z)$ for all $z \in Z$, i.e., $\chi \in \hat{Z}_w$. Let E be the simple $L^1(G)$ -module corresponding to χ in the sense of (3). By theorem 7, there exists a simple $L^1(G_\tau)$ -module F' such that E can be embedded into $\text{ind}_{G_\tau}^G F'$. From (9) it follows that F' is isomorphic to F , hence $\text{ind} F$ contains the simple module E . The uniqueness of E is also an immediate consequence of (9).

The last statements of the theorem were already established in (4) and (3).

Let's summarize the results of §4 and §5. Let G be an exponential Lie group with nilradical N . The canonical map from the set $\mathcal{S} = \mathcal{S}(L^1(G))$ of isomorphism classes of simple $L^1(G)$ -modules onto the set $\text{Priv}(L^1(G))$ of primitive ideals is a bijection. To every $E \in \mathcal{S}$ there is associated an G -orbit $G\tau$ in \hat{N} such that $\text{Ann}_{L^1(N)}(E) = k(G\tau)$. This defines a surjection $\mathcal{S} \rightarrow \hat{N}/G$. The fibers of this map can be described as follows. To an G -orbit $\mathcal{Q} = G\tau$ in \hat{N} there is associated a connected subgroup $Z = Z_\tau$, $N \subset Z \subset G_\tau \subset G$, and a weight $w: Z/N \rightarrow \mathbb{R}$ such that the fiber over \mathcal{Q} is in bijective correspondence with the set \hat{Z}_w of continuous characters $\chi: Z/N \rightarrow \mathbb{C}^\times$ which are dominated by w . Moreover, this correspondence is an homeomorphism if \mathcal{S} carries the Fell

topology and \hat{Z}_w is equipped with the compact open topology. To complete the description of \mathcal{S} one has to know the group $Z_{\mathfrak{D}}$ and the weight $w = w_{\mathfrak{D}}$. The first problem was already solved by Dulfo, see [8]. Suppose that \mathfrak{D} corresponds in the Kirillov picture to the orbit Gg in \mathfrak{n}^* . If f is any extension of g to \mathfrak{g} then $Z_{\mathfrak{D}} = G_f N$. The second problem will be treated in §6. We will prove an estimate for the weight w which allows to determine all the continuous characters dominated by w , i.e., the set \hat{Z}_w .

§6. Estimate of the weight w . First, we recall some notations and introduce some new ones. G is an exponential Lie group with nilradical N , \mathfrak{g} and \mathfrak{n} are their Lie algebras. f is a real linear functional on \mathfrak{g} , $g := f|_{\mathfrak{n}} \in \mathfrak{n}^*$. τ is the irreducible representation corresponding to g , realized in the Hilbert space \mathfrak{R} . p is a projection of rank one in $L^1(N)/\ker \tau$, $\tau(p) = \langle \cdot, \xi \rangle \xi$. τ admits an extension $\tilde{\tau}$ to $Z = G_f N$, and there exists a unique continuous function $u: Z \rightarrow L^1(N)/\ker \tau$ with $\tau(u(x)) = \langle \cdot, \xi \rangle \tilde{\tau}(x)^{-1} \xi$. $w: Z \rightarrow \mathbb{R}$ is defined by $w(x) = \|u(x)\|$. w is independent of the choice of $\tilde{\tau}$ because different extensions differ only by a unitary character of Z/N . And w is essentially independent of the choice of p : different p 's give equivalent weights. Let V be an irreducible finite dimensional real G_f -module; of course, V is one or two dimensional. If the G_f -action is given by the homomorphism $\pi: G_f \rightarrow \text{Aut}(V)$ we define $\text{ch}_V: G_f \rightarrow \mathbb{R}$ by $\text{ch}_V(x) = (\det \pi(x) + \det \pi(x)^{-1})^{1/2}$ if V is one-dimensional, and by $\text{ch}_V(x) = \det \pi(x)^{1/2} + \det \pi(x)^{-1/2}$ if V is two-dimensional. Note that if V is two-dimensional and H is a subgroup of G_f such that V decomposes into two one-dimensional H -submodules V_1 and V_2 then $\text{ch}_{V|H} = \text{ch}_{V_1} \text{ch}_{V_2}$. Now let \mathfrak{p} be an G_f -invariant polarization at the point g . Since G is exponential such a polarization exists, see [1]. Next we choose a Jordan-Hölder composition series for the G_f -module $\mathfrak{n}/\mathfrak{p}$, i.e., $V_0 = \mathfrak{n} \supset V_1 \supset V_2 \supset \dots \supset V_n = \mathfrak{p}$, the V_i 's are G_f -invariant, and V_{i-1}/V_i is an irreducible G_f -module. Then we define $\mu: G_f \rightarrow \mathbb{R}$ by $\mu = \prod_{i=1}^n \text{ch}_{V_{i-1}/V_i}$. Of course, μ is independent of the choice of the Jordan-Hölder series. But it is also independent of the choice of the polarization for the following reason:

If V_i^\perp denotes the orthogonal space to V_i with respect to the skew symmetric form associated with g , i.e., $V_i^\perp = \{X \in \mathfrak{n}; g([X, V_i]) = 0\}$ then $V_0 \supset V_1 \supset \dots \supset V_n = \mathfrak{p} = V_n^\perp \supset V_{n-1}^\perp \supset \dots \supset V_1^\perp \supset V_0^\perp = \mathfrak{n}_g$ is a Jordan-Hölder series for the G_f -module $\mathfrak{n}/\mathfrak{n}_g$. Since there is an G_f -invariant nondegenerate duality between V_{i-1}/V_i and V_i^\perp/V_{i-1}^\perp it follows that $\text{ch}_{V_{i-1}/V_i} = \text{ch}_{V_i^\perp/V_{i-1}^\perp}$. Hence $\mu^2 = \prod_{i=1}^{2n} \text{ch}_{W_{i-1}/W_i}$ for any Jordan-Hölder series $W_0 = \mathfrak{n} \supset W_1 \supset \dots \supset W_{2n} = \mathfrak{n}_g$ of the G_f -module $\mathfrak{n}/\mathfrak{n}_g$.

THEOREM 9. *For a continuous complex character η of $Z/N = G_f N / N \cong G_f / G_f \cap N$ into \mathbb{C}^\times the inequality $|\eta| \leq w$ is equivalent to $|\eta| \leq \mu$. Note that μ is constant on cosets modulo $G_f \cap N$ and may be viewed as a function on Z/N .*

Remark. By taking logarithms, the condition $|\eta| \leq \mu$ can be linearized. In

fact, $|\eta| \leq \mu$ is equivalent to

$$\log |\eta(x)| \leq \frac{1}{2} \sum_{i=1}^n |\log \det_{V_{i-1}/V_i} \text{Ad}(x)| \quad \text{for all } x \in G_f.$$

And the latter inequality is equivalent to the existence of real numbers c_1, \dots, c_n with $-1 \leq c_j \leq 1$ and

$$\log |\eta| = \frac{1}{2} \sum_{i=1}^n c_i \log \det_{V_{i-1}/V_i} \text{Ad}.$$

The latter equivalence is a consequence of the following well-known general fact: Let $\beta, \alpha_1, \dots, \alpha_n$ be linear functionals on a finite dimensional real vector space. Then $\beta \leq \sum_{i=1}^n |\alpha_i|$ iff there exist real numbers c_1, \dots, c_n with $-1 \leq c_i \leq 1$ and $\beta = \sum_{i=1}^n c_i \alpha_i$.

Proof. Let's first assume that $|\eta| \leq w$. We are going to show that $|\eta(\exp(tX))| \leq \mu(\exp(tX))$ for $X \in \mathfrak{g}_f$ and $t \in \mathbb{R}$. Let $D = \text{ad}_{\mathfrak{n}}(X) : \mathfrak{n} \rightarrow \mathfrak{n}$ be the associated derivation on \mathfrak{n} . We will reduce the problem to the case that D is semisimple where one can do explicit calculations. To this end, let $D = D_s + D_n$ be the additive Jordan decomposition of D , where D_s is the semisimple part, and let $\exp(tD_s)$ and $\exp(tD_n)$ be the automorphisms of N such that the diagrams

$$\begin{array}{ccc} \mathfrak{n} & \xrightarrow{\text{Exp}(tD_s)} & \mathfrak{n} \\ \exp \downarrow & \text{exp}(tD_s) & \downarrow \text{exp} \\ N & \longrightarrow & N \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathfrak{n} & \xrightarrow{\text{Exp}(tD_n)} & \mathfrak{n} \\ \exp \downarrow & \text{exp}(tD_n) & \downarrow \text{exp} \\ N & \longrightarrow & N \end{array}$$

commute. Then $\exp(tD_s)\exp(tD_n) = \exp(tD_n)\exp(tD_s)$ is equal to the inner automorphism corresponding to $\exp(tX)$. Moreover, we fix an \mathfrak{g}_f -invariant polarization \mathfrak{p} of \mathfrak{n} at g . Then \mathfrak{p} is invariant under D_s and under $\text{Exp}(tD_s)$, and $\mu(\exp(tX)) = \mu(\text{Exp}(tD_s))$. We realize the representation τ in $\mathfrak{R} = L^2_{\mathfrak{p}}(N, \mathbb{C})$ with $P = \exp(\mathfrak{p})$. The one-parameter groups of unitary operators, corresponding to D_n and D_s , are defined by

$$(\tau_n(t)\varphi)(x) = \varphi(\exp(-tD_n)(x))$$

and

$$(\tau_s(t)\varphi)(x) = \delta(t)^{-1/2} \varphi(\exp(-tD_s)(x))$$

where $\delta(t)$ is the determinant of $\text{Exp}(tD_s)$ on $\mathfrak{n}/\mathfrak{p}$. From the irreducibility of τ it follows that $\tilde{\tau}(\exp(tX))$ and $\tau_s(t)\tau_n(t) = \tau_n(t)\tau_s(t)$ differ only by a unitary character $\chi : \tilde{\tau}(\exp(tX)) = \chi(t)\tau_s(t)\tau_n(t)$. Next, we choose elements $b_i, c_i \in L^1(N)/\ker \tau$ with

$$\tau(b_i) = \langle -, \xi \rangle \tau_s(t)^{-1} \xi$$

and

$$\tau(c_t) = \langle -, \xi \rangle \tau_n(t)^{-1} \xi.$$

It is easy to see that $\chi(t)u(\exp(tX)) = b_t^{\exp(tD_n)}c_t$. Recall that for $f \in L^1(N)$ the function $f^{\exp(tD_n)}$ is defined by $f^{\exp(tD_n)}(x) = f(\exp(tD_n)(x))$; and this induces an isometric action on $L^1(N)/\ker \tau$. It follows that $|\eta(\exp(tX))| \leq w(\exp(tX)) = \|u(\exp(tX))\| \leq \|b_t\| \|c_t\|$. In the last step, we will prove that there is a constant B with $\|b_t\| \leq B\mu(\text{Exp}(tD_s)) = B\mu(\exp(tX))$. In the moment, we take this inequality for granted and show how the first part of the theorem follows. μ can be written in the form

$$\mu(\exp(tX)) = \prod_{j=1}^k (e^{\alpha_j t} + e^{-\alpha_j t}) \prod_{j=1}^m \sqrt{e^{2\beta_j t} + e^{-2\beta_j t}}$$

with real numbers $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m$. Suppose that $|\eta(\exp(tX))| = e^{at}$. The desired inequality is equivalent to

$$|a| \leq \sum_{j=1}^k |\alpha_j| + \sum_{j=1}^m |\beta_j|.$$

Assume to the contrary that

$$\Lambda := |a| - \sum_{j=1}^k |\alpha_j| - \sum_{j=1}^m |\beta_j| > 0.$$

From $e^{\pm at} \leq B\mu(\exp(tX))\|c_t\|$ it follows that $e^{\lambda t} \leq \|c_t\|$ for all λ with $-\Lambda \leq \lambda \leq \Lambda$. Let $L^1(\mathbb{R}, L^1(N)/\ker \tau)$ be the generalized L^1 -algebra for the action of \mathbb{R} on $L^1(N)/\ker \tau$ which is given by the one-parameter automorphism group $\exp(tD_n)$, see above. Then $L^1(\mathbb{R}, L^1(N)/\ker \tau)$ is symmetric because it is a quotient of $L^1(\mathbb{R} \ltimes N)$ and $\mathbb{R} \ltimes N$ is nilpotent, see [22]. On the other hand, $p * L^1(\mathbb{R}, L^1(N)/\ker \tau) * p$ is isomorphic to $L^1(\mathbb{R}, v)$ where the weight v is given by $v(t) = \|c_t\|$. This algebra has non-hermitian multiplicative linear functionals (given by characters $e^{\lambda t}, 0 < \lambda \leq \Lambda$), and is, therefore, not symmetric. So, we have obtained a contradiction.

It remains to estimate $\|b_t\|$. We do this for a particularly chosen rank one projection p in $L^1(N)/\ker \tau$ which is essentially given by the Gauß function. We start with a simple lemma, compare [1] for similar statements.

LEMMA 1. *Let \mathfrak{n} be a nilpotent Lie algebra, let N be the corresponding simply connected Lie group, and let $D : \mathfrak{n} \rightarrow \mathfrak{n}$ be a semisimple derivation. Suppose that \mathfrak{h} is an D -invariant subalgebra of \mathfrak{n} . Then there exist D -irreducible subspaces $\mathfrak{v}_1, \dots, \mathfrak{v}_m$ of \mathfrak{n} with $\mathfrak{n} = \mathfrak{v}_1 \oplus \dots \oplus \mathfrak{v}_m \oplus \mathfrak{h}$ such that the map $(X_1, \dots, X_m, Y) \rightarrow \exp X_1 \dots \exp X_m \exp Y$ from $\mathfrak{v}_1 \oplus \dots \oplus \mathfrak{v}_m \oplus \mathfrak{h}$ into N is a diffeomorphism.*

Proof (by induction on $\dim \mathfrak{n}/\mathfrak{h}$). Let \mathfrak{c} be the normalizer of \mathfrak{h} in \mathfrak{n} . \mathfrak{c} is D -stable, and from Engel's theorem it follows that \mathfrak{h} is strictly contained in \mathfrak{c} .

Then choose a minimal nonzero D -invariant central ideal $\mathfrak{m}/\mathfrak{h}$ in $\mathfrak{c}/\mathfrak{h}$. \mathfrak{m} can be decomposed into $\mathfrak{m} = \mathfrak{v}_m \oplus \mathfrak{h}$ with an D -invariant subspace \mathfrak{v}_m . It is easy to see that the map $(X, Y) \rightarrow \exp(X)\exp(Y)$ from $\mathfrak{v}_m \oplus \mathfrak{h}$ into $\exp \mathfrak{m}$ is a diffeomorphism. Then use the induction hypothesis for the algebra \mathfrak{m} .

Of course, we apply lemma 1 to $\mathfrak{h} = \mathfrak{p}$ and the derivation D_s and get $\mathfrak{n} = \mathfrak{v}_1 \oplus \dots \oplus \mathfrak{v}_m \oplus \mathfrak{p}$ with the properties listed above. Then we identify $\mathfrak{R} = L^2_p(N)$ with $L^2(\mathfrak{v})$, $\mathfrak{v} := \mathfrak{v}_1 \oplus \dots \oplus \mathfrak{v}_m$, via the map $\varphi \rightarrow \varphi'$ from $L^2(\mathfrak{v})$ in \mathfrak{R} given by

$$\varphi'(\exp X_1 \cdot \dots \cdot \exp X_m \exp Y) = e^{-ig(Y)}\varphi(X_1 + \dots + X_m)$$

for $X_j \in \mathfrak{v}_j$, $Y \in \mathfrak{p}$. Next we fix identifications of \mathfrak{v}_j with \mathbf{R} or \mathbf{C} , respectively, such that D_s acts by multiplication with a real or complex number, and define $\xi \in L^2(\mathfrak{v})$ by $\xi(X_1 + \dots + X_m) = e^{-\sum_{j=1}^m |X_j|^2}$. By a theorem of Howe, [11], there exists $p \in L^1(N)/\ker \tau$ with $\tau(p) = \langle -, \xi' \rangle \xi'$. As one will see soon, it is useful to know the matrix coefficients $\langle \tau_s(t)\xi', \xi' \rangle = \langle \sigma(t)\xi, \xi \rangle$ if σ denotes the representation on $L^2(\mathfrak{v})$ corresponding to τ_s . From the definition of τ_s and the identification $\varphi \rightarrow \varphi'$ one deduces very easily that $(\sigma(t)\varphi)(X_1 + \dots + X_m) = \delta(t)^{-1/2} \cdot \varphi[\text{Exp}(-tD_s)(X_1 + \dots + X_m)]$. Recall that $\delta(t)$ is the determinant of $\text{Exp}(tD_s)$ on $\mathfrak{n}/\mathfrak{p} \cong \mathfrak{v}$. Hence

$$\begin{aligned} \langle \tau_s(t)\xi', \xi' \rangle &= \langle \sigma(t)\xi, \xi \rangle \\ &= \delta(t)^{-1/2} \int_{\mathfrak{v}} \xi \left(\sum_{j=1}^m \text{Exp}(-tD_s)X_j \right) \xi \left(\sum_{j=1}^m X_j \right) dX_1 \dots dX_m \\ &= \delta(t)^{-1/2} \prod_{j=1}^m \int_{\mathfrak{v}_j} e^{-|X|^2 - |\text{Exp}(-tD_s)X|^2} dX. \end{aligned}$$

The integrals can easily be computed, and one finds that $\langle \tau_s(t)\xi', \xi' \rangle = \mu(\text{Exp}(tD_s))^{-1} B_0$ with some positive constant B_0 . Recall that we have to estimate $\|b_t\|$ where $b_t \in L^1(N)/\ker \tau$ is now given by $\tau(b_t) = \langle -, \xi' \rangle \tau_s(t)^{-1} \xi'$. Corresponding to the one-parameter group $\exp(tD_s)$ of automorphisms we define an isometric action $\mathbf{R} \times L^1(N) \rightarrow L^1(N)$ by

$$f^{(t)}(x) = \det \text{Exp}(tD_s) f(\exp(tD_s)(x)).$$

One easily checks that $\tau(f^{(t)}) = \tau_s(-t)\tau(f)\tau_s(t)$. This action induces an action on $L^1(N)/\ker \tau$, also denoted by $(t, a) \rightarrow a^{(t)}$. A straightforward calculation shows that $b_t = \langle \tau_s(t)\xi', \xi' \rangle^{-1} p^{(t)} * p$. Hence

$$\begin{aligned} \|b_t\| &= B_0^{-1} \mu(\text{Exp}(tD_s)) \|p^{(t)} * p\| \\ &\leq \mu(\text{Exp}(tD_s)) B_0^{-1} \|p\|^2. \end{aligned}$$

This finishes the proof of one part of the theorem.

For the other part we first state a proposition, then we show how the theorem follows, and finally we prove the proposition. Moreover, the proof of the proposition contains some information about realizations of simple modules as spaces of functions.

PROPOSITION. *Let H be an exponential Lie group with Lie algebra \mathfrak{h} . Let \mathfrak{m} be a nilpotent ideal in \mathfrak{h} with $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{m}$, let $h \in \mathfrak{h}^*$ and $g := h|_{\mathfrak{m}}$. Suppose that $\mathfrak{h}_h + \mathfrak{m} = \mathfrak{h}$. Let τ be the unitary representation of $M = \exp \mathfrak{m}$ corresponding to g . Let τ be realized in the Hilbert space \mathfrak{R} , and let $\mathfrak{R}_0 \subset \mathfrak{R}$ be the corresponding simple $L^1(M)$ -module. Let $\mu: H/M \rightarrow \mathbb{R}$ be constructed to $\mathfrak{m}/\mathfrak{m}_g$ as above, and let η be a complex character of H/M with $|\eta| \leq \mu$. Then there exists an extension $\tilde{\tau}$ of τ to H , a strongly continuous representation π of H by isometries in a Banach space E , and a non-zero bounded operator $T: \mathfrak{R}_0 \rightarrow E$ with $\eta(x)T\tilde{\tau}(x)|_{\mathfrak{R}_0} = \pi(x)T$ for all $x \in H$. Moreover, the subspace $\pi(L^1(H))T\mathfrak{R}_0$ of E is a simple $L^1(H)$ -module.*

Of course, we apply the proposition to $\mathfrak{h} = \mathfrak{g}_f + \mathfrak{n}$, $\mathfrak{m} = \mathfrak{n}$ and $h = f|_{\mathfrak{n}}$. Note that in general \mathfrak{h}_h will be strictly bigger than \mathfrak{g}_f . But since they differ only by elements in the nilradical, considering $\mathfrak{n}/\mathfrak{n}_g$ as an H_h - or as an G_f -module yields the same “ μ -function” on H/N . Let $\eta, \tau, \tilde{\tau}, E, \pi$ and T be as in the proposition. Suppose that $u(x)$ (and then w) is constructed with respect to $\tilde{\tau}$, i.e. $\tau(p) = \langle -, \xi \rangle \xi$, $\tau(u(x)) = \langle -, \xi \rangle \tilde{\tau}(x)^{-1} \xi$, $w(x) = \|u(x)\|$ where $p, u(x) \in L^1(N)/\ker \tau$. Denote by $\tau': H \rightarrow \text{Aut}(\mathfrak{R}_0)$ the restriction of $\tilde{\tau}$ to \mathfrak{R}_0 ; $\tau'|_N$ is an isometric representation. It is obvious that $\tau'(x)\tau'(u(x)) = \tau'(p)$ for all $x \in H$. If we apply T to $\eta(x)\tau'(p) = \eta(x)\tau'(x)\tau'(u(x))$ we obtain $\eta(x)T\tau'(p) = \pi(x)T\tau'(u(x))$. Taking the operator norms yields $|\eta(x)| \|T\tau'(p)\| \leq \|T\| \|\tau'(u(x))\| \leq \|T\| w(x)$ and hence $|\eta(x)| \leq Cw(x)$ with a positive constant C . But then $|\eta(x)| \leq w(x)$ as can be seen by taking powers of x .

Proof of the Proposition. The proposition is proved by induction on the dimension of \mathfrak{h} . Let \mathfrak{w} be the center of \mathfrak{m} . $\mathfrak{w} \cap \ker h$ is an ideal in \mathfrak{h} . If $\mathfrak{w} \cap \ker h \neq 0$ the proposition follows from the induction hypothesis applied to the algebra $\mathfrak{h}/\mathfrak{w} \cap \ker h$. Hence we may assume that \mathfrak{w} is one dimensional and $h(\mathfrak{w}) \neq 0$. Since $h([\mathfrak{h}_h, \mathfrak{w}]) = 0$ it follows that \mathfrak{w} is central in \mathfrak{h} . Let $\mathfrak{a}/\mathfrak{w} \subset \mathfrak{m}/\mathfrak{w}$ be a minimal nonzero ideal in $\mathfrak{h}/\mathfrak{w}$. The dimension of \mathfrak{a} is two or three, and \mathfrak{a} is abelian (this follows from the fact that \mathfrak{h} is exponential).

Let \mathfrak{m}' be the centralizer of \mathfrak{a} in \mathfrak{m} , let $\mathfrak{h}' = \mathfrak{h}_h + \mathfrak{m}'$, $h' = h|_{\mathfrak{h}'}$, and $g' = g|_{\mathfrak{m}'}$. We collect some simple well-known facts:

- (1) $\mathfrak{h}_h \cap \mathfrak{m} \subseteq \mathfrak{m}'$,
- (2) $\mathfrak{m}' = \mathfrak{m} \cap \mathfrak{h}'$, $\mathfrak{h}'/\mathfrak{m}'$ is commutative,
- (3) $\mathfrak{m}_g \subset \mathfrak{m}_{g'} \subset \mathfrak{m}'$,
- (4) $\mathfrak{m}_{g'} = \mathfrak{m}'_{g'} = \mathfrak{m}_g + \mathfrak{a}$, $\mathfrak{a} \cap \mathfrak{m}_g = \mathfrak{w}$,
- (5) $\dim \mathfrak{m}/\mathfrak{m}' = \dim \mathfrak{m}'_{g'}/\mathfrak{m}_g = \dim \mathfrak{a}/\mathfrak{w}$,
- (6) $\mathfrak{h}'_{h'} = \mathfrak{h}_h + \mathfrak{a}$
- (7) If \mathfrak{p} is an $\mathfrak{h}'_{h'}$ -invariant polarization at $g' \in (\mathfrak{m}')^*$ then \mathfrak{p} is an \mathfrak{h}_h -invariant polarization at $g \in \mathfrak{m}^*$. In particular, if σ is the irreducible representation of $M' = \exp \mathfrak{m}'$ corresponding to g' then $\tau = \text{ind}_{M'}^M \sigma$.

The homogeneous space $X = H/H'$, $H' = \exp \mathfrak{h}'$, can be identified with the (one- or two-dimensional) vector group M/M' via the map $M \ni x \rightarrow xH'$. Denote by ν the measure on X obtained from the Lebesgue measure on M/M' . Of course, ν is invariant under translations by elements in M . But translations by elements $a \in H'$ correspond to conjugation on M/M' . Hence we find that ν is quasi-invariant, namely

$$d\nu(ax) = \det \text{Ad}_{m/m'}(a) d\nu(x) \quad \text{for } a \in H, x \in X.$$

Let $\delta = \det \text{Ad}_{m/m'}$. If μ' is the “ μ -function” on $H'/M' \cong H/M$ corresponding to m'/m'_g , then it follows easily from the definition of μ (and from (7)) that $\mu = \mu' \text{ch}_{m/m'}$, i.e.,

$$\mu = \mu'(\delta^{1/2} + \delta^{-1/2}) \quad \text{or} \quad \mu = \mu' \sqrt{\delta + \delta^{-1}}.$$

Now we decompose the given character η , $\eta = \eta' \eta''$, such that $|\eta'| \leq \mu'$, $|\eta''| \leq \delta^{1/2} + \delta^{-1/2}$ and $\eta'' = 1$ on $\ker \delta$ (this is possible by the remark after the theorem). Denote by \mathfrak{S} the representation space of σ . We apply the induction hypothesis to $\mathfrak{h}', h', m', \eta'$ and find an extension $\tilde{\sigma}$ of σ to H' , a strongly continuous isometric representation π' of H' in E' , and a nonzero bounded operator $T' : \mathfrak{S}_0 \rightarrow E'$ with $\eta'(a) T' \sigma'(a) = \pi'(a) T'$ for $a \in H'$ where σ' , of course, denotes the restriction of $\tilde{\sigma}$ to \mathfrak{S}_0 , the simple $L^1(M')$ -module contained in \mathfrak{S} .

The induced representation $\tau = \text{ind}_{M'}^M \sigma$ in $\mathfrak{R} = L^2_M(M, \mathfrak{S})$ can be extended to a representation $\tilde{\tau}$ of H by the formula

$$[\tilde{\tau}(ay)\lambda](x) = \delta(a)^{-1/2} \tilde{\sigma}(a) \lambda(y^{-1} a^{-1} xa)$$

for $a \in H'$, $x, y \in M$ and $\lambda \in \mathfrak{R}$.

Starting from E' we will construct for complex numbers z , $-1 \leq \text{Re } z \leq 1$, a family of isometric strongly continuous representations π_z of H in Banach spaces \bar{E}_z like in [24] or [27]. Let E_z be the space of all continuous functions $\varphi : H \rightarrow E'$ such that $\varphi(ah) = \delta(h)^{1/2(z+1)} \pi'(h)^{-1} \varphi(a)$ for $a \in H$ and $h \in H'$ and such that φ has a compact support modulo H' . Define a norm on E_z by

$$\|\varphi\|_z = \left[\int_X \|\varphi(a)\|^q \delta(a)^{-1} d\nu(a) \right]^{1/q}$$

where $q \text{Re } z = 2 - q$ and $\dot{a} = aH' \in X$. H acts in E_z by $(a\varphi)(x) = \varphi(a^{-1}x)$; and this action extends to an isometric strongly continuous representation π_z in the completion \bar{E}_z of E_z . In the case $\text{Re } z = -1$ which corresponds to $q = \infty$ the norm $\|\cdot\|_z$ must be modified to $\|\varphi\|_z = \|\varphi\|_\infty = \sup_{a \in H} \|\varphi(a)\|$; then \bar{E}_z consists of continuous functions vanishing at infinity modulo H' .

Next, we define embeddings $T_z : \mathfrak{R}_0 \rightarrow \bar{E}_z$ by

$$(T_z \lambda)(xh) = \delta(h)^{1/2(z+1)} \pi'(h)^{-1} [T' \lambda(x)]$$

for $\lambda \in \mathfrak{R}_0$, $x \in M$, $h \in H'$. Of course, this definition has to be justified, in particular one has to show that $\lambda(x) \in \mathfrak{S}_0$. To this end, let \mathcal{C} be the space of all continuous functions Ψ from M into the Banach space \mathfrak{S}_0 with the properties:

(i) $\Psi(xy) = \sigma'(y)^{-1}\Psi(x)$ for $y \in M'$, $x \in M$.

(ii) The function $M/M' \rightarrow \mathfrak{R}$, defined by $x \rightarrow \|\Psi(x)\|_{\mathfrak{S}_0}$, vanishes at infinity.

Then $\|\Psi\|_{\mathcal{C}}$ is defined by $\|\Psi\|_{\mathcal{C}} = \sup_{x \in M} \|\Psi(x)\|_{\mathfrak{S}_0}$. Similarly, we form the spaces $L_M^q(M, \mathfrak{S}_0)$ for $1 \leq q < \infty$, and put $\mathcal{S} = \mathcal{C} \cap L_M^1(M, \mathfrak{S}_0) = \bigcap_q \mathbf{C} \cap L_M^q(M, \mathfrak{S}_0)$. \mathcal{S} may be considered as a subspace of \mathfrak{R} . \mathcal{S} is invariant under $L^1(M)$, and since \mathfrak{R}_0 is the smallest $L^1(M)$ -invariant subspace of \mathfrak{R} one finds that \mathfrak{R}_0 is contained in \mathcal{S} . Moreover, all the embeddings from the Banach space \mathfrak{R}_0 into \mathcal{C} and into $L_M^q(M, \mathfrak{S}_0)$ are bounded.

Let's return to the operator T_z . Since $\lambda(x)$ is contained in \mathfrak{S}_0 for $\lambda \in \mathfrak{R}_0$ and $x \in M$ one may apply T' to $\lambda(x)$. Next we show that $T_z\lambda$ is a well defined function on H , i.e., if $xh = yk$ with $x, y \in M$ and $h, k \in H'$ then $(T_z\lambda)(xh) = (T_z\lambda)(yk)$. The assertion is equivalent to $\pi'(h)^{-1}T'\lambda(x) = \pi'(k)^{-1}T'\lambda(y)$ or to $\pi'(kh^{-1})T'\lambda(x) = T'\lambda(y)$. But $kh^{-1} = y^{-1}x \in H' \cap M = M'$, hence $\pi'(kh^{-1})T'\lambda(x) = \pi'(y^{-1}x)T'\lambda(x) = T'\sigma'(y^{-1}x)\lambda(x) = T'\lambda(xx^{-1}y) = T'\lambda(y)$. It is obvious that $T_z\lambda$ has the correct transformation property for right translations with elements in H' . It remains to show that $T_z\lambda$ has the right integrability property (we do this only for $\text{Re } z \neq -1$, the case $\text{Re } z = -1$ is similar) and that T_z is bounded. Let $q \text{Re } z = 2 - q$. Then

$$\begin{aligned} \|(T_z\lambda)(xh)\|^q \delta(xh)^{-1} &= \delta(h)^{-1} \delta(h)^{q/2(\text{Re } z + 1)} \|T'\lambda(x)\|^q \\ &= \|T'\lambda(x)\|^q \leq \|T'\|^q \|\lambda(x)\|_{\mathfrak{S}_0}^q. \end{aligned}$$

Hence

$$\|T_z\lambda\|_z \leq \|T'\| \left\{ \int_{M/M'} \|\lambda(x)\|_{\mathfrak{S}_0}^q dx \right\}^{1/q} \leq \|T'\| C_q \|\lambda\|_{\mathfrak{R}_0}$$

with a positive constant C_q , because \mathfrak{R}_0 is continuously embedded into $L_M^q(M, \mathfrak{S}_0)$.

Denote again by τ' the restriction to \mathfrak{R}_0 of the extended representation $\tilde{\tau}$ of H in \mathfrak{R} . A straightforward computation (using the explicit formula for $\tilde{\tau}$) shows that $\gamma(a)T_z\tau'(a) = \pi_z(a)T_z$ for all $a \in H$ where $\gamma = \gamma_z$ is given by $\gamma(a) = \eta'(a)\delta(a)^{-\frac{1}{2}z}$. For a suitably chosen z one gets $\eta'' = \delta^{-\frac{1}{2}z}$ and $\eta = \gamma_z$. The proposition is proved.

Remark. The (proof of the) proposition contains some insight how the simple modules can be realized as subspaces of spaces of functions if the stabilizer of the functional in question is big enough, i.e., $\mathfrak{h} = \mathfrak{h}_n + \mathfrak{m}$. In this remark I will point out how the general case can be reduced to such a situation. Let \mathfrak{g} be an exponential Lie algebra with nilradical \mathfrak{n} , let $f \in \mathfrak{g}^*$, $g = f|_{\mathfrak{n}}$, and let B be the skew-symmetric form on \mathfrak{g} associated with f . $\mathfrak{g}_g + \mathfrak{n}/\mathfrak{g}_f + \mathfrak{n}$ is isomorphic to

$\eta = \mathfrak{g}_g / \mathfrak{g}_g \cap (\mathfrak{g}_f + \mathfrak{n})$, and B induces on η a nondegenerate form because \mathfrak{g}_g is the orthogonal space to $\mathfrak{g}_f + \mathfrak{n}$. Let $c / \mathfrak{g}_g \cap (\mathfrak{g}_f + \mathfrak{n})$ be a maximal isotropic subspace of η or, equivalently, c is maximal with respect to $c \subseteq \mathfrak{g}_g \cap c^\perp$. Then put $\mathfrak{h} = c + \mathfrak{n}$, and $\mathfrak{h} = f|_{\mathfrak{h}}$. \mathfrak{h} has the property that $\mathfrak{h} = \mathfrak{h}_h + \mathfrak{h}$, in fact c is contained in \mathfrak{h}_h . Obviously, $\dim(\mathfrak{g}_g + \mathfrak{n})/\mathfrak{h} = \dim \mathfrak{h}/(\mathfrak{g}_f + \mathfrak{h})$. Now, let F be a simple $L^1(H)$ -module with $\text{Ann}_{L^1(N)}(F) = \ker \tau$, τ as usual. Then the induced module $\text{ind}_H^G F = L_H^2(G, F)$ contains a unique simple $L^1(G)$ -module E with $\text{Ann}_{L^1(N)}(E) = k(G\tau)$. In this way, all such simple $L^1(G)$ -modules are obtained. But this is not a parametrization: different F 's may yield isomorphic E 's. More precisely, if E and E' are obtained from F and F' , respectively, then E is isomorphic to E' iff $\text{Ann}_{L^1(G/N)}(F) = \text{Ann}_{L^1(G/N)}(F')$.

Next, I will discuss the relations of the results on simple modules to questions of the symmetry or $*$ -regularity of L^1 -group algebras. Recall that an involutive Banach algebra \mathcal{A} is $*$ -regular if the canonical map from the space $\text{Priv}(C^*(\mathcal{A}))$ of primitive ideals in the enveloping C^* -algebra onto the space $\text{Priv}_*(\mathcal{A})$ of kernels of irreducible involutive representations of \mathcal{A} is an homeomorphism where both spaces are equipped with the Jacobson topology. $*$ -regular L^1 -group algebras were investigated by Boidol in several papers. Finally, in [4], he found a characterization of those connected Lie groups which have a $*$ -regular L^1 -group algebra. In the case of an exponential Lie group G , Boidol proved in [3] that $L^1(G)$ is $*$ -regular if and only if every real linear functional f on the Lie algebra \mathfrak{g} of G satisfies the equivalent conditions of the following lemma.

LEMMA 2. *Let \mathfrak{g} be a solvable Lie algebra, and let \mathfrak{m} be a nilpotent ideal in \mathfrak{g} with $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{m}$. Let $f \in \mathfrak{g}^*$ and $\mathfrak{g} = f|_{\mathfrak{m}}$. Let \mathfrak{w} be the smallest ideal in $\mathfrak{g}_f + \mathfrak{m}$ such that $\mathfrak{g}_f + \mathfrak{m}/\mathfrak{w}$ is nilpotent, let \mathfrak{b} be the largest ideal in \mathfrak{g} with $f(\mathfrak{b}) = 0$ and let \mathfrak{a} be the largest ideal in \mathfrak{m} with $f(\mathfrak{a}) = 0$. Then are equivalent*

- (i) $f(\mathfrak{w}) = 0$.
- (ii) \mathfrak{g}_f acts nilpotently on $\mathfrak{g}/\mathfrak{b}$.
- (iii) \mathfrak{g}_f acts nilpotently on $\mathfrak{m}/\mathfrak{a}$ (note that the maximality of \mathfrak{a} implies that \mathfrak{a} is invariant under \mathfrak{g}_f).
- (iv) \mathfrak{g}_f acts nilpotently on $\mathfrak{m}/\mathfrak{m}_g$.

Remark. (ii) shows that the conditions are independent of the choice of \mathfrak{m} . (iv) establishes the relation to the μ -function.

Proof. (iii) \Rightarrow (i) From (iii) it follows that $(\mathfrak{g}_f + \mathfrak{m})/\mathfrak{a}$ is nilpotent. Hence \mathfrak{w} is contained in \mathfrak{a} and $f(\mathfrak{w}) \subset f(\mathfrak{a}) = 0$.

(i) \Rightarrow (ii) It suffices to show that \mathfrak{g}_f acts nilpotently on $\mathfrak{m} + \mathfrak{b}/\mathfrak{b} \cong \mathfrak{m}/\mathfrak{m} \cap \mathfrak{b}$. Since \mathfrak{w} is an ideal in \mathfrak{g} it follows from (i) that \mathfrak{w} is contained in \mathfrak{b} . Therefore, $\mathfrak{m}/\mathfrak{m} \cap \mathfrak{b}$ is a quotient of $\mathfrak{m}/\mathfrak{m} \cap \mathfrak{w}$, and \mathfrak{g}_f acts nilpotently on the latter space by construction.

(ii) \Rightarrow (iii) \mathfrak{g}_f acts nilpotently on $\mathfrak{m} + \mathfrak{b}/\mathfrak{b} \cong \mathfrak{m}/\mathfrak{b} \cap \mathfrak{m}$, and $\mathfrak{b} \cap \mathfrak{m}$ is contained in \mathfrak{a} .

(iii) \Rightarrow (iv) \mathfrak{a} is contained in \mathfrak{m}_g .

(iv) \Rightarrow (iii) Let $X \in \mathfrak{g}_f$ and put $D = \text{ad}_m X : \mathfrak{m} \rightarrow \mathfrak{m}$. Let $\mathfrak{b} = \bigcap_{n \in \mathbb{N}} D^n(\mathfrak{m})$ or,

equivalently, $\delta = D_s(m)$ if D_s is the semisimple part in the additive Jordan decomposition of D . One verifies easily that $\delta + [\delta, \delta]$ is an ideal in m . From $X \in \mathfrak{g}_f$ it follows that $f(\delta) = 0$. The assumption (iv) implies that $\delta \subset m_g$ and consequently $f([\delta, \delta]) = 0$. Hence $\delta + [\delta, \delta]$ is contained in \mathfrak{a} . In particular, δ is contained in \mathfrak{a} which means that X acts nilpotently on m/\mathfrak{a} .

It turns out that for exponential Lie groups symmetry and $*$ -regularity of the L^1 -group algebras are equivalent.

THEOREM 10. *For an exponential Lie group G there are equivalent:*

- (S) $L^1(G)$ is symmetric.
- (R) $L^1(G)$ is $*$ -regular.
- (F) Every real functional f on the Lie algebra \mathfrak{g} of G satisfies the equivalent conditions (i)–(iv) of lemma 2.

Proof. As mentioned above, the equivalence of (R) and (F) was proved by Boidol. By the way, (R) \Rightarrow (F) or better: non(F) \Rightarrow non(R), follows easily from the results of this paper. The implication (F) \Rightarrow (R) requires some additional work.

(S) \Rightarrow (F). This implication was already proved in [26] using the results of [24]. Actually, suppose that \mathfrak{g} does not satisfy (F). Then there exists a functional f on \mathfrak{g} such that the μ -function for n/n_g is not constant where n is the nilradical of \mathfrak{g} and $g = f|_n$. Consequently, there exist non-trivial real characters η with $\eta \leq \mu$. These characters give rise to simple $L^1(G)$ -modules which cannot be embedded into Hilbert representations, even their annihilators are not involutive ideals.

(F) \Rightarrow (S). (F) implies that for every $f \in \mathfrak{g}^*$ the set $\hat{Z}_w = (G_f N)_w$ consists only of unitary characters because the μ -function is always constant. But this means that every simple $L^1(G)$ -module can be embedded into a Hilbert module. Hence $L^1(G)$ is symmetric.

Let's conclude the paper with three open problems.

(A) Describe the Fell topology on $\text{Priv}(L^1(G))$ for exponential Lie groups G . Note that under the assumptions of Theorem 10, $\text{Priv}(L^1(G)) = \text{Priv}_*(L^1(G)) = \text{Priv } C^*(G)$. And all these spaces are homeomorphic, equipped with the Fell or with the Jacobson topology. Moreover, in this case $\text{Priv}(C^*(G))$ is homeomorphic to the orbit space \mathfrak{g}^*/G , see [3]. This problem involves, in general, the determination of the space $\text{Priv}(C^*(G))$ for arbitrary exponential Lie groups G which is not yet solved. It is conjectured that $\text{Priv}(C^*(G))$ is homeomorphic to the orbit space \mathfrak{g}^*/G .

(B) Is $\text{Priv}_*(L^1(G))$ contained in $\text{Priv}(L^1(G))$ for all connected Lie groups G ?

(C) Describe $\text{Priv}(L^1(G))$ for general solvable Lie groups G . A possible way to attack this problem is to solve first problem 2 in §3 on the orbits of tori in the unitary dual of nilpotent Lie groups. Between writing this paper and its acceptance problem 2 was solved to the affirmative.

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