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The spectral radius of the Coxeter transformations for a generalized Cartan matrix

Claus Michael Ringel
Fakultät für Mathematik, Universität Bielefeld, Universitätsstrasse 25, D-33615 Bielefeld, Germany

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A ***generalized Cartan matrix*** A of size n is an $n \times n$-matrix A with integer coefficients A_{ij} such that the following conditions are satisfied for all $i \neq j$

$$A_{ii} = 2, \quad A_{ij} \leqq 0, \quad \text{and} \quad A_{ij} \neq 0 \Longleftrightarrow A_{ji} \neq 0$$

(see e.g. [K]). We consider the n-dimensional real vectorspace $\mathbb{R}^n$ with its canonical basis $e(1), \ldots, e(n)$. Given a generalized Cartan matrix A of size n, let R_i be the reflection on $\mathbb{R}^n$ defined by

$$e(j)R_i = e(j) - A_{ji}e(i)$$

for all i, j. The product $C = C(A, \pi) = R_{\pi(1)} \ldots R_{\pi(n)}$ where π is a permutation of $I = \{1, 2, \ldots, n\}$, is called a ***Coxeter transformation*** for A.

The eigenvalues of the Coxeter transformations for a generalized Cartan matrix are of interest in many branches of mathematics: let us just mention Lie theory [C], the study of singularities [A], and the representation theory of associative algebras [PT, Z].

The ***spectral radius*** $\rho(L)$ of a linear transformation L of $\mathbb{R}^n$ is the maximum of the absolute values of the eigenvalues of L; the ***multiplicity*** of an eigenvalue λ of L is by definition its multiplicity as a root of the characteristic polynomial.

Theorem. *Let A be a generalized Cartan matrix which is connected and neither of finite nor of affine type. Let C be a Coxeter transformation for A. Then $\rho(C) > 1$, and $\rho(C)$ is an eigenvalue of multiplicity one, whereas any other eigenvalue λ of C satisfies $|\lambda| < \rho(C)$.*

Partial results have been known before. Let π be a permutation of I with $C = C(A, \pi)$. The quiver $Q(A, \pi)$ of A has I as set of vertices, and there is an arrow $i \to j$ provided $\pi(i) < \pi(j)$ and $A_{ij} \neq 0$. The result is known [A, SS] in case the quiver $Q(A, \pi)$ is bipartite (i.e. any vertex is a sink or a source), thus in particular in case it is a tree (with some orientation). For A symmetrizable, it has been shown in [DR] that $\rho(C)$ is an eigenvalue of C and that any other eigenvalue λ of C satisfies $|\lambda| \leqq \rho(C)$. Recently, Takane [T] has considered this case in more detail; in particular, she has shown that the eigenspace for the eigenvalue $\rho(C)$ is one-dimensional.

For the proof, we may assume that $Q(A,\pi)$ is not a tree, and our aim will be to apply the Perron-Frobenius theorem: we will exhibit a basis $b(1),\dots,b(n)$ of $\mathbb{R}^n$ such that C expressed in this basis becomes a primitive non-negative matrix.

Quivers

Recall that a ***quiver*** $Q = (Q_0, Q_1)$ (without multiple arrows) is just given by a set Q_0 and a subset $Q_1 \subseteq Q_0 \times Q_0$, the elements of Q_0 are called ***vertices***, those of Q_1 ***arrows***, given an element $(x,y) \in Q_1$, where $x,y \in Q_0$, one usually writes $x \to y$, and calls it an arrow starting in x and ending in y.

Let Q be a quiver. For any vertex y of Q, we denote by y^+ the set of all vertices z with $y \to z$, and by y^- the set of vertices x with $x \to y$. A vertex y with $y^- = \varnothing$ is called a ***source***, a vertex y with $y^+ = \varnothing$ is called a ***sink***. A sequence $(i_0,\dots,i_m)$ of vertices in Q is called a ***path*** of length m starting in i_0 and ending in i_m, or a path from i_0 to i_m, provided $i_{r-1} \to i_r$ for any $1 \leqq r \leqq m$. The quiver Q is said to be ***directed*** provided there does not exist a path of length at least 1 starting and ending in the same vertex. Vertices x, y with $x \to y$ or $y \to x$ are called ***neighbours***. The quiver Q is said to be ***connected*** provided for any partition $Q_0 = Q_0' \cup Q_0''$ with nonempty subsets Q_0', Q_0'', there exist $x \in Q_0', y \in Q_0''$ which are neighbours. We also will need the notion of a cycle: A sequence $(i_1,\dots,i_m)$ of pairwise different vertices in Q is called a ***cycle***, provided $m \geqq 3$, and for any $1 \leqq r \leqq m$, there is an arrow $i_{r-1} \to i_r$ or $i_r \to i_{r-1}$ (with $i_0 = i_m$).

Let $Q = (Q_0, Q_1)$ be a quiver, and fix some source x of Q. The quiver $\sigma_x Q$ has the same set $(\sigma_x Q)_0 = Q_0$ of vertices, and the set of arrows $(\sigma_x Q)_1$ is obtained from Q_1 by replacing any element $(x,y) \in Q_1$ by (y,x). Note that the vertex x is a sink for $\sigma_x Q$. A sequence $(x_1,\dots,x_m)$ is called a ***source sequence*** provided x_1 is a source for Q, and for $1 < i \leqq m$, the vertex x_i is a source for $\sigma_{x_{i-1}} \cdots \sigma_{x_1} Q$. If $(x_1,\dots,x_m)$ is a source sequence for Q, and J is a subset of Q_0 with $x_i \notin J$ for all $1 \leqq i \leqq m$, we say that $\omega = \sigma_{x_m} \dots \sigma_{x_1}$ is an ***admissible change of orientation outside of*** J.

The following lemma is essentially due to [BGP].

Lemma 1 *Let Q be a connected finite directed quiver. Let $x \in Q_0$, and let J be the set of vertices y such that there exists a path from x to y. Then there exists an admissible change of orientation ω outside of J so that x is the unique source of ωQ.*

Proof. If y is a source of Q different from x, then y cannot belong to J. If a source y different from x exists, we define inductively a source sequence $(y = y_1, y_2, \dots, y_t)$ using only vertices $y_i \notin J$. We claim that this process has to stop after a finite number of steps. Let $J_0 = J$, and for $m \in \mathbb{N}_1$, define J_m as the set of vertices z of Q which either belong to J_{m-1}, or else there exists an arrow $z \to y$ or $y \to z$ with $y \in J_{m-1}$. Then $Q_0 = \bigcup_m J_m$. By induction, one shows that a vertex $y \in J_m$ can occur at most m times in a source sequence outside of J. If $(y_1, \ldots, y_t)$ is a source sequence outside of J with t maximal, then $\sigma_{y_t} \ldots \sigma_{y_1} Q$ cannot have a source different from x. On the other hand, $\sigma_{y_t} \ldots \sigma_{y_1} Q$ still is directed, thus it has to have a source. This shows that x is a source, and the only one, of $\sigma_{y_t} \ldots \sigma_{y_1} Q$.

Conventions

Later, it will be convenient to relabel the elements of I. Thus, from now on I will just be a set of cardinality n. We use it as index set for the rows and columns of A; *always, $A = (A_{ij})$ will be a generalized Cartan matrix indexed by I.* In this way, the reflections R_x, where $x \in I$, are linear transformations on $\mathbb{R}^I$.

We will work with a fixed bijection $\pi: \{1,2,\ldots,n\} \to I$. Using π, we define the Coxeter transformation $C = C(A,\pi) = R_{\pi(1)} \ldots R_{\pi(n)}$. Note that π also defines a quiver $Q = Q(A,\pi)$ with vertex set I and with arrows $x \to y$ provided $\pi^{-1}(x) < \pi^{-1}(y)$ and $A_{xy} \neq 0$. It is not difficult to see that the Coxeter transformation $C(A,\pi)$ only depends on $Q = Q(A,\pi)$ and not on π itself, thus we may write $C(A,Q) = C(A,\pi)$. We should mention that for any admissible change of orientation ω, the Coxeter transformations $C(A,Q)$ and $C(A,\omega Q)$ are similar.

In order to avoid an abundance of minus signs, we denote by $\alpha = (\alpha_{ij})$ the negative of A, thus $\alpha_{ij} = -A_{ij}$, for all $i, j \in I$.

Given an element $c \in \mathbb{R}^I$, we denote by c_i $(i \in I)$ its coordinates. For any subset J of I, and $x \in \mathbb{R}^I$, we denote by $x|J$ the vector with $(x|J)_j = x_j$ for $j \in J$, and $(x|J)_j = 0$ otherwise.

Grips

Let $Q = Q(A,\pi)$. A ***grip*** for (A,Q) will be a path $(i_0, \ldots, i_t)$ in Q such that the following conditions are satisfied:

(1) i_0 is the only source of Q,

(2) $\sum_{x \in I} \alpha_{x i_0} \geqq 1$,

(3) Either $\alpha_{i_1 i_0} \geqq 2$, and $\alpha_{i_{t-1}, i_t} \geqq 2$, or else there exists a path $(i_0 = x_0, x_1, \ldots, x_{s-1}, x_s = i_t)$ from i_0 to i_t with $x_1 \neq i_1$, and $x_{s-1} \neq i_{t-1}$.

(4) For $0 < r < t$, there is only one path starting in i_0 and ending in i_r, and only one path starting in i_r and ending in i_t.

Proposition 1 *Assume that Q is connected and contains a cycle. Then either A is of type $\tilde{\mathbb{A}}$ or else there exists an admissible change of orientation ω so that $(A, \omega Q)$ has a grip.*

Proof. Let $(i_0, \ldots, i_{m-1})$ be a cycle and let $i_r = i_{r'}$ provided $r \equiv r' \bmod m$. At least one of the vertices z on the cycle must satisfy $\sum_{x \in I} \alpha_{xz} \geqq 1$, since otherwise the connected component containing the cycle has to be of type $\tilde{\mathbf{A}}$. We can assume $z = i_0$. Using an admissible change of orientation, we can assume that i_0 is the unique source of Q. There exists a maximal u such that $(i_0, i_1, \ldots, i_u)$ is a path, since Q is directed. Thus, i_u is the endpoint of the two arrows $i_{u-1} \to i_u$ and $i_{u+1} \to i_u$. Take $1 \leqq t \leqq u$ minimal such that there are at least two arrows ending in i_t, say $i_{t-1} \to i_t$ and $i_x \to i_t$. Since i_0 is the unique source, there is a path starting in i_0 and ending in i_x. By construction, there is only one path starting in i_0 and ending in i_r, where $0 < r < t$.

Assume that we have chosen z on our given cycle so that $t = t(z)$ is minimal. Then we claim that for $0 < r < t$, there is also only one path starting in i_r and ending in i_t. For, assume there exists some $0 < r < t$ and a path $(y_0, y_1 \ldots, y_k)$ from i_r to i_t different from $(i_r, i_{r+1}, \ldots, i_{t-1}, i_t)$. Take r maximal, thus $y_1 \neq i_{r+1}$. Then i_r has at least three neighbours, namely i_{r-1}, i_{r+1} and y_1. Let J be the set of vertices y such that there exists a path from i_r to y. There exists an admissible change of orientation ω outside of J which makes i_r into the only source. All paths starting at i_r remain untouched; in particular, the paths $(i_r, i_{r+1}, \ldots, i_t)$ and $(i_r = y_0, y_1 \ldots, y_m = i_t)$ still do exist in ωQ. This shows that i_t still is the endpoint of two arrows. Since $t(i_r) \leqq t - r < t$, we obtain a contradiction to the minimality of t.

The cone

Let us assume now that (A, Q) has a grip $G = (i_0, i_1, \ldots, i_t)$. To simplify our notation, we change the notation for the vertices of Q, the vertex i_r with $0 \leqq r \leqq t$ will just be denoted by r.

Here is the basis we want to consider: For i not in G, let $b(i) = e(i)$, whereas for i in G, let $b(i) = \sum_{j=i}^{t} e(j)$. Let $\mathcal{K}$ be the set of linear combinations $\sum_{i \in I} \lambda_i b(i)$ with $\lambda_i \geqq 0$, we call it the ***cone*** generated by the vectors $b(i)$, it is a closed subset of $\mathbb{R}^I$. Since $b(i)$, $i \in I$ is a basis of $\mathbb{R}^I$, the interior $\mathcal{K}^\circ$ of $\mathcal{K}$ is the set of linear combinations $\sum_{i \in I} \lambda_i b(i)$ with $\lambda_i > 0$. The cone $\mathcal{K}$ may be described alternatively as follows: A vector $c \in \mathbb{R}^I$ belongs to $\mathcal{K}$ if and only if $c_i \geqq 0$ for all i, and, in addition, $c_0 \leqq c_1 \leqq \ldots \leqq c_t$. Similarly c belongs to $\mathcal{K}^\circ$ if and only if $c_i > 0$ for all i, and, in addition, $c_0 < c_1 < \ldots < c_t$.

Given $c, d \in \mathbb{R}^I$, we will write $c \preceq d$ provided $d - c \in \mathcal{K}$.

We will need several other vectors in $\mathcal{K}$. First of all, let $m(I) = \sum_{i \in I} e(i)$; of course, $m(I)$ is the sum of $b(0)$ and the basis vectors $e(i)$, with i not belonging to the grip, thus $m(I) \in \mathcal{K}$.

If $w = (i_0, \ldots, i_m)$ is a path, we set $\alpha_w = \alpha_{i_0 i_1} \alpha_{i_1 i_2} \ldots \alpha_{i_{m-1} i_m}$, note that for a path (i) of length 0, we have $\alpha_{(i)} = 1$. For any vertex $i \in I$, we define $p(i)$ as the vector with components $p(i)_j = \sum \alpha_w$ where the sum ranges over all paths starting in i and ending in j. For any arrow $x \to y$ in I, we have $p(i)_x \leqq p(i)_y$, therefore $p(i) \in \mathcal{K}$, for any $i \in I$.

The operation of C on $\mathcal{K}$

We note the following: Let i be a vertex with $0 < i < t$. Since the only arrow ending in i is the arrow $i-1 \to i$, we can write $C = S_0S_1 \dots S_{t-1}C_1S_tC_2$, where C_1 is a suitable product of the reflections S_x, with x not belonging to the grip, but with a path from x to t, whereas C_2 is a suitable product of the remaining reflections (those of the form S_x, with no path from x to t).

Lemma 2 *If x does not belong to G, then $b(x)C$ belongs to $\mathcal{K}$.*

Proof. Note that for any Coxeter transformation $C' = C(A, \pi')$ and y a vertex of $Q' = Q(A, \pi')$, the vector $e(y)C'$ is non-negative except in case y is a source in Q'. Since the only source of Q belongs to G, we see that $b(x)C$ has to be non-negative.

Let $c = b(x)C$. For $0 \leqq i \leqq t$, let $G(i) = \{0, 1, \dots, i\}$. By induction we see that for $0 \leqq j \leqq t-1$, we have

$$b(x)S_0S_1 \dots S_j = b(x) + \sum_{i=0}^{j} \alpha_{xi}\, p(i)|G(j)\,,$$

therefore $c_0 \leqq c_1 \leqq \dots \leqq c_{t-1}$. The value c_t is the sum of the various non-negative summands $c_y\alpha_{yt}$ where $y \to t$. In particular, one of the summands will be $c_{t-1}\alpha_{t-1,t}$, thus $c_t \geqq c_{t-1}\alpha_{t-1,t} \geqq c_{t-1}$.

Lemma 3 *For all $0 \leqq i \leqq t$, we have $b(i)C \in \mathcal{K}$. If $1 \leqq i \leqq t$, then $b(i-1) \preceq b(i)C$. We have $b(0) \preceq b(0)C$ if and only if either $\alpha_{10} \geqq 2$, or else $\alpha_{10} = 1$ and $0 \to t$. In case $b(0) \not\preceq b(0)C$, there exists an arrow $x \to t$ with $b(x) \preceq b(0)C$.*

Proof. Let $c = b(i)C$. First, we show that $c_0 \leqq c_1 \leqq \dots \leqq c_{t-1}$.

In case $1 \leqq i \leqq t$, we have $c_j = \alpha_{t0}\, p(0)_j$ for $0 \leqq j < i-1$, and $c_{i-1} = \alpha_{t0}\, p(0)_{i-1} + \alpha_{i,i-1}$; for $i \leqq j < t$, we have

$$c_j = c_{j-1}\alpha_{j-1,j} + \alpha_{j+1,j} - 1\,,$$

thus $c_j \geqq c_{j-1}$. In this way, we see that $c_0 \leqq c_1 \leqq \dots \leqq c_{t-1}$, that $c_{i-1} > c_{i-2}$, for $i \geqq 2$, and that $c_0 > 0$, for $i = 1$.

In case $i = 0$, we have $c_0 = \alpha_{10} + \alpha_{t0} - 1$, thus $c_0 \geqq 0$ (and $c_0 > 0$ if and only if $\alpha_{10} + \alpha_{t0} \geqq 2$). For $1 \leqq j < t$, we have

$$c_j = c_{j-1}\alpha_{j-1,j} + \alpha_{j+1,j} - 1\,,$$

thus $c_j \geqq c_{j-1}$.

Note that if x does not belong to the grip, and if there is a path from x to t, then $c_x \geqq 0$ (more precisely, c_x is the sum of two kinds of non-negative summands: one of the summands is $c_0\, p(0)_x$, the others occur in case there are arrows $x \to t$ and then they are of the form α_{tx}).

In order to see that $c_t \geqq c_{t-1}$, we consider two cases: In case t is endpoint of an additional arrow $x \to t$, where $x \neq t-1$, we have

$$c_x = (b(i)S_0S_1 \ldots S_{t-1}C_1)_x \geqq \alpha_{tx} \geqq 1 \ ,$$

and therefore

$$c_t = (b(i)S_0S_1 \ldots S_{t-1}C_1S_t)_t \geqq c_{t-1}\alpha_{t-1,t} + c_x\alpha_{xt} - 1 \geqq c_{t-1} \ .$$

Otherwise, condition (3) asserts that $\alpha_{10} \geqq 2$ and $\alpha_{t-1,t} \geqq 2$. The first condition shows that for $i = 0$, we have $c_0 \geqq 1$, thus $c_{t-1} \geqq 1$. For $1 \leqq i \leqq t$, we have seen before that $c_{t-1} \geqq c_{i-1} = \alpha_{i,i-1} \geqq 1$. Since for all $0 \leqq i \leqq t$, we have $c_{t-1} \geqq 1$, it follows that

$$c_t \geqq c_{t-1}\alpha_{t-1,t} - 1 \geqq c_{t-1} \ .$$

Altogether, we have shown that $c_i \geqq 0$ for all $i \in I$ and that $c_0 \leqq c_1 \leqq \ldots \leqq c_t$, thus $c \in \mathcal{K}$.

Note that for $c \in \mathcal{K}$, we have $b(0) \preceq c$ if and only if $c_0 > 0$. On the one hand, this shows that $b(0) \preceq b(1)C$, since we have shown before that $(b(1)C)_0 > 0$. Similarly, we see that $b(0) \preceq b(0)C$ if and only if $\alpha_{10} + \alpha_{t0} \geqq 2$, if and only if $\alpha_{10} \geqq 2$ or else $t \rightarrow 0$. Also, for $i \geqq 2$, we have shown that $c_{i-2} > c_{i-1}$, and since $c \in \mathcal{K}$, it follows that $b(i-1) \preceq b(i)C$.

Finally, assume that $\alpha_{10} + \alpha_{t0} = 1$, thus $\alpha_{10} = 1$ and $\alpha_{t0} = 0$. Since $\alpha_{10} = 1$, there exists an arrow $x \rightarrow t$ with $x \neq t-1$, since $\alpha_{t0} = 0$, we see that $x \neq 0$, thus x does not belong to the grip, and therefore $b(x) = e(x)$. On the other hand, $(b(0)C)_x > 0$, thus $b(x) \preceq b(0)C$. This completes the proof.

As a consequence we see:

Proposition 2 *The Coxeter transformation C maps $\mathcal{K}$ into itself.*

The primitivity

We have seen that with respect to the basis $b(i)$, $i \in I$, the Coxeter transformation is non-negative. It remains to be shown that some power of C is actually positive with respect to this basis.

Lemma 4 *We have $m(I) + e(t) \preceq m(I)C$.*

Proof. Let $c = m(I)C$. Then $c_0 = -1 + \sum_{x \neq 0}\alpha_{x0} \geqq 2$. We claim that for any $i \in I$ which is not a sink, we have $c_i \geqq c_0$. We show this by induction: let $i \neq 0$, and assume that i is not a sink. We have

$$c_i = -1 + \sum_{j \in i^-} c_j\alpha_{ji} + \sum_{j \in i^+} \alpha_{ji}.$$

Since i is neither a sink nor a source, none of the sets i^-, i^+ is empty; in particular, the last summand is greater or equal to 1. On the other hand, for $j \in i^-$, we have by induction $c_j \geqq c_0$. Thus, $c_i \geqq c_0$. Similarly, given a sink i, there is an arrow $j \rightarrow i$, and by the previous considerations, $c_j \geqq c_0$. It follows that $c_i \geqq -1 + c_j \geqq -1 + c_0 \geqq 1$. This shows that $m(I) \preceq m(I)C$.

In order to see that $m(I)+e(t) \preceq c$, we have to show that $c_{t-1} < c_t$. In case there is only one arrow ending in t, we have $\alpha_{t-1,t} \geqq 2$, thus

$$c_t = -1 + c_{t-1}\alpha_{t-1,t} + \sum_{j \in t^+} \alpha_{jt} \geqq -1 + 2 \cdot c_{t-1} > c_{t-1} .$$

In case there are at least two arrows ending in t, say $t-1 \to t$ and $x \to t$, we get in the same way

$$c_t \geqq -1 + c_{t-1} + c_x > c_{t-1} .$$

The ***height*** of Q is by definition the length of the longest path in Q. Let h be the height of Q.

Lemma 5 *For any $i \in I$, we have $m(I) \preceq b(i)C^{h+2}$.*

Proof. By Lemma 4 we know that $m(I) \preceq m(I)C$, thus we only have to show that $m(I) \preceq b(i)C^{r_i}$ for some $0 \leqq r_i \leqq h+2$.

First, consider the case where either $\alpha_{10} \geqq 2$, or else $\alpha_{10} = 1$ and $0 \to t$, thus $b(0) \preceq b(0)C$ according to Lemma 3. Given any vertex $i \in I$, let t_i be the smallest length of a path from 0 to i. Then clearly $(b(i)C^{t_i})_0 \geqq 1$, thus $b(0) \preceq b(i)C^{t_i}$, since $b(i)C^{t_i}$ belongs to $\mathcal{K}$. On the other hand, we claim that $m(I) \preceq b(0)C$. By assumption, $b(0) \preceq b(0)C$. If $x \to y$ in I, and y does not belong to the grip, then clearly $(b(0)C)_x \leqq (b(0)C)_y$, and therefore $m(I) \preceq b(0)C$. Altogether, we see that $m(I) \preceq b(i)C^{t_i+1}$, and $t_i + 1 \leqq h+1$.

Thus, we can assume that $\alpha_{10} = 1$ and that t is not a neighbour of 0. Let y be a neighbour of 0. We claim that for any $i \in I$ there exists $0 \leqq t_i \leqq h$ such that $e(y) \preceq b(i)C^{t_i}$.

First, assume that there exists a path from y to i, say of length t_i, with t_i being minimal. Then clearly $t_i \leqq h-1$, and $e(y) \preceq b(i)C^{t_i}$.

Second, assume that $i \neq 0$, and that there does not exist a path from y to i. Let t_i be the minimal length of a path from 0 to i. Then $t_i \leqq h$, and $e(y) \preceq b(i)C^{t_i}$.

Finally, consider the case of $i = 0$. According to Lemma 3, there exists some $x \to t$ with $b(x) \preceq b(0)C$. Since 0 is not a neighbour of t, we have $x \neq 0$. Let l_x be the smallest length of a path from 0 to x, thus $l_x \leqq h-1$. Using the previous considerations, $b(y) \preceq b(x)C^{l_x}$. Thus $b(y) \preceq b(0)C^{t_x}$, where $t_x = l_x + 1 \leqq h$.

Consider the case that $\alpha_{x0} = 1$ for all arrows $0 \to x$. Then, there is a path $x_0 \to x_1 \to \ldots \to x_s$ from $0 = x_0$ to $t = x_s$ different from the grip. Also, besides 1 and x_1 there has to be an additional neighbour y of 0. In this case

$$b(0) + e(x_1) + \ldots e(x_s) \preceq e(y)C$$

and

$$m(I) \preceq (b(0) + e(x_1) + \ldots e(x_s))C ,$$

thus $m(I) \preceq e(y)C^2$. Altogether, we see that $m(I) \preceq b(i)C^{t_i+2}$ for all $i \in I$.

Finally, we have to consider the case where $\alpha_{y0} \geqq 2$, for some $y \neq 1$ (and $\alpha_{10} = 1$). Then $m(I) \preceq e(y)C$, and therefore $m(I) \preceq b(i)C^{l_i+1}$ for all $i \in I$.

Proposition 3 *The transformation C^{h+t+2} maps $\mathcal{K}$ into $\mathcal{K}^\circ$.*

Proof. According to Lemma 5, $m(I) \preceq b(i)C^{h+2}$. Using Lemma 4 and Lemma 3, we see by induction that $m(I) + \sum_{j=t-r+1}^{t} b(j) \preceq m(I)C^r$. Altogether,

$$\sum_{i \in I} b(i) \preceq m(I)C^t .$$

This completes the proof.

Proof of the theorem

Let A be a generalized Cartan matrix which is connected and neither of finite nor of affine type. According to [A, SS] we only have to consider the case where Q_A contains a cycle. By Proposition 1, there exists an admissible change of orientation ω so that $(A, \omega Q)$ has a grip. Since the Coxeter transformations $C(A,Q)$ and $C(A,\omega Q)$ are similar, we may replace Q by ωQ. Let $C = C(A,Q)$. We have constructed a basis $b(i)$, $i \in I$, such that the cone $\mathcal{K} = \mathcal{K}(b(i) | \, i \in I)$ is mapped under C into itself, and under some power of C into its interior. The Perron-Frobenius theorem [G, S] shows that there exists a unique eigenvector v inside $\mathcal{K}^\circ$, that the corresponding eigenvalue is just the spectral radius $\rho = \rho(C)$, that the multiplicity of this eigenvalue is 1, and that any other eigenvalue λ satisfies $|\lambda| < \rho$. It follows from $m(I) \preceq m(I)C$ that $\rho \geqq 1$. On the other hand, it is well-known [BMW] that there exists an eigenvector with non-negative coordinates and with eigenvalue 1 only in case A is of affine type, thus $\rho > 1$. This completes the proof.

Remarks. 1. In order to present a proof of the theorem for generalized Cartan matrices with cycles, one only has to consider grips $(i_0, \ldots, i_t)$ such that there is an additional path from i_0 to i_t. In this case, some of our considerations can be deleted. We have dealt with the more general situation in order to outline that corresponding cones also do exist for certain generalized Cartan matrices A without cycles.

2. We may delete the last condition in the definition of a grip and will obtain similar properties for the corresponding cone. This condition was introduced only in order to facilitate the proof.

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