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Directing projective modules

By

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Let A be an Artin algebra. The A-modules which we consider are always left modules of finite length. If X, Y, Z are A-modules, the composition of maps $f: X \to Y$ and $g: Y \to Z$ is denoted by $fg: X \to Z$. The category of (finite length) A-modules is denoted by A-mod. If X, Y are indecomposable A-modules, we denote by $\operatorname{rad}(X, Y)$ the set of non-invertible maps from X to Y. A *path* in A-mod is a sequence $(X_0, \ldots, X_s)$ of (isomorphism classes of) indecomposable A-modules X_i, $0 \leqq i \leqq s$ such that $\operatorname{rad}(X_{i-1}, X_i) \neq 0$ for all $1 \leqq i \leqq s$. We will say that $(X_0, \ldots, X_s)$ is a path from X_0 to X_s of length s, and we write $X \preceq X'$, or $X \preceq_A X'$ to indicate that a path from X to X' exists. If $s \geqq 1$, and $X_0 = X_s$, then the path $(X_0, \ldots, X_s)$ is called a *cycle*. A indecomposable A-module is called *directing* if X does not occur in a cycle.

Our first aim will be to extend the definition of a directing module to decomposable modules. We show that an indecomposable projective A-module P is directing if and only if the radical of P is directing. In case the top of P is injective it follows that P is directing if and only if the radical of P is directing as a module over the factor algebra of A by the trace ideal of P.

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1. Directing modules. Let $\tau = \tau_A$ be the Auslander-Reiten translation on A-mod. The kernel of a map f will be denoted by $\operatorname{Ker} f$, its image by $\operatorname{Im} f$.

Lemma. *Let $f: X \to Y$, and $g: Y \to Z$ be maps with $fg = 0$, and assume that there is no direct summand Y' of Y with $\operatorname{Im} f \subseteqq Y' \subseteqq \operatorname{Ker} g$. Then there exists an indecomposable non-projective module W such that $\operatorname{Hom}(X, \tau W) \neq 0$, and $\operatorname{Hom}(W, Z) \neq 0$.*

Proof. Recall that a map $g: Y \to Z$ is called *right minimal* provided $\operatorname{Ker} g$ does not contain a non-zero direct summand of Y.

First, let us show that we may assume that both f, g are non-zero and that g is right minimal. For, let Y_1 be a maximal direct summand of Y contained in the kernel of g, and let $X_1 = f^{-1}(Y_1)$. In case $X_1 = X$, we have $\operatorname{Im} f \subseteqq Y_1 \subseteqq \operatorname{Ker} g$, with Y_1 a direct summand of Y, contrary to our assumption. Thus X_1 is a proper submodule of X. Let $Y = Y_1 \oplus Y_2$, let $f = [f_1, f_2]$, and $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, where $f_i: X \to Y_i$, $g_i: Y_i \to Z$. By definition, both f_2 and g_2 are non-zero, g_2 is right minimal, and $f_2 g_2 = 0$. Thus, we replace f, g, Y by f_2, g_2, Y_2.

Let $K = \operatorname{Ker} g$, with inclusion map $u: K \to Y$. Since $\operatorname{Im} f$ is contained in K, and $f \neq 0$, there is an indecomposable direct summand K_1 of K with $\operatorname{Hom}(X, K_1) \neq 0$. Let $m_1: K_1 \to K$ be the inclusion map. Since g is right minimal, K_1 is not a direct summand of Y, in particular we see that K_1 cannot be injective. Let $0 \to K_1 \to E \to W \to 0$ be an almost split sequence, and denote the map $K_1 \to E$ by h, the map $E \to W$ by e. Since $m_1 u$ is not a split monomorphism, there exists $v: E \to Y$ with $hv = m_1 u$, and therefore also $v': W \to Z$ with $ev' = vg$. We claim that $v' \neq 0$. Otherwise, $vg = 0$, thus there is $v'': E \to K$ such that $v'' u = v$. But $hv'' u = hv = m_1 u$ yields that $hv'' = m_1$, since u is a monomorphism. But with $hv'' = m_1$ also h is split mono, impossible. This contradiction shows that $\operatorname{Hom}(W, Z) \neq 0$. Thus, we have found an indecomposable non-projective module W with $\operatorname{Hom}(W, Z) \neq 0$, and $\operatorname{Hom}(X, \tau W) = \operatorname{Hom}(X, K_1) \neq 0$.

Corollary. *An indecomposable module X is directing if and only if there does not exist an indecomposable non-projective module W such that $X \lesssim \tau W$ and $W \lesssim X$.*

Proof. If there exists an indecomposable non-projective module W such that $X \lesssim \tau W$ and $W \lesssim X$, then we have a cycle containing X. Conversely, assume there exists a cycle $(X_0, \ldots, X_s)$ with $X = X_0 = X_s$, say with non-zero maps $f_i: X_{i-1} \to X_i$, and write $f_j = f_i$ in case $j \equiv i \pmod{s}$. There is some $t \geqq 1$ with $f_1 \cdots f_t \neq 0$, but $f_1 \cdots f_{t+1} = 0$. We apply the Lemma to $f = f_1 \cdots f_t$, and $g = f_{t+1}$, and conclude that there exists an indecomposable non-projective module W such that $X = X_0 \lesssim \tau W$ and $W \lesssim X_{t+1}$, and, of course, $X_{t+1} \lesssim X$.

We use this characterization of indecomposable directing modules in order to extend the definition as follows: an arbitrary (not necessarily indecomposable) module M will be called *directing* provided there do not exist indecomposable direct summands M_1, M_2 of M, and an indecomposable non-projective module W such that $M_1 \lesssim \tau W$ and $W \lesssim M_2$. (General directing modules have been considered already by Bakke in [1]; directing modules which are in addition sincere have been called *partial slice modules* in [3]).

Remark. We may define the notion of a directing object in any abelian category $\mathscr{A}$ which has almost split sequences: we say that the object M of $\mathscr{A}$ is directing if and only if there do not exist indecomposable direct summands M_1, M_2 of M, and an almost split sequence $0 \to U \to V \to W \to 0$ such that $M_1 \lesssim U$ and $W \lesssim M_2$. If we want to emphasize that we consider paths in $\mathscr{A}$, we may write $\lesssim_{\mathscr{A}}$ instead of $\lesssim$. Assume that $\mathscr{A}$ is an exact abelian subcategory of A-mod which also has almost split sequences. If M is a directing A-module which belongs to $\mathscr{A}$, then M is directing when considered as an object of $\mathscr{A}$. For let M_1, M_2 be indecomposable direct summands of M, and let $0 \to U \to V \to W \to 0$ be an almost split sequence in $\mathscr{A}$ such that $M_1 \lesssim U$ and $W \lesssim M_2$. Since we assume that $\mathscr{A}$ is an exact subcategory, the given almost split sequence shows that $\operatorname{Ext}_A^1(W, U) \neq 0$, thus $\operatorname{Hom}(U, \tau_A W) \neq 0$. Altogether we see that $M_1 \lesssim U \lesssim \tau_A W$ and $W \lesssim M_2$ in A-mod, thus M cannot be directing as an A-module.

Directing modules are very special. The main properties can be found in the following three Propositions. Given any module M, its *support algebra* is the factor algebra of A modulo the ideal generated by all idempotents which annihilate M.

Proposition 1. *Let M be directing, and let B be the support algebra of M. Then* End (M) *is hereditary,* $\operatorname{Ext}^1(M, M) = 0$, *and M, as a B-module, is a partial tilting module.*

Proof. Given indecomposable direct summands M_i, M_j of M, then $\operatorname{Ext}^1(M_i, M_j) = 0$, since otherwise $\operatorname{Hom}(M_j, \tau M_i) \neq 0$, contrary to the assumption that M is directing. Thus we have $\operatorname{Ext}^1(M, M) = 0$.

Since B-mod is an exact abelian subcategory having almost split sequences, we see that M is also directing as a B-module.

We claim that the projective dimension of any indecomposable summand M_i of M as a B-module is at most 1. Otherwise, there is an indecomposable injective B-module I with $\operatorname{Hom}(I, \tau_B M_i) \neq 0$. Since B is the support algebra of M, there exists some direct summand M_j of M with $\operatorname{Hom}(M_j, I) \neq 0$, thus we obtain $M_j \lesssim I \lesssim \tau_B M_i$, impossible. This shows that M as a B-module is a partial tilting module. In the terminology of [3], the B-module M is a partial slice module, thus End M is hereditary.

The next proposition collects the information on paths $(X_1, \ldots, X_s)$, where the X_i are direct summands of a directing module.

If $(X_0, \ldots, X_s)$ is a path, we say that a path $(Y_0, \ldots, Y_t)$ is a *refinement* of $(X_0, \ldots, X_s)$ if there is an order-preserving function $\pi: \{0, \ldots, s\} \to \{0, \ldots, t\}$ such that $X_i = Y_{\pi(i)}$ and $\pi(0) = 0$, $\pi(s) = t$.

We recall that for indecomposable A-modules X, Y, the set $\operatorname{rad}^2(X, Y)$ consists of all finite sums of maps of the form fg, where $f \in \operatorname{rad}(X, C)$, $g \in \operatorname{rad}(C, Y)$, with C indecomposable.

Proposition 2. *Let $(X_0, \ldots, X_s)$ be a path, and assume that there does not exist an indecomposable non-projective module W with $X_0 \lesssim \tau W$ and $W \lesssim X_s$. Let $X = \bigoplus_{i=0}^{s} X_i$. Then the following assertions hold:*

(a) *The module X is directing.*

(b) *If $f_i: X_{i-1} \to X_i$ are non-zero maps, for $1 \leqq i \leqq s$, then $f_1 \cdots f_s \neq 0$.*

(c) $\operatorname{Hom}(X_i, \tau X_j) = 0$, *for all i, j.*

(d) *The number s is bounded by the number of isomorphism classes of simple A-modules.*

(e) *The path can be refined to a path $(Y_0, \ldots, Y_t)$ such that* $\operatorname{rad}^2(Y_{i-1}, Y_i) = 0$ *for* $1 \leqq i \leqq t$.

(f) *Assume we have* $\operatorname{rad}^2(X_{i-1}, X_i) = 0$, *for $1 \leqq i \leqq s$, and let $0 \neq f_i \in \operatorname{rad}(X_{i-1}, X_i)$. If $f_1 \cdots f_s = gh$ for some maps $g: X_0 \to Z$, $h: Z \to X_s$ with Z indecomposable, then there exists some i with $0 \leqq i \leqq s$ and an isomorphism $\eta: Z \to X_i$ such that $g\eta = f_1 \cdots f_i$.*

Proof. (a) Assume there exists an indecomposable non-projective module W such that $X_i \lesssim \tau W$ and $W \lesssim X_j$, for some i, j. Then $X_0 \lesssim X_i \lesssim \tau W$ and $W \lesssim X_j \lesssim X_s$, contrary to the assumption.

(b) This is a direct consequence of Lemma 1.

(c) If $\operatorname{Hom}(X_i, \tau X_j) \neq 0$, then $X_0 \lesssim X_i \lesssim \tau W$, and $W \lesssim X_s$, with $W = X_j$, contrary to our assumption.

(d) This follows from Proposition 1.

(e) According to (d) there exists a refinement which cannot be further refined. But if $(Y_0, \ldots, Y_t)$ is a path which cannot be refined, then necessarily $\operatorname{rad}^2(Y_{i-1}, Y_i) = 0$.

(f) Let us assume that $\operatorname{rad}^2(X_{i-1}, X_i) = 0$, and let $0 \neq f_i \in \operatorname{rad}(X_{i-1}, X_i)$ for $1 \leqq i \leqq s$. Let us assume that $f_1 \cdots f_s = gh$ for some maps $g: X_0 \to Z$, $h: Z \to X_s$, where Z is indecomposable. If h is an isomorphism, let $i = s$, and $\eta = h$. Thus let us assume that h is not an isomorphism. We use induction on s. For $s = 1$ the map g has to be an isomorphism, thus let $\eta = g^{-1}$. Consider the case $s \geqq 2$. Let $\begin{bmatrix} f_s \\ f_s' \end{bmatrix}$ be a sink map for X_s, thus $h = uf_s + vf_s'$, for some maps u, v. Then $f_1 \cdots f_s = gh = guf_s + gvf_s'$ shows that the map $[f_1 \cdots f_{s-1} - gu, gv]$ factors through the kernel K of $\begin{bmatrix} f_s \\ f_s' \end{bmatrix}$. However, either X_s is projective, and $K = 0$, or else X_s is non-projective, and $K = \tau X_s$. In the latter case, the basic assumption gives $\operatorname{Hom}(X_0, \tau X_s) = 0$, thus always $[f_1 \cdots f_{s-1} - gu, gv] = 0$, and therefore $f_1 \cdots f_{s-1} = gu$. The assertion now follows by induction.

Proposition 3. *Let M be a directing A-module. Let M_i $(i \in I)$ be a complete set (one from each isomorphism class) of indecomposable A-modules M_i such that there are indecomposable direct summands M_i', M_i'' of M with $M_i' \precsim M_i \precsim M_i''$. Then I is finite and $\bar{M} = \bigoplus_{i \in I} M_i$ is directing.*

Proof. Let M_i, M_j belong to the set. Assume there is some indecomposable non-projective module W with $M_i \precsim \tau W$, and $W \precsim M_j$. Then we obtain a path $M_i' \precsim M_i \precsim \tau W \precsim W \precsim M_j \precsim M_j''$, where M_i', M_j'' are direct summands of M, impossible. It follows by Parts (e) and (d) of Proposition 2 that I is finite, since the Auslander-Reiten quiver of any Artin algebra is locally finite. Also we see that $\bar{M}$ is directing.

2. Indecomposable projective modules and their radicals.

Theorem 1. *Let P be an indecomposable projective module. Then the following are equivalent:*
(a) *P is directing.*
(b) rad *P is directing.*
(c) *Each indecomposable direct summand of* rad *P is directing.*

Proof. Clearly, if $(X_0, \ldots, X_s)$ is a cycle with $P = X_0 = X_s$, then we can factor any non-invertible map $X_{s-1} \to X_s = P$ through $\operatorname{rad} P$, thus we can refine the path in order to contain some indecomposable summand M_1 of $\operatorname{rad} P$, thus M_1 is not directing. This shows that (c) implies (a). Trivially we have (b) implies (c).

In order to consider the missing implication, we will use the following Lemma.

Lemma. *Let $f: \operatorname{rad} P \to Y$, $g: Y \to Z$ be non-zero maps with $fg = 0$, and assume that Z is indecomposable, and g is right minimal. Then $P \precsim Z$.*

Proof. Since we assume that g is right minimal, the restriction of g to any non-zero direct summand of Y is non-zero.

If $\operatorname{Hom}(P, Y) \neq 0$, then any indecomposable direct summand Y_1 of Y with $\operatorname{Hom}(P, Y_1) \neq 0$ yields $P \precsim Y_1 \precsim Z$.

Thus we can assume that $\mathrm{Hom}(P, Y) = 0$. Let $S = P/\mathrm{rad}\, P$. The map $f: \mathrm{rad}\, P \to Y$ induces from the canonical exact sequence $0 \to \mathrm{rad}\, P \to P \to S \to 0$ an exact sequence $0 \to Y \to E \to S \to 0$, and we denote the map $Y \to E$ by m, the map $E \to S$ by p. Since $\mathrm{Hom}(P, Y) = 0$, the induced exact sequence cannot split, and also $\mathrm{Hom}(Y, S) = 0$.

Take an indecomposable direct summand E' of E with $\mathrm{Hom}(P, E') \neq 0$, say $E = E' \oplus C$, with inclusion map $u: E' \to E$. The restriction up of p to E' is non-zero, whereas the restriction of p to C is zero. Let $m': Y' \to E'$ be the kernel of $up: E' \to S$, thus Y is isomorphic to $Y' \oplus C$, and there is an inclusion map $v: Y' \to Y$ with $vm = m'u$. Note that $Y' \neq 0$, since otherwise the sequence $0 \to Y \to E \to S \to 0$ would split. Since Y' is a non-zero direct summand of Y, we see that $vg \neq 0$.

Now we consider the exact sequence $0 \to Z \to F \to S \to 0$, induced from $0 \to Y \to E \to S \to 0$ by the map $g: Y \to Z$, and we denote the map $Z \to F$ by m'. Since our sequence is in fact induced via the zero map fg, it follows that m' is a split monomorphism. Thus, there exists a map $g': E \to Z$ with $mg' = g$. Note that the restriction ug' of g' to E' is a non-zero map, since $m'ug' = vmg' = vg \neq 0$.

Altogether, we see that $\mathrm{Hom}(E', Z) \neq 0$, thus $P \leqq E' \leqq Z$. This completes the proof of the Lemma.

In order to complete the proof of the Theorem, let P be an indecomposable projective module, and assume there are indecomposable direct summands M_1, M_2 of rad P and an indecomposable non-projective module W such that $M_1 \leqq \tau W$ and $W \leqq M_2$. Let $(X_0, \ldots, X_s)$ be a path with $X_0 = M_1$, and $X_s = \tau W$, and take non-zero maps $f_i: X_{i-1} \to X_i$, for $1 \leqq i \leqq s$. If $f_1 \cdots f_s = 0$, take t maximal with $f = f_1 \cdots f_t \neq 0$, and $g = f_{t+1}$. The Lemma yields $P \leqq X_{t+1}$, thus $P \leqq X_{t+1} \leqq \tau W \leqq W$. If $f_1 \cdots f_s \neq 0$, let $m: \tau W \to V$ be the source map for τW, and $g: V \to W$ its cokernel. In this case, we apply the Lemma to $f = f_1 \cdots f_s m$, and g, in order to conclude that $P \leqq W$. Always, we have $P \leqq W \leqq M_2 \prec P$, thus P is not directing. This completes the proof of Theorem.

Remark. Let X be an indecomposable directing module and let $E \to X$ be the sink map. Then E need not to be directing. Consider for example the simple injective module $I(4)$ in example 1 (see Section 4).

3. An inductive criterion. Let P an indecomposable projective A-module, let $S = P/\mathrm{rad}\, P$. There are two possible ways of replacing A-mod by a related module category B-mod deleting P. First of all, we may factor out the *trace ideal* I of P, thus I is the sum of all images of maps $P \to {}_A A$. Let $B = A/I$, thus we may identify B-mod with the full subcategory $\mathscr{X}$ of A-mod given by all A-modules M with $\mathrm{Hom}(P, M) = 0$. Note that we have $\mathrm{Hom}(P, M) = 0$ if and only if S is not a composition factor of M. Also, we may consider some projective module P' such that P and P' have no indecomposable direct summand in common, but every indecomposable projective module is a direct summand of $P \oplus P'$. Let $C = \mathrm{End}\, P'$. Then the category C-mod is equivalent to the full subcategory $\mathscr{Y}$ of all A-modules M such that S does not occur as a composition factor of soc M or top M. Note that always $\mathscr{X} \subseteq \mathscr{Y}$.

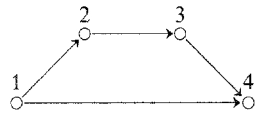
The abelian subcategory $\mathscr{X}$ (but usually not $\mathscr{Y}$) is an exact subcategories and it is closed under extensions. By the remark in Section 1, we see that a directing A-module which belongs to $\mathscr{X}$, is directing also as an object of $\mathscr{X}$.

In case P is directing, $\operatorname{End}(P)$ is a division ring, thus obviously $\operatorname{rad} P$ belongs to $\mathscr{X}$. In addition, for P directing, rad P will be a directing object of $\mathscr{X}$. We are interested to know under what conditions an indecomposable projective module P with $\operatorname{End}(P)$ a divison ring, and such that $\operatorname{rad} P$ is directing as an object of $\mathscr{X}$, is directing itself.

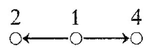
We will present a positive answer in case S is injective, so that $\mathscr{X} = \mathscr{Y}$. (In this case, the algebra A is sometimes said to be a *one-point extension* $A = B[N]$ of B by the B-module $N = \operatorname{rad} P$).

But first we show in an example that in general the conditions above are not sufficient to ensure that P is directing.

Example 1. For this let A be given as the path algebra over the field k of the following quiver modulo the ideal generated by all paths of length two:

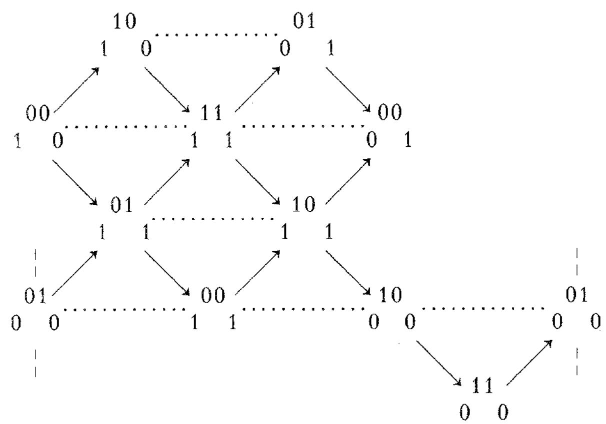


We denote by e_1, e_2, e_3, e_4 the idempotents of A corresponding to the vertices of the quiver. We denote by $S(i)$ the simple module corresponding to the vertex i, by $P(i)$ its projective cover and by $I(i)$ its injective hull. Note that we consider left modules, thus $S(1)$ is simple projective. We consider the indecomposable projective A-module $P(3)$. Note that $\operatorname{End}(P(3)) \cong k$ and $\operatorname{rad} P(3) = S(2)$. Let $e = e_1 + e_2 + e_4$. Then $C = eAe \cong B = A/Ae_3 A$ is a hereditary algebra with quiver



In particular we see that $S(2)$ is a directing B-module.

We denote the indecomposable A-modules by their dimension vectors. The Auslander-Reiten quiver is given as follows, where the horizontal dotted lines indicate the Auslander-Reiten translation, while identification is along the vertical dashed lines.



So we see that $P(3)$ is not directing, since we have a path

$$P(3) \to P(4) \to I(1) \to S(2) \to P(3).$$

Theorem 2. *Let P be indecomposable projective, and assume $S = P/\operatorname{rad} P$ is injective. Let I be the trace ideal of P in A, and $B = A/I$. Then P is directing if and only if* $\operatorname{rad} P$ *is directing as a B-module.*

Proof. If P is directing, then $\operatorname{rad} P$ is directing as an A-module, thus as a B-module.

Before we consider the converse implication, let us recall the following: Given an A-module X, we denote by ιX the maximal B-submodule of X, thus $X/\iota X$ is a direct sum of copies of S. Note that if X is an indecomposable A-module and $\iota X \neq X$, and Y is an indecomposable direct summand of ιX, then $\operatorname{Hom}(\operatorname{rad} P, Y) \neq 0$. (For, $\operatorname{Hom}(\operatorname{rad} P, Y)$ maps onto $\operatorname{Ext}^1(S, Y)$, and the latter group has to be non-zero.)

Now, let $\operatorname{rad} P$ be a directing B-module. First, we show: Let X be an indecomposable B-module, let $X' \to X$ be its sink map in A-mod, and assume $X \lesssim_B Z$ for some indecomposable direct summand Z of $\operatorname{rad} P$. Then X' is a B-module. For the proof, we distinguish two cases: If X is a projective B-module, then $X' = \operatorname{rad} X$ is a submodule of X, thus also a B-module. As second case, we assume that X is non-projective as a B-module, thus also non-projective as an A-module. Then $\iota\tau_A X = \tau_B X$ (see [4] or [5]). We claim that $\iota\tau_A X = \tau_A X$. Otherwise $\operatorname{Hom}(\operatorname{rad} P, \tau_B X) \neq 0$, by the preceeding remark. Let Z' be an indecomposable direct summand of $\operatorname{rad} P$ with $\operatorname{Hom}(Z', \tau_B X) \neq 0$, then we obtain the path $Z' \lesssim_B \tau_B X \lesssim_B X \lesssim_B Z$, contrary to our assumption that $\operatorname{rad} P$ is directing in B-mod. But $\iota\tau_A X = \tau_A X$ means that $\tau_A X$ is a B-module, and therefore also X'.

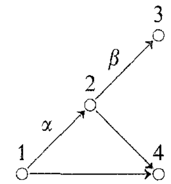
Let us assume that there exists a path $(X_0, \ldots, X_{s+1})$, where $X_0 = P = X_{s+1}$. We may assume that X_s is a direct summand of $\operatorname{rad} P$, therefore $s \geqq 2$. Note that if $\operatorname{Hom}(P, X_t) \neq 0$, for some $2 \leqq t < s$, we may delete $X_1, \ldots, X_{t-1}$ from the path, thus we can assume that $\operatorname{Hom}(P, X_i) = 0$, for $2 \leqq i \leqq s$.

First of all, we show that the length $s + 1$ of such paths is bounded. Let $f: X_1 \to X_2$ be a non-zero map. Note that f cannot vanish on ιX_1, since otherwise the image of f would be a direct sum of copies of S, but S does not occur as a composition factor of X_2. Let Y be an indecomposable direct summand of ιX_1, say with inclusion map $u: Y \to X_1$ such that $uf \neq 0$. According to the remark above, there exists an indecomposable direct summand M_1 of $\operatorname{rad} P$ such that $\operatorname{Hom}(M_1, Y) \neq 0$. Then we see that we obtain a path $(M_1, Y, X_2, \ldots, X_s)$ of length s in B-mod starting and ending in a direct summand of $\operatorname{rad} P$. According to Section 1, the length of such paths is bounded.

On the other hand, we claim that we may replace the path $(X_0, \ldots, X_s)$ by a similar one with s increased by 1. Namely, let $X_2' \to X_2$ be the sink map for X_2. We can factor f through X_2'. In particular, there exists an indecomposable direct summand Z of X_2' such that $\operatorname{Hom}(X_1, Z) \neq 0$. Since there exists an irreducible map $Z \to X_2$, we have $\operatorname{rad}(Z, X_2) \neq 0$. Also, since X_2 is an indecomposable B-module and a predecessor of the direct summand X_s of $\operatorname{rad} P$, we know that X_2' is a B-module, thus X_1 and Z cannot be isomorphic, thus $\operatorname{rad}(X_1, Z) \neq 0$. Altogether we obtain a path $(P, X_1, Z, X_2, \ldots, X_s, P)$ with similar properties as the given one, and with s being increased by 1. This contradiction completes the proof.

4. When are all indecomposable projective modules directing? Theorem 2 may be used in order to construct algebras so that all indecomposable projective modules are directing. However, we should remark that starting with an algebra B such that all indecomposable projective B-modules are directing, and a directing B-module M, some of the indecomposable projective B-modules may cease to be directing when considered as modules over the one-point extension algebra $A = B[M]$, as the following example shows:

Example 2. Let A be given as the path algebra over the field k by:



with relation $\alpha\beta = 0$.

Let B be the support algebra of $S(1)$, $S(2)$ and $S(3)$. Then all indecomposable projective B-modules are directing. But $P(3)$ is not a directing A-module. This follows directly from Theorem 2. In fact $\operatorname{rad} P(3) = S(2)$, and $S(2)$ is a simple regular module over the tame hereditary algebra C obtained from A by factoring out the trace ideal of $P(3)$. So $S(2)$ is not a directing C-module. Note that $\operatorname{rad} P(4)$ is a directing B-module, so $P(4)$ is a directing A-module.

We point out that in the preceding example all indecomposable injective A-modules are directing.

Given an Artin algebra A, we may consider its *quiver* $Q(A)$. Recall that $Q(A)$ is defined as follows: the vertices of $Q(A)$ are the isomorphism classes $[S]$ of the simple A-modules S, and there is an arrow $[S'] \to [S]$ provided $\operatorname{Ext}^1(S, S') \neq 0$. (In this way, for a finite-dimensional basic k-algebra A over an algebraically closed field k the path algebra of $Q(A)$ will map onto A; note that some publications (for example [4]) call the opposite of $Q(A)$ the quiver of A.) We will label the vertices of $Q(A)$ by numbers or letters; given such a label a, we denote by $S(a)$ a representative of the isomorphism class a.

Note that an algebra A such that all indecomposable projective A-modules are directing, necessarily has a directed quiver $Q(A)$.

Let A be an algebra with directed quiver $Q(A)$. A labelling $\{a_1, \ldots, a_n\}$ of the vertices of $Q(A)$ will be called *admissible*, provided $\operatorname{Ext}^1(S(a_i), S(a_j)) \neq 0$ implies that $i > j$. Of course, any admissible labelling allows to reconstruct A as a succession of one-point extensions: Let $A_t = A(a_1, \ldots, a_t)$ be the support algebra of $\bigoplus_{i=1}^{t} S(a_i)$. Then $N_t = \operatorname{rad} P(a_{t+1})$ is an A_t-module, and $A_{t+1} = A_t[N_t]$.

We also consider a partial order on the vertices of $Q(A)$ by defining $a \lesssim b$ if there is a path in $Q(A)$ from a to b. Let a be a vertex of $Q(A)$, then we define A^a as the support algebra of $\bigoplus_{a \not\lesssim b} S(b)$. Then $\operatorname{rad} P(a)$ is an A^a-module. Note that for vertices a, b of $Q(A)$ with $a \lesssim b$ we have a path from $P(a)$ to $P(b)$ in A-mod, so $P(a) \lesssim_A P(b)$.

Theorem 3. *Let A be an algebra with directed quiver $Q(A)$. Then the following are equivalent:*

(a) *All indecomposable projective A-modules are directing.*

(b) *For any admissible labelling $a_1, \ldots, a_n$ of the vertices of $Q(A)$, the radical of $P(a_{t+1})$ is a directing $A(a_1, \ldots, a_t)$-module.*

(c) *For all vertices a of $Q(A)$, the A^a-module* $\operatorname{rad} P(a)$ *is directing.*

Proof. To show that (a) implies (b) let $P(a_{t+1})$ be a directing A-module, then it is also a directing $A(a_1, \ldots, a_{t+1})$-module, thus $\operatorname{rad} P(a_{t+1})$ is a directing $A(a_1, \ldots, a_t)$-module.

Let a be a vertex of $Q(A)$. Then there exists an admissible labelling $a_1, \ldots, a_n$ of the vertices such that A^a is of the form A_t for some t and $a = a_{t+1}$. This shows that (b) implies (c).

To show the missing implication assume that there exists an indecomposable projective A-module P which is not directing. Let $S = P/\operatorname{rad} P$. Let $(X_0, \ldots, X_s)$ be a path in A-mod with $X_0 = P = X_s$. We can assume that for any sink $[S']$ in $Q(A)$, the simple module S' appears as a composition factor of at least one of the X_i. We claim that we can assume that $[S]$ is a sink in $Q(A)$. For, if $[S]$ is not a sink, let $[S']$ be a sink with a path from $[S]$ to $[S']$. Let P' be a projective cover of S', then $P \prec P'$. By assumption, $\operatorname{Hom}(P', X_i) \neq 0$ for some i, thus we obtain a path $P' \leqq X_i \leqq P \prec P'$, thus we may consider P' instead of P.

If $[S] = [S(a)]$ is a sink in $Q(A)$, then $A = A^a[\operatorname{rad} P(a)]$. According to Theorem 2, $\operatorname{rad} P(a)$ cannot be a directing A^a-module. This completes the proof.

Let us stress that Example 2 shows that it is not sufficient to know that for *one* admissible labelling $a_1, \ldots, a_n$ of the vertices of $Q(A)$, the radical of $P(a_{t+1})$ is a directing $A(a_1, \ldots, a_t)$-module in order to conclude that the indecomposable projective A-modules are directing.

Let A be an Artin algebra. Then A is called *representation-finite* if there are only a finite number of isomorphism classes of indecomposable A-modules. A representation-finite Artin algebra A is said to be *representation-directed* if all indecomposable A-modules are directing, or equivalently if the Auslander-Reiten quiver does not contain an oriented cycle. The following result is due to Bautista and Smalø [2], we are going to present an alternative proof.

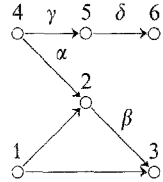
Proposition 4. *Let A be representation-finite. Then A is representation-directed if and only if all indecomposable projective A-modules are directing.*

Proof. Suppose that all indecomposable projective A-modules are directing and assume that there is an indecomposable A-module $X = X_0$ which is not directing. Let $(X_0, \ldots, X_s)$ be a cycle, which we may assume to be a cycle of the Auslander-Reiten quiver. Since there is no indecomposable projective on this cycle, also $(\tau X_0, \ldots, \tau X_s)$ is a cycle. Since A is representation-finite we infer that X_0 is τ-periodic. So we may assume that the given cycle is of the form $(X, E_1, \tau^- X, E_2, \ldots, E_r, \tau^{-r} X = X)$ for some $r \in \mathbb{N}$. Let P be an indecomposable projective A-module with $\operatorname{Hom}(P, X) \neq 0$, and let $(P = Y_0, Y_1, \ldots, Y_n = X)$ be a path from P to X, which we may assume to be a path in the Auslander-Reiten quiver. We now construct inductively for all $i \geqq 0$ a path $(\tau^i X = \tau^i Y_n, \tau^{i-1} Y_{n-1}, \ldots, \tau Y_{n-i+1}, Y_{n-i})$. For $i = 0$ there is nothing to show. Let $(\tau^i X = \tau^i Y_n, \tau^{i-1} Y_{n-1}, \ldots, \tau Y_{n-i+1}, Y_{n-i})$ be the path from $\tau^i X$ to Y_{n-i}. All modules on this path are not directing. Thus there is no projective module on this path. Applying τ to this path yields a path from $\tau^{i+1} X$ to τY_{n-i}. Combining this with the arrow $\tau Y_{n-i} \to Y_{n-(i+1)}$ gives now the required path from $\tau^{i+1} X$ to $Y_{n-(i+1)}$. This shows $P \leqq X \leqq \tau^n X \leqq P$, a contradiction.

The converse implication is clear.

The following example shows that in general the components of the Auslander-Reiten quiver containing indecomposable directing projective modules may contain indecomposable modules which are not directing.

Example 3. Let A be given as the path algebra over the field k by



with relations $\alpha\beta = \gamma\delta = 0$.

Then all indecomposable projective A-modules are directing, as can be seen by using Theorem 2. However the component of the Auslander-Reiten quiver containing P(6) contains modules which are not directing. One may take for example $S(2)$. Note that we have irreducible maps from $I(4)$ to $S(2)$ and to $S(5) = \operatorname{rad} P(6)$.

Finally, let us add the following remark:

Proposition 5. *The A-module ${}_AA$ is directing if and only if A is hereditary.*

Proof. In case A is hereditary, any indecomposable module X with $X \precsim P$ for some indecomposable projective module P is projective itself, thus ${}_AA$ is directing.

Conversely, assume that A is not hereditary. Then there exists an indecomposable projective A-module P with an indecomposable submodule U which is not projective. Since U is non-projective, we can form τU, and there is some indecomposable projective module P' with $\operatorname{Hom}(P', \tau U) \neq 0$. Since P, P' are direct summands of ${}_AA$, we see that ${}_AA$ cannot be directing.

Added in proof.*) There is a recent preprint by A. Skowroński and M. Wenderlich: *Artin algebras with directing indecomposable projective modules.* It contains parallel results and further interesting investigations.

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