JOURNAL OF ALGEBRA 169, 625–647 (1994)

Dense Lie Group Homomorphisms

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Communicated by Georgia Benkart

Received June 18, 1992

INTRODUCTION

This article gives a survey of the basic facts on the structure and properties of dense Lie group homomorphisms, i.e., continuous (analytic)homomorphisms $\varphi: H \to G$ between connected Lie groups such that the image $\varphi(H)$ is dense in G. The dense embeddings, i.e., *injective* continuous homomorphisms, from vector spaces into tori are widely known. Everybody "feels" that in some sense this should be the only possible case. One goal of the present paper is to make this philosophy as precise as possible. A considerable amount of information concerning this theme has accumulated during the past 40 or 50 years; it is scattered in the literature, and we supply a selection of references. However, the material is only partly covered by the standard textbooks. In particular, the role of maximal tori, especially the fact that the corresponding homogeneous spaces are simply connected, has not been sufficiently exploited.

The treatment presented here is conceived as a closed unit which should be intelligible on the basis of fundamental Lie group theory alone, such as can, for instance, be found in Hochschild's book [13], to which we shall refer freely.

Our organization and analysis of the available relevant literature leads to some new insights, primarily the description of all dense embeddings of a given Lie group into other Lie groups, which was formulated as a problem by Goto (see [10]).

The first section of this article contains the basic general properties of dense Lie group homomorphisms. It also includes some immediate conse-

0021-8693/94 $6.00

quences of the considerations, like the Malcev criterion for a Lie subgroup to be closed.

In the second section we describe all dense embeddings of a given connected Lie group H into other Lie groups. Actually, this classification is reduced to an abelian Lie group extension problem. The main goal is to clarify the phenomena of dense embeddings and to deepen our understanding of them. Each of the first two sections contains a main theorem, namely, 1.6 in the first and 2.5 in the second. The latter one is the basis for the solution of Goto's problem.

The first two sections are independent of the Levi decomposition. In other words, they do *not* consider the question how the semisimple part and the radical of the "smaller" group H contribute to the "completion" G. This question is studied in the short third section. In addition, some examples are treated.

As compact subgroups of connected Lie groups will be important in the sequel we recall some of the basic facts. Their importance for the study of closures of subgroups was already observed by Malcev, [14–17], who formulated the main properties of maximal compact subgroups in full generality. Each connected Lie group G contains three distinguished *conjugacy classes* of compact subgroups, namely:

(A) the maximal compact subgroups which are automatically connected;

(B) the connected centers of maximal compact subgroups;

(C) the maximal compact connected abelian subgroups, the maximal tori.

Each of the three classes is preserved by quotient homomorphisms $G \to G/N$ with (almost) connected kernels N. In case (A) this is more or less explicitly stated in Thm. 3.7 of Chap. XV in [13]. For the other classes it follows from the structure theory of compact Lie groups and from the theory of maximal tori in compact connected Lie groups (see [13]).

Each of the three classes could serve as a base to study dense Lie group homomorphisms. Class (C) has the most advantages for our purposes. If T is a maximal torus then the homogeneous space G/T is simply connected (see exercise 1 at the end of Chap. XV in [13]) while for members S of class (B) the homogeneous space G/S has a finite fundamental group, nontrivial in general; see the proof of 1.5 below. Moreover, maximal tori are maximal in the set of all compact abelian subgroups of G. By the way, there exist *finite* abelian subgroups in compact connected Lie groups which are maximal in the set of all abelian subgroups.

Throughout the paper I will use the common convention that a small German letter denotes the Lie algebra of a Lie group named by the corresponding Latin capital. The commutator subgroup of a group is marked as G', the derived subalgebra of a Lie algebra as $\mathfrak{g}'$. The adjoint representation of a Lie algebra is denoted by ad, the adjoint representation of a Lie group by Ad. If several groups are around occasionally a subscript at Ad is added. The closure of a subset of a topological space is marked by a bar. For instance, $\mathrm{Ad}_G(U)^-$ means the closure in Aut $(\mathfrak{g})$ of the image of the subset U in the Lie group G.

1. Basic Facts on Dense Lie Group Homomorphisms

The following little lemma will be useful at several places of this article.

1.1. Lemma. *Let H and G be locally compact groups and assume that H is a countable union of compact subsets. Let $\varphi\colon H \to G$ be a continuous homomorphism with dense image. If T is a closed subgroup of G such that $G = \varphi(H)T$ and if V is a closed subgroup of G containing T then the following conclusions hold.*

(i) *The induced map $\varphi^{-1}(V)/\varphi^{-1}(T) \to V/T$ is a homeomorphism.*

(ii) $\varphi(\varphi^{-1}(V))^- = V$.

Proof. (i) follows from a usual Baire category argument, one may apply, for instance, Thm. 2.5 on p. 7 in [13]: V/T is a homogeneous space for the acting group $\varphi^{-1}(V)$.

(ii) Clearly, $W := \varphi(\varphi^{-1}(V))^-$ is contained in V. There is a commutative diagram

$$\begin{array}{ccc} H/\varphi^{-1}(V) & \xrightarrow{\ j\ } & G/W \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle p} & \\ & G/V & \end{array}$$

with the obvious continuous maps i, j, p. Since j is dense and since by the first part of the lemma i is a homeomorphism, such a diagram is possible only if j and p are homeomorphisms, too. In particular, p is bijective whence $W = V$.

Contrary to general Lie homomorphisms, the dense homomorphisms behave well w.r.t. connected normal Lie subgroups or, equivalently, w.r.t. ideals on the infinitesimal level; compare Thm. 2.1 on p. 190 in [13].

1.2. *Let $\varphi\colon H \to G$ be a dense Lie group homomorphism between connected Lie groups H and G, and let $\mathfrak{n}$ be an ideal in $\mathfrak{h}$. If $d\varphi$ denotes the differential of φ then $d\varphi(\mathfrak{n})$ is an ideal in $\mathfrak{g}$.*

The reason is simply that $\mathfrak{n}$ being $\mathrm{Ad}_H(H)$-invariant the space $d\varphi(\mathfrak{n})$ is $\mathrm{Ad}_G(\varphi(H))^-$ invariant and hence, by density, $\mathrm{Ad}_G(G)$-invariant.

The following theorem, see [5, 16], roughly says that the image of a dense Lie group homomorphism exhausts the whole group except for possibly a toroidal subgroup.

1.3. THEOREM. *Let H and G be connected Lie groups, and let $\varphi: H \to G$ be a dense Lie group homomorphism. If $Z_0(K)$ is the connected center of any of the maximal compact subgroups K of G then $G = \varphi(H)Z_0(K)$. In particular, the different maximal compact subgroups of G are conjugate by elements in $\varphi(H)$.*

Proof. W.l.o.g. we may assume that φ is the inclusion. By 1.2 the subalgebra $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. Since $\mathrm{Ad}(H)$ act trivially on $\mathfrak{g}/\mathfrak{h}$, $\mathrm{Ad}(G)$ acts trivially, too. Therefore, $\mathfrak{g}/\mathfrak{h}$ is abelian, $\mathfrak{g}'$ is contained in $\mathfrak{h}$.

Let $\pi: \tilde{G} \to G$ be a universal covering group of G. For a Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ denote by $\langle \exp \mathfrak{p} \rangle$ the Lie subgroup of $\tilde{G}$ corresponding to $\mathfrak{p}$. If $\mathfrak{k}$ is the Lie algebra of K then $\langle \exp(\mathfrak{k} + \mathfrak{h}) \rangle = \langle \exp \mathfrak{k} \rangle \langle \exp \mathfrak{h} \rangle$ is closed in $\tilde{G}$ as $\mathfrak{k} + \mathfrak{h}$ is an ideal in $\mathfrak{g}$ (see Chap. XII in [13], the case at hand is particularly simple). The inverse image $\pi^{-1}(K)$ is connected as $\tilde{G}/\pi^{-1}(K)$ is simply connected being diffeomorphic to G/K which is an Euclidean space. It follows that $\pi^{-1}(K) = \langle \exp \mathfrak{k} \rangle$. This implies in particular that the kernel of π is contained in $\langle \exp(\mathfrak{k} + \mathfrak{h}) \rangle$. Therefore, $\pi\langle \exp(\mathfrak{k} + \mathfrak{h}) \rangle$ is a closed subgroup of G containing H, hence it coincides with G which gives $\mathfrak{g} = \mathfrak{k} + \mathfrak{h}$. By Thm. 1.3 in Chap. XIII of [13] the "compact Lie algebra" $\mathfrak{k}$ decomposes as $\mathfrak{k} = \mathfrak{k}' + \mathfrak{z}(\mathfrak{k})$. As $\mathfrak{g}'$ is contained in $\mathfrak{h}$ one obtains $\mathfrak{g} = \mathfrak{h} + \mathfrak{z}(\mathfrak{k})$, whence the theorem.

From the theorem we derive two corollaries, for the first one compare [13, p. 190].

1.4. COROLLARY. *Let $\varphi: H \to G$ be a dense Lie group homomorphism between connected Lie groups H and G. If $d\varphi$ denotes the differential of φ then $d\varphi(\mathfrak{h}') = \mathfrak{g}'$.*

Proof. Again we may assume that φ is the inclusion. By 1.2, both $\mathfrak{h}$ and $\mathfrak{h}'$ are ideals in $\mathfrak{g}$. As $\mathrm{Ad}(H)$ acts trivially on $\mathfrak{h}/\mathfrak{h}'$, also $\mathrm{Ad}(G)$ does so which implies that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}'$. This together with the fact that $\mathfrak{g}$ is a sum of $\mathfrak{h}$ and an abelian subalgebra, namely, the center of $\mathfrak{k}$, K as above, gives the claim.

1.5. COROLLARY. *Let $\varphi: H \to G$ be a dense Lie group homomorphism between connected Lie groups H and G. If U is a closed subgroup of G containing $Z_0(K)$ for some maximal compact subgroup K of G then $\varphi(\varphi^{-1}(U))^- = U$. If, in addition, U is connected then $\varphi^{-1}(U)$ has only finitely many components.*

Proof. The first statement follows immediately from the theorem and 1.1. The second statement can be reduced to the case $U = Z_0(K)$: $\varphi^{-1}(U)/\varphi^{-1}(Z_0(K))$ is connected being homeomorphic to $U/Z_0(K)$, by 1.1 (i). If $\varphi^{-1}(Z_0(K))$ has only finitely many components one can deduce the same property for $\varphi^{-1}(U)$ by considering the fibration $\varphi^{-1}(Z_0(K)) \to \varphi^{-1}(U) \to \varphi^{-1}(Z_0(K))$.

In the case $U = Z_0(K)$ we first observe that the fundamental group $\pi_1(G/Z_0(K))$ is finite for the following reasons. The group $\pi_1(G/Z_0(K))$ is the same as $\pi_1(K/Z_0(K))$. But $K/Z_0(K)$ is isomorphic to $K'/Z_0(K) \cap K'$, and the latter semisimple compact Lie group has a finite fundamental group. Since $H/\varphi^{-1}(Z_0(K))$ is homeomorphic to $G/Z_0(K)$ it has a finite fundamental group, too. Therefore, the covering $H/[\varphi^{-1}(Z_0(K))]_0 \to H/\varphi^{-1}(Z_0(K))$ has to be finite which shows that $\varphi^{-1}(Z_0(K))$ has only a finite number of components.

If we replace $Z_0(K)$ by a maximal torus T (which is larger in the sense that each T contains $Z_0(K)$ for a certain K) in G we obtain the following result, which collects some of the most important properties of dense Lie group homomorphisms.

1.6. THEOREM. *Let H and G be connected Lie groups, and let $\varphi: H \to G$ be a dense continuous homomorphism. For any maximal torus T in G one has $G = \varphi(H)T$. In particular, the different maximal tori in G are conjugate by elements in $\varphi(H)$. If a connected Lie subgroup U of G contains one of the maximal tori of G then U is closed, the inverse image $\varphi^{-1}(U)$ is connected, $\varphi^{-1}(U)$ contains a maximal torus of H and $\varphi(\varphi^{-1}(U))^- = U$.*

Proof. The equation $G = \varphi(H)T$ is an immediate consequence of 1.3. From this equation and from the fact that maximal tori are conjugate in G, one concludes that maximal tori are conjugate by elements in $\varphi(H)$. Let U be as in theorem, suppose that U contains the maximal torus T. Applying the first part of the theorem to the inclusion $U \to U^-$, it follows that $U^- = UT = U$. To see that $\varphi^{-1}(U)$ is connected one may argue as in the proof of 1.5. Using the sharper fact that the fundamental group of G/T is trivial instead of merely being finite one obtains the sharper result. Let S be any maximal torus in H. Then $\varphi(S)$ is contained in a maximal torus of G, hence there exists $x \in H$ such that $\varphi(S) \subset \varphi(x)T\varphi(x)^{-1}$. The maximal torus $x^{-1}Sx$ of H is contained in $\varphi^{-1}(U)$. Again the equation $U = \varphi(\varphi^{-1}(U))^-$ follows from 1.1.

First of all we draw two well-known consequences from this theorem; see, for instance, [13, pp. 189/192].

1.7. COROLLARY. *The center $Z(H)$ of a connected Lie group H is contained in a connected closed abelian subgroup.*

Proof. Let T be a maximal torus in the group $G := H/Z(H)$. The inverse image U of T is a connected nilpotent Lie group. Hence the adjoint group of U in Aut(u) is unipotent. Being compact in addition it must be trivial.

1.8. COROLLARY (Malcev Criterion). *A connected Lie subgroup H of a connected Lie group G is closed if the closure in G of every one-parameter subgroup of H lies in H.*

Proof. Clearly, we may assume that H is dense in G. Let T be a maximal torus in G. By 1.6, $G = HT$ and $A := T \cap H$ is a connected subgroup of H such that $\bar{A} = T$. As an abelian connected Lie group A is a product of the maximal torus S in A and a vector group V. Take any "line" L in V. The closure $\bar{L}$ is contained in T, hence compact. On the other hand, by assumption $\bar{L}$ is contained in H. It follows that $\bar{L}$ is contained in A. Being compact $\bar{L}$ is part of S. Hence L is zero, V is zero, $A = S$ is compact, and $\bar{A} = T = S$ is contained in H which implies $G = HT = H$.

Let $\varphi: H \to G$ be a dense embedding of connected Lie groups, let T be a maximal torus in G and T_H be a maximal torus in H. Then the number $n = \dim T - \dim T_H$, which is equal to $\dim T/\varphi(T_H)$ if, as we can arrange, $\varphi(T_H)$ is contained in T, is a rough measure for the "size" of the embedding φ. A somewhat refined version of the proof of 1.6 gives another description of n in terms of the fundamental groups of the groups in question.

1.9. THEOREM. *If $\varphi: H - G$ is a dense continuous embedding of connected Lie groups then the following assertions hold true:*

(a) *The homomorphism $\pi_1(\varphi): \pi_1(H) \to \pi_1(G)$ is injective.*

(b) *If $\mathrm{im}(\pi_1(\varphi))$ denotes the image of $\pi_1(\varphi)$ then $\pi_1(G)/\mathrm{im}(\pi_1(\varphi))$ is a free abelian group whose rank equals the above introduced number $n = \dim T - \dim T_H$.*

(c) *The embedding φ is an isomorphism iff n is equal to zero; i.e., $\pi_1(\varphi)$ is an isomorphism.*

Proof. The theorem will be reduced to the abelian case. Therefore, we first recall the following well known facts. Let $j: A \to B$ be a continuous embedding of connected abelian Lie groups, and let T_A and T_B be the maximal tori in A and B, respectively. Then $\pi_1(A)$ is free abelian of rank $\dim T_A$, in fact $\pi_1(A)$ is canonically isomorphic to $\mathrm{Hom}(\mathbb{T}, T_A) = \mathrm{Hom}(\mathbb{T}, A)$. Moreover, $\pi_1(j)$ is injective and $\pi_1(A)/\mathrm{im}(\pi_1(j))$ is free abelian of rank $\dim T_A - \dim T_B$, actually there is an exact sequence

$$0 \to \mathrm{Hom}(\mathbb{T}, T_A) \xrightarrow{\pi_1(j)} \mathrm{Hom}(\mathbb{T}, T_B) \to \mathrm{Hom}(\mathbb{T}, T_B/j(T_A)) \to 0.$$

Now let T be a maximal torus in G, let $A := \varphi^{-1}(T)$ which is connected by 1.6, and denote by $\varphi'\colon A \to T$ the induced embedding. Also from 1.6, we conclude that φ induces a homeomorphism $\varphi''\colon H/A \to G/T$. Using $\pi_1(G/T) = 0 = \pi_1(H/A)$ the homotopy sequence gives a commutative diagram

$$\begin{array}{ccccccc} \pi_2(H/A) & \to & \pi_1(A) & \to & \pi_1(H) & \to & 0 \\ \downarrow{\scriptstyle \pi_2(\varphi'')} & & \downarrow{\scriptstyle \pi_1(\varphi')} & & \downarrow{\scriptstyle \pi_1(\varphi)} & & \\ \pi_2(G/T) & \to & \pi_1(T) & \to & \pi_1(G) & \to & 0 \end{array}$$

with exact rows. Since $\pi_2(\varphi'')$ is an isomorphism, the injectivity of $\pi_1(\varphi')$ implies that $\pi_1(\varphi)$ is injective, whence (a).

Also from the diagram one reads off that the kernel of the composition of $\pi_1(T) \to \pi_1(G)$ with the quotient homomorphism $\pi_1(G) \to \pi_1(G)/\mathrm{im}(\pi_1(\varphi))$ is precisely the image of $\pi_1(\varphi')$. Hence $\pi_1(G)/\mathrm{im}(\pi_1(\varphi))$ is isomorphic to $\pi_1(T)/\mathrm{im}(\pi_1(\varphi'))$. If T_H is the maximal torus of A then T_H is a maximal torus of H. By what we have seen above $\pi_1(T)/\mathrm{im}(\pi_1(\varphi'))$ is a free abelian group whose rank equals $\dim T - \dim T_H$, whence (b).

Claim (c) is obvious in view of (b) and of 1.6.

In the spirit of our program to relate arbitrary dense embeddings to the embeddings of vector groups into tori we deduce from 1.6 the following consequence.

1.10. Proposition. *Let $\varphi\colon H \to G$ be a dense continuous embedding of connected Lie groups H and G. There exists a vector subgroup V of H such that $\varphi(V)^-$ is a torus, $G = \varphi(H)\varphi(V)^-$ and $\varphi^{-1}(\varphi(V)^-) = V$.*

Remark. If one gives up the property that $\varphi^{-1}(\varphi(V)^-) = V$, by Kronecker's theorem one may even choose V to be a one-parameter group. In this form it is occasionally called Malcev's theorem; compare [10, 16]. This point of view will be discussed in 1.13.

Proof. Again, let T be a maximal torus in G, and let T_H be the maximal torus in the connected abelian Lie group $A := \varphi^{-1}(T)$. There exists a torus S in T such that T is a direct product of S and $\varphi(T_H)$ because the dual sequence $0 \to (T/\varphi(T_H))^\wedge \to \hat{T} \to \varphi(T_H)^\wedge \to 0$ splits. The group $V := \varphi^{-1}(S)$ satisfies the requirements of 1.10.

Instead of 1.10 one might like to have a better result of the form: For each dense embedding $\varphi\colon H \to G$ there exist a vector subgroup V of H and a normal subgroup N of H such that H is a semidirect product of V and N, such that $\varphi(N)$ is closed and that G is a semidirect product of the torus $\varphi(V)^-$ and $\varphi(N)$. There is an obvious obstruction against such a result, namely $G' = \varphi(H')$ has to lie in the closed group $\varphi(N)$, hence the closure

$(G')^-$ has to be contained in $\varphi(H)$. Below, we will give an example which shows that this is not always true.

The next theorem tells that the desired splitting holds true if $(G')^-$ is contained in $\varphi(H)$. Such a situation typically arises in the following manner: Take a faithful linear representation $\varphi: H \to GL_n(\mathbb{R})$ of the connected Lie group H, and let $G := \varphi(H)^-$. Then $G' = \varphi(H')$ is closed in $GL_n(\mathbb{R})$; see Thm. 4.5 in Chap. XVIII of [13]. The theorem was first obtained by Goto (see [8, 10]) under a slightly stronger assumption. It will not be used in the sequel.

1.11. EXAMPLE. Let H be a connected semisimple Lie group such that the center $Z(H)$ is isomorphic to $\mathbb{Z}^2$, for instance one may take the product of two copies of the universal covering group of $SL_2(\mathbb{R})$. Choose an injective (dense) homomorphism $\gamma: Z(H) \to \mathbb{R}$ and form

$$G := (H \times \mathbb{R})/\{(x, \gamma(x))/x \in Z(H)\}.$$

Clearly, $\varphi: H \to G$, $h \to [(h, 0)]$, is a dense embedding, and $\varphi(H') = \varphi(H) = G'$ is dense in G.

1.12. THEOREM. *Let $\varphi: H \to G$ be a dense embedding of connected Lie groups. If $\varphi(H)$ contains the closure of the commutator group G' $(= \varphi(H'))$ then there exist a closed connected normal subgroup N of H and a closed vector group V in H such that*

(a) *H is a semidirect product of V and N,*

(b) *$\varphi(N)$ is closed in G,*

(c) *$\varphi(V)^-$ is a torus group,*

(d) *G is a semidirect product of $\varphi(V)^-$ and $\varphi(N)$.*

Proof. The normal (coabelian) subgroup W of H is defined by $\varphi(W) = \overline{G'}$, it coincides with $(H')^-$ in view of 1.4. Choose maximal tori T_W, T_H, and T_G in W, H, and G, respectively, such that $T_W \subset T_H$ and $\varphi(T_H) \subset T_G$. Then T_H is the maximal torus in $\varphi^{-1}(T_G)$ and $\varphi^{-1}(T_G) \cap W = T_W$ is contained in T_H. Since $(G')^-$ is connected the image $T_G(G')^-/(G')^-$ of T_G in $G/(G')^-$ is *the* maximal torus in the connected abelian group $G/(G')^-$. Hence $G/T_G(G')^-$ is a vector group. Actually, $T_G(G')^-$ is the smallest group within the set of all closed normal subgroups M of G such that G/M is a vector group, in particular $T_G(G')^-$ is independent of the choice of T_G. Correspondingly, $H/T_H W$ is a vector group. The homomorphism φ induces a dense homomorphism $\varphi': H/T_H W \to G/T_G(G')^-$ between vector groups, hence φ' is surjective and open. Then decompose the vector group $H/T_H W$ into a direct product of the vector group $\ker\varphi'$ and a chosen complementary vector group B. The desired group N is defined as the preimage of B under the quotient map $H \to H/T_H W$. It is easy to see that

$\varphi(N)$ is closed in G. From the construction follows $G = \varphi(N)T_G$. Using $\varphi^{-1}(T_G) \cap W \subset T_H$ one deduces $\varphi(N) \cap T_G = \varphi(T_H)$. As in 1.10, we now choose a torus S in T_G such that T_G is the direct product of S and $\varphi(T_H)$, and put $V := \varphi^{-1}(S)$. It is easily checked that V and N satisfy the requirements of the theorem.

Theorem 1.6 can also be used to "determine" all dense Lie subgroups of a given group G.

1.13. Proposition. *Let G be given connected Lie group. Choose a maximal torus T in G and define $N := T(G')^-$. Actually, N is independent of T; it is the smallest closed normal subgroup M such that G/M is a vector group. For a connected Lie subgroup H of G there are equivalent:*

(i) *H is dense in G.*

(ii) $\mathfrak{h} + \mathfrak{t} = \mathfrak{g}$, $\mathfrak{h}' = \mathfrak{g}'$, $\exp(\mathfrak{t} \cap \mathfrak{h})^- = T$.

(iii) $\mathfrak{h} + \mathfrak{n} = \mathfrak{g}$, $\mathfrak{h} \supset \mathfrak{g}'$, $\exp(\mathfrak{t} \cap \mathfrak{h})^- = T$.

All the Lie algebras of minimal dense connected Lie subgroups H are obtained in the following manner: In case $N = (G')^-$, i.e., $T \subset (G')^-$ one chooses a subspace $\mathfrak{w}$ of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{w}$ and one puts $\mathfrak{h} = \mathfrak{g}' + \mathfrak{w}$. In the other case one chooses $\mathfrak{w}$ as above and in addition one chooses a one-dimensional subspace $\mathfrak{r}$ of $\mathfrak{t}$ such that $\langle \exp \mathfrak{r}\rangle^- = T$. Then one puts $\mathfrak{h} = \mathfrak{g}' + \mathfrak{r} + \mathfrak{w}$. In particular, all minimal dense connected Lie subgroups have the same dimension.

Proof. For the properties of N compare the proof of 1.12.

(i) $\Rightarrow$ (ii) is an easy consequence of 1.6 and 1.4.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) Let $K = \overline{H} = \langle \exp \mathfrak{h}\rangle^-$. From the assumptions $\mathfrak{h} \supset \mathfrak{g}'$ and $\exp(\mathfrak{t} \cap \mathfrak{h})^- = T$ it follows that N is contained in K. Since $\mathfrak{h} + \mathfrak{n} = \mathfrak{g}$ one concludes $\mathfrak{k} = \mathfrak{g}$.

It is clear that the "constructed" Lie subalgebras lead to minimal dense subgroups. Now, let a minimal dense connected Lie subgroup H be given. We consider only the case $N \neq (G')^-$. By 1.6, $U := T \cap H$ is a dense connected subgroup of T. It is not hard to see that there exists a one-dimensional subspace $\mathfrak{r}$ of $\mathfrak{u}$ such that $(\exp \mathfrak{r})^- = T$. From (iii) follows the existence of a subspace $\mathfrak{w}$ of $\mathfrak{h}$ such that $\mathfrak{n} \oplus \mathfrak{w} = \mathfrak{g}$. The subalgebra $\mathfrak{g}' + \mathfrak{r} + \mathfrak{w}$ is contained in $\mathfrak{h}$, and it delivers a dense subgroup. Hence, $\mathfrak{h} = \mathfrak{g}' + \mathfrak{r} + \mathfrak{w}$ by the minimality.

2. Construction of Dense Embeddings

The first section was concluded by a description of all dense subgroups of a given Lie group G. In the present section we are mainly concerned with the "dense extensions" of a given Lie group H up to equivalence. The equivalence relation is, of course, the following.

2.1. DEFINITION. Dense embeddings φ_j: $G \to G_j$, $j = 1, 2$, of connected Lie groups are called equivalent if there exists a (unique) isomorphism σ: $G_1 \to G_2$ such that $\sigma\varphi_1 = \varphi_2$.

More precisely, we will reduce the classification problem to an extension problem for connected abelian Lie groups. In the considerations, an eminent role will be played by the closure $\mathrm{Ad}_H(H)^-$ of the adjoint group in $\mathrm{Aut}(\mathfrak{h})$. Clearly, the group $\mathrm{Ad}_H(H)$ only depends on $\mathfrak{h}$, it is the group of inner automorphisms of $\mathfrak{h}$. One of the main theorems tells that $\mathrm{Ad}_H(H)^-$ is an invariant of dense embeddings φ: $H \to G$ in the sense that the linear groups $\mathrm{Ad}_H(H)^-$ and $\mathrm{Ad}_G(G)^-$ are canonically isomorphic.

First we apply the results of the first section to the dense homomorphism Ad: $H \to \mathrm{Ad}_H(H)^-$.

2.2. THEOREM. *Let H be a connected Lie group, let T be a maximal torus in $\mathrm{Ad}_H(H)^-$, and let $U = \mathrm{Ad}_H^{-1}(T)$ be the inverse image of T. Then U is a connected abelian subgroup of H, $\mathrm{Ad}_H(U)$ is dense in T, and $\mathrm{Ad}_H(H)^- = \mathrm{Ad}_H(H)T$. The elements $t \in T$ act trivially on the Lie algebra $\mathfrak{u}$ of U. If T_1 is another maximal torus in $\mathrm{Ad}_H(H)^-$ then $U_1 = \mathrm{Ad}_H^{-1}(T_1)$ is conjugate to U. For each $s \in \mathrm{Ad}_H(H)^-$ there is a unique automorphism $\kappa(s)$ of H whose differential is s; κ is a continuous homomorphism from $\mathrm{Ad}_H(H)^-$ into* $\mathrm{Aut}(H)$.

Proof. In view of 1.6, from the statements in the second sentence we only need to show that U is abelian. Since $\ker \mathrm{Ad}_H = Z(H)$, $U/Z(H)$ is abelian, hence U is nilpotent. Therefore, $\mathrm{Ad}_U(U) = \mathrm{Ad}_H(U)|_{\mathfrak{u}}$ is a group of unipotent linear transformations. On the other hand, $\mathrm{Ad}_H(U) \subset T$ consists of semisimple transformations. It follows that $\mathrm{Ad}_U(U)$ is trivial, U is abelian. A similar argument was already used in the proof of 1.7 and, of course, 1.7 follows from the present theorem. The density of $\mathrm{Ad}_H(U)$ in T implies that T acts trivially on $\mathfrak{u}$. As T_1 and T are conjugate by an element in $\mathrm{Ad}_H(H)$ their preimages are conjugate in H.

Concerning κ we only note that $s \in \mathrm{Ad}_H(H)^-$ induces an automorphism $\tilde{s}$ of $\tilde{H}$ where π: $\tilde{H} \to H$ is a universal covering of H. Since $\tilde{s}$ can be approximated by inner automorphisms it leaves the central subgroup $\ker \pi$ pointwise fixed and hence it induces an automorphism $\kappa(s)$ of H. Alternatively one could argue that $\mathrm{Aut}(H)$ considered as a subset of $\mathrm{Aut}\,\mathfrak{h}$ is a closed subgroup containing $\mathrm{Ad}_H(H)$.

Remark. Some parts of 2.2 were obtained in [10] by different considerations. In that paper groups U of the form $\mathrm{Ad}_H^{-1}(T)$ are called "generalized maximal tori" or gm tori for short. Occasionally we will use this notation, too.

The next proposition gives a characterization of the Lie algebras of gm tori. The characterization is of a purely local type. This proposition as well as the following one are not needed in the sequel; they are only included to complete the picture.

2.3. PROPOSITION. *Let* $\mathfrak{g}$ *be the Lie algebra of a connected Lie group G. For a subalgebra* $\mathfrak{w}$ *of* $\mathfrak{g}$ *there are equivalent:*

(a) *There exists a maximal torus T in* $\mathrm{Ad}_G(G)^-$ *such that* $\mathfrak{w}$ *is the Lie algebra of* $\mathrm{Ad}_G^{-1}(T)$.

(b) *The subalgebra* $\mathfrak{w}$ *is maximal in the set of subalgebras* $\mathfrak{v}$ *of* $\mathfrak{g}$ *satisfying*

(i) $\mathfrak{v}$ *is abelian.*

(ii) *For each* $x \in \mathfrak{v}$ *the operator* $\mathrm{ad}(x)$ *is semisimple and it has a purely imaginary spectrum.*

Proof. Obvious.

In the classification of pointed generating invariant cones in Lie algebras an eminent rôle is played by the so-called compactly embedded Cartan algebras; see [11] or [12]. Those are Cartan algebras which satisfy conditions (i) and (ii) above; their existence is a necessary condition for the existence of pointed generating invariant cones. In general it is only true that an algebra $\mathfrak{w}$ as in 2.3 (b) is contained in a Cartan algebra.

2.4. PROPOSITION. *Let* $\mathfrak{g}$ *be finite dimensional real Lie algebra, and let* $\mathfrak{w}$ *and* $\mathfrak{w}_1$ *be two subalgebras as in* 2.3. (b).

(a) *The algebras* $\mathfrak{w}$ *and* $\mathfrak{w}_1$ *are conjugate by an element in the inner automorphism group of* $\mathfrak{g}$, *i.e., by an element in* $\langle \mathrm{Exp}\ \mathrm{ad}(\mathfrak{g})\rangle$.

(b) *The centralizer* $\mathfrak{c}$ *of* $\mathfrak{w}$ *in* $\mathfrak{g}$ *is solvable.*

(c) *There exists a Cartan subalgebra* $\mathfrak{h}$ *containing* $\mathfrak{w}$. *Each such* $\mathfrak{h}$ *is contained in the centralizer* $\mathfrak{c}$.

(d) *If* $\mathfrak{h}$ *and* $\mathfrak{h}_1$ *are Cartan subalgebras containing* $\mathfrak{w}$ *and* $\mathfrak{w}_1$, *respectively, then* $\mathfrak{h}$ *and* $\mathfrak{h}_1$ *are conjugate by an element in* $\langle \mathrm{Exp}\ \mathrm{ad}(\mathfrak{g})\rangle$.

Remark. The above Cartan algebras should be compared with the so-called standard Cartan algebras introduced by Goto in [9].

Proof. (a) follows immediately from 2.3 and 2.2 as $\langle \mathrm{Exp}\ \mathrm{ad}(\mathfrak{g})\rangle$ is nothing but $\mathrm{Ad}_G(G)$ for any connected Lie group G with Lie algebra $\mathfrak{g}$.

Concerning (b), let us assume that $\mathfrak{c}$ is not solvable. Then $\mathfrak{c}$ contains a simple subalgebra $\mathfrak{s}$. If $\mathfrak{s}$ is a "compact" algebra for each $x \in \mathfrak{s}$ the operator $\mathrm{ad}_{\mathfrak{g}}(x)$ is semisimple and has a purely imaginary spectrum. As x commutes with $\mathfrak{w}$ from the maximality of $\mathfrak{w}$ it follows that x is contained in $\mathfrak{w}$, hence $\mathfrak{s}$ is contained in $\mathfrak{w}$, a contradiction. If $\mathfrak{s}$ is not compact it contains a subalgebra $\mathfrak{l}$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. By the well known representation theory of $\mathfrak{sl}_2(\mathbb{R})$ it is clear that $\mathfrak{l}$ contains a non-zero element h such that $\mathrm{ad}_{\mathfrak{g}}(h)$ is semisimple with purely imaginary spectrum. Again h has to be in $\mathfrak{w}$, hence to commute with $\mathfrak{l}$ which is absurd.

(c) is a special case of Prop. 10, p. 24 in [1]. For the convenience of the reader we repeat the simple argument. Let $\mathfrak{h}$ be any Cartan algebra of the Lie algebra $\mathfrak{c}$. As $\mathfrak{w}$ is central in $\mathfrak{c}$ it has to be contained in $\mathfrak{h}$. We claim that $\mathfrak{h}$ is a Cartan algebra of the Lie algebra $\mathfrak{g}$. There is only to show that $\mathfrak{h}$ coincides with the normalizer $\mathfrak{n}$ of $\mathfrak{h}$ in $\mathfrak{g}$. Let $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{d}$ be an $\mathrm{ad}\mathfrak{w}$-stable vector space decomposition of $\mathfrak{n}$. In particular, $[\mathfrak{w}, \mathfrak{d}]$ is contained in $\mathfrak{d}$. On the other hand, $[\mathfrak{w}, \mathfrak{d}]$ is contained in $[\mathfrak{h}, \mathfrak{n}]$, hence in $\mathfrak{h}$. Therefore, $[\mathfrak{w}, \mathfrak{d}]$ is zero, $\mathfrak{d}$ is contained in $\mathfrak{c}$. It follows that $\mathfrak{n}$ is contained in $\mathfrak{c}$, hence $\mathfrak{n}$ is the normalizer of $\mathfrak{h}$ in $\mathfrak{c}$ which is $\mathfrak{h}$ as $\mathfrak{h}$ is a Cartan algebra. Conversely, if $\mathfrak{h}$ is any Cartan algebra of $\mathfrak{g}$ containing $\mathfrak{w}$ then for any $x \in \mathfrak{w}$ the operator $\mathrm{ad}_{\mathfrak{h}} x = \mathrm{ad}_{\mathfrak{g}} x|_{\mathfrak{h}}$ is nilpotent and semisimple, hence $\mathrm{ad}_{\mathfrak{h}}\mathfrak{w} = [\mathfrak{w}, \mathfrak{h}] = 0$ which means that $\mathfrak{h}$ is contained in $\mathfrak{c}$.

(d) By (a) we may assume that $\mathfrak{w} = \mathfrak{w}_1$. From (c) it follows that $\mathfrak{h}$ and $\mathfrak{h}_1$ are contained in $\mathfrak{c}$. But Cartan algebras in solvable Lie algebras are conjugate under inner automorphisms; see Theorème 3, p. 31 in [1].

After this digression we return to the study of dense embeddings $\varphi: H \to G$. For such an embedding we introduce two notations. Since $d\varphi(\mathfrak{h})$ is an ideal in $\mathfrak{g}$ it is invariant under $\mathrm{Ad}_G(G)^-$. Hence there is a unique continuous homomorphism R_φ: $\mathrm{Ad}_G(G)^- \to \mathrm{Aut}(\mathfrak{h})$ such that $d\varphi(R_\varphi(s)(Y)) = s(d\varphi(Y))$ for $s \in \mathrm{Ad}_G(G)^-$ and $Y \in \mathfrak{h}$. As φ has a dense image R_φ clearly takes its values in $\mathrm{Ad}_H(H)^-$. Besides others in the next theorem it will be proved that R_φ is actually an isomorphism from $\mathrm{Ad}_G(G)^-$ onto $\mathrm{Ad}_H(H)^-$. The composition $R_\varphi \circ \mathrm{Ad}_G$ is denoted by A_φ. Evidently, $A_\varphi \circ \varphi = \mathrm{Ad}_H$, hence A_φ is a dense homomorphism from G into $\mathrm{Ad}_H(H)^-$.

2.5. THEOREM. *Let $\varphi: H \to G$ be a dense embedding of connected Lie groups H and G. Let T be a maximal torus in $Ad_H(H)^-$, let $U = \mathrm{Ad}_H^{-1}(T)$ and let $V = A_\varphi^{-1}(T)$.*

(a) *The subgroup V of G is abelian and connected, $A_\varphi(V)$ is dense in T.*

(b) *$\varphi^{-1}(V) = U$, $\varphi(U)^- = V$. In particular, φ induces a dense embedding from U into V.*

(c) *If T_V denotes the maximal torus of V then T_V is a maximal torus of G. Hence $G = \varphi(H)T_V$.*

(d) *The homomorphism R_φ: $\mathrm{Ad}_G(G)^- \to \mathrm{Ad}_H(H)^-$ is an isomorphism. Consequently, $R_\varphi^{-1}(T)$ is a maximal torus in $\mathrm{Ad}_G(G)^-$ and, therefore, $V = A_\varphi^{-1}(T) = \mathrm{Ad}_G^{-1}(R_\varphi^{-1}(T))$ is a gm torus in G.*

(e) *The linear group* $\mathrm{Ad}_G(G)$ *in* $\mathrm{Aut}(\mathfrak{g})$ *is closed if and only if* $A_\varphi(V) = T$. *In particular, if* $\mathrm{Ad}_H(H)$ *is closed then* $\mathrm{Ad}_G(G)$ *is closed.*

Proof. The equation $\varphi^{-1}(V) = U$ is an immediate consequence of $A_\varphi \circ \varphi = \mathrm{Ad}_H$. By 1.6 applied to the dense homomorphism A_φ and the subgroup T of $\mathrm{Ad}_H(H)^-$, one obtains that V is connected, that $A_\varphi(V)^- = T$ and that V contains a maximal torus of G, say S. Applying 1.6 once more, now to the dense embedding φ and the subgroup V, we get that $G = \varphi(H)S$ and that φ induces a dense embedding from $U = \varphi^{-1}(V)$ into V. In particular, it follows that V is abelian, hence S is the maximal torus T_V of V, and (a), (b), (c) are proved.

Next, we claim that R_φ is injective. Let $s \in \mathrm{Ad}_G(G)^-$ be given. Then s induces an automorphism $\kappa_G(s)$ of G, by 2.2. If $R_\varphi(s)$ is the identity then $\kappa_G(s)$ is the identity on $\varphi(H)$. Since $\varphi(H)$ is dense, $\kappa_G(s)$ is the identity on G, hence s is the identity.

Concerning the surjectivity of R_φ we first observe that it is sufficient to show that the image of R_φ contains T as $\mathrm{Ad}_H(H)^- = T\,\mathrm{Ad}_H(H)$. To a given $t \in T$ there exists a sequence (v_k) in V such that $(A_\varphi(v_k))$ converges to t. Since V is abelian $\mathrm{Ad}_G(v_k)$ acts trivially on $\mathfrak{v}$. From $\mathfrak{g} = d\varphi(\mathfrak{h}) + \mathfrak{v}$ one easily deduces that $\mathrm{Ad}_G(v_k)$ converges to an element $s \in \mathrm{Aut}(\mathfrak{g})$ with the property that $R_\varphi(s) = t$.

Since R_φ is an isomorphism, $\mathrm{Ad}_G(G)$ is closed iff $R_\varphi(\mathrm{Ad}(G)) = \mathrm{Ad}_H(H)^-$ or $A_\varphi(G) = \mathrm{Ad}_H(H)^-$. Evidently, the latter equality is equivalent to $A_\varphi(V) = T$.

Let us summarize: Any chosen maximal torus T in $\mathrm{Ad}_H(H)^-$ defines a connected abelian closed subgroup U of H and a dense homomorphism α: $U \to T$, namely $\alpha = \mathrm{Ad}_H|U$. Any dense embedding φ: $H \to G$ gives rise to a connected abelian closed subgroup V of G and a dense embedding j: $U \to V$, namely the restriction of φ to U, such that α can be (uniquely) extended to a continuous homomorphism γ: $V \to T$, i.e., $\gamma \circ j = \alpha$, namely $\gamma = A_{\varphi|V}$. The main point is that also the converse is true. Suppose there is given a triple (W, j, γ) consisting of a connected abelian Lie group W, a dense embedding j: $U \to W$ and a continuous extension γ: $W \to T$ of α. Then there exists a connected Lie group G and a dense embedding φ: $H \to G$ such that the given triple is equal (up to equivalence) to the triple derived from φ. Clearly, two triples (W_k, j_k, γ_k), $k = 1, 2$, are called equivalent if the embeddings j_k: $U \to W_k$ are equivalent in the sense of 2.1. Note that an isomorphism σ: $W_1 \to W_2$ with $\sigma j_1 = j_2$ also satisfies $\gamma_2\sigma = \gamma_1$.

To construct the group G one forms the semidirect product $W \ltimes H$ with multiplication $(w_1, h_1)(w_2, h_2) = (w_1 w_2, \kappa\gamma(w_2^{-1})(h_1)h_2)$ where $\kappa = \kappa_H$ is the map constructed in 2.2. Clearly, $\Delta = \Delta(U) := \{(j(x)^{-1}, x) | x \in U\}$ is a closed subset of $W \ltimes H$. Using 2.2, in particular the fact that T acts trivially on $\mathfrak{u}$, one verifies easily that Δ is a central subgroup of $W \ltimes H$. Define $G := W \ltimes H/\Delta$, φ: $H \to G$ is the obvious embedding. Of course, $\varphi(H)$ is dense in G. We put $V := A_\varphi^{-1}(T)$, and we claim that (W, j, γ) is equivalent to $(V, \varphi, A_\varphi|_V)$. First one observes that if $[w, h]$ denotes the Δ-coset of $(w, h) \in W \ltimes H$ then $A_\varphi([w, h])$ equals $\gamma(w) \circ \mathrm{Ad}_H(h)$. Hence $A_\varphi^{-1}(T) = (W \ltimes U)/\Delta$. It is easy to verify that σ: $W \to V = (W \ltimes U)/\Delta$ defined by $\sigma(w) = [w, e]$ establishes the desired equivalence. Similarly one can show that the procedure: "Start with a dense embedding φ: $H \to G$, from the associated triple $(V, \varphi, A_\varphi|_V)$ and apply the construction to this triple" gives back φ up to equivalence. We have proved the following.

2.6. THEOREM. *Let H be a connected Lie group. Let T be a maximal torus in $\mathrm{Ad}_H(H)^-$, let $U = \mathrm{Ad}_H^{-1}(T)$ and denote by α: $U \to T$ the dense homomorphism obtained as restriction of Ad_H. There is a bijective correspondence between (equivalence classes of) dense embeddings from H into connected Lie groups and (equivalence classes of) triples (V, j, γ) where j is a dense embedding from U into the abelian connected Lie group V and γ: $V \to T$ satisfies $\gamma j = \alpha$.*

Namely, corresponding to a dense embedding φ: $H \to G$, we have the triple $(V, \varphi|_U, A_\varphi|_V)$, where $V = A_\varphi^{-1}(T)$; conversely, corresponding to the triple (V, j, γ), we have the natural embedding of H in $(V \ltimes H)/\Delta(U)$, where $\Delta(U) = \{(j(x)^{-1}, x) | x \in U\}$.

(If φ: $H \to G$ is a dense embedding, then $d\varphi(\mathfrak{h})$ is an ideal in $\mathfrak{g}$, so G acts on $\mathfrak{h}$. This action defines the homomorphism A_φ: $G \to \mathrm{Ad}_H(H)^-$.)

The "classification" of dense embeddings is reduced to an abelian problem concerning a given (dense) homomorphism α: $U \to T$. This problem can be reduced a little further to see "how many" triples (V, j, γ) exist. To this end, one fixes a vector group complement U_v to the maximal torus T_U of U. Starting from the data

D	vector subspace of U_v
S	torus group
i: $D \to S$	dense embedding
β: $S \to T$	homomorphism such that $\beta i = \alpha|_D$

one can construct a triple (V, j, γ) in the following manner.

Choose a vector space complement F to D in U_v, define $V := T_U \times S \times F$, define j: $U \to V$ by $j(tdf) = (t, i(d), f)$ for $t \in T_U$, $d \in D, f \in F$, and define γ: $V \to T$ by $\gamma(t, s, f) = \alpha(t)\beta(s)\alpha(f)$. The equivalence class of the constructed (V, j, γ) is independent of the choice of F. In fact, in this way one obtains all possible triples (V, j, γ) up to equivalence. More precisely, the following remark holds true. Its easy proof is omitted.

2.7. *Remark*. Let U be a connected abelian Lie group and let $\alpha: U \to T$ be a (dense) continuous homomorphism into the torus group T. Choose a vector group complement U_v to T_U in U, and consider quadruples (D, S, i, β) with the properties stated above. Two such quadruples (D_k, S_k, i_k, β_k) are called equivalent if $D_1 = D_2$ and if the embeddings i_1 and i_2 are equivalent in the sense of 2.1. The equivalence classes of such quadruples are in bijective correspondence to the equivalence classes of triples (V, j, γ) with the usual properties.

In this sense, all dense embeddings of Lie groups can be reduced to the standard example, namely dense embeddings of vector groups into tori. Moreover, the choice of gm tori in order to describe all dense embeddings of a given connected Lie group H is in a sense canonical. Of course, to "produce" the extra elements in a dense extension of H one does not need the maximal torus T_U in a gm torus U of H, but the next theorem tells that one needs, so to speak, all the other elements in U, compare problem 3 on p. 730 in [10].

2.8. THEOREM. *Let H be a connected Lie group and suppose that M is a connected Lie subgroup such that $G = \varphi(H)\varphi(M)^-$ for each dense embedding φ from H into a connected Lie group G. Then there exists a gm torus U in H such that $M \cap U$ is a connected Lie subgroup of H and such that $U = (M \cap U)T_U$ if T_U denotes the maximal torus of U. The group $M \cap U$ need not to be closed in H but it contains a closed vector group U_v such that U is the direct product of U_v and T_U.*

Proof. The assertion in the last sentence follows from the previous one; this is an easy exercise in abelian Lie group theory.

Enlarging an arbitrarily chosen maximal torus in $\mathrm{Ad}_H(M)^-$ to a maximal torus in $\mathrm{Ad}_H(H)^-$ one finds a maximal torus T in $\mathrm{Ad}_H(H)^-$ such that $T \cap \mathrm{Ad}_H(M)^-$ is a maximal torus in $\mathrm{Ad}_H(M)^-$. We claim that the gm torus $U = \mathrm{Ad}_H^{-1}(T)$ has the required properties. Applying 1.6 to the obvious dense homomorphism $M \to \mathrm{Ad}_H(M)^-$, one obtains that $M \cap \mathrm{Ad}_H^{-1}(T \cap \mathrm{Ad}_H(M)^-) = U \cap M$ is a closed connected Lie subgroup of the Lie group M. Hence $U \cap M$ is a connected Lie subgroup of H and of U. Again elementary abelian Lie group theory shows that $M_U := (U \cap M)T_U$ is a closed (connected Lie) subgroup of U. For alter use, we note here that by 1.1 the homogeneous spaces $M/M \cap U$ and $\mathrm{Ad}_H(M)^-/T \cap \mathrm{Ad}_H(M)^-$ are homeomorphic.

So far the particular property of M played no rôle; everything we said is true for any connected Lie subgroup of H. Using this particular property together with the appropriate choice of U made above we now prove:

(*) For any dense embedding $\varphi: H \to G$ into a connected Lie group G the equalities $G = \varphi(H)\varphi(U \cap M)^-$ and $\varphi(U)^- = \varphi(U)\varphi(U \cap M)^-$ hold true.

Let $x \in G$ be given. By assumption there exists $h \in H$ and a sequence (m_k) in M such that $\varphi(m_k)$ converges in G and such that $x = \varphi(h) \lim_{k\to\infty} \varphi(m_k)$. As $\mathrm{Ad}_H = A_\varphi \circ \varphi$, where A_φ is the homomorphism defined in front of 2.5, the sequence $\mathrm{Ad}_H(m_k)$ converges in $\mathrm{Ad}_H(M)^-$. Using the above homeomorphism the cosets of m_k converge in $M/M \cap U$. Hence there exists a sequence (z_k) in $M \cap U$ such that $(m_k z_k)$ converges in M (w.r.t. the internal Lie group topology of M and a fortiori in the topology of H). As $\varphi(m_k)$ converges, clearly $\varphi(z_k^{-1})$ converges and we find $x = \varphi(h) \lim_{k\to\infty} \varphi(m_k z_k z_k^{-1}) = \varphi(h)\varphi(\lim_{k\to\infty} m_k z_k) \lim_{k\to\infty} \varphi(z_k^{-1}) \in \varphi(H)\varphi(U \cap M)^-$. The second asserted equation in (*) follows from the first one using $U = \varphi^{-1}(\varphi(U)^-)$; see 2.5 (b).

Now assuming that, on the contrary to the theorem, $M_U = (U \cap M)T_U$ is a proper subgroup of U we are going to construct a particular dense embedding $\varphi: H \to G$ which violates (*). To this end, we choose a closed vector group B in U such that U is the direct product of B and M_U. Given B we choose a dense embedding i from B into a torus S as well as a continuous homomorphism $\beta: S \to T$ such that $\beta \circ i = \mathrm{Ad}_H$. Of course, such a triple (i, β, S) exists. One may simple take any injective homomorphism χ from B into a torus R. Form $\chi \times \mathrm{Ad}_H$: $B \to R \times T$, let S be the closure of $(\chi \times \mathrm{Ad}_H)(B)$, let i be the corestriction of $\chi \times \mathrm{Ad}_H$ to S, and let β be the restriction of the second projection $R \times T \to T$ to S.

Given (i, β, S) let $V = M_U \times S$, and define $j: U \to V$ and $\gamma: V \to T$ by $j(wb) = (w, i(b))$ and $\gamma(w, s) = \mathrm{Ad}_H(w)\beta(s)$ for $w \in M_U$, $b \in B$ and $s \in S$. With the triple (V, j, γ) we carry out the above construction; i.e., we form the semidirect product $V \ltimes H$ with the multiplication $(v_1, h_1)(v_2, h_2) = (v_1 v_2, \kappa_H \gamma(v_2^{-1})(h_1)h_2)$, we put $G = V \ltimes H/N$ where $N = \{(j(x)^{-1}, x) | x \in U\}$, and we define $\varphi(h) = \pi(e, h)$ where $\pi: V \times H \to G$ denotes the quotient homomorphism. One easily verifies that $\varphi(M_U)$ is closed in G, hence $\varphi(U \cap M)^- \subset \varphi(M_U) \subset \varphi(U)$ and $\varphi(U)\varphi(U \cap M)^- = \varphi(U) = U \ltimes U/N$. But $\varphi(U)^- = V \ltimes U/N$ is strictly larger as i is not onto. This contradicts the assumption that M_U is a proper subgroup of U.

Knowing all dense extensions of a given group it is easy to tell when no proper extension exists. The following criterion was explicitly formulated in [7], where the corresponding groups are called absolutely closed, but it follows also from the results of van Est [3].

2.9. THEOREM. *A connected Lie group H allows no proper dense embedding if and only if the preimage $\mathrm{Ad}_H^{-1}(T)$ of a maximal torus T in $\mathrm{Ad}_H(H)^-$ is compact. The latter property is equivalent to*: $\mathrm{Ad}_H(H)$ *is closed and $Z(H)$ is compact.*

Note that the criterion consists of a local part; namely, $\mathrm{Ad}_H(H)$ is closed, and of a global one.

Proof. Let T be a maximal torus in $\mathrm{Ad}_H(H)^-$, let $U = \mathrm{Ad}_H^{-1}(T)$ and let

U_v be a vector group complement to the maximal torus T_U in U. If U_v is zero, H allows no proper dense embedding by the above considerations. If U_v is non zero, for each non-zero subspace D of U_v one can construct at least one triple (S, i, β) in the sense of 2.6./2.7. It follows that the non-existence of proper dense extensions is equivalent to the compactness of U. But if U is compact then clearly $Z(H)$ as a subgroup of U is compact. Since $\mathrm{Ad}_H(U)$ is dense in T compactness of U implies that T is contained in $\mathrm{Ad}(H)$, hence $\mathrm{Ad}_H(H)^- = \mathrm{Ad}_H(H)T = \mathrm{Ad}_H(H)$. On the other hand, if $\mathrm{Ad}_H(H)$ is closed Ad_H defines a surjective homomorphism from U onto the compact group T with kernel $Z(H)$. Therefore, the compactness of U is equivalent to the criterion stated in the theorem.

Of course, semisimple Lie groups with finite center satisfy the criterion of the theorem. But also some solvable Lie groups like the "$ax + b$ – group" do so. In other words, the criterion is that gm tori are maximal tori. Above we observed that every gm torus is contained in the Lie subgroup corresponding to a Cartan algebra. Hence the criterion is satisfied if the Lie algebra of a maximal torus is a Cartan algebra. This fact can also be proved directly, i.e., without using 2.3/4.

2.10. Proposition. *Let S be a maximal torus in the connected Lie group G. Suppose that $\mathfrak{s}$ is a Cartan algebra of $\mathfrak{g}$, i.e., $\mathfrak{s}$ coincides with its normalizer in $\mathfrak{g}$. Then any dense embedding of G into another connected Lie group is an isomorphism.*

Proof. Let T be a maximal torus in $\mathrm{Ad}_G(G)^-$ containing $\mathrm{Ad}_G(S)$, and let $U = \mathrm{Ad}_G^{-1}(T) \supset S$. As we know U is connected and abelian, hence $U = S$ because $\mathfrak{s}$ is a Cartan algebra. But the compactness of U implies that G does not allow dense extensions.

Theorem 2.9 gives rise to the question whether dense embeddings can be described simpler in the case of a closed adjoint group. The next proposition says that this is indeed the case and explains why we could construct in 1.11 a dense embedding of $SL_2(\mathbb{R})^\sim \times SL_2(\mathbb{R})^\sim$ in a different fashion than suggested by 2.6/7. Observe that the proposition applies in particular to semisimple groups. The class of groups, or rather Lie algebras, with closed adjoint group was intensively studied by van Est, [2–4], the proposition follows from his results. Generalizations to arbitrary connected locally compact groups are discussed in [6].

2.11. Proposition. *Let $\varphi: H \to G$ be a dense embedding of connected Lie groups. Suppose that $\mathrm{Ad}_H(H)$ is closed in $\mathrm{Aut}(\mathfrak{h})$. Then also $\mathrm{Ad}_G(G)$ is closed in $\mathrm{Aut}(\mathfrak{g})$. If $Z(H)$ and $Z(G)$ are the centers of H and G, resp., then $\varphi(Z(H))^- = Z(G)$ and $G = \varphi(H)Z_0(G)$. Each dense extension of H is obtained in the following manner: Choose a closed subgroup D of $Z(H)$ and a dense embedding $\iota: D \to W$ where W is a connected abelian Lie* *group. Then form* $W \times H/\{(\iota(x)^{-1}, x) | x \in D\}$ *and take the obvious embedding of H into this group.*

Remark. One may ask whether the claim $\varphi(Z(H))^- = Z(G)$ is true without the assumption "$\mathrm{Ad}_H(H)$ is closed." This is not the case as can be shown by examples.

Proof. We observed already in 2.5 that $\mathrm{Ad}_G(G)$ is closed. From the fact that in the present case R_φ is an isomorphism from $\mathrm{Ad}_G(G)$ onto $\mathrm{Ad}_H(H)$ it follows that every inner automorphism I_x, $x \in G$, of G equals $I_{\varphi(h)}$ for a suitable $h \in H$. Hence $G = \varphi(H)Z(G)$ which implies $G = \varphi(H)Z_0(G)$ as $Z(G)/Z_0(G)$ is countable. Lemma 1.1 gives that φ induces a dense embedding from $\varphi^{-1}(Z(G)) = Z(H)$ into $Z(G)$.

We omit the details for the "constructive" part, we remark only that for a given dense embedding $\psi: H \to M$ we choose $D = \psi^{-1}(Z_0(M))$.

As an illumination and application of 2.6 we construct two dense embeddings with certain properties.

2.12. PROPOSITION. (a) *To each connected Lie group H there exists a dense embedding* φ *into a connected Lie group G such that G allows no further dense embeddings. In particular,* $\mathrm{Ad}_G(G)$ *is closed in* $\mathrm{Aut}(\mathfrak{g})$.

(b) *Let* $\varphi: H \to G$ *be a dense embedding of connected Lie groups. Suppose there is given a covering* $p: H_1 \to H$. *Then there exist a covering* $q: G_1 \to G$ *of G and a dense embedding* $\varphi_1: H_1 \to G_1$ *such that* $q\varphi_1 = \varphi p$.

Proof. (a) As usual let T be a maximal torus in $\mathrm{Ad}_H(H)^-$ and let $U = \mathrm{Ad}_H^{-1}(T)$. Then one can construct a torus S, a dense embedding $j: U \to S$ and a continuous homomorphism $\gamma: S \to T$ such that $\mathrm{Ad}_{H|U} = \gamma j$. Namely, similar to 2.8, one takes any dense embedding ι from U into a torus R and defines S to be the closure of $\{(\iota(x), \mathrm{Ad}_H(x)) | x \in U\}$ in $R \times T$; γ is the restriction of the second projection. Then one applies the construction of 2.6 to the triple (S, j, γ) and obtains the desired embedding $\varphi: H \to G$. There are several ways to see that G allows no dense embeddings, one of them is to apply 2.9: Since R_φ is an isomorphism the preimage $R_\varphi^{-1}(T)$ of T is a maximal torus in $\mathrm{Ad}_G(G)^-$. The Ad_G-preimage of this maximal torus is precisely $A_\varphi^{-1}(T) = (S \ltimes U)/\Delta \cong S$, hence compact.

(b) It is not hard to see that the assertion is true in the abelian case. One way is to use universal coverings and to apply the following easy lemma:

Let $\mathfrak{a}$ be a vector subspace of the vector space $\mathfrak{b}$. Let K be a discrete subgroup of $\mathfrak{b}$ such that $\mathfrak{a} + K$ is dense in $\mathfrak{b}$. To a given subgroup D of $\mathfrak{a} \cap K$ there exists a subgroup E of K such that $E \cap \mathfrak{a} = D$ and that $E + \mathfrak{a}$ lies dense in $\mathfrak{b}$.

Now, we consider the general case. Let, as usual, T be a maximal torus in $\mathrm{Ad}_H(H)^-$, let $U = \mathrm{Ad}_H^{-1}(T)$, let $U_1 = \mathrm{Ad}_{H_1}^{-1}(T) = p^{-1}(U)$, and let $V =$

$A_\varphi^{-1}(T) \subset G$. By the "abelian proposition" there exist a connected Lie group V_1, a covering $q: V_1 \to V$, and a dense embedding $\varphi_1: U_1 \to V_1$ such that $q \circ \varphi_1 = \varphi|_U \circ p|_{U_1}$. There is also a map, namely, $A_\varphi q$, from V_1 into the maximal torus T of $\mathrm{Ad}_{H_1}(H_1) = \mathrm{Ad}_H(H)^-$ which acts as group of automorphisms of H_1 as well. Hence we may apply the construction of 2.6 to the triple $(V_1, \varphi_1, A_\varphi q)$ and we obtain an extension G_1 of H_1, $G_1 = V_1 \ltimes H_1/\Delta$ with the obvious embedding $\varphi_1: H_1 \to G_1$ where $\Delta = \{(\varphi_1(x)^{-1}, x) | x \in U_1\}$ is a central subgroup. The homomorphism $q: G_1 \to G$ is defined by $q([v_1, h_1]) = q(v_1)\varphi(p(h_1))$. It is easy to verify that q is a well-defined homomorphism, that q is a covering and that $q\varphi_1 = \varphi p$.

Remark 1. In the above proof of (b) one could replace V by a maximal torus S in G, U by $\varphi^{-1}(S)$ and U_1 by $p^{-1}\varphi^{-1}(S)$. Each connected abelian Lie group V_1 together with a covering $q: V_1 \to S$ and a dense embedding $\varphi_1: p^{-1}\varphi^{-1}(S) \to V_1$ such that $q \circ \varphi_1 = \varphi|_{\varphi^{-1}(S)} \circ p|_{p^{-1}\varphi^{-1}(S)}$ delivers a solution G_1 in a similar way as well. The latter approach is technically a little easier. I preferred the first one in order to exhibit the direct relation to 2.6.

Remark 2. The solution G_1 is by no means unique. There exist dense embeddings φ_1 from $\mathbb{R}^2$ into $\mathbb{R} \times \mathbb{T}^2$ such that φ_1 followed by the projection q of $\mathbb{R} \times \mathbb{T}^2$ onto $\mathbb{T}^3$ is a dense embedding, too. Hence $\varphi = q\varphi_1$ leads to at least two coverings, namely q and the identity of $\mathbb{T}^3$. Even more evident, the solution in (a) is far from being unique as the construction shows. This should be compared with the fact that with each Lie algebra there may be associated an essentially unique Lie algebra with closed inner automorphism group (see [2–4]). Closely related, in [18] one finds dense embeddings of a given Lie group into Lie groups with closed adjoint groups and with certain additional properties.

3. The Levi Decomposition and Dense Homomorphisms

Let $\varphi: H \to G$ be a dense homomorphism of connected Lie groups. The "additional" elements of G, i.e., the elements outside the image, should be essentially a solvable phenomenon. And indeed, G is a product of $\varphi(H)$ and the radical of G, see below. On the other hand, as the example of 1.11 shows, at least the center of the semisimple part of H may contribute additional elements. Therefore, we study the product S_H of the center of H with the radical, and we investigate how it behaves when dense homomorphisms are applied. It turns out that this product is of finite index in the largest normal solvable subgroup of H. As Section 2 shows, an important role is played by the dense homomorphism $\mathrm{Ad}_H: H \to \mathrm{Ad}_H(H)^-$. Therefore, we pay some attention to this case. At the first glance, the largest normal solvable subgroup of H has the better behaviour under dense homomorphisms than its subgroup S_H of finite index. We include some examples to show that this is indeed the case. But S_H has the advantage that it is a smaller group which actually is also able to produce via completion all additional elements of G.

3.1. DEFINITION. For any connected Lie group H the largest connected normal solvable (closed) Lie subgroup of H is denoted by $\mathrm{Rad}(H)$, the product of the center $Z(H)$ and of $\mathrm{Rad}(H)$ is denoted by S_H. A maximal connected semisimple Lie subgroup L is called a Levi factor of H.

Observe that Levi factors L need not to be closed; see 1.11. The equality $H = L\,\mathrm{Rad}(H)$ is true for any Levi factor L.

3.2. PROPOSITION. *Let H be a connected Lie group.*

(i) *If L is a Levi factor of H then* $\mathrm{Rad}(H)Z(L)$ *is a closed normal subgroup. In fact, it is the largest closed normal solvable subgroup of H; in particular it is independent of the choice of L.*

(ii) *The group* S_H *is closed and normal in H, it is a subgroup of* $\mathrm{Rad}(H)Z(L)$ *of finite index.*

(iii) H/S_H *is a semisimple Lie group with finite center.*

(iv) $\mathrm{Rad}(H/Z(H)) = S_H/Z(H)$.

Proof. Let $Q = H/\mathrm{Rad}(H)$, the quotient map $p\colon H \to Q$ induces a covering from the semisimple Lie group L onto Q which implies that $p(Z(L)) = Z(Q)$. From $p(Z(L)) = Z(Q)$ and $H = L\,\mathrm{Rad}(H)$, one deduces $p^{-1}(Z(Q)) = \mathrm{Rad}(H)Z(L)$. From this equation, (i) follows using the fact that $Z(Q)$ is the largest normal solvable subgroup of the semisimple Lie group Q.

As $p(Z(H)) \subset Z(Q)$ one has the inclusions $\mathrm{Rad}(H) \subset S_H = \mathrm{Rad}(H)Z(H) \subset p^{-1}(Z(Q))$. Since $p^{-1}(Z(Q))/\mathrm{Rad}(H)$ is discrete (isomorphic to $Z(Q)$), S_H is closed in H. Clearly, S_H is normal in H. To see that S_H is a finite index in $p^{-1}(Z(Q))$ one considers $\psi\colon L \to \mathrm{Aut}(\mathfrak{r})$, where $\mathfrak{r}$ is the Lie algebra of $\mathrm{Rad}(H)$, defined by $\psi(x) = \mathrm{Ad}(x)|_{\mathfrak{r}}$. The image $\psi(Z(L))$ is the center of the linear semisimple Lie group $\psi(L)$, hence finite; see Prop. 4.1 in Chap. XVIII of [13]. The intersection $K := \ker\psi \cap Z(L)$ is contained in $Z(H)$, and $\mathrm{Rad}(H)K$ is of finite index in $\mathrm{Rad}(H)Z(L)$. Therefore, $\mathrm{Rad}(H)Z(H)$ is of finite index in $\mathrm{Rad}(H)Z(L)$.

(iii) is an immediate consequence, (iv) is obvious.

3.3. THEOREM. *Let* $\varphi\colon H \to G$ *be a dense continuous homomorphism between connected Lie groups H and G.*

(a) $\mathrm{Rad}(G) \subset \varphi(S_H)^- \subset S_G$. *The subgroup* $\varphi(S_H)^-$ *is of finite index in* S_G.

(b) $G = \varphi(H)\mathrm{Rad}(G) = \varphi(H)\varphi(S_H)^-$.

(c) *If* $\ker\varphi$ *is contained in* S_H *then* S_H *is a subgroup of* $\varphi^{-1}(S_G)$ *of finite index.*

(d) *If* $\ker \varphi = Z(H)$ *then* $G = \varphi(H)\varphi(\text{Rad } H)^-$, $\text{Rad}(G) = \varphi(\text{Rad } H)^-$ *and* $\text{Rad}(G)$ *is of finite index in* S_G. *In particular, this applies to* $G = \text{Ad}_H(H)^-$ *and* $\varphi = \text{Ad}_H$.

(e) *If* T_r *is a maximal torus in* $\text{Ad}_H(\text{Rad } H)^-$ *then* $\text{Ad}_H(H)^- = \text{Ad}_H(H)T_r$. *There exists a maximal torus* T *in* $\text{Ad}_H(H)^-$ *such that* $T = T_r\text{Ad}_H(U)$ *where* $U = \text{Ad}_H^{-1}(T)$.

Proof. Of course, $\varphi(Z(H)) \subset Z(G)$ and $\varphi(\text{Rad } H) \subset \text{Rad } G$, hence $\varphi(S_H)^- \subset S_G$ as S_G is closed. The homomorphism φ induces a dense homomorphism from H/S_H in $G/\varphi(S_H)^-$. As H/S_H is semisimple with finite center this homomorphism has to be onto, hence $G/\varphi(S_H)^-$ is semisimple with finite center as well. From the latter fact one easily deduces that $\text{Rad}(G)$ is contained in $\varphi(S_H)^-$ and that $\varphi(S_H)^-$ is of finite index in S_G. By 1.4 any Levi factor of G is contained in $\varphi(H)$, hence $G = \varphi(H)\text{Rad}(G)$. From (a) follows $G = \varphi(H)\varphi(S_H)^-$.

Concerning (c) one first observes that S_H is always a subgroup of $\varphi^{-1}(S_G)$, compare (a). Since $\ker \varphi$ and S_G are solvable, $\varphi^{-1}(S_G)$ is a solvable closed normal subgroup of H, hence contained in $\text{Rad}(H)Z(L)$ for any Levi factor L of H. As S_H is of finite index in $\text{Rad}(H)Z(L)$, it has to be of finite index in $\varphi^{-1}(S_G)$ as well.

(d) is an easy consequence of (a) and (b): By (a), $\text{Rad}(G) \subset \varphi(S_H)^- = \varphi(\text{Rad } H)^- \subset S_G$, and $\varphi(S_H)^- = \varphi(\text{Rad } H)^-$ is of finite index in S_G. Since $\varphi(\text{Rad } H)$ is contained in $\text{Rad}(G)$ the latter group has to coincide with $\varphi(\text{Rad } H)^-$. The equality $G = \varphi(H)\varphi(\text{Rad } H)^-$ now follows from (b).

Concerning (e) we first apply 1.6 to the dense homomorphism $\text{Rad}(H) \to \text{Ad}_H(\text{Rad } H)^-$ obtained by restricting Ad_H. One gets $\text{Ad}_H(\text{Rad } H)^- = \text{Ad}_H(\text{Rad } H)T_r$. Using (d) one concludes that $\text{Ad}_H(H)^- = \text{Ad}_H(H)T_r$. Let T be a maximal torus in $\text{Ad}_H(H)^-$ such that $T \cap \text{Ad}_H(\text{Rad } H)^- = T_r$. Any such T does the job.

As an illustration we will consider two further examples. The examples will also show that some equalities one might guess are not true in general, in particular one has:

(A) $\varphi(S_H)^-$ is possibly a proper subgroup of S_G, even if φ is a dense embedding or $\varphi = \text{Ad}_H$ and $G = \text{Ad}_H(H)^-$.

(B) S_H is possibly a proper subgroup of $\varphi^{-1}(\varphi(S_H)^-)$ even if φ is a dense embedding or $\varphi = \text{Ad}_H$ and $G = \text{Ad}_H(H)^-$.

(C) $\varphi^{-1}(\varphi(S_H)^-)$ is possibly a proper subgroup of $\varphi^{-1}(S_G)$ even if φ is a dense embedding or $\varphi = \text{Ad}_H$ and $G = \text{Ad}_H(H)^-$.

(D) For dense embeddings $\varphi: H \to G$ the center $Z(G)$ might be strictly larger than $\varphi(Z(H))^-$, compare the remark after 2.11.

Moreover, we note that the example of 1.11 shows

(E) For dense embeddings $\varphi: H \to G$ the radical of G might be strictly larger than $\varphi(\mathrm{Rad}\, H)^-$.

3.4. EXAMPLE. Let $X = (\mathbb{R}^2 \otimes \mathbb{C}) \oplus \mathbb{R} \oplus \mathbb{R}$, the tensor product is taken over the reals. The group $SL_2(\mathbb{R}) \times \mathbb{R}$ acts on X by

$$(A, r) \cdot (u \otimes v, w, z) = (Au \otimes e^{ir}v, w, z + rw)$$

where $Au = (a_{11}u_1 + a_{12}u_2, a_{21}u_1 + a_{22}u_2)$ if $A = \binom{a_{11}a_{12}}{a_{21}a_{22}}$, $u = (u_1, u_2)$. We form the semidirect product H of $SL_2(\mathbb{R}) \times \mathbb{R}$ with X, i.e., the multiplication is given by

$$\begin{aligned}(A, r, u \otimes v, w, z)&(B, s, u' \otimes v', w', z')\\ &= (AB, r + s, B^{-1}u \otimes e^{-is}v + u' \otimes v', w + w', z + z' - sw).\end{aligned}$$

One quickly computes that $Z(H) = \{(0, 0, z) \in X \mid z \in \mathbb{R}\}$. Then clearly $S_H = \mathrm{Rad}(H) = \mathbb{R} \ltimes X$. Next, one observes that the adjoint group of H is closed, hence isomorphic to $H/Z(H) =: G$. The group G is a semidirect product of $SL_2(\mathbb{R}) \times \mathbb{R}$ with $Y = (\mathbb{R}^2 \otimes \mathbb{C}) \oplus \mathbb{R}$ where the action is given by

$$(A, r) \cdot (u \otimes v, w) = (Au \otimes e^{ir}v, w).$$

The center of G is the union of $\{(\mathrm{id}, 2\pi k, 0, w) \mid k \in \mathbb{Z}, w \in \mathbb{R}\}$ and $\{(-\mathrm{id}, (2k + 1)\pi, 0, w) \mid k \in \mathbb{Z}, w \in \mathbb{R}\}$, hence $S_G = \{\pm \mathrm{id}\} \times \mathbb{R} \ltimes Y$.

We see that $\mathrm{Ad}(S_H)^- = \mathrm{Ad}(S_H) = \mathrm{Ad}(\mathrm{Rad}\, H) = \mathrm{Rad}(G)$ is a proper subgroup of S_G and that $\mathrm{Ad}^{-1}(\mathrm{Ad}(S_H)^-)$ is a proper subgroup of $\mathrm{Ad}^{-1}(S_G)$; compare (A) and (C).

3.5. EXAMPLE. Let α and β be two real numbers which are linearly independent over the rationals. The group $SU(2) \times \mathbb{R}$ acts on $\mathbb{C}^3$ by

$$(A, r) \cdot (z_1, z_2, z_3) = (e^{i\alpha r}(a_{11}z_1 + a_{12}z_2), e^{i\alpha r}(a_{21}z_1 + a_{22}z_2), e^{i\beta r}z_3)$$

if $A = \binom{a_{11}a_{12}}{a_{21}a_{22}}$. Again we form the corresponding semidirect product $H = (SU(2) \times \mathbb{R}) \ltimes \mathbb{C}^3$. One quickly shows that the center of H is trivial, so $S_H = \mathrm{Rad}(H) = \mathbb{R} \ltimes \mathbb{C}^3$.

Also $SU(2) \times \mathbb{T} \times \mathbb{T}$ acts on $\mathbb{C}^3$ by

$$(A, t_1, t_2) \cdot (z_1, z_2, z_3) = (t_1(a_{11}z_1 + a_{12}z_2), t_1(a_{21}z_1 + a_{22}z_2), t_2 z_3),$$

and we form the semidirect product

$$G = (SU(2) \times \mathbb{T} \times \mathbb{T}) \ltimes \mathbb{C}^3.$$

The embedding $r \to (e^{i\alpha r}, e^{i\beta r})$ from $\mathbb{R}$ into $\mathbb{T} \times \mathbb{T}$ induces a dense embedding φ from H into G. The center $Z(G)$ consists of two points, $(-id, -1, 1, 0, 0, 0)$ is the non-trivial element. So, $\varphi(Z(H))^-$ is a proper subgroup of $Z(G)$; compare (D). Clearly, $S_G = (\{\pm id\} \times \mathbb{T} \times \mathbb{T}) \ltimes \mathbb{C}^3$. We see that $\varphi(S_H)^- = \mathrm{Rad}(G)$ is a proper subgroup of S_G and $\varphi^{-1}(\varphi(S_H)^-)$ is strictly contained in $\varphi^{-1}(S_G)$; compare (A) and (C).

The adjoint group of G is closed being a semidirect product of the (closed) unipotent group $\mathrm{Ad}_G(\mathbb{C}^3)$ and the compact group $\mathrm{Ad}_G(SU(2) \times \mathbb{T} \times \mathbb{T})$. The adjoint group $\mathrm{Ad}_G(G)$ is isomorphic to $G/Z(G)$. As we known from 2.5 the embedding φ induces an isomorphism R_φ from $\mathrm{Ad}_G(G)^- = \mathrm{Ad}_G(G) = G/Z(G)$ onto $\mathrm{Ad}_H(H)^-$. In other words, $\mathrm{Ad}_H(H)^-$ can be identified with $G/Z(G)$, and the composition ψ: $H \to G/Z(G)$ of φ with the quotient map $G \to G/Z(G)$ is essentially Ad_H: $H \to \mathrm{Ad}_H(H)^-$. Observe that ψ is injective as well. We see that $S_H = \mathrm{Rad}\, H = \mathbb{R} \ltimes \mathbb{C}^3$ is strictly contained in $\psi^{-1}(\psi(S_H)^-) = (\{\pm \mathrm{id}\} \times \mathbb{R}) \ltimes \mathbb{C}^3$; compare (B).

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