Annals of Mathematics, **140** (1994), 503–556

Unitary representations of Lie groups and operators of finite rank

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Besides being a principal object of study in harmonic analysis, the convolution algebra of L^1-functions on a locally compact group is traditionally used for investigating (unitary) group representations by methods of associative algebra. With the development of the theory of C^*-algebras this role was more and more taken over by the group C^*-algebras. A considerable amount of information concerning group C^*-algebras, in particular of Lie groups, has accumulated. This article may be regarded as part of a program to use the obtained insights and methods in order to derive results in terms of more classical objects, as L^1-functions or, in the context of Lie groups, even smooth functions, briefly, to prove regularity theorems.

To make this more concrete, let me describe two such regularity questions not to be treated in this paper. The topology on the unitary dual $\widehat{G}$ of a given locally compact group G can be characterized as follows. A set A in $\widehat{G}$ is closed if and only if for each $\pi \in \widehat{G} \setminus A$ there exists an $f \in C^*(G)$ such that $\pi(f) \neq 0$ and $\rho(f) = 0$ for all $\rho \in A$. One may ask whether one can always find an $f \in L^1(G)$ with these properties. It turns out that for some groups this is possible, for instance for groups with polynomially growing Haar measure; for others it is not, for instance for noncompact semisimple Lie groups. More information on this topic can be found in [2].

Secondly, as Pukanszky has shown, [40], each primitive quotient $C^*(G)/P$ of the group C^*-algebra of a connected Lie group G has a unique faithful trace tr_P. *A priori* it is not clear that there is an L^1-function f on G such that $0 < \mathrm{tr}_P(f) < \infty$. But Charbonnel has proved [5], [6], that one can find such an f even in $\mathcal{D}(G) = C_c^\infty(G)$.

In this article we shall solve the following problem in the affirmative. Let π be a continuous irreducible unitary representation of a connected Lie group H, and suppose that $\pi\left(C^*(H)\right)$ contains the compact operators on the representation space $\mathfrak{H}_\pi$; i.e., the norm closure of $\pi\left(L^1(H)\right)$ contains the compact operators. Is it true that $\pi\left(L^1(H)\right)$ contains an operator of rank one? Actually, we shall do somewhat better. We shall construct a smooth function f on H such that $\pi(f)$ is an operator of rank one and such that $f\,r$

is integrable for each representative function r of H. The smoothness of f is a minor point; this can be arranged for by convolving with test functions; see 3.2 below. More crucial is the decay of f. I guess that in some sense the obtained result is optimal. At least it is not true in general that one may choose f to be compactly supported. In case of the Heisenberg group and of the nonabelian two-dimensional Lie group this is not possible.

The above problem has some history. Clearly the semisimple case follows from the work of Harish-Chandra, from the "finiteness of the K-types"; see also (6.3) below. In this case one may indeed choose f to be compactly supported. The nilpotent case was settled by Dixmier, [9], using the symbolic calculus. It also follows from a paper of Howe, [26], where the image of the space of Schwartz functions on H under π was determined. The case of exponential groups was treated by Ludwig, [32], and myself, [37]. While Ludwig followed in some sense the work of Howe, my approach using some symbolic calculus and the symmetry of certain L^1-group algebras was more closely related to that of Dixmier. Recently, Du Cloux, [13], has successfully considered the case of, say, almost algebraic groups in creating a theory of "smooth representations" of such groups. Like the present article and like Ludwig he followed Howe's approach for nilpotent groups.

A major difficulty when treating this problem is how to *use effectively* the assumption that $\pi(C^*(H))$ contains the compact operators. To some extent one has to deal with all continuous irreducible unitary representations, whence the title. In doing so I learned a lot from some papers of Dixmier, [8], [10], [11], in particular the use of algebraic groups, and from some papers of Duflo, [14], [15], [16], in particular the use of the metaplectic representation. The difficulty is overcome by associating with each connected Lie group H a group in the so-called class $[MB]$; see Section 8. The associated group has, via Takai duality, more or less the same representation theory as the original group. It has the advantage that the orbits in the unitary dual of the nilradical are locally closed. The definition of the class $[MB]$ was arranged that way to contain the stabilizers of points in that unitary dual as well. By an inductive procedure the problem is finally reduced to two-step nilpotent Lie groups, not necessarily connected. According to [31], in this case one understands very well what the assumption on the existence of compact operators in the image means.

The paper is organized as follows. After a brief discussion of representative functions on groups the class $[MB]$ is introduced in the second section. Moreover, there is established the existence of certain cross sections which behave well with respect to representative functions. In the third section the main Theorem 3.1, the existence of operators of finite rank in the image of L^1, is formulated for groups in the class $[MB]$, together with some comments.

Sections 4 through 7 are devoted to the proof of that theorem. In the final section it is shown how connected Lie groups can be treated by means of groups in $[MB]$.

Let me finish this introduction with a technical remark. I shall always assume that the various left Haar (Lebesgue) measures are suitably normalized so that Weil's formula and the Fourier inversion formula hold true without additional constants. Since it is a long paper this remark is repeated occasionally.

1. Representative functions

In this introductory section we recall some elementary facts on representative functions needed later. More substantial information can be found in a series of papers by Hochschild and Mostow, [24] is the first, [25] is the last article of this series, and also in the books [21], [22], [34]. As we will later construct functions with a certain decay with respect to all representative functions, the real ones are good enough for our purposes: The real and the imaginary parts of a complex representative function are again representative functions, hence they do not deliver anything new. Of course, the following remarks apply to complex representative functions as well.

Let G be a topological group. The group G acts on the algebra $C(G)$ of continuous real-valued functions by left and right translations, $(\lambda(x)f)(y) = f(x^{-1}y)$, $(\rho(x)f)(y) = f(yx)$.

PROPOSITION 1.1. *Let f be a real-valued continuous function on the topological group G. Then the following are equivalent.*

(i) *The linear span of $\{\rho(x)f \mid x \in G\}$ is finite-dimensional.*

(ii) *The linear span of $\{\lambda(x)f \mid x \in G\}$ is finite-dimensional.*

(iii) *The linear span of $\{\lambda(x)\rho(y)f \mid x,y \in G\} = \{\rho(y)\lambda(x)f \mid x,y \in G\}$ is finite-dimensional.*

(iv) *There exist continuous functions $\alpha_1,\beta_1,\alpha_2,\dots,\alpha_n,\beta_n$ on G such that $f(xy) = \alpha_1(x)\beta_1(y) + \cdots + \alpha_n(x)\beta_n(y)$ for all $x,y \in G$.*

(v) *There exist functions $\alpha_1,\beta_1,\alpha_2,\dots,\beta_n$ on G such that $f(xy) = \alpha_1(x)\cdot\beta_1(y) + \cdots + \alpha_n(x)\beta_n(y)$ for all $x,y \in G$.*

(vi) *There exist a continuous representation ω of G in a finite-dimensional real vector space $\mathcal{V}$, a vector ξ in $\mathcal{V}$ and a vector η in the linear dual $\mathcal{V}'$ such that $f(x) = \langle\omega(x)\xi,\eta\rangle$ for all $x \in G$.*

Definition 1.2. Functions f satisfying (i) - (vi) are called (real) representative functions on G. Evidently, the collection of these functions forms an algebra.

Proof. Suppose that f satisfies (i). Choose a maximal linearly independent system $\alpha_1, \dots, \alpha_n$ in the set $\{\rho(x)f \mid x \in G\}$. Then $\alpha_1, \dots, \alpha_n$ is a basis of the span of $\{\rho(x)f \mid x \in G\}$, say $\mathcal{V}$. In particular, for each $y \in G$ there exist uniquely determined real coefficients $\beta_1(y), \dots, \beta_n(y)$ such that $\rho(y)f = \beta_1(y)\alpha_1 + \dots + \beta_n(y)\alpha_n$. Evaluation at x gives $f(x\,y) = \beta_1(y)\alpha_1(x) + \dots + \beta_n(y)\alpha_n(y)$. The functions α_j are continuous as translates of f. We want to see that the β_j are continuous, too. To this end define a map $Z: G \longrightarrow \mathbb{R}^n$ by $Z(x) = (\alpha_1(x), \dots, \alpha_n(x))$. We claim that $\{Z(x) \mid x \in G\}$ spans $\mathbb{R}^n$. If not, there would exist $m < n$ and vectors $W_1, \dots, W_m \in \mathbb{R}^n$, $W_j = (w_{j1}, \dots, w_{jn})$ as well as functions $\gamma_1, \dots, \gamma_m \colon G \longrightarrow \mathbb{R}$ such that $Z(x) = \sum_{j=1}^m \gamma_j(x) W_j$ for all $x \in G$, i.e., $\alpha_\ell = \sum_{j=1}^m w_{j\ell}\gamma_j$ contradicting the fact that the α's span a space of dimension n. Now choose $a_1, \dots, a_n \in G$ such that $Z(a_1), \dots, Z(a_n)$ is a basis of $\mathbb{R}^n$. For each $y \in G$ one gets a system of linear equations

$$f(a_j y) = \beta_1(y)\alpha_1(a_j) + \dots + \beta_n(y)\alpha_n(a_j)$$

for $j = 1, \dots, n$. The "unknown vector" $(\beta_1(y), \dots, \beta_n(y))$ is uniquely determined by this system. Since the functions $y \mapsto f(a_j y)$ are continuous, Cramer's rule shows the continuity of the β_j.

We proved (i) $\Longrightarrow$ (iv). Concerning (i) $\Longrightarrow$ (vi) let, as above, $\mathcal{V} = L_{\mathbb{R}}(\alpha_1, \dots, \alpha_n)$, define $\omega(x) \in \mathrm{GL}(\mathcal{V})$ by $\omega(x)\alpha = \rho(x)\alpha$ and define $\eta \in \mathcal{V}'$ by $\langle \alpha, \eta \rangle = \alpha(e)$. It is easy to see that $f(x) = \langle \omega(x)f, \eta \rangle$ for all $x \in G$. It remains to show that ω is continuous. The continuity of the β_j shows that the function $x \mapsto \omega(x)f$ from G into $\mathcal{V}$ is continuous because $\omega(x)f = \beta_1(x)\alpha_1 + \dots + \beta_n(x)\alpha_n$. Since G is a topological group then for each $a \in G$ the function $x \mapsto \omega(x\,a)f$ is continuous. But as $\omega(x\,a)f = \omega(x)(\omega(a)f)$ and as the vectors $\{\omega(a)f \mid a \in G\}$ span $\mathcal{V}$ it follows that ω is a continuous representation.

It is easy to see that (iv) implies (iii) and it is evident that (iv) implies (v). Hence we have (i) $\Longrightarrow$ (iii), (iv), (v), (vi). The opposite implications are clear; therefore, (i), (iii), (iv), (v) and (vi) are equivalent. Also (iii) $\Longrightarrow$ (ii) is clear. If f satisfies (ii) then $f^\vee, f^\vee(x) = f(x^{-1})$, satisfies (i), hence $f^\vee$ has property (iii) which implies that f satisfies (iii).

As a corollary of the proof one obtains the following lemma.

LEMMA 1.3. *Let X and Y be topological spaces, and suppose that $f\colon X \times Y \longrightarrow \mathbb{R}$ is a function, which is separately continuous in each variable and which is a tensor; i.e., there are functions $\alpha_1, \dots, \alpha_n\colon X \to \mathbb{R}$ and $\beta_1, \dots, \beta_n\colon Y \to \mathbb{R}$ with $f(x,y) = \alpha_1(x)\beta_1(x) + \dots + \alpha_n(x)\beta_n(y)$ for all $x \in X, y \in Y$. Then the α's and β's can be chosen as continuous functions.*

Proof. If n is taken to be minimal with respect to the possible representations of f then the $\alpha_1, \dots, \alpha_n$ and the $\beta_1, \dots, \beta_n$ are linearly independent in the linear space of all functions on X and Y, respectively. The argument above shows that there exist $a_1, \ldots, a_n \in X$ such that $(\alpha_j(a_k))$ is a nonsingular $n \times n$ matrix. Hence for each $y \in Y$ the $\beta_1(y), \ldots, \beta_n(y)$ are determined by the system

$$f(a_j, y) = \alpha_1(a_j)\beta_1(y) + \cdots + \alpha_n(a_j)\beta_n(y)$$

of linear equations, $1 \leq j \leq n$. As above Cramer's rule gives that the $\beta_1, \ldots, \beta_n$ are continuous. Of course, the analogous argument applies to the α's.

For later use we state the following easy lemma.

LEMMA 1.4. *Let G be a topological group, and let H be an open subgroup of finite index. Let f be a continuous function on G such that the right translates $\rho(h)f$, $h \in H$, span a finite-dimensional space. Then f is a representative function on G.*

Proof. Let $\mathcal{V} = \mathrm{L}_{\mathbb{R}}\{\rho(h)f; h \in H\}$ and let S be a set of representatives for the H-cosets, i.e., each element $x \in G$ can be uniquely written as $x = s\,h$ with $s \in S$, $h \in H$. Then every right translate $\rho(x)f = \rho(s\,h)f = \rho(s)\rho(h)f$ is contained in the finite-dimensional space $\sum_{s \in S} \rho(s)\mathcal{V}$.

This section is concluded by the following remark which later on will simplify some arguments.

Remark 1.5. For each finite-dimensional continuous representation ψ of G in the real vector space $\mathcal{V}$ there exists a positive representative function r dominating all the representative functions associated with ψ; i.e., for $\xi \in \mathcal{V}, \eta \in \mathcal{V}'$ there exists a constant C depending on ξ and η such that

$$|\langle \psi(x)\xi, \eta \rangle| \leq \mathrm{C}\, r(x)$$

for all $x \in G$. Moreover, r may be chosen to be submultiplicative; i.e., $r(x\,y) \leq r(x)r(y)$ for all $x, y \in G$.

Proof. Choose a basis of $\mathcal{V}$ and define $\psi_{jk} : G \longrightarrow \mathbb{R}$, $1 \leq j, k \leq \dim \mathcal{V} =: n$, to be the entries of the matrix associated with $\psi(x)$ with respect to the chosen basis. Then let $s = \sum_{j,k=1}^{n} \psi_{jk}^2 + \overset{\vee}{\psi}{}_{jk}^2$ which evidently is a representative function of G. If the chosen basis is declared to be an orthonormal basis for a euclidean structure on $\mathcal{V}$ then $s(x)$ is essentially the sum of the squares of the operator norms of $\psi(x)$ and $\psi(x)^{-1}$. More precisely, there is a positive constant E such that

$$\mathrm{E}\, s(x) \leq \|\psi(x)\|^2 + \|\psi(x)^{-1}\|^2 \leq \frac{1}{\mathrm{E}} s(x)$$

for all x in G. From this interpretation of s it follows easily that s dominates all representative functions associated with ψ and that s is submultiplicative up to a positive constant D: $s(xy) \leq \mathrm{D}\, s(x)\, s(y)$ for all $x, y \in G$. Replacing s by $r = \mathrm{D}\, s$ we find a representative function which is submultiplicative and still dominates the representative functions associated with ψ.

The remark means in particular that some of the representative functions are so-called weight functions; for more detail see, for example, [41].

2. Cross sections

Later, cross sections with certain properties with respect to representative functions will play an important role. Also we introduce here the crucial class $[MB]$ of auxiliary non-connected Lie groups which will be used to study connected groups.

PROPOSITION 2.1. *Let H be a connected Lie group with Lie algebra $\mathfrak{h}$. Suppose that H admits a locally faithful continuous representation in a finite-dimensional real vector space. Let $\mathfrak{w}$ be an ideal in $\mathfrak{h}$ contained in the nilradical of $\mathfrak{h}$, i.e., in $\mathfrak{n} = [\mathfrak{h},\mathfrak{r}] = [\mathfrak{h},\mathfrak{h}] \cap \mathfrak{r}$ where $\mathfrak{r}$ denotes the radical of $\mathfrak{h}$. Then:*

(a) *The image $W = \exp \mathfrak{w}$ of $\mathfrak{w}$ under the exponential map is a closed normal subgroup of H, and* exp *induces a diffeomorphism from $\mathfrak{w}$ onto W.*

Moreover, there exists a C^∞-cross section $s\colon H/W \to H$ with the following properties:

(b) *There exist $A_1,\ldots,A_n \in \mathfrak{w}$ and real functions $\alpha_1,\ldots,\alpha_n$, $\beta_1,\ldots,\beta_n$ on H/W such that $s(xy) = s(x)s(y)\exp\left(\sum_{j=1}^n \alpha_j(x)\beta_j(y)A_j\right)$ for all $x,y \in H/W$.*

(c) *If $p\colon H \to H/W$ denotes the quotient map then $s(xp(z)) = s(x)s(p(z))$ for all $x \in H/W$ and all z in the center $Z(H)$ of H. Moreover, $s(p(e)) = e$ and $s(p(Z(H))) \subset Z(H)$. Hence s induces a homomorphism from $p(Z(H))$ into $Z(H)$.*

Proof. Let $\psi\colon H \to \mathrm{GL}(\mathcal{V})$ be a continuous representation in the finite-dimensional real space $\mathcal{V}$ with discrete central kernel. Then $H/\ker\psi$ allows a faithful representation with closed image; see [21, Chap. XVIII]. Hence we may assume from the beginning that $\psi(H)$ is closed in $\mathrm{GL}(\mathcal{V})$. Since $\mathfrak{w}$ is contained in $\mathfrak{n}$ the differential $\mathrm{d}\psi$ induces an isomorphism from $\mathfrak{w}$ onto a Lie algebra consisting of nilpotent matrices. From this fact (a) follows easily. Moreover, ψ induces an isomorphism from W onto $\psi(W)$.

To construct the cross section s we form the "scindable hull" $\mathfrak{g}$ of $\mathrm{d}\psi(\mathfrak{h})$ in $\mathrm{End}(\mathcal{V})$ in the sense of [4, Chap. VII, § 5]. It is the smallest Lie algebra in $\mathrm{End}(\mathcal{V})$ containing $\mathrm{d}\psi(\mathfrak{h})$ such that with each element X in $\mathfrak{g}$ the semi-simple and the nilpotent parts in the additive Jordan decomposition of X are contained in $\mathfrak{g}$. The algebra $\mathfrak{g}$ is a semidirect sum $(\mathfrak{s} \times \mathfrak{t}) \ltimes \mathfrak{u}$ where $\mathfrak{s}$ is a semisimple algebra, $\mathfrak{t}$ consists of semisimple linear transformations on $\mathcal{V}$ such that $[\mathfrak{t},\mathfrak{t}] = 0 = [\mathfrak{t},\mathfrak{s}]$, and $\mathfrak{u}$ consists of nilpotent transformations. In the notation of [4], $\mathfrak{u}$ is just $\mathfrak{n}_{\mathcal{V}}$, is uniquely determined, and is the set of all nilpotent transformations in the radical ($= \mathfrak{t} \ltimes \mathfrak{u}$) of $\mathfrak{g}$. Of course, $d\psi(\mathfrak{w})$ is contained in $\mathfrak{u}$, and $d\psi(\mathfrak{w})$ is an ideal in $\mathfrak{g}$. Let $G \leq \mathrm{GL}(\mathcal{V})$ be the Lie group corresponding to $\mathfrak{g}$ (in its internal Lie group topology). The group G contains $\psi(H)$ as a closed (co-abelian) normal subgroup.

The image $\psi(Z(H))$ of the center $Z(H)$ of H is contained in the center of G because if $\psi(x)$, $x \in H$, acts, via conjugation, trivially on $d\psi(\mathfrak{h})$ it acts on $\mathfrak{g}$ trivially as well by the construction of the scindable hull. Also, if for some $x \in H$ the image $\psi(x)$ is central in G then x is in $Z(H)$ because ψ is locally faithful. Hence $Z(H) = \psi^{-1}(Z(G))$. Let $S = \langle \exp \mathfrak{s} \rangle$, $T = \exp \mathfrak{t}$ and $U = \exp \mathfrak{u}$. Then G is a semidirect product of ST and U, and in particular, the groups ST and U have a trivial intersection. The center $Z(G)$ consists of all products zu where z is in the center of ST and z acts by conjugation trivially on U (and on $\mathfrak{u}$); i.e., z is in the center of G, and u is in the center of U and fixed under conjugation with elements in ST. The reason for this is that the center of ST acts by conjugation semisimply on $\mathfrak{u}$ while U acts unipotently.

Denote by $\mathfrak{z}(\mathfrak{u})^{ST}$ the space of elements in the center $\mathfrak{z}(\mathfrak{u})$ of $\mathfrak{u}$ which are fixed under the action of ST, and choose a subspace $\mathfrak{a}$ of $\mathfrak{z}(\mathfrak{u})^{ST}$ such that $\mathfrak{z}(\mathfrak{u})^{ST} + d\psi(\mathfrak{w})$ is a direct sum of $\mathfrak{a}$ and $d\psi(\mathfrak{w})$. Moreover, choose an ST-invariant vector subspace $\mathfrak{v}$ of $\mathfrak{u}$ such that $\mathfrak{u}$ is a direct sum of $\mathfrak{v}$ and $\mathfrak{a}+d\psi(\mathfrak{w})$. This is possible because $\mathfrak{u}$ is a semisimple ST-module under conjugation. Then put $C = ST \exp \mathfrak{v} \exp \mathfrak{a}$. This is a set of representatives for the $\psi(W)$-cosets in G. More precisely, by the well known properties of the exponential map for nilpotent groups (applied to U), for each $x \in H$ there exist *unique* elements $c(x) \in C$ and $\gamma(x) \in W$ such that $\psi(x) = c(x)\psi(\gamma(x))$; moreover γ is a C^∞-map. The desired s is given as $s(x) = x\gamma(x)^{-1}$ which is actually a function on H/W. Evidently, s is a cross section against $p\colon H \to H/W$. Very often we will consider s as a function on H as well.

Concerning (b), let x_1, x_2 be given elements in G. We may write

$$\psi(s(x_j)) = b_j \exp X_j \exp Y_j = c(x_j)$$

with uniquely determined $b_j \in ST$, $X_j \in \mathfrak{v}$ and $Y_j \in \mathfrak{a}$. We know that $s(x_1x_2)$ is of the form $s(x_1)s(x_2)w$ with a certain $w \in W$ which we want to compute. On the one hand, $\psi(s(x_1x_2))$ is in C and on the other hand, $\psi(s(x_1x_2))$ equals $\psi(s(x_1))\psi(s(x_2))\psi(w)$. This information determines w. Now

$$\begin{aligned}\psi(s(x_1))\psi(s(x_2))\psi(w) &= b_1 \exp X_1 \exp Y_1 b_2 \exp X_2 \exp Y_2 \psi(w)\\ &= b_1 b_2 \exp\left(b_2^{-1} X_1 b_2\right) \exp\left(X_2\right) \exp\left(Y_1 + Y_2\right) \psi(w)\end{aligned}$$

because Y_1 is contained in $\mathfrak{z}(\mathfrak{u})^{ST}$. Using the Campbell-Hausdorff formula one finds a uniquely determined polynomial function P on $\mathfrak{v} \times \mathfrak{v}$ with values in $\mathfrak{w}$ such that

$$\exp V_1 \exp V_2 \in \exp \mathfrak{v} \exp \mathfrak{v} \exp(d\psi(P(V_1, V_2)))$$

for $V_1, V_2 \in \mathfrak{v}$. Since $\exp(d\psi(Z)) = \psi(\exp Z)$ for $Z \in \mathfrak{w}$ one obtains $w = \exp(-P(b_2^{-1}X_1 b_2, X_2))$. Choosing bases in $\mathfrak{v}$ and $\mathfrak{w}$ one readily sees that the function $(x_1, x_2) \mapsto -P(b_2^{-1}X_1 b_2, X_2)$ has the form claimed in the proposition.

Concerning (c), we first observe that for $h \in Z(H)$ the element $c(h)$ is in $Z(G)$ and $\gamma(h)$ is in $Z(H)$. From the above discussion on the centers it follows that $\psi(h)$ may be written as $\psi(h) = b \exp Z$ with $b \in (ST) \cap Z(G)$ and $Z \in \mathfrak{z}(\mathfrak{u})^{ST}$. The element Z decomposes as $Z = Z_a + Z_w$ with $Z_a \in \mathfrak{a} \subset \mathfrak{z}(\mathfrak{u})^{ST}$ and $Z_w \in \mathfrak{z}(\mathfrak{u})^{ST} \cap d\psi(\mathfrak{w})$. But $c(h) = b \exp Z_a$ is in $Z(G)$ and $\gamma(h)$ is in $Z(H)$ because $\psi(\gamma(h)) = \exp Z_w$ is in $Z(G)$. Moreover, $Cc(h) = C = c(h)C$. The claim (c) is equivalent to the equation $s(xh) = s(x)s(h)$ for $x \in H$, $h \in Z(H)$. By definition, this equation is equivalent to $xh\gamma(xh)^{-1} = x\gamma(x)^{-1}h\gamma(h)^{-1}$ or, as h is central, to $\gamma(hx) = \gamma(h)\gamma(x)$. But from $\psi(x) = c(x)\psi(\gamma(x))$ and $\psi(h) = c(h)\psi(\gamma(h))$ follows

$$\psi(hx) = \psi(h)\psi(x) = c(h)\psi(\gamma(h))c(x)\psi(\gamma(x)) = c(h)c(x)\psi(\gamma(h))\psi(\gamma(x)),$$

as $\psi(\gamma(h))$ is in $Z(G)$, and hence $\psi(\gamma(h))\psi(\gamma(x)) = \psi(\gamma(hx))$ and $c(hx) = c(h)c(x)$ because $c(h)c(x) \in C$.

Definition 2.2. We denote by $[MB]$ the class of Lie groups G which are semidirect products $G = B \ltimes M$ with the following properties:

(i) The group B is a compactly generated abelian Lie group.

(ii) The connected component M_0 of M allows a locally faithful representation in a finite-dimensional real vector space.

(iii) The adjoint group $\mathrm{Ad}(M_0)$ in $\mathrm{Aut}(\mathfrak{m})$, $\mathfrak{m}$ = the Lie algebra of M, is the connected component of an algebraic group.

(iv) The commutators [M,B] are central in G.

(v) The commutators [M,M] are contained in the B-fixed point set M^B.

(R) There exists a closed subgroup D of M, central in G, such that G/D is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ for a certain n.

(Z) There exists an open subgroup L of M, invariant under the action of B such that M/L is finite, $L = M_0 Z(L)$ and L/M_0 is a finitely generated abelian group.

Remarks 2.3.

(a) Condition (iii) is, of course, a local property, i.e., a property of the Lie algebra $\mathfrak{m}$. Actually, by a theorem of Goto's, see [7, Chap. V, § 5, p. 336] or

[23], it means that $\mathfrak{m}$ is the Lie algebra of a real linear algebraic group; i.e., M_0 is locally isomorphic to such a group.

(b) The properties (i) through (Z) are somewhat redundant. For instance it follows from (R) that B is necessarily a compactly generated Lie group. Moreover, the above properties are certainly not minimal for the arguments to go through in the proof of the main Theorem 3.1. For instance, I am almost sure that one can circumvent property (ii). Its presence simplifies some proofs and eliminates tedious discussion of a number of uninteresting cases. Property (ii) is not shared by the Heisenberg group with "compactified center." In any case the class $[MB]$ is good enough to give the theorem for connected Lie groups – and this is the justification for its existence.

(c) From (i), (iv) and (v) it follows that all commutators [G,G] are contained in M^B.

(d) For examples of such groups see the final section.

In the next theorem we extend the cross sections of Proposition 2.1 to cross sections for groups in the class $[MB]$.

THEOREM 2.4. *Let $G = B \ltimes M$ be a Lie group satisfying* (i), (ii), (iv), (v) *and* (Z) *of Definition* 2.2. *Let $\mathfrak{w}$ be a subspace of the nilradical of $\mathfrak{m}$, invariant under the adjoint action of G. Then the following hold true.*

(a) *The image $W = \exp \mathfrak{w}$ is a closed normal subgroup of G, via* exp *diffeomorphic to $\mathfrak{w}$.*

(b) *There exists a C^∞-cross section $\sigma\colon G/W \to G$ such that for each representative function r on G the composite function $r \circ \sigma$ is a representative function on G/W. Moreover, $\sigma(e) = e$, and when σ is viewed as a map from G into G then $\sigma(Z(M)) \subset Z(M)$ and $\sigma(xz) = \sigma(x)\sigma(z)$ for $x \in G$, $z \in Z(M)$.*

(c) *If in addition G satisfies* (iii) *and* (R), *i.e., G belongs to $[MB]$, then $G/W = B \ltimes M/W$ belongs to $[MB]$, too. Actually, if D is a central subgroup of G as in* (R) *then DW is closed, DW/W is central in G/W and G/DW is isomorphic to a closed subgroup of some* $\mathrm{GL}_m(\mathbb{R})$, *i.e., $D' = DW/W$ has the required properties with respect to $B \ltimes M/W$. Also there is a C^∞-cross section $\tau\colon G/WD \to G/D$ such that $r \circ \tau$ is a representative function of G/WD for each representative function r of G/D.*

Proof. Claim (a) was shown in Proposition 2.1. Observe that part (ii) of Definition 2.2 gives that $H = M_0$ allows a locally faithful representation. The assumption that $\mathfrak{w}$ is G-invariant guarantees that W is normal in G. To prove (b) we extend a cross section $s\colon M_0/W \to M_0$ with the properties of Proposition 2.1 step by step. First we choose a subgroup $L = M_0Z(L)$ of M as in (Z). Without loss of generality we may assume that $Z(M)$ is contained in L.

When the cross section s is viewed as a function on M_0, constant on W-cosets, it induces a continuous homomorphism $Z(M_0) \to Z(M_0)$. This homomorphism delivers a continuous homomorphism $\gamma\colon Z(M_0)/Z(M_0)\cap W \to Z(M_0)$ with $s(z) = \gamma(z \cdot (Z(M_0) \cap W)) \in z(Z(M_0) \cap W)$ for $z \in Z(M_0)$. Since $\mathfrak{w}$ is contained in the nilradical, the group $\mathrm{Ad}_{M_0}(W)$ consists of unipotent linear transformations. In particular, $\mathrm{Ad}_{M_0}(W)$ is a simply connected group, which implies that $Z(M_0) \cap W = W \cap \ker \mathrm{Ad}_{M_0}$ is a vector group. From this fact one easily deduces that γ can be extended, there exists a continuous homomorphism $\varepsilon\colon Z(L)/Z(M_0) \cap W \to Z(L)$ such that $\varepsilon(z(Z(M_0) \cap W)) \in z(Z(M_0) \cap W)$ for $z \in Z(L)$ and $\varepsilon(z(Z(M_0) \cap W)) = \gamma(z(Z(M_0) \cap W) = s(z)$ for $z \in Z(M_0)$. Viewing ε as a map on $Z(L)$ we define $\sigma\colon B \ltimes L \to B \ltimes L$ by

$$\sigma(bxz) = bs(x)\varepsilon(z) \quad \text{for } b \in B,\ x \in M_0 \text{ and } z \in Z(L).$$

Using the properties of s and ε we easily check that σ is well-defined, smooth, and constant on W-cosets, and that σ induces a cross section against the quotient map $B \ltimes L \to B \ltimes (L/W)$.

Next we show that σ differs from a homomorphism in a controlled way as in (b) of Proposition 2.1. By (iv) of 2.2 the commutator $b^{-1}yby^{-1}$, $b \in B$, $y \in L$, is central in G; in particular such a commutator is contained in $Z(L)$. Moreover, $b^{-1}yby^{-1}$ equals $b^{-1}\sigma(y)b\sigma(y)^{-1}$ because $\sigma(y)$ differs from y by an element in W, but W is contained, by (v) of 2.2, in the B-fixed point set. By construction of ε the element

$$E(b,y) = \varepsilon(b^{-1}yby^{-1})^{-1}b^{-1}yby^{-1} = \varepsilon(b^{-1}yby^{-1})^{-1}b^{-1}\sigma(y)b\sigma(y)^{-1}$$

is contained in $W \cap Z(M_0)$. The map E on $B \times L$ factors through $B \times (L/L^B)$. By (v) of 2.2, L/L^B is an abelian group, actually a compactly generated abelian Lie group. We conclude that E defines a continuous bicharacter on $B \times (L/L^B)$ with values in $W \cap Z(M_0)$, which is a vector group. Such bicharacters are essentially given by matrices; in particular, we find functions $\alpha_1', \dots, \alpha_m'$ on L, $\beta_1', \dots, \beta_m'$ on B and elements $A_1', \dots, A_m'$ in $\mathfrak{w} \cap \mathfrak{z}(\mathfrak{m})$ such that

$$E(b,y) = \exp \sum_{j=1}^{m} \alpha_j'(y)\beta_j'(b)A_j'.$$

For each element $y \in L = M_0Z(L)$ we choose an element $t(y) \in M_0$ such that $t(y)^{-1}y \in Z(L)$. According to (b) of 2.1 we choose elements $A_1, \dots, A_n$ in $\mathfrak{w}$ and functions $\alpha_1, \dots \alpha_n$, $\beta_1, \dots, \beta_n$ such that

$$s(x_1x_2) = s(x_1)s(x_2) \exp \sum_{j=1}^{n} \alpha_j(x_1)\beta_j(x_2)A_j$$

for $x_1, x_2 \in M_0$. Now let $b_1, b_2 \in B$ and $y_1, y_2 \in L$ be given. We want to compare $\sigma(b_1y_1b_2y_2)$ with $\sigma(b_1y_1)\sigma(b_2y_2)$, hence define $w \in W$ by $\sigma(b_1y_1b_2y_2) = \sigma(b_1y_1)\sigma(b_2y_2)w$. Then we get

$$\sigma(b_1b_2b_2^{-1}y_1b_2y_2) = b_1b_2\sigma(b_2^{-1}y_1b_2y_1^{-1}y_1y_2) = b_1\sigma(y_1)b_2\sigma(y_2)w,$$

hence

$$\sigma(b_2^{-1}y_1b_2y_1^{-1}y_1y_2) = b_2^{-1}\sigma(y_1)b_2\sigma(y_2)w = b_2^{-1}\sigma(y_1)b_2\sigma(y_1)^{-1}\sigma(y_1)\sigma(y_2)w.$$

Since $b_2^{-1}y_1b_2y_1^{-1}$ is central in $B \ltimes L$ and since σ is multiplicative with respect to central elements we find

$$\begin{aligned}\sigma(y_1y_2) &= \varepsilon(b_2^{-1}y_1b_2y_1^{-1})^{-1}b_2^{-1}\sigma(y_1)b_2\sigma(y_1)^{-1}\sigma(y_1)\sigma(y_2)w \\ &= E(b_2, y_1)\sigma(y_1)\sigma(y_2)w = \sigma(y_1)\sigma(y_2)wE(b_2, y_1).\end{aligned}$$

Now write $y_j = t(y_j)z_j$ with $z_j \in Z(L)$ for $j = 1, 2$. Then

$$\begin{aligned}\sigma(y_1y_2) &= \varepsilon(z_1z_2)s(t(y_1)t(y_2)) \\ &= \varepsilon(z_1z_2)s(t(y_1))s(t(y_2))\exp\sum_{j=1}^{n}\alpha_j(t(y_1))\beta_j(t(y_2))A_j\end{aligned}$$

and

$$\sigma(y_1)\sigma(y_2)wE(b_2, y_1) = \varepsilon(z_1)s(t(y_1))\varepsilon(z_2)s(t(y_2))wE(b_2, y_1)$$

whence

$$\begin{aligned}w &= E(b_2, y_1)^{-1}\exp\sum_{j=1}^{n}\alpha_j(t(y_1))\beta_j(t(y_2))A_j \\ &= \exp\left(\sum_{j=1}^{n}\alpha_j(t(y_1))\beta_j(t(y_2))A_j - \sum_{j=1}^{m}\alpha_j'(y_1)\beta_j'(b_2)A_j'\right).\end{aligned}$$

It follows that there exist elements $Q_1, \ldots, Q_k \in \mathfrak{w}$ and real-valued functions $\lambda_1, \ldots, \lambda_k$, $\mu_1, \ldots, \mu_k$ on $B \ltimes L$ such that

$$(*) \qquad \sigma(uv) = \sigma(u)\sigma(v)\exp\sum_{j=1}^{k}\lambda_j(u)\mu_j(v)Q_j$$

for all $u, v \in B \ltimes L$. Even though the construction of the λ_j's and μ_j's does not make them continuous (the function t might be discontinuous) it follows from Lemma 1.3 that one can arrange for them to be continuous.

Finally we extend the cross section σ in the most obvious way to the whole of G. Choose any set F of representatives for the $B \ltimes L$-cosets in G; i.e., each element in G can be uniquely written as fu with $f \in F$ and $u \in B \ltimes L$. Then define $\sigma\colon G \to G$ by $\sigma(fu) = f\sigma(u)$. Clearly, σ delivers a C^∞-cross section $G/W \to G$. We are left to show that $r \circ \sigma$ is a representative function for each function r on G which is of the form $r(x) = \langle \psi(x)\xi, \eta\rangle$ for some continuous finite-dimensional representation ψ of G in $\mathcal{V}$, $\xi \in \mathcal{V}$ and $\eta \in \mathcal{V}'$. By Lemma 1.4 it is sufficient to show that the right translates of $r \circ \sigma$ with elements $u_0 \in B \ltimes L$ are contained in a finite-dimensional space. This means that we have to consider the following functions in $x = fu$, $f \in F$, $u \in B \ltimes L$:

$$\begin{aligned} r(\sigma(xu_0)) &= r(f\sigma(uu_0)) \\ &= \langle \psi(f\sigma(u))\psi(\sigma(u_0))\psi\left(\exp\left(\sum_{j=1}^{k} \lambda_j(u)\mu_j(u_0)Q_j\right)\right)\xi, \eta\rangle \\ &= \langle \psi(\sigma(x)\sigma(u_0)) \exp\left(\sum_{j=1}^{k} \lambda_j(u)\mu_j(u_0) d\psi(Q_j)\right)\xi, \eta\rangle \end{aligned}$$

by means of $(*)$.

But as $\mathfrak{w}$ is contained in the nilradical the above exponential is actually a polynomial. Hence there exist endomorphisms $P_1, \dots, P_q$ on $\mathcal{V}$ and functions $\kappa_1, \dots, \kappa_q$, $\nu_1, \dots, \nu_q$ on $B \ltimes L$ such that

$$\exp\left(\sum_{j=1}^{k} \lambda_j(u)\mu_j(u_0) d\psi(Q_j)\right) = \sum_{j=1}^{q} \kappa_j(u)\nu_j(u_0)P_j$$

for all $u, u_0 \in B \ltimes L$. Extending κ_j to G by $\widetilde{\kappa}_j(fu) = \kappa_j(u)$ one obtains

$$r(\sigma(xu_0)) = \sum_{j=1}^{q} \widetilde{\kappa}_j(x)\langle \psi(\sigma(x))\psi(\sigma(u_0))\nu_j(u_0)P_j\xi, \eta\rangle;$$

i.e., each of the functions $x \to r(\sigma(xu_0))$ is of the form

$$x \longmapsto \sum_{j=1}^{q} \widetilde{\kappa}_j(x)\langle \psi(\sigma(x))\xi_j,\ \eta\rangle$$

with vectors $\xi_1, \dots, \xi_q \in \mathcal{V}$. This set of functions forms a finite-dimensional space.

ad (c) Except for (ii), (iii) and (R) all other properties of the groups in [MB] are evidently satisfied by $B \ltimes (M/W)$. Concerning (iii) it is enough to observe that the adjoint group of $(M/W)_0 = M_0/W$ is obtained from $\mathrm{Ad}_{\mathfrak{m}}(M_0)$ by taking the induced transformations in $\mathfrak{m}/\mathfrak{w}$. Also it is a general theorem, see e.g., [23], that the quotient of an algebraic Lie algebra with respect to an ideal in the nilradical is again an algebraic Lie algebra.

Let D be a closed subgroup of M, central in G, and let $\psi\colon G \to \mathrm{GL}_n(\mathbb{R})$ be a continuous homomorphism with closed image and $\ker\psi = D$. Since $\psi(W)$ is a closed (simply connected) subgroup of $\mathrm{GL}_n(\mathbb{R})$, the group $WD = \psi^{-1}(\psi(W))$

is closed in G. The group $G/D = B \ltimes (M/D)$ evidently satisfies the properties (i), (ii), (iv), (v) and (Z) of 2.2. The Lie algebra of $WD/D \triangleleft G/D$ fulfills the hypothesis of the theorem. Applying part (b) of the theorem to the quotient map $G/D \to G/DW$ we find a cross section $\tau\colon G/WD \to G/D$ with the properties described there. In particular, if $\psi_{ik}(x)$, $x \in G$, are the entries of $\psi(x)$ then there exist representations ρ_{ik} of G/WD in real vector spaces $\mathcal{V}_{ik}$ and vectors $\xi_{ik} \in \mathcal{V}_{ik}$ and $\eta_{ik} \in \mathcal{V}'_{ik}$ such that

$$\psi_{ik}(\tau(\dot{x})) = \langle \rho_{ik}(\dot{x})\xi_{ik}, \eta_{ik}\rangle$$

for $\dot{x} \in G/WD$. Forming the direct sum $\rho = \bigoplus_{i,k} \rho_{ik}$ we see easily that ρ is a faithful representation of G/WD and its image is closed.

Similarly, even more easily, one can show that $(M/W)_0 = M_0/W$ has property (ii).

By the way, both (ii) and (R), can be verified for G/W along the following lines. The group $\psi(W)$, ψ as above, is an algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$. The normalizer H of $\psi(W)$ is an algebraic group, too, containing $\psi(G)$. The quotient $H/\psi(W)$ exists as an affine algebraic group, and it contains $\psi(G)/\psi(W)$ as a closed (in the euclidean topology) subgroup.

3. Statement of the main theorem and remarks

THEOREM 3.1. *Let G be a Lie group in the class* [*MB*], *and let π be a continuous irreducible unitary representation of G such that $\pi(C^*(G))$ contains the compact operators. Suppose that a central subgroup D of G according to* (R) *of Definition* 2.2 *is chosen. Then there exists a C^∞-function f on G such that*

(a) $\int_G |f(x)r(x)|\mathrm{d}x < \infty$ *for all representative functions r of G/D, and*

(b) $\pi(f)$ *is an orthogonal projection of rank* 1.

The following remarks will be used at several places. Their content is, roughly speaking, that to construct functions f with L^1-estimates with respect to representative functions and to construct functions with uniform estimates with respect to representative functions, such that $\pi(f)$ is an operator of rank 1, are more or less equivalent tasks. Also smoothness is no problem at all, since this can be arranged for by convolving with test functions.

Remarks 3.2. These remarks apply to a general Lie group G with a given central subgroup D such that G/D is isomorphic to a closed subgroup of some $\mathrm{GL}_n(\mathbb{R})$.

(A) Let γ be a unitary character of D. If a measurable function g on G satisfies $g(xz) = g(x)\gamma(z)^{-1}$ for all $z \in D$ and $x \in G$ and $\sup_{x\in G/D} |g(x)r(x)| < \infty$

for all representative functions r of G/D (observe that $|g|$ is a function o G/D) then the integral

$$\int_{G/D} |g(x)r(x)|\mathrm{d}x$$

is finite for all representative functions on G/D.

For the next three remarks let π be a continuous irreducible unitary rep resentation of G such that $\pi(z) = \gamma(z)\,\mathrm{Id}$ for $z \in D$.

(B) The following are equivalent:

(1) There exists a measurable function g on G with the propertie described in (A) such that

$$\pi(g) = \int_{G/D} g(x)\pi(x)\mathrm{d}x$$

is a nonzero operator of finite rank. Observe that the integrand $g(x)\pi(x)$ i actually a function on G/D by the transformation property of g.

(2) There exists a measurable function h on G such that $h(xz)$ $h(x)\gamma(z)^{-1}$ for all $z \in D$ and $x \in G$, such that

$$\int_{G/D} |h(x)r(x)|\mathrm{d}x < \infty$$

for all representative functions r of G/D such that $\pi(h)$ is a nonzero operato of finite rank.

If (1) or (2) is satisfied then there exists a C^∞-function k on G wit $k(xz) = k(x)\gamma(z)^{-1}$ for all $x \in G, z \in D$ such that

(a′) $\int_{G/D} |k(x)r(x)|\mathrm{d}x < \infty$ for all representative functions r on G/D,

(b′) $\pi(k)$ is an orthogonal projection of rank one,

(c′) $\sup_{x\in G/D} |k(x)r(x)| < \infty$ for all representative functions on G/D,

(d′) All derivatives $X * k * Y$ where X, Y are in the enveloping algebra $\mathfrak{U}\mathfrak{g}$ satisfy (a′) and (c′) as well.

(C) If there exists a continuous function f on G satisfying (a) and (b) as i the theorem then (1) and (2) are satisfied, hence there exist functions k a above.

(D) If (1) or (2) holds true then there exists a C^∞-function f satisfying (a and (b) of the theorem and in addition

(c) $\sup_{x\in G}|f(x)r(x)| < \infty$ for all representative functions r on G/D, and

(d) all derivatives $X * f * Y$, where $X, Y \in \mathfrak{U}\mathfrak{g}$, satisfy (a) and (c).

Combining (C) and (D) one sees that one might add the properties (c and (d) in the statement of the theorem.

Proof of (A). Let ψ be an isomorphism from G/D onto a closed subgroup H of $\mathrm{GL}_n(\mathbb{R})$. Define β on $\mathrm{GL}_n(\mathbb{R})$ by

$$\beta(x) = \{1 + \sum_{i,k=1}^{n} x_{ik}^2\}^m (\det x)^{-n}$$

where m is some large positive integer and x_{ik} are the entries of the matrix x. Since the left Haar measure on $\mathrm{GL}_n(\mathbb{R})$ is given by $(\det x)^{-n}\mathrm{d}x_{11}\dots\mathrm{d}x_{nn}$, the integral over $\frac{1}{\beta}$ equals

$$\int \{1 + \sum_{i,k=1}^{n} x_{ik}^2\}^{-m} \mathrm{d}x_{11}\dots\mathrm{d}x_{nn}$$

which is finite if m is chosen large enough – and this is assumed from now on.

Hence for almost all x in $\mathrm{GL}_n(\mathbb{R})$ the integral

$$\int_H \frac{1}{\beta(xu)}\Delta(u)\delta(u)^{-1}\mathrm{d}u$$

is finite where $\mathrm{d}u$ denotes the Haar measure of H, Δ and δ are the modular functions of $\mathrm{GL}_n(\mathbb{R})$ and H, respectively; see e.g., [1, p. 95]. Pick such an x. Then evidently the function r_0 on G/D given by $r_0(u) = \beta(x\,\psi(u))\Delta(\psi(u))^{-1}\cdot \delta(\psi(u))$ is a (positive) representative function on G/D satisfying

$$\int_{G/D} r_0(u)^{-1}\mathrm{d}u < \infty.$$

Now let r be any representative function on G/D. Then $|g(x)r(x)| = |g(x)r(x)r_0(x)|r_0(x)^{-1}$ for all $x \in G$. Since $|g\,r\,r_0|$ is uniformly bounded on G/D by assumption and $\frac{1}{r_0}$ is integrable we conclude that $|g(x)r(x)|$ is integrable over G/D.

Proof of (B). The implication (1) $\Longrightarrow$ (2) is obvious in view of (A). One may simply take $h = g$. Actually, (A) is necessary to ensure that $\pi(g)$ exists.

To prove (2) $\Longrightarrow$ (1) let h be as in (2), take any compactly supported continuous function φ on G and let $g = \varphi * h$. Then g satisfies $g(xz) = g(x)\gamma(z)^{-1}$ for all $x \in G, z \in D$. Let r be a representative function of G/D. We want to show that $|g\,r|$ is uniformly bounded. In view of Remark 1.5 we may suppose that r is positive and submultiplicative. Hence there exists a positive constant C such that

$$r(sy) \leq \mathrm{C}\, r(y)$$

for all $y \in G$ and all s in the support S of φ. Let $x \in G$ be given. Then $|g\,r|(x)$ is estimated by $\int_G \mathrm{d}y|\varphi(y)|\,\left|h\left(y^{-1}x\right)\right| r(x) = \int_S \mathrm{d}s|\varphi(s)|\,\left|h\left(s^{-1}x\right)\right| r(x)$. Since

$r(x) \leq \mathrm{C}\, r\left(s^{-1}x\right)$ for all $x \in G$ and $s \in S$ we obtain

$$\begin{aligned} |g\,r|(x) &\leq \mathrm{C} \int_G \mathrm{d}y\, \left|\varphi(y)h\left(y^{-1}x\right)\right| r\left(y^{-1}x\right) \\ &= \mathrm{C} \int_G \mathrm{d}y |\varphi(xy)h(y)| r(y). \end{aligned}$$

Define Φ on G/D by $\Phi(\dot{y}) = \int_D |\varphi(y\,z)|\mathrm{d}z$, where $\dot{y}$ denotes the D-coset of y. Then Φ is a compactly supported continuous function on G/D, hence uniformly bounded by C', say. One gets

$$|g\,r|(x) \leq \mathrm{C} \int_{G/D} \mathrm{d}\dot{y}\, \Phi(\dot{xy}) |h(y)| r(y) \leq \mathrm{CC}' \int_{G/D} \mathrm{d}\dot{y}\, |h(y)| r(y)$$

which is finite by assumption.

Moreover, $\pi(g) = \pi(\varphi)\pi(h)$ is an operator of finite rank and clearly φ can be chosen in such a manner that $\pi(g) \neq 0$.

The equivalence of (1) and (2) is established. We are left to prove the existence of a k with the properties claimed in (B). Let $\mathcal{B}$ denote the set of all measurable functions f on G such that $f(xz) = f(x)\gamma(z)^{-1}$ for all $z \in D$ and $x \in G$ and such that (a′) holds; i.e.,

$$\int_{G/D} |f(x) r(x)| \mathrm{d}\dot{x}$$

is finite for all representative functions of G/D. Using once more that each r may be substituted by a submultiplicative one, and that $r \mapsto \check{r}$ transforms the representative function into itself one deduces that $\mathcal{B}$ forms an involutive algebra where the convolution and involution are given by

$$(f * g)(x) = \int_{G/D} \mathrm{d}\dot{y}\, f(xy)\, g\left(y^{-1}\right)$$

and $f^*(x) = f\left(x^{-1}\right)^{-} \Delta(x)^{-1}$.

It is easy to see that $\mathcal{B}$ contains "many" functions; e.g., for each $\varphi \in \mathcal{C}_c(G)$ the function $\varphi^{\#}$, defined by $\varphi^{\#}(x) = \int_D \varphi(xz)\gamma(z)\mathrm{d}z$, belongs to $\mathcal{B}$.

Now, let h as in (2) be given; i.e., h is contained in $\mathcal{B}$ and $\pi(h)$ is a nonzero operator of finite rank. Then clearly $\mathcal{A} := h^* * \mathcal{B} * h$ is an involutive subalgebra of $\mathcal{B}$. The image $\pi(\mathcal{A})$ is a nonzero *finite-dimensional* involutive subalgebra of $B(\mathfrak{H})$ if $\mathfrak{H}$ denotes the space of π. Since $\pi(\mathcal{A})$ is finite-dimensional it coincides with

$$\pi(h)^* \pi\left(\mathcal{C}_c(G)\right) \pi(h) = \pi(h)^* \pi\left(\mathrm{L}^1(G)\right) \pi(h) = \pi(h)^* \pi\left(C^*(G)\right) \pi(h).$$

If $\mathfrak{K}$ denotes the kernel of $\pi(h)$ then $\pi(\mathcal{A})$ annihilates $\mathfrak{K}$ and transforms the (finite-dimensional) orthogonal complement $\mathfrak{K}^{\perp}$ into itself. In the obvious way one may view $\pi(\mathcal{A})$ as a C^*-subalgebra of $B\left(\mathfrak{K}^{\perp}\right)$.

Using the irreducibility of π one deduces immediately that $\pi(\mathcal{A})$ acts irreducibly on $\mathfrak{K}^{\perp}$. Hence, in the original picture, $\pi(\mathcal{A})$ actually consists of all $T \in B(\mathfrak{H})$ with $T(\mathfrak{K}) = 0$ and $T\left(\mathfrak{K}^{\perp}\right) \subset \mathfrak{K}^{\perp}$. In particular there exists a $k_0 \in \mathcal{A} \subset \mathcal{B}$ such that $\pi(k_0)$ is an orthogonal projection of rank one. Take a test function ψ on G and form $k = \psi^* * k_0 * \psi$. Then k satisfies (a′), also k is equal to $(\psi^*)^\# * k_0 * \psi^\#$ where $\psi^\#$ is defined as above and now the convolution has to be performed in $\mathcal{B}$. The operator $\pi(k)$ equals $\pi(\psi)^*\pi(k_0)\pi(\psi)$, hence $\pi(k)$ is a nonnegative multiple of an orthogonal projection of rank one. Clearly, one may choose ψ such that $\pi(k)$ is an orthogonal projection of rank one. Viewing k, for instance, as the convolution of ψ^* and $k_0 * \psi \in \mathcal{B}$ we saw in the proof of (2) $\Longrightarrow$ (1) that k satisfies the uniform estimate (c′). Concerning (d′) one observes that the derivatives $X * k * Y$ of k are of the form $\varphi_1 * k_0 * \varphi_2$ with test functions φ_1, φ_2 which is, as we have seen, good enough to ensure (a′) and (c′).

Proof of (C). Suppose that the continuous function f on G satisfies the properties (a) and (b) of the theorem. As above form $f^\#(x) = \int_D f(xz)\gamma(z)dz$. Then clearly $h = f^\#$ has the properties stated in (2).

Proof of (D). If (1) or (2) holds true then there exists a function k with the properties (a′) through (d′) of (B). Take a Bruhat function b, see for instance [41], on G with respect to D; i.e., b is a nonnegative continuous function on G such that for each compact set K in G the restriction of b to KD is compactly supported and that $\int_D b(xz)\,dz = 1$ for all $x \in G$. Then put $f_0 = k\,b$. The function f_0 satisfies (a) and $\pi(f_0) = \pi(k)$ is an orthogonal projection of rank one. Choose a test function ψ on G such that $\pi(\psi)^*\pi(f_0)\pi(\psi)$ is an orthogonal projection, too. Then the smooth function $f = \psi^* * f_0 * \psi$ satisfies (a), (b), (c) and (d). This can be seen by reasoning similar to that in (B).

4. Nontrivial kernels of $d\pi$ on the nilradical

In this section we begin with the proof of Theorem 3.1 which proceeds by induction on $\dim M_0$. First we consider the case:

(T) *There exists a nonzero ideal* $\mathfrak{w}$ *in the nilradical* $\mathfrak{n}$ *of* $\mathfrak{m}$, *invariant under the adjoint representation of* G, *such that* π *is trivial on* $W = \exp \mathfrak{w}$. *In other words,* π *is not locally faithful on* $\exp \mathfrak{n}$.

The representation π yields a representation of G/W, also denoted by π. As was shown in Theorem 2.4 the group $G/W = B \ltimes M/W$ belongs to $[MB]$. Given D in G the canonical choice for D' in G/W is $D' = DW/W$. In the proof of (c) of Theorem 2.4 we constructed a $\mathcal{C}^\infty$-cross section $\tau\colon G/WD \longrightarrow G/D$ carrying representative functions of G/D into representative functions of G/WD.

Clearly, we apply the induction hypothesis to the representation π of G/W with given central subgroup D'. As in the Remarks 3.2 we let $\gamma \in \widehat{D}$ be the character corresponding to π. Of course, we will view γ also as a character of $D' = DW/W \simeq D/(W \cap D)$. By (C) and (D) of 3.2 there exists a smooth function f on G/W satisfying

- $f(xz) = f(x)\gamma(z)^{-1}$ for all $x \in G/W, z \in D'$;
- $\pi(f) = \int_{G/DW} f(x)\pi(x)\mathrm{d}x$ is an orthogonal projection of rank one;
- $\int_{G/WD} |f(x)r(x)|\mathrm{d}x < \infty$ for each representative function r of G/WD.

To lift f back to G appropriately we use the above cross section τ. Denote by $p\colon G \longrightarrow G/D$ and by $q\colon G \longrightarrow G/WD$ the quotient maps. For each $x \in G$ the element $\tau(q(x))^{-1}p(x)$ of G/D is contained in WD/D. Choose a nonnegative test function φ on WD/D such that $\int_{WD/D}\varphi(v)\mathrm{d}v = 1$. Then define the (smooth) function F on G by

$$F(x) = f(x)\varphi\left(\tau(q(x))^{-1}p(x)\right)$$

where, of course, the function f on G/W is considered as a function on G in the most obvious way.

The function F satisfies $F(xz) = F(x)\gamma(z)^{-1}$ for all $x \in G, z \in D$. Next we want to show that $\int_{G/D} |F(x)r(x)|\mathrm{d}x$ is finite for each representative function r of G/D. By 1.5 we may assume that r is positive. Choose a continuous representation ρ of G/D in the finite-dimensional real vector space $\mathcal{V}$ and vectors $\xi \in \mathcal{V}$ and η in $\mathcal{V}'$ such that $r(x) = \langle\rho(x)\xi, \eta\rangle$. When we use the cross section τ the integral $\int_{G/D} |F(x)|r(x)\mathrm{d}x$ may be written as

$$\int_{G/WD}\mathrm{d}y \int_{WD/D}\mathrm{d}w\,|F|\,(\tau(y)w)\,r(\tau(y)w).$$

For $y \in G/WD$ and $w \in WD/D$ the value $|F|(\tau(y)w)$ equals $|f|(\tau(y))\varphi(w) = |f|(y)\varphi(w)$. Hence

$$\begin{aligned}\int_{G/D}|F(x)|r(x)\mathrm{d}x &= \int_{G/WD}\mathrm{d}y\int_{WD/D}\mathrm{d}w|f|(y)\varphi(w)\langle\rho(\tau(y))\rho(w)\xi,\eta\rangle \\ &= \int_{G/WD}\mathrm{d}y|f|(y)\langle\rho(\tau(y))\xi',\eta\rangle.\end{aligned}$$

where $\xi' = \int_{WD/D}\mathrm{d}w\varphi(w)\rho(w)\xi$. But the integral $\int_{G/WD}\mathrm{d}y\,|f|(y)\langle\rho(\tau(y))\xi',\eta\rangle$ exists because $y \mapsto \langle\rho(\tau(y))\xi',\eta\rangle$ is a representative function of G/WD.

Using $\int_{WD/D} \varphi(v)\mathrm{d}v = 1$ one easily computes in a similar fashion that $\pi(F) = \pi(f)$. Hence $h = F$ satisfies the requirements of (2) in (B) of 3.2 and by (D) of 3.2 we are done.

5. Induced representations

Next we consider the case Ind), the case of induced representations, which constitutes the main body of this paper. An isomorphism constructed by Green for C^*-algebras will be "concretized" in terms of smooth functions.

Ind) *There exists an abelian ideal* $\mathfrak{w}$ *in the nilradical* $\mathfrak{n}$ *of the Lie algebra* $\mathfrak{m}$ *of* M, *invariant under the adjoint representation of* G, *such that the kernel of* π *in* $\mathrm{L}^1(W)$, $W = \exp \mathfrak{w}$, *is of infinite codimension in* $\mathrm{L}^1(W)$.

The Pontrjagin dual $\widehat{W}$, which is the structure space of the Banach algebra $\mathrm{L}^1(W)$ can be identified with the linear dual $\mathfrak{w}^*$ in the usual manner, the functional ψ on $\mathfrak{w}$ corresponding to $\chi \in \widehat{W}$ given by $\chi(\exp Y) = \mathrm{e}^{-2\pi i \psi(Y)}, Y \in \mathfrak{w}$. The group G acts on $\mathfrak{w}^*$ by $(x\psi)(Y) = \psi\left(\mathrm{Ad}(x)^{-1}(Y)\right)$ and, accordingly, on $\widehat{W}$ by $(x\chi)(w) = \chi\left(x^{-1} w\, x\right)$ for $x \in G, w \in W, \chi \in \widehat{W}, \psi \in \mathfrak{w}^*$ and $Y \in \mathfrak{w}$. From (iii) of 2.2 it follows from a theorem of C. Chevalley, (see [10, p. 183], for an outline of the proof), that the M_0-orbits in $\mathfrak{w}^*$, which are orbits of the "almost algebraic" group $\mathrm{Ad}(M_0)$, are locally closed in $\mathfrak{w}^*$. From (Z) of 2.2 we obtain that the M-orbits in $\mathfrak{w}^*$ are locally closed, too. They are finite unions of M_0-orbits, whose numbers do not exceed the index of L in M.

As B acts by (v) of 2.2 trivially on $\mathfrak{w}^*$, the G-orbits and the M-orbits coincide. Since π is irreducible the kernel of π in $\mathrm{L}^1(W)$ is the kernel (in the hull-kernel sense) of the closure of a G-orbit in $\widehat{W}$. As the G-orbits are locally closed such a closure contains a *unique* relatively open G-orbit $\mathfrak{Y}$, say $\mathfrak{Y} = G\chi_0$. Altogether we obtain $\ker_{\mathrm{L}^1(W)} \pi = k\left(\overline{\mathfrak{Y}}\right)$. Using the above identification we will view $\mathfrak{Y}$ as the subset $G\psi_0$ of $\mathfrak{w}^*$. The M_0-orbit $\mathfrak{X} = M_0\psi_0$ is a relatively open subset of $\overline{\mathfrak{Y}}$. Let H be the stabilizer of ψ_0 in G. The set $\mathfrak{Y}$ is a submanifold of $\mathfrak{w}^*$, diffeomorphic to G/H.

For later use we choose a measurable cross section $\sigma\colon \mathfrak{Y} \longrightarrow G$, i.e., $\sigma(\psi)\psi_0 = \psi$ and $\sigma(\psi_0) = e$, and a coordinate system κ around ψ_0 in $\mathfrak{w}^*$ with the following properties:

(5.1) • κ is a diffeomorphism from the unit cube $Q = \{(x_1, \ldots, x_m); |x_j| < 1\}$ in $\mathbb{R}^m, m = \dim \mathfrak{w}$, onto the open subset $\kappa(Q)$ of $\mathfrak{w}^*$.

- $\kappa(Q) \cap \overline{\mathfrak{Y}} \subset \mathfrak{X}$,
- $\kappa^{-1}(\mathfrak{X}) = \kappa^{-1}\left(\overline{\mathfrak{Y}}\right) = Q^n \times \{0\}$ where $n = \dim \mathfrak{X} = \dim G/H$ and $Q^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; |x_j| < 1\}$,
- $\kappa(0) = \psi_0$,
- σ is smooth on $\kappa(Q) \cap \mathfrak{X}$.

The cube Q^n will be identified with $Q^n \times \{0\} \subset Q$ and similarly the unit cube Q^{m-n} in $\mathbb{R}^{m-n}$ with $\{0\} \times Q^{m-n} \subset Q$. In the sequel we will use the cross section σ effectively only on $\kappa(Q) \cap \mathfrak{X}$. A crucial rôle will be played by the map $\zeta\colon G \times \mathfrak{Y} \longrightarrow H$ defined by

$$\zeta(y, \psi) = \sigma(y\psi)^{-1} y\, \sigma(\psi). \tag{5.2}$$

Next we claim that the stabilizer group H belongs to the class $[MB]$. Since B does not act on $\mathfrak{w}^*$ it is contained in H. Hence H is the semidirect product of B and $H \cap M$. The properties (i), (ii), (iv), (v) and (R) are obviously satisfied by H. Observe that D is contained in H. To prove (Z) for H pick a subgroup L of M according to (Z). Since $Z(L)$ fixes ψ_0 we obtain from $L = M_0 Z(L)$ that $L \cap H = KZ(L)$ where $K = M_0 \cap H$. Moreover, $(L \cap H)_0 = (M \cap H)_0 = K_0$. Since the stabilizer of ψ_0 in the "almost algebraic" group $\mathrm{Ad}_{\mathfrak{m}}(M_0)$, which is nothing but $\mathrm{Ad}_{\mathfrak{m}}(K)$, has only finitely many components (because stabilizers in linear algebraic groups are again algebraic, (see e.g., [7, Chap. III, n°9, p. 208]), and real linear algebraic groups have only finitely many connected components with respect to the euclidean topology, (see e.g., [45])), we deduce that $\mathrm{Ad}_{\mathfrak{m}}(K_0)$ is of finite index in $\mathrm{Ad}_{\mathfrak{m}}(K)$.

Hence $Z(M_0)K_0$ is of finite index in K. Therefore, $Z(L)Z(M_0)K_0 = Z(L)K_0$ is of finite index in $Z(L)K = L \cap H$, which is of finite index in $M \cap H$. Altogether, $Z(L)K_0$ is an open subgroup of finite index in $M \cap H$. Hence as the "L-group" of $B \ltimes (M \cap H)$ we may take $L' = Z(L)K_0$. Clearly L' is B-invariant as L, and hence $Z(L)$, is B-invariant, and as $M \cap H$, and hence $K_0 = (M \cap H)_0$, is B-invariant.

To see that the discrete abelian group L'/K_0 is finitely generated we first observe that the quotient $L'/L' \cap M_0$ of L'/K_0 is isomorphic to a subgroup of L/M_0, hence finitely generated. Therefore, we only have to study $(L' \cap M_0)/K_0$. Since $L' \cap M_0 = K_0 Z(M_0)$, the latter quotient is isomorphic to $Z(M_0)/K_0 \cap Z(M_0)$, which is finitely generated because $Z(M_0)$, as the center of a connected Lie group, is compactly generated.

It remains to show that the adjoint group of $(H \cap M)_0 = K_0$ is the connected component of an algebraic subgroup of $\mathrm{GL}(\mathfrak{k})$ where $\mathfrak{k}$ denotes the Lie algebra of K. The theorems on algebraic groups mentioned above imply that $\mathrm{Ad}_{\mathfrak{m}}(K_0)$ is the connected component of an algebraic subgroup of $\mathrm{GL}(\mathfrak{m})$. Since $\mathrm{Ad}_{\mathfrak{k}}(K_0)$ in $\mathrm{GL}(\mathfrak{k})$ is obtained from $\mathrm{Ad}_{\mathfrak{m}}(K_0)$ by restricting the linear transformations to $\mathfrak{k}$, it has the claimed property. Of course, the Zariski closure of $\mathrm{Ad}_{\mathfrak{m}}(K_0)$ in $\mathrm{GL}(\mathfrak{m})$ leaves $\mathfrak{k}$ invariant as well.

We are going to apply the induction hypothesis to the group H. By Mackey's imprimitivity theorem π may be written as an induced representation, $\pi = \mathrm{ind}_H^G \tau$, for some continuous irreducible unitary representation τ of H with $\tau(w) = \chi_0(w)\,\mathrm{Id}$ for $w \in W$. But we need some more information, namely that $\tau(C^*(H))$ contains the compact operators. This information is supplied by a theorem of Ph. Green, [18], which may be viewed as a C^*-algebraic version of Mackey's imprimitivity theorem.

To be specific, let $G \ltimes W$ be the semidirect product with multiplication $(x,a)(y,b) = (xy, y^{-1}a\,y\,b)$. There is a canonical homomorphism $G \ltimes W \longrightarrow G$ given by $(x,a) \mapsto xa$. This homomorphism delivers a representation ρ of $G \ltimes W, \rho(x,a) = \pi(xa)$. The L^1-group algebra $L^1(G \ltimes W)$ may be viewed as the covariance algebra $L^1(G, L^1(W))$; see [29]. The representation ρ vanishes on $L^1(G, k(\overline{\mathfrak{Y}}))$, hence one obtains a representation of $L^1(G, L^1(W)/k(\overline{\mathfrak{Y}}))$, also denoted by ρ. The ρ-image of the C^*-hull of this algebra, which equals $\pi(C^*(G))$, contains the compact operators.

The C^*-hull is nothing but $C^*(G, \mathcal{C}_\infty(\overline{\mathfrak{Y}}))$ where the action of G on $\mathcal{C}_\infty(\overline{\mathfrak{Y}})$ is induced by the action of G on $\overline{\mathfrak{Y}}$. Moreover, ρ is nontrivial on the ideal $C^*(G, \mathcal{C}_\infty(\mathfrak{Y}))$ in $C^*(G, \mathcal{C}_\infty(\overline{\mathfrak{Y}}))$ where $\mathcal{C}_\infty(\mathfrak{Y})$ is identified with the set of functions in $\mathcal{C}_\infty(\overline{\mathfrak{Y}})$ vanishing on $\overline{\mathfrak{Y}} \setminus \mathfrak{Y}$. Hence $\rho(C^*(G, \mathcal{C}_\infty(\mathfrak{Y})))$ contains the compact operators. By Corollary 2.10 of [18] the algebra $C^*(G, \mathcal{C}_\infty(\mathfrak{Y}))$ is isomorphic to the C^*-tensor product of $C^*(H)$ and the algebra of compact operators on $L^2(G/H)$ for a suitable (quasi-invariant) measure on G/H. This means that the space of ρ, which is the space of π, can be written as the tensor product of a Hilbert space $\mathfrak{H}$ and $L^2(G/H)$ such that the representation ρ considered as a representation of $C^*(H) \otimes \mathfrak{K}(L^2(G/H))$ acts by $\rho(f \otimes A) = \tau'(f) \otimes A$ for $f \in C^*(H), A \in \mathfrak{K}(L^2(G/H))$ and a certain representation τ' of $C^*(H)$. Evidently, $\tau'(C^*(H))$ has to contain the compact operators on $\mathfrak{H}$.

Now one can either show that $\pi = \operatorname{ind}\tau'$ (avoiding the application of the classical imprimitivity theorem) or if one starts from $\pi = \operatorname{ind}\tau$ one can show that one may choose $\mathfrak{H} = \mathfrak{H}_\tau, \tau' = \tau$. In any event, π is induced from a continuous irreducible unitary representation τ of H such that $\tau(w) = \chi_0(w)\,\mathrm{Id}$ for $w \in W$ and $\tau(C^*(H))$ contains the compact operators. Instead of using 2.10 of [18] one could deduce this fact even more directly from Theorem 3.1 in [18]. I preferred the other approach because it contains the program for the treatment of case Ind):

Go through the various identifications above with particularly chosen functions instead of C^*-algebra elements.

The representations π and τ are scalar on D, say $\pi(z) = \gamma(z)\,\mathrm{Id}$ and $\tau(z) = \gamma(z)\,\mathrm{Id}$ for $\gamma \in \widehat{D}$ and all $z \in D$. Applying the induction hypothesis to (H, τ, D) we deduce from 3.2 that there exists a $\mathcal{C}^\infty$-function f on H such that

(5.3) • $f(x\,z) = f(x)\gamma(z)^{-1}$ for all $x \in H, z \in D$;

• $|X * f * Y|\,|r|$ is uniformly bounded on H/D for all representative functions r of H/D and all $X, Y \in \mathfrak{U}\mathfrak{h}$;

• $\tau(f)$ is an orthogonal projection of rank one.

Suppose that left Haar measures on G and H are chosen, let Δ and δ be the respective modular functions. Then take the "measure" $\mu = \mu_{G,H}$ on G/H in the sense of [1, p. 93]; i.e., μ applies to functions T on G satisfying $T(xh) = \delta(h)\Delta(h)^{-1}T(x)$ for $x \in G, h \in H$, and for any compactly supported continuous function S on G one has

$$\int_G S(x)\mathrm{d}x = \oint_{G/H} \mathrm{d}\mu(x) \int_H \mathrm{d}h\, S(xh)\delta(h)^{-1}\Delta(h).$$

We choose a smooth function u on the manifold $\mathfrak{Y}$ such that the (compact) support of u is contained in $\kappa\left(\frac{1}{2}Q^n\right)$ where κ is the coordinate system of (5.1). In particular, the support of u is contained in $\mathfrak{X}$. Moreover, we assume that

$$\oint_{G/H} \mathrm{d}\mu(x)|u(x\psi_0)|^2\, \delta\left(\sigma(x\psi_0)^{-1}x\right) \Delta\left(\sigma(x\psi_0)^{-1}x\right)^{-1} = 1. \tag{5.4}$$

Observe that $\sigma'(x) := \sigma(x\psi_0)^{-1}x \in H$ for all $x \in G$ and that $\sigma'(xh) = \sigma'(x)h$ for all $h \in H$. From the latter equation one deduces easily that the above integrand satisfies the requirements for the application of μ. Then define $F_0\colon G \times \mathfrak{Y} \longrightarrow \mathbb{C}$ by

$$F_0(y,\psi) = \delta(\zeta(y,\psi))^{1/2}\, \Delta(\zeta(y,\psi))^{-1/2}\, u(y\psi)\Delta(\sigma(\psi))\, \overline{u}(\psi)\, f(\zeta(y,\psi)) \tag{5.5}$$

where ζ is as in (5.1). The support of this function F_0 is contained in the closed set $\{(y,\psi); \psi, y\psi \in \operatorname{supp}(u)\}$. On the open neighborhood $\left\{(y,\psi); \psi, y\psi \in \kappa\left(\frac{1}{2}Q^n\right)\right\}$ of this set, the map ζ is smooth, hence F_0 is smooth. Our next goal is to extend F_0 to a function F on the whole of $G \times \mathfrak{w}^*$. This can be easily done by means of the coordinate system κ. Choose any smooth function s on $Q^{m-n} = \left\{(x_{n+1}, \ldots, x_m)\,; |x_j| < 1\right\}$ such that $s(0, \ldots, 0) = 1$ and that the support of s is contained in $\frac{1}{2}Q^{m-n}$. Then put $F = 0$ on $G \times (\mathfrak{w}^* \setminus \kappa(Q))$, and on $G \times \kappa(Q)$ define F by

$$\begin{aligned} F\left(y, \kappa\left(x_1, \ldots, x_m\right)\right) &= F_0\left(y, \kappa\left(x_1, \ldots, x_n\right)\right) s\left(x_{n+1}, \ldots, x_m\right) \\ &= s\left(x_{n+1}, \ldots, x_m\right) \left(\delta^{1/2}\Delta^{-1/2} f\right) \left(\zeta\left(y, \kappa\left(x_1, \ldots, x_n\right)\right)\right) \\ &\quad \times u\left(y\kappa\left(x_1, \ldots, x_n\right)\right) \overline{u}\left(\kappa\left(x_1, \ldots, x_n\right)\right) \Delta\left(\sigma\left(\kappa\left(x_1, \ldots, x_n\right)\right)\right) \end{aligned} \tag{5.6}$$

where, of course, $\kappa\left(x_1, \ldots, x_n\right)$ means $\kappa\left(x_1, \ldots, x_n, 0, \ldots, 0\right)$ in view of the identification of Q^n with $Q^n \times \{0\}$.

Clearly F is smooth. For each $y \in G$ the support of $F(y, -)$ is a compact subset of $\kappa\left(\frac{1}{2}Q\right)$. Moreover, just like f, the function F has a transformation property with respect to D,

$$F(yz, \psi) = \gamma(z)^{-1}\, F(y, \psi)$$

for all $z \in D$, $y \in G$ and $\psi \in \mathfrak{w}^*$. Next we take the inverse Fourier transform of each $F(y,-)$; i.e., F_2 on $G \times \mathfrak{w}$ is given by

$$F_2(y,X) = \int_{\mathfrak{w}^*} \mathrm{d}\psi\, e^{2\pi i \psi(X)} F(y,\psi).$$

Then we define $g\colon G \longrightarrow \mathbb{C}$ by

$$\begin{aligned} g(y) &= \int_{\mathfrak{w}} \mathrm{d}X\, F_2(y\exp(-X),X) \\ &= \int_{\mathfrak{w}} \mathrm{d}X \int_{\mathfrak{w}^*} \mathrm{d}\psi\, e^{2\pi i\psi(X)} F(y\exp(-X),\psi). \end{aligned} \tag{5.7}$$

This step corresponds to the transition $G \ltimes W \longrightarrow G$ in the above considerations. The existence of the integral defining g requires some justification. Actually, we also want some decay of g. Both will later be deduced from the following structure of the derivatives of F.

(5.8) For each linear partial differential operator A on $\mathfrak{w}^*$ with smooth coefficients there exist some natural number J and for $1 \le j \le J$ smooth functions v_j on Q, w_j on $\kappa(Q^n)$, h_j on H and representative functions r_j of H/D such that:

- $h_j(xz) = h_j(x)\gamma(z)^{-1}$ for all $x \in H, z \in D$;
- $|(X * h_j * Y)\,r|$ is uniformly bounded for all $X, Y \in \mathfrak{U}\mathfrak{h}$ and all representative functions r of H/D;
- $(AF)\,(b,\kappa\,(x_1,\ldots,x_m))$

$$= \sum_{j=1}^{J} v_j\,(x_1,\ldots,x_m)\, w_j\,(b\kappa\,(x_1,\ldots,x_n))\,(r_j h_j)\,(\zeta\,(b,\kappa\,(x_1,\ldots,x_n)))$$

for all $(b,x_1,\ldots,x_m) \in G \times Q$ such that $b\kappa\,(x_1,\ldots,x_n) \in \kappa\,(Q^n)$.

Observe that the above partial description of AF is indeed complete because at other points AF vanishes anyway, it even vanishes on a considerably larger set.

To verify (5.8) we need the (total) derivatives of the maps $\psi \mapsto b\psi \in \mathfrak{X}$ and $\psi \mapsto \zeta(b,\psi) \in H$ for $(b,\psi) \in G\times\mathfrak{w}^*$ with $\psi, b\psi \in \kappa\,(Q^n)$. These derivatives are "computed" by using the chain rule and introducing further letters. As usual for a point p on a manifold N we denote by N_p the tangent space at p; for a differentiable map φ between manifolds we denote by $(\mathrm{d}\varphi)_p$ the differential of φ at p (if we do not use particular abbreviations). The Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$ of H and G, respectively, are considered as the tangent spaces at the origin.

(5.9) For $\psi_1 \in \kappa\,(Q^n)$ let $S\,(\psi_1)$ denote the differential at ψ_0 of the map $\psi \mapsto \sigma\,(\psi_1)\,\psi$; $S\,(\psi_1)$ is a linear map from $\mathfrak{X}_{\psi_0}$ into $\mathfrak{X}_{\psi_1}$. For $h \in H$ let $C(h)$ be the differential at ψ_0 of the map $\psi \mapsto h\psi$.

One may identify $\mathfrak{X}_{\psi_0}$ with $\mathfrak{g}/\mathfrak{h}$ in a canonical manner. Under this identification $C(h)$ is nothing but the operator $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)$. We will not use this fact. What we *will* use is that C is a continuous representation of H which is trivial on D. The latter facts can be easily checked directly.

Let $b \in G$ and $\psi_1 \in \kappa(Q^n)$ be given such that $b\psi_1 \in \kappa(Q^n)$. In terms of S and C the differential $T(b, \psi_1)$ of $\psi \mapsto b\psi$ at ψ_1 can be expressed as

$$T(b,\psi_1) = S(b\psi_1) \circ C(\zeta(b,\psi_1)) \circ S(\psi_1)^{-1}. \tag{5.10}$$

This is a linear map from $\mathfrak{X}_{\psi_1}$ into $\mathfrak{X}_{b\psi_1}$. The claim (5.10) follows from the factorization of $\psi \mapsto b\psi$ into the product of $\psi \mapsto \sigma(\psi_1)^{-1}\psi$, $\psi \mapsto \zeta(b,\psi_1)\psi$ and $\psi \mapsto \sigma(b\psi_1)\psi$. Recall that $\zeta(b,\psi) = \sigma(b\psi)^{-1}b\sigma(\psi)$.

Let again b and ψ_1 be as above. We want to find the differential of $\psi \mapsto \zeta(b,\psi)$ at ψ_1, which is a linear map from $\mathfrak{X}_{\psi_1}$ into H_h where $h := \zeta(b,\psi_1) \in H$. For $x \in G$ we denote by L^x the *left* translation *on* G, $L^x(y) = xy$; for $x \in H$ we denote by R^x the *right* translation *on* H, $R^x(y) = yx$. Define the map ε (with values in H, for ψ's close to ψ_1) by

$$\zeta(b,\psi) = \varepsilon(b,\psi)h,$$

i.e.,

$$\begin{aligned}\varepsilon(b,\psi) &= \sigma(b\psi)^{-1}b\,\sigma(\psi)h^{-1} = \sigma(b\psi)^{-1}\sigma(b\psi_1)h\sigma(\psi_1)^{-1}\sigma(\psi)h^{-1}\\ &= \left[\sigma(b\psi_1)^{-1}\sigma(b\psi)\right]^{-1} h\,\sigma(\psi_1)^{-1}\sigma(\psi)h^{-1}\end{aligned}$$

because $b = \sigma(b\psi_1)\,h\sigma(\psi_1)^{-1}$.

From this description of ε one reads off that the differential of $\psi \mapsto \varepsilon(b,\psi)$, first viewed as a function from (a part of) $\mathfrak{X}$ into G, at ψ_1 equals

$$\begin{aligned}&-\left(\mathrm{d}\,L^{\sigma(b\psi_1)^{-1}}\right)_{\sigma(b\psi_1)} \circ (\mathrm{d}\sigma)_{b\psi_1} \circ T(b,\psi_1)\\ &+ \mathrm{Ad}_{\mathfrak{g}}(h) \circ \left(\mathrm{d}\,L^{\sigma(\psi_1)^{-1}}\right)_{\sigma(\psi_1)} \circ (\mathrm{d}\sigma)_{\psi_1}.\end{aligned} \tag{5.11}$$

This is a linear map from $\mathfrak{X}_{\psi_1}$ into $\mathfrak{g}$. Since we know that ε takes its values in H, the linear map has necessarily to take its values in $\mathfrak{h}$. Multiplying the above linear map from the left by $\left(\mathrm{d}R^h\right)_e$ one gets the differential of $\psi \mapsto \zeta(b,\psi)$ at ψ_1.

We turn to the proof of (5.8) and first observe that F itself as defined in (5.6) has the claimed structure. This is clear because $x \mapsto \delta(x)^{1/2}\Delta(x)^{-1/2}, x \in H$, is a representative function of H/D and $(x_1,\ldots,x_n) \mapsto \Delta(\sigma(\kappa(x_1,\ldots,x_n)))$ is a smooth function on Q^n. For the general case it is sufficient to show that if v_j, w_j, r_j and h_j are given as in (5.8) the k^{th} partial derivative of

$$(x_1,\ldots,x_m) \mapsto v_j(x_1,\ldots,x_m)\,w_j(b\kappa(x_1,\ldots,x_n))\,(r_j h_j)(\zeta(b,\kappa(x_1,\ldots,x_n)))$$

has again such a structure. This is evident for $k > n$. For $1 \leq k \leq n$ let e_k be the k^{th} standard basis vector in $\mathbb{R}^n$. As v_j is harmless we are left to differentiate the other factors. Since j is fixed from now on we omit the index j (and use j for other purposes). Using the above formula (5.10) for $T(b,\psi)$ one sees that $\frac{\partial}{\partial x_k} w\left(b\kappa\left(x_1,\ldots,x_n\right)\right)$ at $x=(x_1,\ldots,x_n) \in Q^n$ is obtained by evaluating (of course, as always under the assumption that $b\kappa(x)$ is contained in $\kappa\left(Q^n\right)$)

$$(\mathrm{d}w)_{b\kappa(x)} \circ S(b\kappa(x)) \circ C(\zeta(b,\kappa(x))) \circ S(\kappa(x))^{-1} \circ (\mathrm{d}\kappa)_x$$

at e_k; for the definition of S and C compare (5.9).

Let $X_1,\ldots,X_n$ be a basis of $\mathfrak{X}_{\psi_0}$. For each $x \in Q^n$ there are real coefficients $\lambda_j(x)$, $1 \leq j \leq n$, such that $S(\kappa(x))^{-1}\left((\mathrm{d}\kappa)_x(e_k)\right) = \sum_{j=1}^n \lambda_j(x) X_j$. The λ_j are smooth functions on Q^n depending on k. Applying $C(\zeta(b,\kappa(x)))$ to the obtained expression we find representative functions β_{jp}, $1 \leq j, p \leq n$, of H/D such that

$$\left\{C(\zeta(b,\kappa(x))) \circ S(\kappa(x))^{-1} \circ (\mathrm{d}\kappa)_x\right\}(e_k) = \sum_{j,p=1}^{n} \lambda_j(x)\beta_{jp}(\zeta(b,\kappa(x)))X_p.$$

For any $\psi \in \kappa\left(Q^n\right)$ the linear functional $(\mathrm{d}w)_\psi \circ S(\psi)$ on $\mathfrak{X}_{\psi_0}$ depends smoothly on ψ. Hence there exist smooth functions $\mu_1,\ldots,\mu_n$ on $\kappa\left(Q^n\right)$ such that

$$\left((\mathrm{d}w)_\psi \circ S(\psi)\right)(X_p) = \mu_p(\psi) \quad \text{for } 1 \leq p \leq n.$$

Applying this to $\psi = b\kappa(x)$ one finds

$$\frac{\partial}{\partial x_k} w(b\kappa(x)) = \sum_{j,p=1}^{n} \lambda_j(x)\mu_p(b\kappa(x))\beta_{jp}(\zeta(b\kappa(x))).$$

After multiplying by $v\left(x, x_{n+1},\ldots,x_m\right)(rh)(\zeta(b,\kappa(x)))$ we obtain the desired structure.

Next we compute $\frac{\partial}{\partial x_k} h(\zeta(b,\kappa(x)))$. Using the formula (5.11) for the total derivative of $\psi \mapsto \zeta(b,\psi)$ we obtain this partial derivative by evaluating

$$\begin{aligned}(\mathrm{d}h)_{\zeta(b,\psi)} &\circ \left(\mathrm{d}R^{\zeta(b,\psi)}\right)_e \\ &\circ \Big\{ -\left(\mathrm{d}L^{\sigma(b\psi)^{-1}}\right)_{\sigma(b\psi)} \circ (\mathrm{d}\sigma)_{b\psi} \circ S(b\psi) \circ C(\zeta(b,\psi)) \circ S(\psi)^{-1} \\ &\quad + \mathrm{Ad}_{\mathfrak{g}}(\zeta(b,\psi)) \circ \left(\mathrm{d}L^{\sigma(\psi)^{-1}}\right)_{\sigma(\psi)} \circ (\mathrm{d}\sigma)_\psi \Big\} \circ (\mathrm{d}\kappa)_x\end{aligned}$$

at e_k where we put $\psi = \kappa(x)$.

As above

$$\left[C(\zeta(b,\psi))\circ S(\psi)^{-1}\circ(d\kappa)_x\right](e_k)=\sum_{j,p=1}^{n}\lambda_j(x)\beta_{jp}(\zeta(b,\psi))X_p.$$

Let $Y_1,\dots,Y_t$ be a basis of $\mathfrak{g}$ such that $Y_1,\dots,Y_r$ is a basis of $\mathfrak{h}$. There exist smooth functions $\nu_1,\dots,\nu_t$ on Q^n such that

$$\left[\left(\mathrm{d}L^{\sigma(\psi)^{-1}}\right)_{\sigma(\psi)}\circ(\mathrm{d}\sigma)_\psi\circ(\mathrm{d}\kappa)_x\right](e_k)=\sum_{j=1}^{t}\nu_j(x)Y_j.$$

Applying $\mathrm{Ad}_{\mathfrak{g}}(\zeta(b,\psi))$ to this vector one gets

$$[\mathrm{Ad}_{\mathfrak{g}}(\zeta(b,\psi))\circ\left(\mathrm{d}L^{\sigma(\psi)^{-1}}\right)_{\sigma(\psi)}\circ(\mathrm{d}\sigma)_\psi\circ(\mathrm{d}\kappa)_x](e_k)=\sum_{j=1}^{t}\nu_j(x)\sum_{l=1}^{t}\alpha_{jl}(\zeta(b,\psi))Y_l$$

with representative functions α_{jl} of H/D. Note that $\mathrm{Ad}_{\mathfrak{g}}$ is trivial on D. The map

$$\psi'\mapsto\left(\mathrm{d}L^{\sigma(\psi')^{-1}}\right)_{\sigma(\psi')}\circ(\mathrm{d}\sigma)_{\psi'}\circ S(\psi')$$

from $\kappa(Q^n)$ into $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{X}_{\psi_0},\mathfrak{g})$ depends smoothly on ψ'. Hence there are smooth functions ρ_{pl} on $\kappa(Q^n)$, $1\le p\le n$, $1\le l\le t$, such that

$$\left[\left(\mathrm{d}L^{\sigma(\psi')^{-1}}\right)_{\sigma(\psi')}\circ(\mathrm{d}\sigma)_{\psi'}\circ S(\psi')\right](X_p)=\sum_{l=1}^{t}\rho_{pl}(\psi')Y_l.$$

Applying this formula to $\psi'=b\psi$ we obtain

$$\begin{aligned}&\Big\{-\left(\mathrm{d}L^{\sigma(b\psi)^{-1}}\right)_{\sigma(b\psi)}\circ(\mathrm{d}\sigma)_{b\psi}\circ S(b\psi)\circ C(\zeta(b,\psi))\circ S(\psi)^{-1}\\&\quad+\mathrm{Ad}_{\mathfrak{g}}(\zeta(b,\psi))\circ\left(\mathrm{d}L^{\sigma(\psi)^{-1}}\right)_{\sigma(\psi)}\circ(\mathrm{d}\sigma)_\psi\Big\}((\mathrm{d}\kappa)_x(e_k))\\&=\sum_{l=1}^{t}\sum_{j=1}^{t}\nu_j(x)\alpha_{jl}(\zeta(b,\psi))Y_l-\sum_{l=1}^{t}\sum_{j,p=1}^{n}\lambda_j(x)\beta_{jp}(\zeta(b,\psi))\rho_{pl}(b\psi)Y_l\\&=\sum_{l=1}^{r}\left\{\sum_{q=1}^{t}\nu_q(x)\alpha_{ql}(\zeta(b,\psi))-\sum_{j,p=1}^{n}\lambda_j(x)\beta_{jp}(\zeta(b,\psi))\rho_{pl}(b\psi)\right\}Y_l\end{aligned}$$

since we know that the value has to be in $\mathfrak{h}$.

Using the formula $(Z * h)(\zeta) = \left[(\mathrm{d}h)_\zeta \circ \left(\mathrm{d}R^\zeta\right)_e\right](-Z)$ for all $Z \in \mathfrak{h}$ and all $\zeta \in H$ one finally obtains (by substituting $\psi = \kappa(x)$)

$$\begin{aligned}\frac{\partial}{\partial x_k} h(\zeta(b,\kappa(x))) &= \sum_{l=1}^{r} (Y_l * h)\,(\zeta(b,\kappa(x))) \\ &\cdot \left\{ \sum_{j,p=1}^{n} \lambda_j(x)\beta_{jp}(\zeta(b,\kappa(x)))\rho_{pl}(b\kappa(x)) - \sum_{q=1}^{t} \nu_q(x)\alpha_{ql}(\zeta(b,\kappa(x))) \right\}.\end{aligned} \tag{5.12}$$

Multiplying by $r(\zeta(b,\kappa(x)))w(b\kappa(x))v\,(x,x_{n+1},\dots,x_m)$ one gets a function of the claimed structure. Observe also that $Y_l * h$ satisfies $(Y_l * h)\,(xz) = (Y_l * h)\,(x)\gamma(z)^{-1}$ for all $x \in H, z \in D$.

Finally one has to compute $\frac{\partial}{\partial x_k} r(\zeta(b,\kappa(x)))$. This can be done as above and one finds a formula as in (5.12). The only extra thing to be noted in this case is that $Y_l * r$ is again a representative function of H/D.

From (5.8) we deduce the following basic estimate:

$$\sup_{b\in G,\psi\in\mathfrak{w}^*} |(AF)(b,\psi)\beta(b)| < \infty \tag{5.13}$$

for all representative functions β of G/D and all linear partial differential operators A on $\mathfrak{w}^*$ with smooth coefficients.

To prove (5.13) we may assume again that β is positive and submultiplicative. From the structure of F as defined in (5.6) it follows that we only need to consider pairs (b,ψ) with the property that there exists $(x,x_{n+1},\dots,x_m) \in \frac{1}{2}\overline{Q} = \frac{1}{2}\overline{Q}^n \times \frac{1}{2}\overline{Q}^{m-n}$ with $\psi = \kappa\,(x,x_{n+1},\dots,x_m)$ and $b\kappa(x) \in \kappa\left(\frac{1}{2}\overline{Q}^n\right)$. In view of (5.8) for (b,ψ) as above one has to consider terms of the form

$$\beta(b)\,|v_j\,(x,x_{n+1},\dots,x_m)\,w_j(b\kappa(x))\,(r_jh_j)\,(\zeta(b,\kappa(x)))|$$

where the v_j, w_j, r_j and h_j have the properties stated in (5.8). Clearly v_j and w_j are uniformly bounded on $\frac{1}{2}\overline{Q}$ and on $\kappa\left(\frac{1}{2}\overline{Q}^n\right)$, respectively. From the definition (5.2) of ζ it follows that $b = \sigma(b\kappa(x))\,\zeta(b,\kappa(x))\,\sigma(\kappa(x))^{-1}$.

Since $\sigma\left(\kappa\left(\frac{1}{2}\overline{Q}^n\right)\right)$ is a compact subset of G the submultiplicativity of β yields a constant $K = K_\beta$ such that $\beta(b) \le K\beta(\zeta(b,\kappa(x)))$. Hence up to constants we are left to consider

$$(\beta\, r_j\, h_j)\,(\zeta(b,\kappa(x))).$$

Since $\beta|_H$ is a representative function of H/D this expression stays bounded.

From (5.13) a uniform estimate of the Fourier transforms of the AF's readily follows. Recall that by the index 2 we denote the inverse Fourier transform with respect to the second variable; for instance

$$F_2(b,X) = \int_{\mathfrak{w}^*} \mathrm{d}\psi\, e^{2\pi i\psi(X)} F(b,\psi)$$

for $(b,X) \in G \times \mathfrak{w}$.

(5.14) For each representative function β of G/D and each differential operator A on $\mathfrak{w}^*$ as above there exists a constant $\mathrm{K} = \mathrm{K}(\beta, A)$ such that

$$|(AF)_2(b,X)\beta(b)| \leq \mathrm{K}$$

for all $b \in G, X \in \mathfrak{w}$.

The reason is simply that the uniform norms of $(AF)_2(b,-)\beta(b)$ are estimated by the L^1-norms of $(AF)(b,-)\beta(b)$. And since all the $(AF)(b,-)$ are supported by the fixed compact set $\kappa\left(\frac{1}{2}\overline{Q}\right)$ the latter L^1-norms are estimated, up to a constant, by $\sup_{b,\psi} |(AF)(b,\psi)\beta(b)|$ which is finite by (5.13).

Now we are ready to prove that the function g defined in (5.7) actually exists and that $g\,\beta$ is uniformly bounded for each representative function β of G/D. Recall that g is "defined" as

$$g(b) = \int_{\mathfrak{w}} \mathrm{d}X\, F_2(b\exp(-X), X)$$

for $b \in G$.

Indeed we will show a little more, namely that

$$g'(b) := \int_{\mathfrak{w}} \mathrm{d}X\, |F_2(b\exp(-X), X)|, \quad b \in G, \tag{5.15}$$

exists and $\beta\, g'$ is uniformly bounded. Note that since F_2 satisfies $F_2(b\,z, X) = F_2(b,X)\gamma(z)^{-1}$ for $b \in G$, $z \in D$, $X \in \mathfrak{w}$ one has $g(bz) = \gamma(z)^{-1}g(b)$ while g' is a function on G/D.

Once more we assume that β is positive and submultiplicative. In particular,

$$\beta(b) \leq \beta(b\exp(-X))\,\beta(\exp X) \tag{5.16}$$

for all $X \in \mathfrak{w}$ and $b \in G$. Since $\mathfrak{w}$ is contained in the nilradical we know that $Q(X) := \beta(\exp X)$, $X \in \mathfrak{w}$, is a polynomial function on $\mathfrak{w}$. Choose a positive polynomial function P on $\mathfrak{w}$ such that $\frac{Q}{P}$ is integrable over $\mathfrak{w}$.

Then for $b \in G$,

$$\begin{aligned} g'(b)\beta(b) &= \beta(b) \int_{\mathfrak{w}} \mathrm{d}X \frac{1}{P(X)}\, |P(X)\, F_2(b\exp(-X), X)| \\ &= \beta(b) \int_{\mathfrak{w}} \mathrm{d}X \frac{1}{P(X)}\, |(AF)_2(b\exp(-X), X)| \end{aligned}$$

for a certain differential operator A on $\mathfrak{w}^*$, depending on P. Using (5.16) one gets

$$g'(b)\beta(b) \leq \int_{\mathfrak{w}} \mathrm{d}X \frac{Q(X)}{P(X)} |\beta(b\exp(-X))(AF)_2(b\exp(-X), X)|$$

which is finite because of (5.14).

From (A) of 3.2 it follows that g' is integrable over G/D. Consequently, $|g|$ is integrable over G/D. We claim that

$$\pi(g) = \int_{G/D} g(x)\pi(x)\mathrm{d}x$$

is an orthogonal projection of rank 1, and this will finish the proof of the present case in view of (D) of 3.2.

We think of π as being realized in a space of measurable function $\eta\colon G \longrightarrow \mathfrak{H}_\tau$ such that $\eta(xa) = \Delta(a)^{-1/2}\delta(a)^{1/2}\,\tau(a)^{-1}\,\eta(x)$ for $(x,a) \in G \times H$ and

$$\oint_{G/H} \mathrm{d}\mu(x)(\eta(x), \eta(x))$$

is finite; compare [1, p. 98]. The representation π is given by

$$(\pi(y)\eta)(x) = \eta\left(y^{-1}x\right).$$

Recall that $\tau(z) = \gamma(z)\,\mathrm{Id}$, $z \in D$, and $\tau(w) = \chi_0(w)\,\mathrm{Id}, \tau(\exp X) = e^{-2\pi i\psi_0(X)}\,\mathrm{Id}$ for $w \in W$, $X \in \mathfrak{w}$; in particular, $\gamma = \chi_0$ on $W \cap D$.

Pick a vector $\eta \in \mathfrak{H}_\pi$. To avoid difficulties with sets of measure zero when evaluating, we assume that η is continuous with compact support modulo H. By definition of π and g, see (5.7),

$$\begin{aligned}(\pi(g)\eta)(x) &= \int_{G/D} \mathrm{d}y\, g(xy)\,\eta\left(y^{-1}\right)\\ &= \int_{G/D} \mathrm{d}y \int_{\mathfrak{w}} \mathrm{d}X\, F_2(xy\exp(-X), X)\eta\left(y^{-1}\right).\end{aligned}$$

For fixed $x, y \in G$ we first compute the integral over $WD/D \cong W/W \cap D$:

$$\begin{aligned}&\int_{W/W\cap D} \mathrm{d}w \int_{\mathfrak{w}} \mathrm{d}X\, F_2(x\,y\,w\exp(-X), X)\,\eta\left(w^{-1}y^{-1}\right)\\ &= \int_{W/W\cap D} \mathrm{d}w \int_{\mathfrak{w}} \mathrm{d}X\, F_2(x\,y\,w\exp(-X), X)\,\chi_0\left(y\,w\,y^{-1}\right)\,\eta\left(y^{-1}\right).\end{aligned}$$

Substitute $v = w\exp(-X)$. Observe that for *all* $x, y \in G$ the function at hand is absolutely integrable over $W/W \cap D \times \mathfrak{w}$. Above, following (5.15) and (5.16), the corresponding question was considered for the integral over

$G/D \times \mathfrak{w}$; the cases are very similar. The integral turns out to be equal to

$$\int_{W/D\cap D} \mathrm{d}v\, \chi_0\left(y\,v\,y^{-1}\right) \int_{\mathfrak{w}} \mathrm{d}X\, F_2(x\,y\,v, X)\, e^{-2\pi i (y^{-1}\psi_0)(X)} \eta\left(y^{-1}\right).$$

By the Fourier inversion formula (the functions in question are Schwartz functions with respect to the second variable), if the Lebesgue measures are suitably adapted one gets:

$$\begin{aligned}
&\int_{W/W\cap D} \mathrm{d}v\, \chi_0\left(y\,v\,y^{-1}\right)\; F\left(x\,y\,v, y^{-1}\psi_0\right) \eta\left(y^{-1}\right)\\
&\quad = \int_{W/W\cap D} \mathrm{d}v\, F\left(x\,y\,v, y^{-1}\psi_0\right) \eta\left((y\,v)^{-1}\right)\\
&\quad = \int_{W/W\cap D} \mathrm{d}v\, F_0\left(x\,y\,v, (y\,v)^{-1}\psi_0\right) \eta\left((y\,v)^{-1}\right)
\end{aligned}$$

because $v \in W$ acts trivially on $\mathfrak{w}^*$. Using this result one obtains

$$(\pi(g)\eta)(x) = \int_{G/D} \mathrm{d}y\, F_0\left(x\,y, y^{-1}\psi_0\right) \eta\left(y^{-1}\right),$$

or, by substituting $y \mapsto y^{-1}$,

$$(\pi(g)\eta)(x) = \int_{G/D} \mathrm{d}y\, \Delta(y)^{-1} F_0\left(x\,y^{-1}, y\,\psi_0\right) \eta(y). \tag{5.17}$$

Note that there is no difference between the modular functions of G and of G/D as D is central in G.

The integral in (5.17) will be evaluated using Weil's formula with respect to the subgroup H/D of G/D. To prepare the application of this formula, for fixed $x, y \in G$, we establish the following:

$$\begin{aligned}
&\int_{H/D} \mathrm{d}a\, \Delta(y\,a)^{-1} F_0\left(x\,a^{-1}\,y^{-1}, y\,\psi_0\right) \eta(y\,a)\, \delta(a)^{-1} \Delta(a)\\
&= \int_{H/D} \mathrm{d}a\, \Delta(y)^{-1} \delta(a)^{-1} F_0\left(x\,a^{-1}\,y^{-1}, y\,\psi_0\right) \Delta(a)^{-1/2} \delta(a)^{1/2} \tau(a)^{-1} \eta(y)\\
&= \int_{H/D} \mathrm{d}a\, \Delta(y)^{-1} F_0\left(x\,a\,y^{-1}, y\,\psi_0\right) \Delta(a)^{1/2} \delta(a)^{-1/2} \tau(a)\, \eta(y).
\end{aligned}$$

Using the definition (5.5) of F_0 the integral takes the form

$$\int_{H/D} da\,\Delta(y)^{-1}\Delta(\sigma(y\,\psi_0))\,\overline{u}\,(y\,\psi_0)\,u\,(x\,\psi_0)\left(\delta^{1/2}\Delta^{-1/2}f\right)\left(\zeta\left(x\,a\,y^{-1},y\,\psi_0\right)\right)$$
$$\cdot\,\Delta(a)^{1/2}\delta(a)^{-1/2}\tau(a)\eta(y)$$
$$=\int_{H/D} da\,\Delta\left(y^{-1}\sigma(y\,\psi_0)\right)\overline{u}(y\,\psi_0)\,u(x\,\psi_0)$$
$$\cdot\left(\delta^{1/2}\Delta^{-1/2}f\right)\left(\sigma\left(x\,\psi_0\right)^{-1}x\,a\,y^{-1}\sigma(y\,\psi_0)\right)\Delta(a)^{1/2}\delta(a)^{-1/2}\tau(a)\eta(y).$$

Introducing the new variable $b=\sigma(x\,\psi_0)^{-1}x\,a\,y^{-1}\,\sigma(y\,\psi_0)$, one obtains

$$\int_{H/D} db\,\Delta\left(y^{-1}\,\sigma(y\,\psi_0)\right)\overline{u}(y\,\psi_0)u(x\,\psi_0)\delta\left(y^{-1}\sigma(y\,\psi_0)\right)^{-1}\left(\delta^{1/2}\Delta^{-1/2}f\right)(b)$$
$$\cdot\left(\Delta^{1/2}\delta^{-1/2}\right)\left(x^{-1}\sigma(x\psi_0)b\sigma(y\,\psi_0)^{-1}y\right)\tau\left(x^{-1}\sigma(x\,\psi_0)\,b\,\sigma(y\,\psi_0)^{-1}y\right)\eta(y)$$
$$=\Delta^{1/2}\left(y^{-1}\sigma(y\psi_0)\right)\Delta^{1/2}\left(x^{-1}\sigma(x\,\psi_0)\right)\delta^{-1/2}\left(y^{-1}\sigma(y\,\psi_0)\right)\delta^{-1/2}\left(x^{-1}\sigma(x\,\psi_0)\right)$$
$$\cdot\,\overline{u}(y\,\psi_0))u(x\,\psi_0)\tau\left(x^{-1}\sigma(x\psi_0)\right)\tau(f)\left[\tau\left(\sigma(y\,\psi_0)^{-1}y\right)(\eta(y))\right].$$

Therefore, defining the map $\Gamma=\Gamma_\eta$ from G into $\mathfrak{H}_\tau$ by

$$\Gamma(y)=\left(\Delta^{1/2}\delta^{-1/2}\right)\left(y^{-1}\sigma(y\,\psi_0)\right)\overline{u}(y\,\psi_0)\,\tau\left(\sigma(y\,\psi_0)^{-1}y\right)(\eta(y)) \tag{5.18}$$

we may write

$$\begin{aligned}&\int_{H/D} da\,\Delta(y\,a)^{-1}\,F_0\left(x\,a^{-1}y^{-1},y\,\psi_0\right)\eta(y\,a)\,\delta(a)^{-1}\,\Delta(a)\\&\quad=\left(\Delta^{1/2}\delta^{-1/2}\right)\left(x^{-1}\sigma(x\,\psi_0)\right)u(x\,\psi_0)\,\tau\left(x^{-1}\sigma(x\psi_0)\right)\tau(f)\,\Gamma(y).\end{aligned} \tag{5.19}$$

One readily verifies that

$$\Gamma(y\,a)=\Gamma(y)\delta(a)\,\Delta(a)^{-1}$$

for all $(y,a)\in G\times H$.

Hence $\oint_{G/H}$ is applicable to Γ. Carrying out the integration over y in the integral (5.17) defining $(\pi(g)\eta)(x)$ one deduces from (5.19) that

$$\begin{aligned}(\pi(g)\eta)(x)&=\left(\Delta^{1/2}\delta^{-1/2}\right)\left(x^{-1}\sigma(x\,\psi_0)\right)u(x\,\psi_0)\\&\quad\cdot\tau\left(x^{-1}\sigma(x\,\psi_0)\right)\tau(f)\left(\oint_{G/H} d\mu(y)\,\Gamma(y)\right).\end{aligned} \tag{5.20}$$

From this description of $\pi(g)\eta$ one sees that $\pi(g)$ is an operator of rank one. Actually it is an orthogonal projection of rank one. If ω is a unit vector in the space $\tau(f)(\mathfrak{H}_\tau)$ then $\eta_0 \in \mathfrak{H}_\pi$ defined by

$$\eta_0(x) = \left(\Delta^{1/2}\delta^{-1/2}\right)\left(x^{-1}\sigma(x\zeta_0)\right) u(x\zeta_0)\,\tau\left(x^{-1}\sigma(x\zeta_0)\right)\omega$$

is a unit vector in $\mathfrak{H}_\pi$. This follows from the normalization (5.4) imposed on u.

Using the description (5.20) of $\pi(g)$ one verifies that $\pi(g)\eta = \langle \eta, \eta_0\rangle \eta_0$ for all $\eta \in \mathfrak{H}_\pi$. The easy proof for this is omitted.

Of course, the only reason why we assumed in Ind) that $\ker_{L^1(W)} \pi$ is of infinite codimension was that we wanted to apply the induction hypothesis. If $\ker_{L^1(W)} \pi$ is of finite codimension and the theorem is still true for H, the above considerations work as well and are even much easier. This is formulated as a remark which is also a good transition to the next section.

Remark 5.21. Let $\mathfrak{w}$ be an abelian ideal in the nilradical $\mathfrak{n}$ of the Lie algebra $\mathfrak{m}$, invariant under the adjoint representation of G. Suppose that the kernel of π in $L^1(W)$ is of finite codimension in $L^1(W)$. Then $\ker_{L^1(W)} = k(\mathfrak{Y})$ with some finite G-orbit $\mathfrak{Y}$ in $\widehat{W}$, say $\mathfrak{Y} = G\chi_0$. Let H be the stabilizer of χ_0 in G. Then H is in the class $[MB]$ and there exists a continuous irreducible unitary representation τ of H such that $\pi = \mathrm{ind}_H^G \tau$, $\tau|_W = \chi_0 \,\mathrm{Id}$ and $\tau(C^*(H))$ contains the compact operators. Suppose finally that the theorem is true for (H, τ); i.e., there exists a $\mathcal{C}^\infty$-function f on H such that $\int_H |f(x)\, r(x)| \mathrm{d}x < \infty$ for all representative functions r on H/D and that $\tau(f)$ is an orthogonal projection of rank 1. Then choose a *test function* t on W such that the Fourier integral $\int_W \mathrm{d}w\, t(w)\chi(w)$, $\chi \in \widehat{W}$, is equal to one for $\chi = \chi_0$ and vanishes on the finite set $\mathfrak{Y} \setminus \{\chi_0\}$. Extend f in the most obvious way to a function $\tilde{f}$ on G, and define g on G by

$$g(y) = \int_W \mathrm{d}w\, t(w)\, \tilde{f}\left(y\, w^{-1}\right).$$

It is easy to check (the proof is omitted) that g is integrable against each representative function of G and that $\pi(g)$ is an orthogonal projection of rank one.

6. Zero- and one-dimensional nilradicals

After these preparations we now consider systematically possible abelian ideals in the nilradical $\mathfrak{n}$ of $\mathfrak{m}$ in the traditional manner. There are four cases (see p. 143 in [12]).

(0) $\mathfrak{n} = 0$;

(1) $\mathfrak{n}$ is one-dimensional;

(2) there exists a characteristic abelian ideal in $\mathfrak{n}$ of dimension ≥ 2;

(3) $\mathfrak{n}$ is isomorphic to a Heisenberg algebra of dimension ≥ 3.

The case (0) includes $\mathfrak{m} = 0$, which is the basis for the induction.

First we consider case (2). Let $\mathfrak{w}$ be an abelian characteristic ideal in $\mathfrak{n}$ of dimension ≥ 2. By means of T) and Ind) we may assume that π is locally faithful on $W = \exp \mathfrak{w}$ and that $\ker_{L^1(W)} \pi$ is of finite codimension in $L^1(W)$. As in the final Remark 5.21 of the last section let $\ker_{L^1(W)} \pi = k(\mathfrak{Y})$, where $\mathfrak{Y} = G\chi_0$ is a finite G-orbit in $\widehat{W}$, $H = \mathrm{Stab}(\chi_0)$, $\pi = \mathrm{ind}_H^G \tau$ and $\tau|_W = \chi_0 \mathrm{Id}$. In view of this remark it is sufficient to show that the theorem is true for (H, τ, D). But since $(\ker \tau \cap W)_0 = (\ker \chi_0)_0$ is non-trivial for dimensional reasons case T) applies to (H, τ).

In case (0), $\mathfrak{m}$ is a direct sum of a semisimple algebra $\mathfrak{s}$, the commutator algebra of $\mathfrak{m}$, and the center $\mathfrak{r}$ of $\mathfrak{m}$. The Lie group $S = \langle \exp \mathfrak{s} \rangle$ corresponding to $\mathfrak{s}$ is closed because M_0 allows a locally faithful representation ψ: Since the image $\psi(S) = [\psi(S), \psi(S)]$ is closed one gets $\psi(S) = \psi(\overline{S}) = \psi(S)^-$ whence $\overline{S}$ is contained in $S \ker \psi$. As $\ker \psi$ is countable this gives $\overline{S} = S$. Moreover $S = [M_0, M_0]$ is normal in G, and it consists of B-fixed points by (v) of Definition 2.2. Choose a subgroup L of M according to (Z) of 2.2. Since $\exp \mathfrak{r}$ commutes with M_0 it is contained in the center $Z(L)$ which implies $L = SZ(L)$. Let $K = B \ltimes Z(L)$. By (iv) of 2.2, K is nilpotent of step one or two. Moreover, K is the centralizer of S in $B \ltimes L$, and $B \ltimes L$ is the product of the two normal subgroups S and K.

At this point we interrupt the discussion of case (0) and pass to case (1) because it turns out that there we will find a similar structure. Then both cases will be treated jointly.

In case (1) we may assume in addition, using T) and Ind), that π is non-trivial on $\exp \mathfrak{n}$ and that $\mathfrak{n}$ is central in $\mathfrak{m}$. Let $\mathfrak{r}$ be the radical of $\mathfrak{m}$, let $\mathfrak{s}$ be any Levi factor of $\mathfrak{m}$, and let $\mathfrak{v}$ be any $\mathfrak{s}$-invariant vector space complement of $\mathfrak{n}$ in $\mathfrak{r}$, $\mathfrak{r} = \mathfrak{v} \oplus \mathfrak{n}$. By definition of the nilradical, $[\mathfrak{s}, \mathfrak{r}]$ is contained in $\mathfrak{n}$, in particular $[\mathfrak{s}, \mathfrak{v}]$ is contained in $\mathfrak{n}$ which gives $[\mathfrak{s}, \mathfrak{v}] = 0$. As $\mathfrak{n}$ is central in $\mathfrak{m}$ (or because $\mathfrak{n}$ is one-dimensional), also $[\mathfrak{s}, \mathfrak{n}] = 0$, hence $[\mathfrak{s}, \mathfrak{r}] = 0$. Therefore, $\mathfrak{n} = [\mathfrak{r}, \mathfrak{r}]$, $\mathfrak{r}$ is two step nilpotent, and $\mathfrak{s}$ is *the* Levi factor, namely it coincides with the second derived algebra of $\mathfrak{m}$. As in case (0) one can conclude that $S = \langle \exp \mathfrak{s} \rangle$ is a closed normal subgroup of G consisting of B-fixed points. As in case (0) choose a subgroup L of M according to (Z) of 2.2. The group $\exp(\mathfrak{r})Z(L)$ is precisely the centralizer of S in L, and $K := B \ltimes (\exp(\mathfrak{r})Z(L))$ is the centralizer of S in $B \ltimes L$. Again $B \ltimes L$ is the product of the two normal subgroups S and K. Also in this case K is nilpotent of step two: In view of (iv) of 2.2 it is enough to show that the commutators $[\exp(\mathfrak{r})Z(L), \exp(\mathfrak{r})Z(L)] = [\exp(\mathfrak{r}), \exp(\mathfrak{r})] = \exp(\mathfrak{n})$ are central in K. Since $\mathfrak{n}$ is central in $\mathfrak{m}$, $\exp(\mathfrak{n})$ commutes with $\exp(\mathfrak{r})$; by (v) of 2.2, $\exp \mathfrak{n}$ is fixed by conjugation with elements in B.

To finish the cases (0) and (1) we first have to study the groups $B \ltimes L = SK$ whose structure we know very well and then we have to jump to the finite step from $B \ltimes L$ up to $G = B \ltimes M$. Let me mention that at this point one does not need anymore the central group D and representative functions. We are going to show that there exists a continuous (smooth) function g on G such that $\int_G |g(x)|w(x)dx$ is finite for *each* positive continuous submultiplicative function w on G (called a weight function for short) and that $\pi(g)$ is an orthogonal projection of rank one. In view of 1.5 this is a stronger information than claimed in 3.1.

Let us for short denote by $[BD]$ the class of groups with the above property, i.e.:

(6.1) A locally compact group P belongs to the class $[BD]$ if for each continuous irreducible unitary representation ρ of P such $\rho(C^*(P))$ contains the compact operators there exists a continuous function f on P such that $\int_P |f(x)|w(x)dx < \infty$ for each weight function w on P and that $\pi(f)$ is an orthogonal projection of rank one.

Before proving that groups of the form SK as above (i.e., S is a connected semisimple Lie group, K is a locally compact nilpotent group of step ≤ 2, and K centralizes S) belong to $[BD]$ we show the following:

(6.2) The primitive ideal spaces of their group C^*-algebras satisfy the T_1-axiom; i.e., each such primitive ideal is maximal.

To see this it is clearly sufficient to consider a direct product $P = S \times K$ of a connected semisimple Lie group and a locally compact two step nilpotent group K. Let ρ be a continuous irreducible unitary representation of P. We have to show that $\rho(C^*(P))$ is a simple C^*-algebra. Since S is a type I group, ρ decomposes as a tensor product $\rho = \rho_s \otimes \rho_k$ where ρ_s and ρ_k are continuous irreducible unitary representations of S and K, respectively. Since $\rho_s(C^*(S))$ is precisely the algebra $\mathfrak{K}$ of compact operators on $\mathfrak{H}_{\rho_s}$ it follows that $\rho(C^*(P))$ is the tensor product of $\mathfrak{K}$ and $\rho_k(C^*(K))$. But $\rho_k(C^*(K))$ is simple; see e.g., [38], [31] for more precise information. Hence $\rho(C^*(P)) = \mathfrak{K} \otimes \rho_k(C^*(K))$ is simple, too.

To see that

(6.3) groups of the form SK are in $[BD]$, one may again reduce to the case of a direct product $P = S \times K$. As above a given representation ρ decomposes as $\rho = \rho_s \otimes \rho_k$. Since we now assume that $\rho(C^*(P))$ contains the compact operators we can conclude also that $\rho_k(C^*(K))$ equals the algebra of compact operators. From [31, Theorem 2.9], it follows that there exists a continuous function h on K such that h is integrable against each weight function of K and that $\rho_k(h)$ is an orthogonal projection of rank one. Concerning the S-part we apply the results of Harish-Chandra, [20]. Let L be the Ad-preimage of a maximal compact subgroup of the adjoint group of S. The center $Z(S)$ is contained in L; on $Z(S)$ the representation ρ_s equals a unitary character, say η. The restriction of ρ_s to L decomposes into a direct sum of finite-dimensional irreducible representations with *finite* multiplicity. Let ψ be one of those, and let χ be the product of the character of ψ with $\dim \psi$. Then $\chi(xz) = \chi(x)\eta(z)$ for $x \in L$ and $z \in Z(S)$, and the operator $\rho_s(\overline{\chi}) = \int_{L/Z(S)} \overline{\chi}(x)\rho_s(x)\mathrm{d}x$ is an orthogonal projection of finite rank.

Let b be a (smooth) Bruhat function on L with respect to $Z(S)$; i.e., b is a nonnegative test function (observe that $L/Z(S)$ is compact) on L such that

$$\int_{Z(S)} b(xz)\mathrm{d}z = 1$$

for all $x \in L$. The pointwise product $\overline{\chi}b$ is a test function on L with the property that $\rho_s(\overline{\chi}b)$ is an orthogonal projection of finite rank. Then $(\overline{\chi}b)^* * \mathcal{D}(S) * (\overline{\chi}b)$, where the convolution has to be performed in the measure algebra of S, is an involutive subalgebra of $\mathcal{D}(S)$, whose image under ρ_s is a finite-dimensional algebra. From the irreducibility of ρ_s it follows that there exists a $g \in (\overline{\chi}b)^* * \mathcal{D}(S) * (\overline{\chi}b) \subset \mathcal{D}(S)$ such that $\rho_s(g)$ is an orthogonal projection of rank one. Forming $f = g \otimes h$ one obtains a function on $S \times K$ with the desired properties.

With this information at hand the proof in the cases (0) and (1) is finished by means of the following lemma.

LEMMA 6.4. *Let Q be a locally compact group, and let P be an open subgroup of finite index. Suppose that the primitive ideal space of $C^*(P)$ has the T_1 property, and that P belongs to $[BD]$. Then Q belongs to $[BD]$, too.*

Proof. First we notice that by a result in [36] the primitive ideal space of $C^*(Q)$ satisfies the T_1-axiom as well. Let ρ be a continuous irreducible unitary representation of Q such that $\rho(C^*(Q))$ contains the algebra of compact operators, which means by the T_1-property, that $\rho(C^*(Q))$ equals this algebra. According to [35] the restriction of ρ to P decomposes into a finite sum of continuous irreducible unitary representations of P,

$$\rho|_P = \tau_1 \oplus \cdots \oplus \tau_m,$$

where the τ_j operate in $\mathfrak{H}_j$, say.

For each j, $1 \le j \le m$, and each $f \in \mathrm{L}^1(P)$ the operator $\tau_j(f)$ is compact. Let $\mathcal{A}$ be the set of all measurable functions on P which are integrable against each weight function of P. The set $\mathcal{A}$ is an involutive (as the weight functions are stable under $w \mapsto \overset{\vee}{w}$) subalgebra of $\mathrm{L}^1(P)$ containing $\mathcal{C}_c(P)$. For $1 \le j \le m$ let $\mathfrak{a}_j$ be the set of all $f \in \mathcal{A}$ such that $\tau_j(f)$ is an operator of finite rank. The $\mathfrak{a}_j$'s are two-sided ideals in $\mathcal{A}$, invariant under left and right translations with elements in the group P. Since P belongs to $[BD]$ the ideal $\mathfrak{a}_j$ is not annihilated by τ_j, by the irreducibility of τ_j and the translation-invariance of the $\mathfrak{a}_j$ the sets $\tau_j(\mathfrak{a}_j)(\mathfrak{H}_j)$ are total in $\mathfrak{H}_j$. Put $\mathfrak{a} := \bigcap_{j=1}^m \mathfrak{a}_j$. We claim that there exists an index j, $1 \leq j \leq m$, such that $\tau_j(\mathfrak{a}) \neq 0$. Suppose to the contrary that $\tau_j(\mathfrak{a}) = 0$ for all j. Then take a minimal set I of indices in $\{1, \ldots, m\}$ such that $\bigcap_{j \in I} \mathfrak{a}_j = \mathfrak{a}$; i.e., for proper subsets I' of I one has $\mathfrak{a} \subsetneq \bigcap_{j \in I'} \mathfrak{a}_j$. From our assumptions it follows that the cardinality of I is at least two. Without loss of generality one may suppose that $I = \{1, \ldots, n\}$. Since $(\mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n) * \mathfrak{a}_1$ is contained in $\mathfrak{a}$ we conclude that $\tau_1(\mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n)\tau_1(\mathfrak{a}_1) = 0$. But as $\tau_1(\mathfrak{a}_1)(\mathfrak{H}_1)$ is total in $\mathfrak{H}_1$ this implies that $\tau_1(\mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n) = 0$. In particular, $\mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n$ is contained in $\mathfrak{a}_1$ whence $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n = \mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n$ contradicting the minimality of I.

Hence we know the existence of an $f \in \mathcal{A}$ such that $\tau_j(f)$ is an operator of finite rank for all j and that at least one of the $\tau_j(f)$'s is not zero. The function f may also be considered as a function on Q, $f(Q \setminus P) = 0$. Then f is integrable against each weight function of Q, and $\rho(f)$ is a nonzero operator of finite rank. Denote by $\mathcal{B}$ the convolution algebra of all measurable functions on Q which are integrable against each weight function of Q. Then $f^* * \mathcal{B} * f$ is an involutive subalgebra. The image $\rho(f^* * \mathcal{B} * f)$ is finite-dimensional, hence $\rho(f^* * \mathcal{B} * f) = \rho(f^* * \mathcal{C}_c(Q) * f)$. From the irreducibility of ρ it follows that there exists a $g \in \mathcal{C}_c(Q)$ such that $\rho(f^* * g * f)$ is an orthogonal projection of rank one (compare also 3.2). The function $f^* * g * f$ has the claimed properties.

7. Nilradicals isomorphic to a Heisenberg algebra

Finally suppose that $\mathfrak{n}$ is a Heisenberg algebra of dimension ≥ 3. If $\mathfrak{z}(\mathfrak{n})$ denotes the center of $\mathfrak{n}$ in view of T) we may assume that π is locally faithful on $Z(N) = \exp \mathfrak{z}(\mathfrak{n})$, in view of Ind); we may assume that $\mathfrak{z}(\mathfrak{n})$ is central in $\mathfrak{m}$. From (Z) and (v) of 2.2 it follows that G operates as a finite group on $\mathfrak{z}(\mathfrak{n})$; actually there is at most one nontrivial automorphism in $\mathrm{Ad}(G)|_{\mathfrak{z}(\mathfrak{n})}$. Using 5.21 again we may readily reduce to the case that $\mathrm{Ad}(G)$ operates trivially on $\mathfrak{z}(\mathfrak{n})$; hence $Z(N)$ is central in G. From (ii) of 2.2 it follows that $Z(N)$ is closed in G and that it is isomorphic to $\mathbb{R}$. Since π is irreducible there is a (non-trivial) character χ on $Z(N)$ such that $\pi(z) = \chi(z)\,\mathrm{Id}$ for $z \in Z(N)$. A basis vector Z of $\mathfrak{z}(\mathfrak{n})$ is determined by

(7.1)
$$\chi(\exp(tZ)) = e^{-2\pi i t}$$

for all $t \in \mathbb{R}$.

Choose an arbitrary vector space complement $\mathfrak{v}$ to $\mathfrak{z}(\mathfrak{n})$ in $\mathfrak{n}, \mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}(\mathfrak{n})$, and denote by $\mathrm{Sp}(\mathfrak{v})$ the group of automorphisms of the Lie algebra $\mathfrak{n}$ which leave $\mathfrak{v}$ invariant and fix $\mathfrak{z}(\mathfrak{n})$ pointwise. The group $\mathrm{Sp}(\mathfrak{v})$ is the symplectic group in dimension $\dim \mathfrak{n}/\mathfrak{z}(\mathfrak{n})$. Denote by $\mathrm{Mp}(\mathfrak{v})$ the corresponding metaplectic group and by $\kappa\colon \mathrm{Mp}(\mathfrak{v}) \to \mathrm{Sp}(\mathfrak{v})$ the canonical two-fold covering; see e.g., [14], [27], [44]. The crucial property of $\mathrm{Mp}(\mathfrak{v})$ to be used below is that the unique continuous irreducible representation of $N = \exp \mathfrak{n}$ which equals χ on $Z(N)$ can be extended to a continuous unitary representation of $\mathrm{Mp}(\mathfrak{v}) \ltimes N$ where the action of $\mathrm{Mp}(\mathfrak{v})$ on N is given via κ by the canonical action of $\mathrm{Sp}(\mathfrak{v})$ on N.

Let P denote the group of all elements $x \in G$ such that the restriction $\mathrm{Ad}_{\mathfrak{n}}(x)$ of $\mathrm{Ad}(x)$ to $\mathfrak{n}$ transforms $\mathfrak{v}$ into itself. It is easy to see that every Lie algebra automorphism of $\mathfrak{n}$ leaving $\mathfrak{z}(\mathfrak{n})$ pointwise fixed is a (unique) product of an element in $\mathrm{Sp}(\mathfrak{v})$ and an inner automorphism. Hence $G = PN$ and the intersection $P \cap N$ equals $Z(N)$. Moreover, since B leaves $\mathfrak{n}$ pointwise fixed, P is a semidirect product of B and $K := P \cap M$. Clearly, the given central subgroup D is contained in K. Let K_s be the pullback of $\mathrm{Ad}_{\mathfrak{n}}\colon K \to \mathrm{Sp}(\mathfrak{v})$ and $\kappa\colon \mathrm{Mp}(\mathfrak{v}) \to \mathrm{Sp}(\mathfrak{v})$; i.e., $K_s = \{(x,h) \in K \times \mathrm{Mp}(\mathfrak{v});\ \mathrm{Ad}_{\mathfrak{n}}(x) = \kappa(h)\}$ considered as a (closed) subgroup of the direct product $K \times \mathrm{Mp}(\mathfrak{v})$. Using the fact that B acts trivially on $\mathfrak{n}$ we check easily that the group B acts homomorphically on K_s by $b(x,h)b^{-1} = (bxb^{-1}, h)$ for $b \in B$, $(x,h) \in K_s$. Hence we may form the semidirect product $B \ltimes K_s =: P_s$.

There are canonical homomorphisms $\mu\colon P_s \to \mathrm{Mp}(\mathfrak{v})$ and $\lambda\colon P_s \to P$ given by $\mu(b(x,h)) = h$ and $\lambda(b(x,h)) = bx$ for $b \in B$, $(x,h) \in K_s$. The homomorphism λ has a two element kernel, and the diagram

$$\begin{array}{ccc} P_s & \xrightarrow{\mu} & \mathrm{Mp}(\mathfrak{v}) \\ \lambda\downarrow & & \downarrow\kappa \\ P & \xrightarrow{\mathrm{Ad}_{\mathfrak{n}}} & \mathrm{Sp}(\mathfrak{v}) \end{array} \tag{7.2}$$

commutes. We claim that the group P_s belongs to the class $[MB]$.

Property (i) is clear as B is unchanged (as always). Concerning (ii) we note that λ induces a covering from $(K_s)_0$ onto K_0 of degree 1 or 2. Since M_0 allows a locally faithful representation this is also true for $(K_s)_0$. The group K_0 equals $K \cap M_0$ because the homogeneous space $M_0/K \cap M_0$ is homeomorphic to the simply connected space $N/Z(N)$, which implies that $K \cap M_0$ is connected. Since (iii) of 2.2 is a local property, it is enough to show that $\mathrm{Ad}_{\mathfrak{k}}(K_0)$ is the connected component of an algebraic subgroup of $\mathrm{GL}(\mathfrak{k})$. The group $\mathrm{Ad}_{\mathfrak{m}}(K_0) = \mathrm{Ad}_{\mathfrak{m}}(K \cap M_0)$ is the intersection of $\mathrm{Ad}_{\mathfrak{m}}(M_0)$ and $\{\alpha \in \mathrm{GL}(\mathfrak{m}); \alpha(\mathfrak{v}) = \mathfrak{v}\}$; it is the connected component of an algebraic subgroup of $\mathrm{GL}(\mathfrak{m})$. Since $\mathrm{Ad}_{\mathfrak{k}}(K_0)$ is obtained from $\mathrm{Ad}_{\mathfrak{m}}(K_0)$ by restricting the linear transformations to $\mathfrak{k}$, it has the claimed property (iii).

Concerning (iv) one computes that the commutators in $[B, K_s]$ are of the form $(bxb^{-1}x^{-1}, 1) \in K \times \mathrm{Mp}(\mathfrak{v})$ where $b \in B$, $x \in K$. They are central in $P_s = B \ltimes K_s$ because (iv) holds true for the original group. Property (v) is equally easy. To verify (R) we take, of course, the given central group D and form $D_s := \lambda^{-1}(D)$. It is easy to check that D_s is central in P_s. But P_s/D_s is canonically isomorphic to P/D which is a closed subgroup of G/D, and hence isomorphic to a closed subgroup of some $\mathrm{GL}_n(\mathbb{R})$. Let $L = M_0 Z(L)$ be a subgroup of M as in (Z). Then $Z(L)$ is contained in K, and $L \cap K$ is of finite index in K. From $Z(L) \subset K$ and $K_0 = K \cap M_0$ one deduces that $L \cap K = Z(L)K_0$. It is easy to check that $L_s := \lambda^{-1}(ZL)(K_s)_0 = \lambda^{-1}(L \cap K)$ has the required properties with respect to P_s.

Of course, we plan to apply the induction hypothesis to P_s. To this end we need a representation τ of P_s, suitably related to π. This is obtained as follows. We view the representation π as a representation of $P_s \ltimes N$ where the action of $a \in P_s$ on $x \in N$ is given by $axa^{-1} = \lambda(a)x\lambda(a)^{-1}$. Explicitly, the representation π_s of $P_s \ltimes N$ is defined by $\pi_s(a,x) = \pi(\lambda(a)x)$. The convolution algebra $\mathrm{L}^1(P_s \ltimes N)$ is*-isomorphic to the covariance algebra $\mathrm{L}^1(P_s, \mathrm{L}^1(N))$ where the action of $a \in P_s$ on $\varphi \in \mathrm{L}^1(N)$ is defined by

$$\varphi^a(x) = \varphi(\lambda(a)x\lambda(a)^{-1}). \tag{7.3}$$

For the notion of covariance algebras see [29] where these algebras were called "Verallgemeinerte L^1-Algebren". The multiplication in $\mathrm{L}^1(P_s, \mathrm{L}^1(N))$ is given by

$$(f * g)(a) = \int_{P_s} \mathrm{d}b\, f(ab)^{b^{-1}} * g(b^{-1}).$$

Since $\pi|_{Z(N)} = \chi\,\mathrm{Id}$ the representation π_s factors through $\mathrm{L}^1(P_s, \mathrm{L}^1(N)) \to \mathrm{L}^1(P_s, \mathrm{L}^1(N)_\chi)$ where $\mathrm{L}^1(N)_\chi$ denotes the convolution algebra of all measurable functions φ on N such that $\varphi(xz) = \chi(z)^{-1}\varphi(x)$ for all $x \in N$, $z \in Z(N)$ and $\int_{N/ZN} |\varphi(x)|\,\mathrm{d}x < \infty$. The action of P_s on $\mathrm{L}^1(N)_\chi$ is formally the same as on $\mathrm{L}^1(N)$. The assumption on π implies that the π_s-image of the C^*-hull of $\mathrm{L}^1(P_s, \mathrm{L}^1(N)_\chi)$, which is the C^*-covariance algebra $C^*(P_s, C^*(\mathrm{L}^1(N)_\chi))$, contains the compact operators on $\mathfrak{H}_\pi$. By the way, the C^*-hull $C^*(N)_\chi$ of $\mathrm{L}^1(N)_\chi$ is precisely the algebra of compact operators.

To write down a representation τ of $C^*(P_s)$ explicitly we are going to construct a very concrete orthogonal projection q of rank one in $C^*(N)_\chi$. Actually, q will be a very nice function in $\mathrm{L}^1(N)_\chi$. The metaplectic representation will allow us to identify $q * C^*(P_s, C^*(N)_\chi) * q$ with $C^*(P_s)$, and this identification will be used to define τ.

Choose any continuous irreducible unitary representation α of N such that $\alpha|_{Z(N)} = \chi\,\mathrm{Id}$; α is uniquely determined by this equation up to unitary equivalence. There exists a continuous unitary representation U of $\mathrm{Mp}(\mathfrak{v})$ in $\mathfrak{H}_\alpha$ such that

$$\alpha(\exp(\kappa(h)(X))) = U(h)\alpha(\exp X)U(h)^{-1} \tag{7.4}$$

for all $h \in \mathrm{Mp}(\mathfrak{v})$, $X \in \mathfrak{n}$, and hence

$$\alpha(\lambda(a)x\lambda(a)^{-1}) = U(\mu(a))\alpha(x)U(\mu(a))^{-1}$$

for all $a \in P_s$, $x \in N$.

In order to define q we choose a basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ of $\mathfrak{v}$ such that the $[X_j, Y_j] = Z$ for $1 \leq j \leq n$ are, up to skew-symmetry, the only nonvanishing brackets in $\mathfrak{n}$. Recall (7.1) that the basis vector Z of $\mathfrak{z}(\mathfrak{n})$ is determined by $\chi(\exp(tZ)) = \mathrm{e}^{-2\pi i t}$. The chosen basis can be used to identify $\mathrm{L}^1(\mathbb{R}^{2n}) = \mathrm{L}^1(\mathfrak{v})$ with $\mathrm{L}^1(N)_\chi$: if $\varphi \in \mathrm{L}^1(\mathbb{R}^{2n})$ then define φ' on N by

$$(7.5)\quad \varphi'\left(\exp\left(\sum_{j=1}^{n} x_j X_j + \sum_{j=1}^{n} y_j Y_j\right)\exp(tZ)\right) = \mathrm{e}^{2\pi i t}\varphi(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

This identification defines by transport of structure a multiplication (and an involution) on $\mathrm{L}^1(\mathbb{R}^{2n})$ which is denoted by $\natural$ and is given by the formula

$$(7.6)\qquad (\varphi \natural \psi)(x', y') = \int\int_{\mathbb{R}^{2n}} \mathrm{d}x\mathrm{d}y\, \varphi(x, y)\psi(x' - x, y' - y)\mathrm{e}^{i\pi(x',y')\mathfrak{J}(x,y)^T}$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $\mathfrak{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and $I =$ the unit matrix in n dimensions. The involution is as usual given by $\varphi^*(x, y) = \overline{\varphi}(-x, -y)$.

The operation $\natural$ is called twisted convolution. The identification of $\mathrm{L}^1(N)_\chi$ with $\mathrm{L}^1(\mathbb{R}^{2n})$ was chosen that way to obtain exactly the multiplication law studied in [27] because we want to apply some formulas derived there. Clearly, also, the action of the group P_s is transformed under the above identification. An element $a \in P_s$ acts on $\varphi \in \mathrm{L}^1(\mathbb{R}^{2n})$ via

$$(7.7)\qquad \varphi^a(w^T) = \varphi((\sigma(a)w)^T)$$

where w is a column vector of length $2n$ and σ is a certain continuous homomorphism from P_s into $\mathrm{Sp}_{2n}(\mathbb{R})$, namely $\sigma(a)$ is $\mathrm{Ad}(\lambda(a))|_{\mathfrak{v}}$ written as a matrix using the chosen basis of $\mathfrak{v}$.

According to [27], for a symmetric complex $2n \times 2n$ matrix $\mathfrak{A}$ with positive definite real part, the associated Gaussian function is denoted by $\gamma_{\mathfrak{A}}$,

$$\gamma_{\mathfrak{A}}(w^T) = \mathrm{e}^{-\pi w^T \mathfrak{A} w}$$

for $w \in \mathbb{R}^{2n}$. The $\gamma_{\mathfrak{A}}$'s form a semigroup under twisted convolution, the oscillator semigroup, namely

$$(7.8)\qquad \gamma_{\mathfrak{A}_1} \natural \gamma_{\mathfrak{A}_2} = \det(\mathfrak{A}_1 + \mathfrak{A}_2)^{-\frac{1}{2}} \gamma_{\mathfrak{A}_3}$$

where

$$\mathfrak{A}_3 = \mathfrak{A}_2 - (\mathfrak{A}_2 + \frac{i}{2}\mathfrak{J})(\mathfrak{A}_1 + \mathfrak{A}_2)^{-1}(\mathfrak{A}_2 - \frac{i}{2}\mathfrak{J}).$$

In [27], one finds some other useful descriptions of $\mathfrak{A}_3$. In the following we shall work with $q = \gamma_{\frac{1}{2}\mathfrak{E}}$ where $\mathfrak{E}$ denotes the $2n \times 2n$ unit matrix, and the corresponding, cf.(7.5), function $q' \in \mathrm{L}^1(N)_\chi$. The function q is so to speak an orthogonal projection of rank one in $\mathrm{L}^1(\mathbb{R}^{2n})$; one has

$$q\natural q = q,\ q^* = q,\ q\natural\mathrm{L}^1(\mathbb{R}^{2n})\natural q = \mathbb{C}q \tag{7.9}$$

and the corresponding formulas for q' in $\mathrm{L}^1(N)_\chi$. The first two equations are obvious. The last one is a little more delicate, but still the computation is omitted. It can be done by direct verification or by applying the semigroup law and using the fact that the $\gamma_{\mathfrak{X}}$'s are total in $\mathrm{L}^1(\mathbb{R}^{2n})$. A similar claim can be found in [37], with slightly different normalizations.

The above equations imply that $\alpha(q')$ is an orthogonal projection of rank one, say

$$\alpha(q') = (-,\xi_0)\xi_0$$

with some unit vector $\xi_0 \in \mathfrak{H}_\alpha$. For $a \in P_s$ we then define $\omega(a) \in \mathrm{L}^1(\mathbb{R}^{2n})$ by requiring that

$$\alpha(\omega(a)') = (-,\xi_0)U(\mu(a))^{-1}\xi_0. \tag{7.10}$$

Since α is faithful there exists at most such an $\omega(a)$. Soon we will see that $\omega(a)$ actually exists. The $\omega(a)$'s will be used to identify $q' * \mathrm{L}^1(P_s, \mathrm{L}^1(N)_\chi) * q'$ with an algebra of functions on P_s. For $a \in P_s$ the element q^a, compare (7.7), is also a Gaussian function, namely

$$q^a = \gamma_{\frac{1}{2}\mathfrak{B}} \quad \text{where } \mathfrak{B} = \mathfrak{B}_a = \sigma(a)^T\sigma(a). \tag{7.11}$$

Hence the twisted convolution product $q^a\natural q$ is a nonzero multiple of the Gaussian function $\gamma_{\frac{1}{2}\mathfrak{Y}}$ where

$$\frac{1}{2}\mathfrak{Y} = \frac{1}{2}\mathfrak{E} - \left(\frac{1}{2}\mathfrak{E} + \frac{i}{2}\mathfrak{J}\right)\left(\frac{1}{2}\mathfrak{B} + \frac{1}{2}\mathfrak{E}\right)^{-1}\left(\frac{1}{2}\mathfrak{E} - \frac{i}{2}\mathfrak{J}\right). \tag{7.12}$$

From (7.4) one easily derives $\alpha(q'^a) = \alpha((q^a)') = U(\mu(a))^{-1}\alpha(q')U(\mu(a))$ for all $a \in P_s$. Hence $\alpha(q'^a)\alpha(q') = \alpha((q^a\natural q)')$ is the rank one operator

$$(-,\xi_0)(U(\mu(a))\xi_0,\xi_0)U(\mu(a))^{-1}\xi_0.$$

Since $q^a\natural q$ is different from zero the scalar product $(U(\mu(a))\xi_0,\xi_0)$ does not vanish; hence

$$\omega(a) = (U(\mu(a))\xi_0,\xi_0)^{-1}q^a\natural q.$$

From this description of ω it follows that $\omega(a)$ exists, that $a \mapsto \omega(a)$ is a continuous function from P_s into $\mathrm{L}^1(\mathbb{R}^{2n})$ and that $\omega(a)$ is proportional to $\gamma_{\frac{1}{2}\mathfrak{Y}}$:

$$\omega(a) = \varepsilon(a)\gamma_{\frac{1}{2}\mathfrak{Y}} \tag{7.13}$$

with some continuous complex-valued function ε on P_s. We shall compute $|\varepsilon(a)|$ and the real part of $\mathfrak{Y}$ explicitly. This is good enough to control the growth of $\omega(a)$, for instance in the norm of $\mathrm{L}^1(\mathbb{R}^{2n})$. From the definition of $\omega(a)$ it follows that $\omega(a)^*\natural\omega(a) = q = \gamma_{\frac{1}{2}\mathfrak{E}}$. Since $\omega(a)^* = \overline{\varepsilon(a)}\gamma_{\frac{1}{2}\overline{\mathfrak{Y}}}$ the semigroup law implies

$$|\varepsilon(a)|^2 \det\left(\frac{1}{2}(\mathfrak{Y} + \overline{\mathfrak{Y}})\right)^{-1/2} = 1.$$

Recall (7.12) that

$$\mathfrak{Y} = \mathfrak{E} - (\mathfrak{E} + i\mathfrak{J})(\mathfrak{B} + \mathfrak{E})^{-1}(\mathfrak{E} - i\mathfrak{J});$$

hence $\mathrm{Re}(\mathfrak{Y}) = \mathfrak{E} - (\mathfrak{B} + \mathfrak{E})^{-1} - \mathfrak{J}(\mathfrak{B} + \mathfrak{E})^{-1}\mathfrak{J}$. Using $\mathfrak{J}^{-1} = -\mathfrak{J}$ one obtains

$$\mathfrak{J}(\mathfrak{B} + \mathfrak{E})^{-1}\mathfrak{J} = \mathfrak{J}^{-1}(\mathfrak{B} + \mathfrak{E})^{-1}\mathfrak{J}^{-1} = (\mathfrak{J}\mathfrak{B}\mathfrak{J} - \mathfrak{E})^{-1}.$$

Since $\mathfrak{B}$ is symmetric and symplectic, $\mathfrak{J}\mathfrak{B}\mathfrak{J}$ equals $-\mathfrak{B}^{-1}$, hence

$$\begin{aligned} \mathrm{Re}(\mathfrak{Y}) &= \mathfrak{E} - (\mathfrak{B} + \mathfrak{E})^{-1} + (\mathfrak{B}^{-1} + \mathfrak{E})^{-1} = 2\mathfrak{B}(\mathfrak{E} + \mathfrak{B})^{-1} \\ &= 2\sigma(a)^T\sigma(a)(\mathfrak{E} + \sigma(a)^T\sigma(a))^{-1}, \end{aligned} \tag{7.14}$$

and

$$\begin{aligned} |\varepsilon(a)| &= \det(2\sigma(a)^T\sigma(a)(\mathfrak{E} + \sigma(a)^T\sigma(a))^{-1})^{1/4} \\ &= \det(2(\mathfrak{E} + \sigma(a)^T\sigma(a))^{-1})^{1/4}. \end{aligned}$$

Of course, in the preceding equations the definition (7.11) of $\mathfrak{B}$ was used.

Recall that if $F \in \mathrm{L}^1(P_s, \mathrm{L}^1(N)_\chi)$ and $\varphi \in \mathrm{L}^1(N)_\chi$ then $\varphi * F$ and $F * \varphi$ are given by

$$(\varphi * F)(a) = \varphi^a * (F(a))$$

and

$$(F * \varphi)(a) = F(a) * \varphi \quad \text{for } a \in P_s.$$

In particular, for a function $F \in \mathrm{L}^1(P_s, \mathrm{L}^1(N)_\chi)$ to be contained in $q' * \mathrm{L}^1(P_s, \mathrm{L}^1(N)_\chi) * q'$ means that for $a \in P_s$ the value $F(a)$ has to lie in the space $q'^a * \mathrm{L}^1(N)_\chi * q'$. The latter space is one-dimensional as can be seen by applying the faithful representation α. Actually, $q'^a * \mathrm{L}^1(N)_\chi * q'$ is spanned by $\omega(a)'$. If one defines for, say, $f \in \mathcal{C}_c(P_s)$ the function $\tilde{f}\colon P_s \to \mathrm{L}^1(N)_\chi$ by

$$\tilde{f}(a) = f(a)\omega(a)' \tag{7.15}$$

one can verify by direct computation that the assignment $f \mapsto \tilde{f}$ is an injective $*$-morphism from the convolution algebra $\mathcal{C}_c(P_s)$ into $q' * \mathrm{L}^1(P_s, \mathrm{L}^1(N)_\chi) * q'$.

Indeed, this map extends to an isometric $*$-isomorphism from the Beurling algebra $\mathrm{L}^1_\beta(P_s)$, whose weight β is given by $\beta(a) = \|\omega(a)\|_1$, onto $q' * \mathrm{L}^1(P_s, \mathrm{L}^1(N)_\chi) * q'$. More precise information on this weight will be needed later. Here we only observe that β is symmetric, $\beta(a^{-1}) = \beta(a)$. The straightforward verification of this fact is omitted. When taking the C^*-hulls the weight disappears, we get a C^*-isomorphism from $C^*(P_s)$ onto $q' *$

$C^*(P_s, C^*(N)_\chi) * q'$. Clearly, the function q' may also be viewed as a multiplier on $C^*(P_s, C^*(N)_\chi)$. In particular. we can form $\pi_s(q')$; $\pi_s(q')$ is a certain orthogonal projection in $\mathfrak{H}_\pi$.

The algebra $q' * C^*(P_s, C^*(N)_\chi) * q'$ operates via π_s in $\pi_s(q')(\mathfrak{H}_\pi)$ and annihilates the orthogonal complement $\pi_s(q')(\mathfrak{H}_\pi)^\perp$. This means that we may consider the restriction of π_s to $q' * C^*(P_s, C^*(N)_\chi)) * q'$ as a representation in $\pi_s(q')(\mathfrak{H}_\pi)$. And this representation contains the compact operators in its image in view of our assumptions on π. Using our identification of $q' * C^*(P_s, C^*(N)_\chi) * q'$ with $C^*(P_s)$ we obtain a representation τ of $C^*(P_s)$ in $\pi_s(q')(\mathfrak{H}_\pi)$. The induction hypothesis applied to (P_s, τ, D_s) delivers a C^∞ function f on P_s such that

(7.16) • f is integrable against each representative function of P_s/D_s,

• $\tau(f)$ is an orthogonal projection of rank one.

Moreover, we choose any test function φ in $\mathcal{D}(Z(N))$ such that $\int_{Z(N)} \varphi(t) \cdot \chi(t) \mathrm{d}t = 1$. Then define $F\colon\ P_s \times N \to \mathbb{C}$ by

$$\begin{aligned} &F(a, \exp(x_1X_1 + \cdots + x_nX_n + y_1Y_1 + \cdots + y_nY_n)t) \\ &\quad = f(a)\omega(a)(x_1, \ldots, x_n, y_1, \ldots, y_n)\varphi(t) \end{aligned} \tag{7.17}$$

for $a \in P_s$, x_j, $y_j \in \mathbb{R}$ and $t \in Z(N)$.

Provided that $F \in \mathrm{L}^1(P_s \times N)$ (as will be shown soon) the image of F under the canonical quotient map $\mathrm{L}^1(P_s, \mathrm{L}^1(N)) \to \mathrm{L}^1(P_s, \mathrm{L}^1(N)_\chi)$ is just $\tilde{f}$ (cf. (7.15)). Denote by ν the above introduced surjective homomorphism $P_s \ltimes N \to G$, $\nu(a, x) = \lambda(a)x$. The kernel of ν equals $\{(z, \lambda(z)^{-1}); z \in \lambda^{-1}(Z(N))\}$. Then define $g\colon\ G \to \mathbb{C}$ for $a \in P_s$, $u \in N$ by

$$\begin{aligned} g(\nu(a,u)) &= \int_{\lambda^{-1}(Z(N))} \mathrm{d}z\, F((a,u)\cdot(z,\lambda(z)^{-1})) \\ &= \int_{\lambda^{-1}(Z(N))} \mathrm{d}z\, F(az, u\lambda(z)^{-1}) \end{aligned} \tag{7.18}$$

as $Z(N)$ is central in G.

Again under the assumption that F is integrable it follows that $g \in \mathrm{L}^1(G)$, hence $\pi(g)$ exists, and it is pretty obvious from the construction that $\pi(g)$ annihilates $\pi_s(q')(\mathfrak{H}_\pi)^\perp = \pi(q')(\mathfrak{H}_\pi)^\perp$ and on $\pi(q')(\mathfrak{H}_\pi)$ it coincides with $\tau(f)$.

We are left to show that F is integrable and that g is integrable against each representative function r of G/D. Both claims will readily follow if the integral

$$\int_{P_s} \mathrm{d}a \int_N \mathrm{d}u\, |F(a,u)|\, |r(\nu(a,u))|$$

is finite.

Once more 1.5 is used and r may be assumed to be positive and submultiplicative. Then we are reduced to showing that

$$\int_{P_s} da \int_N du\, |F(a,u)|\, r(\lambda(a)) r(u)$$

is finite or that

$$\int_{P_s} da \int_{\mathbb{R}^{2n}} dx dy \int_{Z(N)} dt\, |f(a)||\omega(a)(x,y)||\varphi(t)| r(\lambda(a)) \cdot r(\exp(x_1X_1 + \cdots + y_nY_n)) r(t)$$

is finite.

The integral over $Z(N)$ causes no problem, it just gives a certain number. Moreover, the function $a \mapsto r(\lambda(a))$ is a representative function on P_s/D_s. If we can show that the integral

$$\int_{\mathbb{R}^{2n}} dx dy\, |\omega(a)(x,y)| r(\exp(x_1X_1 + \cdots + y_nY_n)) \tag{7.19}$$

is dominated by a representative function ρ on P_s/D_s, we are done by the properties (7.16) of f.

To this end, one observes that the function $w = \binom{x^T}{y^T} \mapsto r(\exp(x_1X_1 + \cdots + y_nY_n))$ is a polynomial in w, as $\mathfrak{n}$ is the nilradical of $\mathfrak{m}$. Since each polynomial is dominated by $p(\sum_{j=1}^{2n} w_j^2)$ for an appropriate polynomial p in one variable (with positive coefficients) we are left to compute and to estimate

$$\int_{\mathbb{R}^{2n}} dw\, p\left(\sum_{j=1}^{2n} w_j^2\right) \det\left(\mathfrak{B}(\mathfrak{E}+\mathfrak{B})^{-1}\right)^{1/4} e^{-\pi w^T \mathfrak{B}(\mathfrak{E}+\mathfrak{B})^{-1} w}$$

where $\mathfrak{B} = \sigma(a)^T\sigma(a)$; see (7.11) and (7.14).

This is done by diagonalizing the positive definite matrix $\mathfrak{B}(\mathfrak{E}+\mathfrak{B})^{-1}$. Let $t_1, \ldots, t_{2n}$ be the eigenvalues of $\mathfrak{B}(\mathfrak{E}+\mathfrak{B})^{-1}$, certain numbers between 0 and 1. Then $\det \mathfrak{B}(\mathfrak{E}+\mathfrak{B})^{-1} = \det(\mathfrak{E}+\mathfrak{B})^{-1} = \prod_{j=1}^{2n} t_j$. Hence the integral in question equals

$$\prod_{j=1}^{2n} t_j^{1/4} \int_{\mathbb{R}^{2n}} dw\, p\left(\sum_{j=1}^{2n} w_j^2\right) e^{-\pi \sum_{j=1}^{2n} t_j w_j^2}$$
$$= \prod_{j=1}^{2n} t_j^{-\frac{1}{4}} \int_{\mathbb{R}^{2n}} du\, p\left(\sum_{j=1}^{2n} t_j^{-1} u_j^2\right) e^{-\pi \sum_{j=1}^{2n} u_j^2}.$$

The expression $p\left(\sum_{j=1}^{2n} t_j^{-1} u_j^2\right)$ may be viewed as a polynomial in $t_1^{-1}, \ldots, t_{2n}^{-1}$ whose coefficients are polynomials in $u_1, \ldots, u_{2n}$. Carrying out the integration one ends up with the product of $\prod_{j=1}^{2n} t_j^{-1/4}$ and a polynomial in $t_1^{-1}, \ldots, t_{2n}^{-1}$ with real coefficients. Since $t_j^{-1/4}$ is dominated by t_j^{-1} (as $0 < t_j < 1$) we are left to show that each t_j^{-1} is dominated by a representative function ρ of P_s/D_s. Each t_j^{-1} is an eigenvalue of $\mathfrak{B}^{-1}(\mathfrak{E} + \mathfrak{B}) = \mathfrak{B}^{-1} + \mathfrak{E} = \mathfrak{E} + \sigma(a^{-1})\sigma(a^{-1})^T$. By means of the standard scalar product on $\mathbb{R}^{2n}$ we introduce norms on $\mathbb{R}^{2n}$ and on the real $2n \times 2n$ matrices viewed as operators on $\mathbb{R}^{2n}$. Then

$$t_j^{-1} \leq \|\mathfrak{E} + \sigma(a^{-1})\sigma(a^{-1})^T\|_{\text{op}} \leq 1 + \|\sigma(a^{-1})\|_{\text{op}}^2.$$

Since $\|\sigma(a^{-1})\|_{\text{op}}^2$ is dominated by $\sum_{i,k=1}^{2n} \sigma_{ik}(a^{-1})^2$ where the $\sigma_{ik}(a)$ denote the matrix entries of $\sigma(a)$ we can finish this section and the proof of 3.1 with the remark that the σ_{ik} and hence the $\sigma_{ik}^{\vee}$ are representative functions of P_s/D_s.

□

8. Connected Lie groups

In order not to interrupt the later discussion we start with the following lemma, which is probably known but I could not find it explicitly stated in the literature.

LEMMA 8.1. *Let $\mathcal{V}$ be a finite-dimensional real vector space, and let $\mathfrak{h}$ be a Lie subalgebra of* End $(\mathcal{V})$. *Denote by $\mathfrak{r}$ the radical and by $\mathfrak{n} = [\mathfrak{h},\mathfrak{r}] = [\mathfrak{h},\mathfrak{h}] \cap \mathfrak{r}$ the nilradical of $\mathfrak{h}$. Let $\mathfrak{g}$ be the Lie algebra of the smallest algebraic subgroup of* GL $(\mathcal{V})$ *containing* $\exp \mathfrak{h}$. *Then there exists an abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that*:

(1) *$\mathfrak{g}$ is the direct sum of $\mathfrak{a}$ and $\mathfrak{h}$,*
(2) $[\mathfrak{a},\mathfrak{h}] \subset [\mathfrak{h},\mathfrak{h}]$, $[\mathfrak{a},\mathfrak{r}] \subset \mathfrak{n}$.
(3) *$\mathfrak{a} + \mathfrak{r}$ is the radical of $\mathfrak{g}$.*

Proof. Let $\mathfrak{p}$ be a Cartan subalgebra of $\mathfrak{r}$, and let $\mathfrak{q}$ be the Lie algebra of the smallest algebraic subgroup of GL($\mathcal{V}$) containing exp($\mathfrak{p}$). Of course, $\mathfrak{q}$ is contained in $\mathfrak{g}$. Actually, by the results of [7] (in particular, Prop. 21, Chap. VI, § 4, p. 403), $\mathfrak{q}+\mathfrak{r}$ is the Lie algebra of the smallest algebraic subgroup containing $\exp \mathfrak{r}$. If $\mathfrak{s}$ is any Levi factor in $\mathfrak{h}$ then $\mathfrak{h} = \mathfrak{s} + \mathfrak{r}$ and $\mathfrak{g} = \mathfrak{s} + (\mathfrak{q} + \mathfrak{r}) = \mathfrak{q} + \mathfrak{h}$. From $[\mathfrak{g},\mathfrak{g}] = [\mathfrak{h},\mathfrak{h}]$ and $[\mathfrak{q}+\mathfrak{r},\mathfrak{q}+\mathfrak{r}] = [\mathfrak{r},\mathfrak{r}]$ (compare e.g., [22, Chap. 13]), it follows that $[\mathfrak{q},\mathfrak{h}] \subset [\mathfrak{h},\mathfrak{h}]$ and $[\mathfrak{q},\mathfrak{r}] \subset [\mathfrak{r},\mathfrak{r}]$.

The above considerations reduce the problem of constructing $\mathfrak{a}$ more or less to the case of a nilpotent algebra. Of course, in general "many" of the members of the nilpotent algebra $\mathfrak{q}$ (which is a Cartan subalgebra of the radical

$\mathfrak{q}+\mathfrak{r}$ of $\mathfrak{g}$) will be semisimple linear transformations on $\mathcal{V}$. The nilpotency of $\mathfrak{q}$ is used as follows. Let X be a semisimple element in $\mathfrak{q}$. Then $\mathrm{ad}(X): \mathfrak{q} \longrightarrow q$ is semisimple, too. On the other hand, $\mathrm{ad}(X)$ is nilpotent. Hence X is central in $\mathfrak{q}$, i.e., the set $\mathfrak{c}$ of all semisimple elements in $\mathfrak{q}$ is a central subalgebra of $\mathfrak{q}$.

For $Y \in \mathrm{End}(\mathcal{V})$ the semisimple and the nilpotent part in the additive Jordan decomposition are denoted by Y_s and Y_n, respectively. Since each element in $[\mathfrak{p},\mathfrak{p}]$ ($= [\mathfrak{q},\mathfrak{q}]$ by the above mentioned theorem) is nilpotent we conclude that $finite-dimensional := \{Y_n : Y \in \mathfrak{p}\}$ is an ideal in $\mathfrak{q}$. As $finite-dimensional$ consists of nilpotent transformations it is an algebraic Lie algebra. But also $\mathfrak{c}$ is an algebraic Lie algebra, hence $\mathfrak{c}+finite-dimensional$ is an algebraic Lie algebra. Applying the Jordan decomposition to elements in $\mathfrak{p}$ one concludes that $\mathfrak{p}$ is contained in $\mathfrak{c}+finite-dimensional$ and, therefore, $\mathfrak{q} = \mathfrak{c}+finite-dimensional$. Hence each $Q \in \mathfrak{q}$ may be written as $Q = C+Y_n$ with $C \in \mathfrak{c}$ and $Y \in \mathfrak{p}$, or $Q = (C - Y_s)+Y$, and, consequently, $\mathfrak{q} = \mathfrak{c}+\mathfrak{p}$. The algebra $\mathfrak{c}$ satisfies the requirements imposed on $\mathfrak{a}$ except for (1). Choosing any subspace $\mathfrak{a}$ in $\mathfrak{c}$ such that $\mathfrak{c}+\mathfrak{r}$ is a direct sum of $\mathfrak{a}$ and $\mathfrak{r}$ one finds the desired algebra.

THEOREM 8.2. *Let H be a connected Lie group and let π be a continuous irreducible unitary representation of H such that $\pi(C^*(H))$ contains the compact operators. Then there exists a smooth function f on H such that*

(a) $\int_H |f(x)\,r(x)|\mathrm{d}x < \infty$ *for all representative functions r of H,*
(b) $\pi(f)$ *is an orthogonal projection of rank one,*
(c) $\sup_{x\in H} |f(x)\,r(x)| < \infty$ *for all representative functions r of H,*
(d) *all derivatives $X * f * Y$, where $X,Y \in \mathfrak{U}\mathfrak{h}$, satisfy* (a) *and* (c).

Proof. It is not hard to reduce the theorem to the simply connected case. So let us assume from the beginning that H is simply connected. According to Ado's theorem, we may choose any faithful representation of $\mathfrak{h}$. By means of Lemma 8.1 there exists an abelian Lie algebra $\mathfrak{a}$ acting by derivations on $\mathfrak{h}$ such that

$$(8.3) \qquad [\mathfrak{a},\mathfrak{h}] \subset [\mathfrak{h},\mathfrak{h}],\ [\mathfrak{a},\mathfrak{r}] \subset \mathfrak{n},\ \mathfrak{a} \ltimes \mathfrak{r} \quad \text{is the radical of } \mathfrak{a} \ltimes \mathfrak{h},$$

where again $\mathfrak{r}$ denotes the radical and $\mathfrak{n}$ the nilradical of $\mathfrak{h}$.

Moreover, each connected Lie group with algebra $\mathfrak{a} \ltimes \mathfrak{h}$ has an almost algebraic adjoint group. In particular, this applies to the simply connected group $A \ltimes H$ where A is a simply connected group with algebra $\mathfrak{a}$, i.e., A is a vector group. Taking the direct product of $A \ltimes H$ and the one-dimensional torus $\mathbb{T}$ one obtains a group M which still satisfies (iii) of Definition 2.2.

Let B be the Pontrjagin dual of A. The group B may also be viewed as the set of characters of $A \ltimes H$ vanishing on H. This shows that B does not really depend on the chosen complement A of H in $A \ltimes H$, but rather on the whole group $A \ltimes H$ and on H. On the manifold $B \times (A \ltimes H) \times \mathbb{T} = B \times M$ define a multiplication by

$$(b,x,t)\ (b',x',t') = (b\,b', x\,x', t\,t'b'(x)).$$

The obtained group G is a connected Lie group which is in an obvious sense a semidirect product of B and $M = (A \ltimes H) \times \mathbb{T}$. The following observation might be illuminating in view of the eminent rôle played by the nilradical in the proof of the main Theorem 3.1.

Observation 8.4. The nilradical of the Lie algebra $\mathfrak{m} = (a \ltimes h) \times \mathbb{R}$ of M coincides with the nilradical of $\mathfrak{h}$.

This observation follows immediately from the properties (8.3) of the acting algebra $\mathfrak{a}$. Next we claim that the group $G = B \ltimes M$ belongs to $[MB]$. Property (i) of 2.2 is obvious and also (Z) is trivial as M is connected. Property (iii) was already discussed. Concerning (ii) it is enough to observe that by construction $A \ltimes H$ allows a locally faithful representation. Concerning (iv) and (v) one notices that $[B,M] = \mathbb{T}$ and $[M,M] \subset H$. Evidently, $\mathbb{T}$ is central in G, and H is fixed by B.

It remains to consider (R). To this end, let $K = K_H$ be the representation kernel of H, i.e., the intersection of the kernels of all continuous finite-dimensional representations of H.

(8.5) The subgroup $D := K \times \mathbb{T}$ is the representation kernel of G.

Take any continuous finite-dimensional representation ρ of G. Restricting ρ to $B \ltimes (A \times \mathbb{T})$ which is nothing but the Heisenberg group with compact center, one concludes that ρ is trivial on $\mathbb{T}$. Restricting ρ to H, one sees that ρ is trivial on K. This shows that the representation kernel K_G contains D.

For the reverse inclusion we first observe that any continuous finite-dimensional representation ψ of H in $\mathcal{V}$, say, can be extended to a representation of $A \ltimes H$ (see [21, Th. 2.2, p. 215]), i.e., there exist a finite-dimensional real vector space $\mathcal{W}$ containing $\mathcal{V}$ and a continuous representation ρ of $A \ltimes H$ in $\mathcal{W}$ such that $\rho(H)(\mathcal{V}) = \mathcal{V}$ and $\rho(x)|_{\mathcal{V}} = \psi(x)$ for all $x \in H$.

To apply this theorem one has to verify that if R and N, respectively, denote the groups corresponding to the radical $\mathfrak{r}$ of $\mathfrak{h}$ and the nilradical $\mathfrak{n}$ of $\mathfrak{h}$, then the commutators $[A \ltimes H, R]$ are contained in N. But this is more or less equivalent to 8.4. Since $B \times \mathbb{T}$ is normal in G and $G/(B \times \mathbb{T})$ is canonically isomorphic to $A \ltimes H$ the constructed representation ρ may also be viewed as a representation of G.

Next we note that K_G is contained in $H \times \mathbb{T}$ because $G/(H \times \mathbb{T})$ is the direct product of B and A which clearly allows faithful representations. Now if $D = K \times \mathbb{T}$ were a proper subgroup of K_G, there would exist $x \in H \cap K_G$,

$x \notin K$. If one chose a continuous finite-dimensional representation ψ of H with $\psi(x) \neq 1$ the above extension procedure would lead to a contradiction.

This procedure, from ψ to ρ, also shows the following.

(8.6) For each representative function r of H there exists a representative function s of G such that $s|_H = r$. Indeed, one may even choose s as a representative function of $G/(B \times \mathbb{T})$.

For formal reasons we remark that $D = K \times \mathbb{T} = K_G$ is contained in M. As K_G is contained in the kernel of $\mathrm{Ad}_{\mathfrak{g}}: G \longrightarrow \mathrm{GL}(\mathfrak{g})$, which is in the center of G, it follows that D is central in G. Moreover, it is a general fact of connected Lie groups that G/K_G allows a faithful representation (see [17] or [24]). In addition, such a representation may be chosen with closed image; see Chap. XVIII in [21]. Now it is verified that G belongs to $[MB]$.

Next we have to deal with the given representation π of H in $\mathfrak{H}$. To apply 3.1 we must have an associated representation ρ of G. This representation is given by Takai duality, [43]. But we need the explicit form.

First the representation π is extended to a representation π' of $H \times \mathbb{T}$ by $\pi'(h,t) = \pi(h)\bar{t}$. Then π' is induced up to $A \ltimes H \times \mathbb{T}$ and delivers a representation of $A \ltimes H \times \mathbb{T}$ in $\mathrm{L}^2(A, \mathfrak{H})$. This representation can be canonically extended to a representation of G; explicitly

$$(\rho(b,a,h,t)\xi)(a') = b(a')t^{-1}\pi\left(a'^{-1} a\, h\, a^{-1} a'\right)\left(\xi\left(a^{-1}a'\right)\right) \tag{8.7}$$

for $a, a' \in A$, $b \in B$, $h \in H$, $t \in \mathbb{T}$ and $\xi \in \mathrm{L}^2(A, \mathfrak{H})$. Here the abelian groups $A, B, \mathbb{T}$ are written multiplicatively.

Let $\chi: \mathbb{T} \longrightarrow \mathbb{T}$ be defined by $\chi(t) = t^{-1}$ and denote by $\mathrm{L}^1(G)_\chi$ the convolution algebra of all measurable functions $f: G \to \mathbb{C}$ such that $f(xt) = \chi(t)^{-1} f(x)$ for all $x \in G$, $t \in \mathbb{T}$ and $\int_{G/\mathbb{T}} |f(x)| \mathrm{d}x < \infty$. Accordingly one may form $C^*(G)_\chi$ as the C^*-completion of the involutive algebra $\mathrm{L}^1(G)_\chi$ as well as $\mathrm{L}^1(B \ltimes A \times \mathbb{T})_\chi$ and $C^*(B \ltimes A \times \mathbb{T})_\chi$. In the case at hand one may even view $L^1(G)_\chi$ as a subset of $\mathrm{L}^1(G)$, indeed as an ideal. The same applies to $C^*(G)_\chi$.

The Takai duality says that $C^*(G)_\chi$ is isomorphic to the tensor product of $C^*(H)$ and $C^*(B \ltimes A \times \mathbb{T})_\chi$, the latter being canonically isomorphic to the algebra of compact operators on $\mathrm{L}^2(A)$. Hence there is a bijective correspondence between the representations of $C^*(H)$ and the representations of $C^*(G)_\chi$, which can be identified with the continuous unitary representations τ of G such that $\tau|_{\mathbb{T}} = \chi\,\mathrm{Id}$. Of course, the representation ρ of G written above is nothing but the representation corresponding to π. In particular, ρ is irreducible and $\rho(C^*(G)_\chi) = \rho(C^*(G))$ contains the algebra of compact operators.

As in case Ind) in Section 5 our task will be to apply the above isomorphism of C^*-algebras to particularly chosen functions and to show that one obtains functions with the desired properties. With some additional minor effort the following computations could be used to produce a proof without applying Takai's theorem explicitly.

For any $u \in \mathrm{L}^1(A)$ and $b \in B = \widehat{A}$ define $u^b \in \mathrm{L}^1(A)$ by

$$u^b(a) = b(a)^{-1}u(a). \tag{8.8}$$

Fix any test function u on A with $\|u\|_2 = 1$. Then define $q\colon B \times A \times \mathbb{T} \longrightarrow \mathbb{C}$ by

$$q(b,a,t) = t\left(u^b * u^*\right)(a) = t\int_A \mathrm{d}a'\, u(a\,a')\, b(a\,a')^{-1}\,\overline{u}(a'). \tag{8.9}$$

It is not hard to show that q is an L^1-function. Actually, $(b,a) \mapsto \left(u^b * u^*\right)(a)$ is a Schwartz function on $B \times A$ as can be seen by taking Fourier transforms. One readily verifies that $q^* = q$. Our next goal is to compute $\rho(q)$. It turns out that $\rho(q)$ is an orthogonal projection if the Lebesgue measure on B is suitably normalized (the Lebesgue measure on A is considered fixed already; this was used in the normalization $\|u\|_2 = 1$). For $\xi \in \mathrm{L}^2(A, \mathfrak{H})$ and $a'' \in A$ one has

$$(\rho(q)\xi)(a'') = \int_B \mathrm{d}b \int_A \mathrm{d}a' \int_A \mathrm{d}a\, b(a'^{-1}a)^{-1} u(a''a'^{-1}a)\overline{u}(a)\,\xi(a').$$

Carrying out the integration over a and b one obtains

$$\int_B \mathrm{d}b \int_A \mathrm{d}a\, b(a'^{-1}a)^{-1}\, u(a''a'^{-1}a)\overline{u}(a) = u(a'')\overline{u}(a'). \tag{8.10}$$

This can for instance be seen by taking the Fourier transform with respect to the variable a'. Hence

$$(\rho(q)\xi)(a'') = u(a'') \int_A \mathrm{d}a'\,\overline{u}(a')\xi(a'). \tag{8.11}$$

Using $\|u\|_2 = 1$ one easily deduces that $\rho(q)$ is the orthogonal projection onto $\rho(q)\mathrm{L}^2(A,\mathfrak{H}) = \{u(-)\xi_0; \xi_0 \in \mathfrak{H}\}$ which can be identified with $\mathfrak{H}$.

For $\mathbf{\Phi} \in \mathrm{L}^1(G)_\chi$ we are going to compute the operator $\rho(q)\,\rho(\mathbf{\Phi})\rho(q)$. We know that this operator maps a vector of the form $u(-)\xi_0$ onto a vector of the form $u(-)\eta_0$. We shall compute η_0 in terms of $\mathbf{\Phi}$ and ξ_0. If $\eta = \rho(\mathbf{\Phi})\xi$ and $a' \in A$ then

$$\begin{aligned}\eta(a') &= \int_B \mathrm{d}b \int_A \mathrm{d}a \int_H \mathrm{d}h\, b(a')\, u\left(a^{-1}a'\right) \mathbf{\Phi}(b,a,h,1)\pi\left(a'^{-1}a\,h\,a^{-1}a'\right)\xi_0\\ &= \int_B \mathrm{d}b \int_A \mathrm{d}a \int_H \mathrm{d}h\, \mathbf{\Phi}\left(b, a'a^{-1}, h, 1\right) b(a')\, u(a)\pi\left(a^{-1}h\,a\right)\xi_0.\end{aligned}$$

Observe that the product of the Haar measures on $B, A, H, \mathbb{T}$ delivers a Haar measure on G. The substitution $k = a^{-1}ha$ leads to

$$\eta(a') = \int_B \mathrm{d}b \int_A \mathrm{d}a \int_H \mathrm{d}k\, \alpha(a)\, b(a')\, u(a)\, \mathbf{\Phi}\left(b, a'a^{-1}, a\,k\,a^{-1}, 1\right) \pi(k)\xi_0,$$

where the homomorphism $\alpha\colon\ A \to \mathbb{R}_+$ is, by definition, given by

$$\mathrm{d}(a\,k\,a^{-1}) = \alpha(a)\mathrm{d}k. \tag{8.12}$$

The desired vector $\eta_0 \in \mathfrak{H}$ is obtained as

$$\begin{aligned}\eta_0 &= \int_A \mathrm{d}a'\, \overline{u}(a')\, \eta(a')\\ &= \int_A \mathrm{d}a' \int_B \mathrm{d}b \int_A \mathrm{d}a \int_H \mathrm{d}k\, \overline{u}(a')\, \alpha(a)\, b(a')\, u(a)\\ &\quad \cdot \mathbf{\Phi}\left(b, a'a^{-1}, a\,k\,a^{-1}, 1\right) \pi(k)\xi_0.\end{aligned} \tag{8.13}$$

Hence define $\varphi\colon\ H \longrightarrow \mathbb{C}$ by

$$\varphi(k) = \int_A \mathrm{d}a' \int_B \mathrm{d}b \int_A \mathrm{d}a\, \overline{u}(a')\, \alpha(a)\, b(a')\, u(a)\, \mathbf{\Phi}\left(b, a'a^{-1}, a\,k\,a^{-1}, 1\right). \tag{8.14}$$

It is not hard to see that $\varphi \in \mathrm{L}^1(H)$ for each $\mathbf{\Phi} \in \mathrm{L}^1(G)_\chi$. Equation (8.13) now reads as

$$\eta_0 = \pi(\varphi)\xi_0.$$

Without proof we remark that restricting the assignment $\mathbf{\Phi} \mapsto \varphi$ to $\mathbf{\Phi}$'s in $q * \mathrm{L}^1(G)_\chi * q$ one obtains a bounded $*$-morphism from $q * \mathrm{L}^1(G)_\chi * q$ into $\mathrm{L}^1(H)$. This $*$-morphism extends to an isomorphism from $q * C^*(G)_\chi * q$ onto $C^*(H)$. Observe that q is contained in the multiplier algebra of $C^*(G)_\chi$ as well. From these facts one could also deduce that ρ is irreducible and that $\rho(C^*(G))$ contains the compact operators (these are the only consequences of the Takai duality theorem to be used).

According to the main Theorem 3.1 let h be a C^∞-function on G such that

$$\int_G |h(x)s(x)|\mathrm{d}x < \infty$$

for all representative functions s of G and that $\rho(h)$ is an orthogonal projection of rank one. It may happen that $\rho(q)\,\rho(h)\,\rho(q)$ is zero. This could be overcome by varying q, i.e., varying u. But it can also be overcome by varying h. From the irreducibility of ρ it follows that there exists a test function ψ on G such that

$$\rho(q)\rho\left(\psi^* * h^* * h * \psi\right)\rho(q)$$

is an orthogonal projection of rank one. The function

$$g := \psi^* * h^* * h * \psi \tag{8.15}$$

also satisfies

$$\int_G |g(x)s(x)|\mathrm{d}x < \infty$$

for all representative functions s of G ($g\,s$ is also uniformly bounded in view of 3.2).

Then define $F\colon G \longrightarrow \mathbb{C}$ by $F(x) = \int_{\mathbb{T}} g(x\,t)t^{-1}\mathrm{d}t$ and apply the above map $\mathbf{\Phi} \mapsto \varphi$ to F. The resulting function f on H is given by

(8.16)

$$f(y) = \int_A \mathrm{d}a' \int_B \mathrm{d}b \int_A \mathrm{d}a \int_{\mathbb{T}} \mathrm{d}t\,\overline{u}(a')\alpha(a)b(a')u(a)t^{-1}g(b, a'a^{-1}, aya^{-1}, t)$$

for $y \in H$.

From the construction it is clear that f is a smooth L^1-function and that $\pi(f)$ is an orthogonal projection of rank one. Next we claim that $|fr|$ is integrable for each representative function r of H. In (8.6) we have seen that there exists a representative function s of $G/(B \times \mathbb{T})$ such that $s|_H = r$. Without loss of generality we may assume that s is positive and submultiplicative.

The integral $\int_H |f(y)|s(y)\mathrm{d}y$ is dominated by

$$\begin{aligned}&\int_H \mathrm{d}y \int_A \mathrm{d}a' \int_B \mathrm{d}b \int_A \mathrm{d}a \int_{\mathbb{T}} \mathrm{d}t\, s(y)|\overline{u}(a')\alpha(a)b(a')u(a)t^{-1}g(b, a'a^{-1}, aya^{-1}, t)| \\ &\quad = \int_H \mathrm{d}x \int_A \mathrm{d}a' \int_B \mathrm{d}b \int_A \mathrm{d}a \int_{\mathbb{T}} \mathrm{d}t\, s(a^{-1}xa)|\overline{u}(a')u(a)g(b, a'a^{-1}, x, t)|.\end{aligned}$$

Introducing the new variable $a'' = a'a^{-1}$ instead of a one obtains that $\int_H |f(y)|s(y)\mathrm{d}y$ is dominated by

$$\begin{aligned}&\int_H \mathrm{d}x \int_A \mathrm{d}a' \int_B \mathrm{d}b \int_A \mathrm{d}a'' \int_{\mathbb{T}} \mathrm{d}t\, s(a'a^{-1}a''xa''^{-1}a')|\overline{u}(a')u(a'a''^{-1})g(b, a'', x, t)| \\ &\leq \int_H \mathrm{d}x \int_A \mathrm{d}a' \int_B \mathrm{d}b \int_A \mathrm{d}a \int_{\mathbb{T}} \mathrm{d}t\, s(a'^{-1})s(a^{-1}a')s(ax)|\overline{u}(a')u(a'a^{-1})g(b, a, x, t)|.\end{aligned}$$

Since u is compactly supported there exists a constant C such that

$$\int_A \mathrm{d}a'\, s(a'^{-1})s(a^{-1}a')|\overline{u}(a')u(a'a^{-1})| \leq \mathrm{C}$$

for all $a \in A$. Hence the integral in question is dominated by

$$\mathrm{C}\int_H \mathrm{d}x \int_B \mathrm{d}b \int_A \mathrm{d}a \int_{\mathbb{T}} \mathrm{d}t\, s(ax)|g(b, a, x, t)| = \mathrm{C}\int_G \mathrm{d}y\, s(y)|g(y)|$$

which is finite.

From 3.2 it follows that there exists a smooth function on H satisfying (a) through (d) of the theorem. We just remark that the constructed f already satisfies (a) through (d). □

Concluding remarks

The following remarks are of a different nature, some are more technical, others indicate possible further investigations.

(I) Revisiting the whole proof one may say, simplifying somewhat, that there are essentially two sources for finite-rank operators. One is the "largeness" of compact subgroups in semisimple Lie groups with finite center. The other one is the structure of (idempotents in) the covariance algebra $L^1(G, U)$ where G is a locally compact group and U is an appropriate involutive subalgebra of $\mathcal{C}_\infty(G)$ endowed with its own Banach algebra norm. These algebras were used in several papers of Leptin and myself. For further details concerning these algebras see for instance [30], in particular Theorem 4.

(II) In (5.1) the parametrization κ of the G-orbit $\mathfrak{Y}$ and the cross section $\sigma : \mathfrak{Y} \to G$ were chosen independently. Of course, both could be derived from a common source, namely from a fixed vector space complement of $\mathfrak{h}$ in $\mathfrak{g}$ and the exponential map. It did not seem wise to me to relate σ and κ that way - and not to use the relation. For other purposes it is possibly more appropriate, for instance if one wishes to follow the action of the universal enveloping algebras through the various inductive constructions.

(III) Apparently, concerning the existence of finite-rank operators, finite extensions $H \leq G$ are more difficult than I expected at first glance. Above, I dealt twice with such a situation, first in the case where H is the stabilizer of a character in an abelian normal subgroup (compare (5.21)), and secondly in 6.4 where further properties of H were assumed. I do not know in general whether the existence of finite rank operators is stable under finite extensions. To be specific, I do not know if 8.2 remains true when H is only supposed to have finitely many connected components.

(IV) It was shown by Moore and Rosenberg, [33], that the primitive ideal space of the group C^*-algebra of a connected Lie group has the property that one-point sets are open in their closure. Moreover, the structure of the corresponding simple subquotients can be determined, [38]. Possibly, similar results hold true for groups in the class $[MB]$ - and are perhaps even faster to prove when using the methods of this paper.

(V) My original purpose was (see (VI) below) for a given representation π to find an f in L^1 such that $\pi(f)$ is a projection of rank 1. But it turned out that the inductive proof goes through when demanding sharper properties of the function to be constructed. Again simplifying, the reason is that Schwartz functions behave well when taking Fourier transforms. Indeed we constructed an f in something like a Schwartz space, namely in the Fréchet algebra of all $\mathcal{C}^\infty$-functions g satisfying

$$\int_G |(X * g * Y)(z) r(z)| \, dz < \infty$$

for all $X, Y \in \mathfrak{U}\mathfrak{g}$ and all representative functions r, endowed with the obvious seminorms. We used this algebra more or less explicitly at several places, for instance in Section 3. Of course, even in the case of $G = \mathbb{R}^n$ this algebra does not coincide with the usual Schwartz space because the exponentials are representative functions. This could be "repaired" by allowing only a distinguished algebra of representative functions in the above definition, for instance coefficients of rational representations in the case of algebraic groups as in [13]. In other words, variations of the theme are possible (we used in § 3 coefficients, which were constant on cosets of a given central subgroup).

Instead of just constructing a particular function, as we did, one might ask more ambitiously for a description of the image of such Fréchet algebras on general Lie groups under irreducible unitary representations as is available in the case of nilpotent groups, [26], and partly in the case of exponential groups, [32]. In this regard I am not too optimistic for the near future. But I have the feeling that such algebras might be of some use in other aspects of harmonic analysis, for instance in order to obtain information on the asymptotic behaviour of matrix coefficients as in [28].

(VI) Let π be a continuous irreducible unitary representation of a connected Lie group H such that $\pi(C^*(H))$ contains the compact operators on the representation space $\mathfrak{H}$. Arguing as in [9] one can easily deduce from 8.2 that $\mathfrak{H}$ contains a unique algebraically irreducible $\mathrm{L}^1(H)$-submodule. In particular, $\ker_{\mathrm{L}^1(H)} \pi$ is a primitive ideal in the sense of algebra. A couple of years ago I could prove (still unpublished) that for solvable H and *any* continuous irreducible unitary representation π of H the annihilator $\ker_{\mathrm{L}^1(H)} \pi$ is always a primitive ideal. Moreover, the map $\ker_{C^*(H)} \pi \mapsto \ker_{\mathrm{L}^1(H)} \pi = \ker_{C^*(H)} \pi \cap \mathrm{L}^1(H)$ is injective in this case. To avoid misunderstandings: I do not claim that $\ker_{\mathrm{L}^1(H)} \pi$ is C^*-dense in $\ker_{C^*(H)} \pi$, even though I do not know of any counterexample. Presumably similar results are true for arbitrary connected Lie groups. I hope to return to this circle of questions soon.

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(Received June 17, 1992)