The Module Theoretical Approach to Quasi-hereditary Algebras

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Quasi-hereditary algebras were introduced by L.Scott [S] in order to deal with highest weight categories as they arise in the representation theory of semi–simple complex Lie algebras and algebraic groups. Since then, also many other algebras arising naturally, such as the Auslander algebras, have been shown to be quasi-hereditary. It seems to be rather surprising that the class of quasi-hereditary algebras, defined in purely ring–theoretical terms, has not been studied before in ring theory.

The central concept of the theory of quasi-hereditary algebras are the notions of a standard and a costandard module; these modules depend in an essential way, on a (partial) ordering of the set of all simple modules. So we start with a finite dimensional algebra A, and a partial ordering of the simple A–modules, in order to define the standard modules $\Delta(i)$ and the costandard modules $\nabla(i)$; we have to impose some additional conditions on their endomorphism rings, and on the existence of some filtrations, in order to deal with quasi-hereditary algebras. This is the content of the first chapter. The second chapter collects some properties of quasi-hereditary algebras, in particular those needed in later parts of the paper. The third chapter presents the process of standardization: here, we give a characterization of the categories of Δ–filtered modules over quasi-hereditary algebras. In fact, we show that given indecomposable A–modules $\Theta(1), \dots, \Theta(n)$ over a finite–dimensional algebra such that $\mathrm{rad}(\Theta(i), \Theta(j)) = 0$, and $\mathrm{Ext}^1(\Theta(i), \Theta(j)) = 0$ for all $i \geq j$, the category $\mathcal{F}(\Theta)$ of all A–modules with a Θ–filtration is equivalent (as an exact category) to the category of all Δ–filtered modules over a quasi-hereditary algebra.

The forth and the fifth chapter consider cases when the standard modules over a quasi-hereditary algebra have special homological properties: first, we assume that any $\Delta(i)$ has projective dimension at most 1, then we deal with the case that the dominant dimension of any $\Delta(i)$ is at least 1. Both these properties, as well as their duals, are satisfied for the Auslander algebras of a uniserial algebra, and we are going to present the Auslander–Reiten quivers of the category of Δ–filtered modules for some examples.

These notes give a unified treatment of some basic results of Cline–Parshall–Scott [PS,CPS], Dlab–Ringel [DR2], Donkin [D], Ringel [R2] and Soergel [So]; they are intended as a guideline for understanding further investigations in [PS] and [R4]. We will not cover those developments of the theory of quasi-hereditary algebras which are formulated in terms of the internal ring structure (these results are rather easily available in the literature, some of the references are listed in the bibliography at the end of the paper).

1. Definition of a quasi-hereditary algebra

Let k be a field, and A a finite–dimensional k–algebra. We denote by A–mod the category of all (finite–dimensional left) A–modules. If Θ is a class of A–modules (closed under isomorphisms), $\mathcal{F}(\Theta)$ denotes the class of all A–modules M which have a Θ–filtration, i.e. a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{t-1} \supseteq M_t \supseteq \cdots \supseteq M_m = 0$ such that all factors $M_{t-1}/M_t, \quad 1 \leq t \leq m$, belong to Θ.

Let $E(\lambda), \quad \lambda \in \Lambda$, be the simple A–modules (one from each isomorphism class), and we assume that the index set Λ is endowed with a partial ordering. If M is an A–module, we denote the Jordan–Hölder multiplicity of $E(\lambda)$ in M by $[M : E(\lambda)]$. For each $\lambda \in \Lambda$, let $P(\lambda)$ be the projective cover, and $Q(\lambda)$ the injective hull of $E(\lambda)$. Denote by $\Delta(\lambda) = \Delta_A(\lambda) = \Delta_\Lambda(\lambda)$ the maximal factor module of $P(\lambda)$ with composition factors of the form $E(\mu)$ where $\mu \leq \lambda$; these modules $\Delta(\lambda)$ are called the *standard* modules, and we set $\Delta = \{\Delta(\lambda) | \lambda \in \Lambda\}$. Similarly, denote by $\nabla(\lambda) = \nabla_A(\lambda) = \nabla_\Lambda(\lambda)$ the maximal submodule of $Q(\lambda)$ with composition factors of the form $E(\mu)$ where $\mu \leq \lambda$; in this way, we obtain the set $\nabla = \{\nabla(\lambda) | \lambda \in \Lambda\}$ of *costandard* modules. Let us point out that $\nabla(\lambda)$ is the dual of a corresponding standard module: Let $D = \mathrm{Hom}_k(-, k)$ be the duality with respect to the base field. Let A^o be the opposite algebra of A, with simple modules $E_{A^o}(\lambda) = DE_A(\lambda)$ (note that we use the same index set!). Then $\nabla_A(\lambda) = D\Delta_{A^o}(\lambda)$. It follows that any statement on standard modules yields a corresponding statement for costandard modules, we often will refrain from stating the dual results explicitly.

Note that the only module M with $\mathrm{Hom}(\Delta(\lambda), M) = 0$ for all $\lambda \in \Lambda$ is the zero module $M = 0$. [For $M \neq 0$, let $E(\lambda)$ be a submodule, then $\mathrm{Hom}(\Delta(\lambda), M) \neq 0$.] Dually, the only module M with $\mathrm{Hom}(M, \nabla(\lambda)) = 0$ is the zero module.

Given a set $\mathcal{X}$ of A–modules, then for any A–module M, we denote by $\eta_{\mathcal{X}} M$ the *trace* of $\mathcal{X}$ in M, it is the maximal submodule of M generated by $\mathcal{X}$.

The standard modules may be characterized as follows:

Lemma 1.1. *For any A-module M, and $\lambda \in \Lambda$ the following assertions are equivalent:*
(i) $M \cong \Delta(\lambda)$,
(ii) $\operatorname{top} M \cong E(\lambda)$, *all composition factors of M are of the form* $E(\mu)$, *with* $\mu \leq \lambda$, *and* $\operatorname{Ext}^1(M, E(\mu)) = 0$ *for all* $\mu \leq \lambda$,
(iii) $M \cong P(\lambda)/\eta_{\{P(\mu)|\, \mu \not\leq \lambda\}} P(\lambda)$.

Lemma 1.2. *Let M be an A-module, and* $\lambda, \mu \in \Lambda$. *Then:*
(a) $\operatorname{Hom}(\Delta(\lambda), M) \neq 0$ *implies* $[M : E(\lambda)] \neq 0$.
(b) $\operatorname{Hom}(\Delta(\lambda), \Delta(\mu)) \neq 0$ *implies* $\lambda \leq \mu$.
(c) $\operatorname{Hom}(\Delta(\lambda), \nabla(\mu)) \neq 0$ *implies* $\lambda = \mu$.

For the proof of (c), we use both (a) and its dual statement.

The sets Δ and ∇ depend, in an essential way, on the given partial ordering of Λ. We will say that two partially ordered sets Λ, Λ' used as index sets for the simple A-modules are *equivalent* provided the sets $\{\Delta_\Lambda(\lambda) |\, \lambda \in \Lambda\}$ and $\{\Delta_{\Lambda'}(\lambda) |\, \lambda \in \Lambda'\}$ coincide, and $\{\nabla_\Lambda(\lambda) |\, \lambda \in \Lambda\}$ and $\{\nabla_{\Lambda'}(\lambda) |\, \lambda \in \Lambda'\}$ coincide.

In general, the standard, and the costandard modules will change when we refine the ordering. In order to avoid this to happen, we usually will consider only adapted orderings in the sense of the following definition: A partial ordering Λ of the set of simple A-modules $\{E(\lambda) |\, \lambda \in \Lambda\}$ is said to be *adapted*, provided the following condition holds: for every A-module M with $\operatorname{top} M \cong E(\lambda_1)$ and $\operatorname{soc} M \cong E(\lambda_2)$, where λ_1 and λ_2 are incomparable, there is some $\mu \in \Lambda$ with $\mu > \lambda_1$ and $\mu > \lambda_2$ such that $[M : E(\mu)] \neq 0$. [Observe that we may weaken the condition as follows: we only have to require the existence of some $\mu \in \Lambda$ with $\mu > \lambda_1$ *or* $\mu > \lambda_2$ such that $[M : E(\mu)] \neq 0$. Indeed, in case the weaker condition is satisfied, assume that there exists a module M with top $E(\lambda_1)$, socle $E(\lambda_2)$, where λ_1 and λ_2 are incomparable, and such that there is no $\mu \in \Lambda$ with $\mu > \lambda_1, \mu > \lambda_2$ and $[M : E(\mu)] \neq 0$. We may assume that M is of smallest possible length, and we know that there is at least a μ with $[M : E(\mu)] \neq 0$ and either $\mu > \lambda_1$ or $\mu > \lambda_2$. Assume we have $\mu > \lambda_1$. Now M has a submodule M' with top $E(\mu)$. Note that μ cannot be comparable with λ_2. The minimality of M implies that there is ν such that $\nu > \mu, \nu > \lambda_2$, and $[M' : E(\nu)] \neq 0$. But $\nu > \mu > \lambda_1, \nu > \lambda_2$, and $[M : E(\nu)] \neq 0$, contrary to our assumption.] As an example of non-adapted partial orderings, the reader should have in mind any non-semisimple algebra with the discrete ordering of the simple modules.

If Λ' is a refinement of Λ, and Λ is adapted, then clearly $\Delta_{\Lambda'}(\lambda) = \Delta_\Lambda(\lambda)$ and $\nabla_{\Lambda'}(\lambda) = \nabla_\Lambda(\lambda)$ for all $\lambda \in \Lambda$, thus Λ and Λ' are equivalent, and Λ' also is adapted. Thus, for Λ adapted, we always may assume that we deal with a total ordering. In such a case, we may replace Λ by the equivalent index set $\{1, 2, \ldots, n\}$ with its natural ordering.

In case we deal with an adapted partial ordering, we may reformulate Lemma 1.1 as follows:

Lemma 1.1′. *For any A-module M, and $\lambda \in \Lambda$, where Λ is adapted, the following assertions are equivalent:*
(i) $M \cong \Delta(\lambda)$,
(ii) $\operatorname{top} M \cong E(\lambda)$, *all composition factors of M are of the form $E(\mu)$, with $\mu \leq \lambda$, and* $\operatorname{Ext}^1(M, E(\mu)) \neq 0$ *implies* $\mu > \lambda$,
(iii) $M \cong P(\lambda)/\eta_{\{P(\mu) \mid \mu > \lambda\}} P(\lambda)$.

As an immediate consequence, we see:

Lemma 1.3. *Assume Λ is adapted. Let M be an A-module, and $\lambda, \mu \in \Lambda$. Then*
(a) If $\operatorname{Ext}^1(\Delta(\lambda), M) \neq 0$, *then* $[M : E(\mu)] \neq 0$, *for some* $\mu > \lambda$.
(b) If $\operatorname{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0$, *then* $\lambda < \mu$.
(c) $\operatorname{Ext}^1(\Delta(\lambda), \nabla(\mu)) = 0$.

For (a), note that $\operatorname{Ext}^1(\Delta(\lambda), M) \neq 0$ implies that $\operatorname{Ext}^1(\Delta(\lambda), E) \neq 0$ for some composition factor E of M. Let $E = E(\mu)$, then $\mu > \lambda$, according to Lemma 1.1′(ii). For the proof of (b), assume $\operatorname{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0$. Then $[\Delta(\mu) : E(\nu)] \neq 0$, for some $\nu > \lambda$, according to (a). However $[\Delta(\mu) : E(\nu)] \neq 0$ implies $\nu \leq \mu$, therefore $\lambda < \nu \leq \mu$. Similarly, for the proof of (c), we use (a) in order to see that $\operatorname{Ext}^1(\Delta(\lambda), \nabla(\mu)) \neq 0$ implies $\lambda < \mu$. But in this case, the duality also yields the dual statement $\lambda > \mu$, so we obtain a contradiction.

The main interest will lie on the subcategory $\mathcal{F}(\Delta)$ of all A-modules with a Δ-filtration. First of all, let us point out that usually $\mathcal{F}(\Delta)$ is closed under direct summands:

Lemma 1.4. *Let $\Lambda = \{1, 2, \ldots, n\}$, with the canonical ordering. For any A-module M, and $0 \leq i \leq n$, let $\eta_i M = \eta_{\{P(j) \mid j > i\}} M$. Then M belongs to $\mathcal{F}(\Delta)$ if and only if for all $1 \leq i \leq n$, the factor $\eta_{i-1} M / \eta_i M$ is a direct sum of copies of $\Delta(i)$.*

Proof: Assume M belongs to $\mathcal{F}(\Delta)$. According to Lemma 1.3 (b), there is a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$ such that for all $1 \leq i \leq n$, the factor M_{i-1}/M_i is isomorphic to a direct sum of copies of $\Delta(i)$. It follows that $M_i = \eta_i M$.

Lemma 1.5. *Assume Λ is adapted. Then $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms.*

Proof: We may assume that $\Lambda = \{1, 2, \ldots, n\}$ with its canonical ordering. Let X, Y belong to $\mathcal{F}(\Delta)$, and let $f : X \to Y$ be an epimorphism, say with kernel K. Clearly $f(\eta_i X) = \eta_i Y$ for all i, since any map $P(j) \to Y$ lifts to X. For any $1 \le i \le n$, we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \eta_i X \cap K & \xrightarrow{\iota} & \eta_i X & \xrightarrow{f} & \eta_i Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \iota_X} & & \downarrow{\scriptstyle \iota_Y} & & \\
0 & \longrightarrow & \eta_{i-1} X \cap K & \xrightarrow{\iota} & \eta_{i-1} X & \xrightarrow{f} & \eta_{i-1} Y & \longrightarrow & 0
\end{array}$$

with exact rows (the maps being the canonical inclusions or induced by f.) As cokernels of the vertical maps, we obtain the exact sequence $0 \to K_i \to X\langle i\rangle \to Y\langle i\rangle \to 0$ with $K_i = \eta_{i-1}X \cap K/\eta_i X \cap K$, $X\langle i\rangle = \eta_{i-1}X/\eta_i X$, and $Y\langle i\rangle = \eta_{i-1}Y/\eta_i Y$. Now, both $X\langle i\rangle$, and $Y\langle i\rangle$ are direct sums of copies of $\Delta(i)$, thus K_i as a submodule of $X\langle i\rangle$ has only composition factors of the form $E(j)$, with $j \le i$. Since we know that $\mathrm{Ext}^1(\Delta(i), E(j)) = 0$, for $j \le i$, it follows that the cokernel sequence splits, thus K_i is a direct sum of copies of $\Delta(i)$. In this way, we see that K has the filtration $K = \eta_0 X \cap K \supseteq \eta_1 X \cap K \supseteq \cdots \supseteq \eta_n X \cap K = 0$, with factors in $\mathcal{F}(\Delta)$, thus K belongs to $\mathcal{F}(\Delta)$.

We consider now the case that the endomorphism rings of standard modules and of costandard modules are division rings. Note that modules with a division ring as endomorphism ring are called *Schurian*.

Lemma 1.6. *The following statements are equivalent, for any $\lambda \in \Lambda$:*
(i) $\Delta(\lambda)$ *is a Schurian module.*
(ii) $[\Delta(\lambda) : E(\lambda)] = 1$.
(iii) If M is an A-module with top and socle isomorphic to $E(\lambda)$, and $[M : E(\mu)] \neq 0$ only for $\mu \le \lambda$, then $M \cong E(\lambda)$.
*(ii)** $[\nabla(\lambda) : E(\lambda)] = 1$.
*(i)** $\nabla(\lambda)$ *is a Schurian module.*

With these preparations, we are able to present the definition of a quasi-hereditary algebra.

Theorem 1. *Assume that Λ is adapted, and that all standard modules are Schurian. Then the following conditions are equivalent:*
(i) $\mathcal{F}(\Delta)$ *contains* ${}_AA$.
(ii) $\mathcal{F}(\Delta) = \{X \mid \mathrm{Ext}^1(X, \nabla) = 0\}$.
(iii) $\mathcal{F}(\Delta) = \{X \mid \mathrm{Ext}^i(X, \nabla) = 0$ *for all* $i \ge 1\}$.
(iv) $\mathrm{Ext}^2(\Delta, \nabla) = 0$

An algebra A with an adapted partial ordering Λ, whose standard modules are Schurian and such that the equivalent conditions of Theorem 1 are satisfied, is said to be ***quasi-hereditary***. The usual definition is (i), or an equivalent form in terms of "heredity chains", see [S, PS, CPS, DR2], and Soergel [So] presented the last condition (iv). In fact, the decisive implication (iv) $\Longrightarrow$ (ii) can be traced back to Donkin [D]. Condition (iv) obviously is self–dual, thus we may add the dual form of the remaining conditions:

*(i)** $\mathcal{F}(\nabla)$ *contains* $D(A_A)$.
*(ii)** $\mathcal{F}(\nabla) = \{Y \mid \operatorname{Ext}^1(\Delta, Y) = 0\}$.
*(iii)** $\mathcal{F}(\nabla) = \{Y \mid \operatorname{Ext}^i(\Delta, Y) = 0$ *for all* $i \geq 1\}$.

Since under the assumption that Λ is adapted, $\mathcal{F}(\Delta)$ is closed under direct summands, and under kernels of surjective maps, the condition (i) may be reformulated that $\mathcal{F}(\Delta)$ contains all projective modules, or also that $\mathcal{F}(\Delta)$ is resolving. Recall that a full subcategory of A–mod is said to be *resolving* provided it is closed under extensions, kernels of surjective maps, and contains all projective modules. Of course, there is the dual concept of a *coresolving* subcategory (closed under extensions, cokernels of injective maps, and containing all injective modules). So, condition (i)* may be reformulated that $\mathcal{F}(\nabla)$ contains all injective modules, or also that $\mathcal{F}(\nabla)$ is coresolving.

Proof of Theorem 1: (iii) implies (iv): this is trivial.

(iv) implies (ii): According to Lemma 1.3 (c), we know that any module X in $\mathcal{F}(\Delta)$ satisfies $\operatorname{Ext}^1(X, \nabla) = 0$. We are going to prove the converse. We may assume that $\Lambda = \{1, 2, \dots, n\}$. Let X be a module with $\operatorname{Ext}^1(X, \nabla) = 0$. Let i be minimal with $\eta_i X = 0$. By induction on i, we are going to show that X belongs to $\mathcal{F}(\Delta)$. For $i = 0$, we deal with the zero module, so nothing has to be shown. So assume $i \geq 1$. Let $X' = \eta_{i-1}X$, and $X'' = X/X'$.

First, let us show that X'' belongs to $\mathcal{F}(\Delta)$. For $s < i$, we have $\operatorname{Hom}(X', \nabla(s)) = 0$, since X' is generated by $P(i)$, and $[\nabla(s) : E(i)] = 0$. For $s > i$, we have $\operatorname{Hom}(X', \nabla(s)) = 0$, since $\eta_{s-1}X' = 0$, whereas $\nabla(s)$ is cogenerated by $Q(s)$. The exact sequence $0 \to X' \to X \to X'' \to 0$ induces for any s an exact sequence

$$\operatorname{Hom}(X', \nabla(s)) \to \operatorname{Ext}^1(X'', \nabla(s)) \to \operatorname{Ext}^1(X, \nabla(s)).$$

We have seen that the first term is zero for $s \neq i$, and by assumption, the last term is always zero, thus $\operatorname{Ext}^1(X'', \nabla(s)) = 0$ for $s \neq i$. The same is true for $s = i$, according to the dual of Lemma 1.1', since the composition factors of X'' are of the form $E(j)$, with $j < i$. Since $\operatorname{Ext}^1(X'', \nabla) = 0$, and $\eta_{i-1}X'' = 0$, we know by induction that X'' belongs to $\mathcal{F}(\Delta)$.

Next, let us note that $\operatorname{Ext}^1(X', \nabla) = 0$. Namely, the exact sequence $0 \to X' \to X \to X'' \to 0$ yields for any s an exact sequence

$$\operatorname{Ext}^1(X, \nabla(s)) \to \operatorname{Ext}^1(X', \nabla(s)) \to \operatorname{Ext}^2(X'', \nabla(s)),$$

again, the first term is zero by assumption, the last term is zero according to condition (iv), and the fact that X'' belongs to $\mathcal{F}(\Delta)$.

Since X' is generated by $P(i)$, and $\eta_i X' = 0$, it follows that there exists an exact sequence $0 \to K \to Z \to X' \to 0$, where Z is a direct sum of copies of $\Delta(i)$, and K is contained in the radical of Z. In particular, $\eta_{i-1}K = 0$. The exact sequence $0 \to K \to Z \to X' \to 0$ yields an exact sequence

$$\mathrm{Hom}(Z, \nabla(s)) \to \mathrm{Hom}(K, \nabla(s)) \to \mathrm{Ext}^1(X', \nabla(s)),$$

the last term is always zero, as we have shown, the first term is zero at least for $s \neq i$, according to Lemma 1.2 (c), thus we see that $\mathrm{Hom}(K, \nabla(s)) = 0$, for $s \neq i$. The same is true for $s = i$, since the composition factors of K are of the form $E(j)$ with $j < i$, and $\nabla(i)$ is cogenerated by $Q(i)$. However, $\mathrm{Hom}(K, \nabla) = 0$ implies $K = 0$, as we have remarked before Lemma 1.1. This shows that X' is isomorphic to a direct sum of copies of $\Delta(i)$, thus both X' and X'' belong to $\mathcal{F}(\Delta)$, and therefore also X.

(ii) implies (i): this again is trivial.

(i) implies (iii): We assume that ${}_AA$ belongs to $\mathcal{F}(\Delta)$. We show that any module X in $\mathcal{F}(\Delta)$ satisfies $\mathrm{Ext}^i(X, \nabla) = 0$ for all $i \geq 1$. According to Lemma 1.3 (c), we know it for $i = 1$, thus consider some $i \geq 2$. Take any exact sequence $0 \to X' \to P \to X \to 0$, where P is a free A-module. Since both P and X belong to $\mathcal{F}(\Delta)$, also X' belongs to $\mathcal{F}(\Delta)$, according to Lemma 1.5. On the other hand, $\mathrm{Ext}^i(X, \nabla(\mu)) = \mathrm{Ext}^{i-1}(X', \nabla(\mu))$, and, by induction, the latter group is zero. Thus we see

$$\mathcal{F}(\Delta) \subseteq \{X \mid \mathrm{Ext}^i(X, \nabla) = 0 \text{ for all } i \geq 1\} \subseteq \{X \mid \mathrm{Ext}^1(X, \nabla) = 0\},$$

in particular, the first inclusion shows that (iv) is satisfied. Since we know already that (iv) implies (ii), we see that all inclusions are equalities. This finishes the proof.

2. Some properties of quasi-hereditary algebras

Let A be a quasi-hereditary algebra with respect to Λ. We are going to give bounds for the Loewy length, and the projective or injective dimension of modules in terms of Λ. Also, we will consider the Jordan–Hölder multiplicities of selected modules.

If Λ' is a subset of Λ, let $\mathrm{h}(\Lambda')$ be the maximal number m such that there exists a chain $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ in Λ, with all $\lambda_i \in \Lambda'$.

Given an A-module M, its *support* $\operatorname{supp} M$ is the set of all $\lambda \in \Lambda$, such that $[M : E(\lambda)] \neq 0$.

Lemma 2.1. *Let M be an A-module, and $h = \mathrm{h}(\operatorname{supp} M)$. Then the Loewy length of M is at most $2^{h+1} - 1$.*

Remark: The algebra A does not have to be quasi-hereditary in order that the assertion holds. It is sufficient to know that Λ is adapted, and that the standard modules are Schurian.

Proof: We use induction on h. For $h = -1$, we have $M = 0$, and the zero module has Loewy length zero. Assume now that $h \geq 0$, and let $\mu_1, \ldots, \mu_t$ be the maximal elements of $\operatorname{supp} M$. Let M' be the smallest submodule of M such that none of the elements μ_i, with $1 \leq i \leq t$, belongs to the support of M/M'. Let M'' be the largest submodule of M' such that none of the elements μ_i, with $1 \leq i \leq t$ belongs to its support. By induction, the Loewy length of both M/M' and of M'' is at most $2^h - 1$. We claim that $N = M'/M''$ is semisimple. Otherwise, there are submodules $N'' \subset N' \subseteq N$ such that N'/N'' has length at least two, and simple top $E(\mu_r)$ and simple socle $E(\mu_s)$, with $r, s \in \{1, \ldots, t\}$. Note that N'/N'' is a factor module of $\Delta(\mu_r)$, since μ_r is maximal in $\operatorname{supp} N'/N''$, and Λ is adapted (see Lemma 1.1'). However, $[N'/N'' : E(\mu_s)] \neq 0$ shows that $\mu_s \leq \mu_r$. Thus $\mu_s = \mu_r$, since μ_s is maximal. But since $\Delta(\mu_r)$ is standard, we cannot have $[N'/N'' : E(\mu_r)] \geq 2$. This contradiction shows that N has to be semisimple, and therefore the Loewy length of M is at most $2 \cdot (2^h - 1) + 1 = 2^{h+1} - 1$.

Given $\lambda \leq \mu$ in Λ, we denote by $[\lambda, \mu]$ the subset $\{\nu \mid \lambda \leq \nu \leq \mu\}$, the interval between λ and μ. Similarly, $[\lambda, -] = \{\nu \mid \lambda \leq \nu\}$ is the principal filter generated by λ, and $[-, \mu] = \{\nu \mid \nu \leq \mu\}$ is the principal ideal generated by μ.

Lemma 2.2. *Let $\lambda \in \Lambda$. Then*

$$\operatorname{proj.dim.} \Delta(\lambda) \leq \mathrm{h}([\lambda, -]), \quad \operatorname{proj.dim.} E(\lambda) \leq \mathrm{h}(\Lambda) + \mathrm{h}([-, \lambda]).$$

Consequently, the projective dimension of a module in $\mathcal{F}(\Delta)$ is bounded by $\mathrm{h}(\Lambda)$, *and the global dimension of A is bounded by* $2\,\mathrm{h}(\Lambda)$.

Proof: If λ is maximal, then $\Delta(\lambda)$ is projective. So assume that λ is not maximal. There is a submodule U of $P(\lambda)$ such that $\Delta(\lambda) = P(\lambda)/U$, and U is filtered with factors some standard modules $\Delta(\mu)$ with $\mu > \lambda$. Note that for $\mu > \lambda$, we have $\mathrm{h}([\mu, -]) < \mathrm{h}([\lambda, -])$, thus by induction we have

$$\operatorname{proj.dim.} U \leq \max_{\mu > \lambda} \operatorname{proj.dim.} \Delta(\mu) \leq \max_{\mu > \lambda} \mathrm{h}([\mu, -]) < \mathrm{h}([\lambda, -]).$$

It follows that $\operatorname{proj.dim.} \Delta(\lambda) \leq 1 + \operatorname{proj.dim.} U \leq \mathrm{h}([\lambda, -])$.

If λ is minimal, then $E(\lambda) = \Delta(\lambda)$, thus by the previous considerations, $\operatorname{proj.dim.} E(\lambda) \leq \mathrm{h}(\Lambda)$. Assume that λ is not minimal. The composition factors of $\operatorname{rad} \Delta(\lambda)$ are of the form $E(\nu)$ with $\nu < \lambda$, thus by induction proj. dim. $E(\nu) \leq \mathrm{h}(\Lambda) + h([-,\nu]) < \mathrm{h}(\Lambda) + h([-,\lambda])$, and proj. dim. $\Delta(\lambda) \leq \mathrm{h}(\Lambda)$. This shows that proj. dim. $E(\lambda) \leq \mathrm{h}(\Lambda) + \mathrm{h}([-,\lambda])$.

In order to consider the various Jordan–Hölder multiplicities of a module as a vector with integer coefficients, it seems to be convenient to replace Λ by a totally ordered refinement. Thus, let $\Lambda = \{1, 2, \ldots, n\}$, with the canonical ordering. For any A–module M, we may consider the n-tuple $\mathbf{dim}M$ with coordinates $(\mathbf{dim}M)_j = [M : E(j)]$, called the *dimension vector* of M, and we consider $\mathbf{dim}M$ as an element of the Grothendieck group $K_0(A) = \mathbb{Z}^n$. The sets Δ and ∇ yield $n \times n$–matrices $\mathbf{dim}\Delta, \mathbf{dim}\nabla$, (the rows with index i being $\mathbf{dim}\Delta(i), \mathbf{dim}\nabla(i)$, respectively).

Lemma 2.3. *Let $\Lambda = \{1, 2, \ldots, n\}$, and assume the standard modules are Schurian. Then both matrices* $\mathbf{dim}\Delta, \mathbf{dim}\nabla$ *are unipotent lower triangular matrices.*

In particular, we see that under the assumptions of Lemma 2.3, both the dimension vectors $\mathbf{dim}\Delta(\lambda)$, as well as the dimension vectors $\mathbf{dim}\nabla(\lambda)$, form a $\mathbb{Z}$-basis of $K_0(A)$. The basis $\mathbf{dim}\Delta(\lambda)$ of $K_0(A)$ will be called the *standard* basis, the basis $\mathbf{dim}\nabla(\lambda)$ the *costandard* basis of $K_0(A)$. If M is an A–module, the coefficients of $\mathbf{dim}M$ expressed in the standard basis will be denoted by $[M : \Delta(i)]$, its coefficients in terms of the costandard basis will be $[M : \nabla(i)]$, thus

$$\mathbf{dim}M = \sum_{i=1}^{n} [M : \Delta(i)]\mathbf{dim}\Delta(i) = \sum_{i=1}^{n} [M : \nabla(i)]\mathbf{dim}\nabla(i).$$

For M in $\mathcal{F}(\Delta)$, the number of copies of $\Delta(i)$ in any Δ–filtration of M is $[M : \Delta(i)]$. Namely, if d_i is the number of copies of $\Delta(i)$ in some Δ–filtration of M, then $\mathbf{dim}M = \sum d_i \mathbf{dim}\Delta(i)$, therefore $d_i = [M : \Delta(i)]$.

For any $\lambda \in \Lambda$, let $d_\lambda = \dim_k \mathrm{End}(\Delta(\lambda))$.

Lemma 2.4. *The restriction of the functor* $\mathrm{Hom}(-, \nabla(\lambda))$ *to* $\mathcal{F}(\Delta)$ *is exact, and for* $M \in \mathcal{F}(\Delta)$, *we have*

$$\dim_k \mathrm{Hom}(M, \nabla(\lambda)) = d_\lambda \cdot [M : \Delta(\lambda)].$$

Proof: We know from 1.3 (c) that $\mathrm{Ext}^1(\mathcal{F}(\Delta), \nabla(\lambda)) = 0$, thus the restriction of the functor $\mathrm{Hom}(-, \nabla(\lambda))$ to $\mathcal{F}(\Delta)$ is exact. Also, by 1.2 (c), we know that $\mathrm{Hom}(\Delta(\mu), \nabla(\lambda)) = 0$ for $\lambda \neq \mu$, thus given a Δ–filtration of M, the functor $\mathrm{Hom}(-, \nabla(\lambda))$ will pick out those factors which are of the form $\Delta(\lambda)$.

The following equality is sometimes called Bernstein–Gelfand–Gelfand reciprocity law:

Lemma 2.5. *For all λ, μ in Λ*

$$[P(\mu) : \Delta(\lambda)] \cdot d_\lambda = [\nabla(\lambda) : E(\mu)] \cdot d_\mu.$$

Proof: Lemma 2.4 with $M = P(\mu)$ shows that the left hand side is equal to $\dim_k \operatorname{Hom}(P(\mu), \nabla(\lambda))$. However, for any A–module N, we have $\dim_k \operatorname{Hom}(P(\mu), N) = [N : E(\mu)] \cdot d_\mu$.

Consider the case where $d_\lambda = 1$ for all $\lambda \in \Lambda$ (for example, if the base field k is algebraically closed), then we can reformulate the reciprocity law as follows: recall that the Cartan matrix $C(A)$ of A is, by definition, the matrix whose columns are just the transpose of the vectors $\mathbf{dim}P(\lambda)$. Then

$$C(A) = (\mathbf{dim}\Delta)^t \cdot (\mathbf{dim}\nabla);$$

in particular, the determinant of the Cartan matrix is equal to 1.

3. Standardization

Let $\mathcal{C}$ be an abelian k–category and $\Theta = \{\Theta(\lambda) | \lambda \in \Lambda\}$ a finite set of objects of $\mathcal{C}$. The set Θ is said to be *standardizable* provided the following conditions are satisfied:

(F) $\dim_k \operatorname{Hom}(\Theta(\lambda), \Theta(\mu)) < \infty$, and $\dim_k \operatorname{Ext}^1(\Theta(\lambda), \Theta(\mu)) < \infty$, for all $\lambda, \mu \in \Lambda$.

(D) The quiver with vertex set Λ which has an arrow $\lambda \to \mu$ (and just one) provided $\operatorname{rad}(\Theta(\lambda), \Theta(\mu)) \neq 0$ or $\operatorname{Ext}^1(\Theta(\lambda), \Theta(\mu)) \neq 0$, has no (oriented) cycles.

(Here, $\operatorname{rad}(\Theta(\lambda), \Theta(\mu))$ denotes the set of non–invertible maps $\Theta(\lambda) \to \Theta(\mu)$.) Let us point out that condition (D) asserts, in particular, that all the objects $\Theta(\lambda)$ are Schurian, and do not have self–extensions.

Given a standardizable set Θ indexed by Λ, then condition (D) defines a partial ordering on Λ, with $\lambda \leq \mu$ provided there exists a chain of arrows $\lambda = \lambda_0 \to \lambda_1 \to \cdots \to \lambda_m = \mu$. The set Λ with this ordering may be called the *weight set* for Θ.

As before, we denote by $\mathcal{F}(\Theta)$ the full subcategory of all objects in $\mathcal{C}$ having a Θ–filtration.

Theorem 2. *Let Θ be a standardizable set of objects of an abelian category $\mathcal{C}$. Then there exists a quasi-hereditary algebra A, unique up to Morita equivalence, such that the subcategory $\mathcal{F}(\Theta)$ of $\mathcal{C}$ and the category $\mathcal{F}(\Delta_A)$ of all Δ_A–filtered A–modules are equivalent.*

Proof: Without loss of generality, we may refine Λ to a total ordering, thus we may assume that $\Lambda = \{1, 2, \ldots, n\}$, with its natural ordering.

First, let us observe that for any $1 \le i \le n$, there exists an indecomposable Ext–projective object $P_\Theta(i)$ of $\mathcal{F}(\Theta)$ with an epimorphisms $P_\Theta(i) \to \Theta(i)$ with kernel in $\mathcal{F}(\Theta)$. In order to show the existence, we fix some i. We want to construct inductively indecomposable objects $P(i,m)$, with $i \le m \le n$, such that there is an exact sequence $0 \to K(i,m) \to P(i,m) \to \Theta(i) \to 0$, with $K(i,m)$ in $\mathcal{F}(\Theta(i+1), \ldots, \Theta(m))$ and such that $\mathrm{Ext}^1(P(i,m), \Theta(j)) = 0$, for $1 \le j \le m$. Let $P(i,i) = \Theta(i)$, and therefore $K(i,i) = 0$; the condition (D) shows that the Ext–condition is satisfied. Now assume $i < m$, and that $P(i, m-1)$ and $K(i, m-1)$ are already defined. Let $d(i,m) = \dim \mathrm{Ext}^1(P(i,m-1), \Theta(m))_{\mathrm{End}(\Theta(m))}$, then there is a "universal extension"

$$0 \to d(i,m)\Theta(m) \to P(i,m) \to P(i,m-1) \to 0$$

(the induced map $\mathrm{Hom}(d(i,m)\Theta(m), \Theta(m)) \to \mathrm{Ext}^1(P(i,m-1), \Theta(m))$ being surjective). It is easy to see that $\mathrm{Ext}^1(P(i,m), \Theta(j)) = 0$, for all $j \le m$. Also, since $\mathrm{Hom}(\Theta(m), P(i,m-1)) = 0$, it follows that $P(i,m)$ is indecomposable. We define $K(i,m)$ as the kernel of the composition of the given maps $P(i,m) \to P(i,m-1)$ and $P(i,m-1) \to \Theta(i)$, thus $K(i,m)$ is an extension of $d(i,m)\Theta(m)$ by $K(i,m-1)$. This finishes the induction step. We define $P_\Theta(i) = P(i,n)$.

Given any object $X \in \mathcal{F}(\Theta)$, we claim that there exists an exact sequence $0 \to X' \to P_0(X) \to X \to 0$, with $P_0(X) \in \mathrm{add}\, P_\Theta$, and X' again in $\mathcal{F}(\Theta)$. For $X = \Theta(i)$, we take $P_0(\Theta(i)) = P(i)$, and we proceed by induction: assume there is given an object X in $\mathcal{F}(\Theta)$ with a non–zero proper submodule U such that both U and $Y = X/U$ belong to $\mathcal{F}(\Theta)$. By induction, there are epimorphisms $\epsilon_U : P_0(U) \to U$ and $\epsilon_Y : P_0(Y) \to Y$ such that $P_0(U), P_0(Y)$ belong to $\mathrm{add}\, P_\Theta$, whereas the kernels U' of ϵ_U, and Y' of ϵ_Y belong to $\mathcal{F}(\Theta)$. Let $\iota : U \to X$ be the inclusion map, and $\pi : X \to Y$ the projection. Since $\mathrm{Ext}^1(P_0(Y), U) = 0$, there is some $\alpha : P_0(Y) \to X$ such that $\alpha\pi = \epsilon_Y$. Then $[\epsilon_U \iota, \alpha] : P_0(U) \oplus P_0(Y) \to X$ is surjective, and its kernel is an extension of U' by Y'.

Let $P = \bigoplus_{i=1}^n P_\Theta(i)$, and A its endomorphism ring. We consider the functor $F = \mathrm{Hom}(P, -) : \mathcal{C} \to A$–Mod. Condition (F) asserts that A is a finite–dimensional algebra, and that the images $F(X)$, where X belongs to $\mathcal{F}(\Theta)$, are finite dimensional A–modules. Finally, since $\mathrm{Ext}^1(P, X) = 0$, for $X \in \mathcal{F}(\Theta)$, we see that F is exact on exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$, with $X \in \mathcal{F}(\Theta)$.

Let $P_A(i) = F(P_\Theta(i))$, and consider for $1 \le i \le n$, the modules $\Delta(i) = F(\Theta(i))$. Since F is exact on exact sequences of $\mathcal{C}$ whose objects lie inside $\mathcal{F}(\Theta)$, it follows that F maps $\mathcal{F}(\Theta)$ into $\mathcal{F}(\Delta)$. We claim that the restriction of F to $\mathcal{F}(\Theta)$ is fully faithful. Of course, this is true for the restriction of F to $\operatorname{add} P_\Theta$. Let X be in $\mathcal{F}(\Theta)$. As we have seen above, there is a map $\delta_X : P_1(X) \to P_0(X)$ in $\mathcal{C}$ such that its cokernel is X, and such that the kernel X'' and the image X' of δ_X both belong to $\mathcal{F}(\Theta)$. We denote the projection map by $\epsilon_X : P_0(X) \to X$. Note that under F, the exact sequences $P_1(X) \to P_0(X) \to X \to 0$ goes to a projective presentation of $F(X)$. Now assume X, Y in $\mathcal{F}(\Theta)$ are given. Let $f : X \to Y$ be a map with $F(f) = 0$. Since $\operatorname{Ext}^1(P_0(X), Y') = 0$, and $\operatorname{Ext}^1(P_1(X), Y'') = 0$, it follows that there are maps $f_0 : P_0(X) \to P_0(Y)$, and $f_1 : P_1(X) \to P_1(Y)$ such that $f_0\epsilon_Y = \epsilon_X f$, and $f_1\delta_Y = \delta_X f_0$. Since $F(f) = 0$, there is a map g' : $F(P_0(X)) \to F(P_1(Y))$ such that $gF(\delta_Y) = F(f_0)$. However, $g' = F(g)$ for some $g : P_0(X) \to P_1(Y)$, and $f_0 = g\delta_Y$, using the fact that the restriction of F to $\operatorname{add} P_\Theta$ is fully faithful. Hence $\epsilon_X f = f_0\epsilon_Y = g\delta_Y\epsilon_Y = 0$, thus $f = 0$. This shows that the restriction of F to all of $\mathcal{F}(\Theta)$ is faithful. In order to show that it is full, let $f' : F(X) \to F(Y)$ be a map. Since there are given projective presentations, we obtain maps $f_i' : F(P_i(X)) \to F(P_i(Y))$, for $i = 0, 1$ such that $f_0'F(\epsilon_Y) = F(\epsilon_X)f'$, and $f_1'F(\delta_Y) = F(\delta_X)f_0'$. Since the restriction of F to $\operatorname{add} P_\Theta$ is fully faithful, we can write $f_i' = F(f_i)$, with maps $f_i : P_i(X) \to P_i(Y)$, and we have $f_1\delta_Y = \delta_X f_0$. Since $\delta_X f_0 \epsilon_Y = 0$, there is $f : X \to Y$ in $\mathcal{C}$ such that $f_0\epsilon_Y = \epsilon_X f$. Under F we obtain $F(\epsilon_X)F(f) = F(f_0\epsilon_Y) = F(\epsilon_X)f'$, therefore $F(f) = f'$, since $F(\epsilon_X)$ is an epimorphism. Thus, the restriction of F to $\mathcal{F}(\Theta)$ is also full.

As a consequence, we see that A is quasi-hereditary relative to the ordering $\{1, 2, \ldots, n\}$, and that the modules $\Delta(i)$ are the standard modules. For, $P_A(i) = F(P_\Theta(i))$ has a Δ–filtration, the upper factor being $\Delta(i)$, the remaining ones being of the form $\Delta(j)$, with $j > i$. In particular, $\Delta(i)$ has simple top $E(i)$. Since $\operatorname{Hom}(P_A(j), \Delta(i)) \neq 0$ only for $j \leq i$, it follows that $\Delta(i)$ is the maximal factor module of $P_A(i)$ with composition factors of the form $E(j)$, where $j \leq i$.

It remains to be seen that any A–module in $\mathcal{F}(\Delta)$ is of the form $F(X)$ with X in $\mathcal{F}(\Theta)$. Let M be a non–zero module in $\mathcal{F}(\Delta)$, let U be a submodule isomorphic to some $\Delta(i)$, such that also M/U belongs to $\mathcal{F}(\Delta)$. Denote by $\iota : \Delta(i) \to M$ a monomorphism with image U, and by $\pi : M \to M/U$ the projection map. By induction, there is an object Y in $\mathcal{F}(\Theta)$ such that $F(Y) = M/U$. As we know, there is an epimorphism $\epsilon_Y : P_0(Y) \to Y$ with $P_0(Y) \in \operatorname{add} P_\Theta$, such that its kernel Y' also belongs to $\mathcal{F}(\Theta)$. Let $u : Y' \to P_0(Y)$ be the inclusion map. Since $F(P_0(Y))$ is projective, there is a map $\alpha : F(P_0(Y)) \to M$ such that $\alpha\pi = F(\epsilon_Y)$. Then $[\iota, \alpha] : \Delta(i) \oplus F(P_0(Y)) \to M$ is surjective, and its kernel is easily seen to be isomorphic to $F(Y')$, with kernel map of the form $[\phi, F(u)] : F(Y') \to \Delta(i) \oplus F(P_0(Y))$ where $\phi : F(Y') \to \Delta(i)$ is some map. Since the objects $F(Y')$, and $\Delta(i) = F(\Theta(i))$ are images under F, and F is full, there is a map $h : Y' \to \Theta(i)$ with $F(h) = \phi$. With u also $[h, u] : Y' \to \Theta(i) \oplus P_0(Y)$ is a monomorphism, let X be its cokernel. Since $u = [h, u]\begin{bmatrix}0\\1\end{bmatrix}$, the cokernel X of $[h, u]$ maps onto the cokernel Y of u, say by $e : X \to Y$, and the kernel of e is the same as the kernel of $\begin{bmatrix}0\\1\end{bmatrix}$, thus just $\Theta(i)$. In this way, we see that X as an extension of $Y \in \mathcal{F}(\Theta)$ and $\Theta(i)$ belongs to $\mathcal{F}(\Theta)$. The exact sequence

$$0 \longrightarrow Y' \xrightarrow{[h,u]} \Theta(i) \oplus P_0(Y) \longrightarrow X \longrightarrow 0$$

goes under F to an exact sequence, since Y' belongs to $\mathcal{F}(\Theta)$, thus $F(X)$ is isomorphic to the cokernel of $F([h, u]) = [\phi, F(u)]$, thus to M. This finishes the proof.

Note that if Θ is a standardizable set of an abelian k-category $\mathcal{C}$, this set is also standardizable when considered in the opposite category $\mathcal{C}^o$, of course then its weight set will be changed to the opposite partially ordered set. It follows that for any statement dealing with standardizable sets, there also is a corresponding dual statement.

In particular, we see that given a standardizable set Θ of modules, the category $\mathcal{F}(\Theta)$ always has sufficiently many Ext–projective modules. (This is a consequence of Theorem 2 as stated, but actually, it was the first step in its proof, and one may refer to the proof in order to get further properties of the Ext–projective objects from the construction presented there.) By duality, the category $\mathcal{F}(\Theta)$ also has sufficiently many Ext–injective modules.

Of course, given a quasi-hereditary algebra A, the set of standard modules is a standardizable set, its weight set will be called the *weight set of* A. Here, the Ext–projective objects of $\mathcal{F}(\Delta)$ are just the projective modules. The Ext–injective objects of $\mathcal{F}(\Delta)$ have been considered in [R2]. The indecomposable Ext–injective A–modules have been denoted by $T(1), \ldots, T(n)$, where $\Delta(\lambda)$ is embedded into $T(\lambda)$, with $T(\lambda)/\Delta(\lambda) \in \mathcal{F}(\{\Delta(\mu) | \mu < \lambda\})$, and $T = \bigoplus_\lambda T(\lambda)$ has been called the *characteristic module* of A.

If A is quasi-hereditary, then also the set of costandard modules is a standardizable set, its weight set is the opposite of the weight set of A. Note that the Ext–projective modules in $\mathcal{F}(\nabla)$ will belong to $\mathcal{F}(\Delta)$, according to Theorem 1,(ii), and they are Ext–injective in $\mathcal{F}(\Delta)$. Thus, the indecomposable Ext–projective modules in $\mathcal{F}(\nabla)$ are just the modules $T(1), \ldots, T(n)$. Note that the construction of $P_\nabla(\lambda)$ shows that $[P_\nabla(\lambda) : E(\lambda)] = 1$, and that $[P_\nabla(\lambda) : E(\mu)] \neq 0$ only in case $\mu \leq \lambda$ (here, $\leq$ is the given ordering of the weight set of A.) This implies that $P_\nabla(\lambda) = T(\lambda)$. Altogether, we see:

Proposition 3.1. *Let A be a quasi-hereditary algebra. Then there are indecomposable modules $T(\lambda), \lambda \in \Lambda$, and exact sequences*

$$0 \to \Delta(\lambda) \to T(\lambda) \to X(\lambda) \to 0,$$

and

$$0 \to Y(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0,$$

where $X(\lambda)$ is filtered with factors $\Delta(\mu), \mu < \lambda$, and $Y(\lambda)$ is filtered with factors $\nabla(\mu), \mu < \lambda$, such that the module $T = \bigoplus_{\lambda\in\Lambda} T(\lambda)$ satisfies

$$\operatorname{add} T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla).$$

The characteristic module $T = \bigoplus_{\lambda\in\Lambda} T(\lambda)$ is a generalised tilting and cotilting module; a general context for the existence of such a module has been exhibited by Auslander and Reiten [AR]; they also show:

Proposition 3.2. *Let T be the characteristic module*

$$\mathcal{F}(\Delta) = \{X \in A\text{-mod} \mid \operatorname{Ext}^i(X,T) = 0 \textit{ for all } i \geq 1\},$$

and

$$\mathcal{F}(\nabla) = \{Y \in A\text{-mod} \mid \operatorname{Ext}^i(T,Y) = 0 \textit{ for all } i \geq 1\}.$$

A short version of the proof of Auslander and Reiten may be found in [R2].

Note that any subset of a standardizable set again is standardizable. In particular, given the set Δ of standard modules over a quasi-hereditary algebra, any subset will be standardizable and we obtain corresponding quasi-hereditary algebras. Of course, starting with an ideal of Λ, we will obtain just one of the factor algebras A/I, where I is an ideal belonging to a heredity chain of A. Similarly, starting with a filter of Λ, we will obtain a quasi-hereditary algebra of the form eAe, where e is an idempotent such that the ideal generated by e belongs to a heredity chain. These two extreme cases have been considered in [PS] and in [DR2].

4. Standard modules of small projective dimension

In this section, we are going to consider quasi-hereditary algebras with the property that all the standard modules have projective dimension at most 1, and also those satisfying the dual property that all costandard modules have injective dimension at most 1. In fact, in case both properties are satisfied, we will see that a category rather similar to $\mathcal{F}(\Delta)$ can be described very nicely.

Lemma 4.1. *Let A be a quasi-hereditary algebra. Then the following conditions are equivalent:*

(i) The projective dimension of any standard module is at most 1.
(ii) The projective dimension of the characteristic module T is at most 1.
(iii) The subcategory $\mathcal{F}(\nabla)$ is closed under factor modules.
(iv) All divisible modules belong to $\mathcal{F}(\nabla)$.

Recall that a module is said to be *divisible,* provided it is generated by an injective module. Dually, the *torsionless* modules are those which are cogenerated by projective modules. There is the following dual statement:

Lemma 4.1*. *Let A be a quasi-hereditary algebra. Then the following conditions are equivalent:*

(i) The injective dimension of any costandard module is at most 1.
(ii) The injective dimension of the characteristic module T is at most 1.
(iii) The subcategory $\mathcal{F}(\Delta)$ is closed under submodules.
(iv) All torsionless modules belong to $\mathcal{F}(\Delta)$.

Let us remark that quasi-hereditary algebras satisfying the latter conditions have been studied rather carefully in [DR5]; there, one may find additional equivalent properties.

Proof of Lemma 4.1. (i) implies (ii): This is trivial, since T belongs to $\mathcal{F}(\Delta)$.

(ii) implies (iii): Let $M \in \mathcal{F}(\nabla)$, and let N be a submodule of M. We apply $\mathrm{Ext}^1(T,-)$ to the exact sequence $0 \to N \to M \to M/N \to 0$ and obtain a surjective map $\mathrm{Ext}^1(T,M) \to \mathrm{Ext}^1(T,M/N)$, since $\mathrm{proj.dim.}\,T \leq 1$. The first group is zero, since $M \in \mathcal{F}(\nabla)$, thus $\mathrm{Ext}^1(T,M/N) = 0$. We use Proposition 3.2 in order to conclude that $M/N \in \mathcal{F}(\nabla)$, again taking into account that $\mathrm{proj.dim.}\,T \leq 1$.

(iii) implies (iv): This is trivial, since the injective modules belong to $\mathcal{F}(\nabla)$.

(iv) implies (i): Let Y be an arbitrary A-module, we want to show that $\mathrm{Ext}^2(\Delta(\lambda),Y) = 0$. Let $0 \to Y \to Q(Y) \to Y' \to 0$ be exact, with $Q(Y)$ injective. Then $\mathrm{Ext}^2(\Delta(\lambda),Y) \cong \mathrm{Ext}^1(\Delta(\lambda),Y')$. Now, Y' is divisible, thus by assumption Y' belongs to $\mathcal{F}(\nabla)$, therefore $\mathrm{Ext}^1(\Delta(\lambda),Y') = 0$. This completes the proof.

Note that under the equivalent conditions exhibited in Lemma 4.1, the characteristic module T is a tilting module in the sense of [HR1], thus it defines a torsion pair $(\mathcal{G}(T),\mathcal{H}(T))$, where

$$\mathcal{G}(T) = \{Y \in A\text{-mod} \mid \mathrm{Ext}^1(T,Y) = 0\},$$

and

$$\mathcal{H}(T) = \{Y \in A\text{-mod} \mid \mathrm{Hom}(T,Y) = 0\}.$$

The following lemma is an immediate consequence of Proposition 3.2 and Lemma 4.1.

Lemma 4.2. *Assume that the projective dimension of any standard module is at most 1. Then $\mathcal{G}(T) = \mathcal{F}(\nabla)$.*

Theorem 3. *Assume that the projective dimension of any standard module and the injective dimension of any costandard module is at most 1. Let ϕ be the endofunctor of A–mod defined by $\phi(M) = M/\eta_T M$. This functor ϕ induces an equivalence between $\mathcal{F}(\Delta)/\langle T\rangle$ and $\mathcal{H}(T)$.*

By definition, the category $\mathcal{F}(\Delta)/\langle T\rangle$ has the same objects as $\mathcal{F}(\Delta)$, the morphisms being residue classes of maps in $\mathcal{F}(\Delta)$, two maps $f, g : X \to Y$ belong to the same residue class if and only if $f - g$ factors through a direct sum of copies of T. Similarly, we may consider A–mod$/\langle T\rangle$, and $\mathcal{F}(\Delta)/\langle T\rangle$ is a full subcategory. Note that the isomorphism classes of indecomposable objects in A–mod$/\langle T\rangle$ are just the isomorphism classes of the indecomposable A–modules which do not belong to $\operatorname{add} T$.

Proof: We know that $(\mathcal{G}(T), \mathcal{H}(T))$ is a torsion pair. Now $\eta_T M$ is the torsion submodule of M, thus $M/\eta_T M$ belongs to $\mathcal{H}(T)$. Of course, $\eta_T T = T$, thus $\phi(T) = 0$, therefore ϕ induces a functor A–mod$/\langle T\rangle \to \mathcal{H}(T)$, which we also denote by ϕ. We want to show that the restriction of ϕ to $\mathcal{F}(\Delta)/\langle T\rangle$ is fully faithful and dense.

First of all, let Y belong to $\mathcal{H}(T)$. Take a universal extension $0 \to mT \to \tilde{Y} \to Y \to 0$ of Y by copies of T. In the corresponding long exact sequence

$$\operatorname{Hom}(mT, T) \to \operatorname{Ext}^1(Y, T) \to \operatorname{Ext}^1(\tilde{Y}, T) \to \operatorname{Ext}^1(mT, T)$$

the connecting homomorphism is surjective. Since $\operatorname{Ext}^1(T, T) = 0$, it follows that $\operatorname{Ext}^1(\tilde{Y}, T) = 0$. Our assumption that the injective dimension of T is at most 1 and Proposition 3.2 imply that $\tilde{Y}$ belongs to $\mathcal{F}(\Delta)$. Of course, the image of mT in $\tilde{Y}$ is just $\eta_T\tilde{Y}$, thus $\phi(\tilde{Y}) = Y$. This shows that our functor is dense.

Given M in $\mathcal{F}(\Delta)$, we claim that $\eta_T M$ always belongs to $\operatorname{add} T$. As a submodule of $M \in \mathcal{F}(\Delta)$, it also belongs to $\mathcal{F}(\Delta)$, since $\operatorname{inj.dim.} T \leq 1$; as a module in $\mathcal{G}(T)$, it belongs to $\mathcal{F}(\nabla)$, thus to $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \operatorname{add} T$.

Let $M_1, M_2 \in \mathcal{F}(\Delta)$, and let $f : M_1 \to M_2$ be a map. Assume that $\phi(f) : M_1/\eta_T M_1 \to M_2/\eta_T M_2$ is the zero map, thus f maps into $\eta_T M_2$, thus f factors through a module in $\operatorname{add} T$. This shows that $\phi : \mathcal{F}(\Delta)/\langle T\rangle \to \mathcal{H}(T)$ is faithful.

In order to show that $\phi : \mathcal{F}(\Delta)/\langle T\rangle \to \mathcal{H}(T)$ is full, let us consider again $M_1, M_2 \in \mathcal{F}(\Delta)$, and let $g : M_1/\eta_T M_1 \to M_2/\eta_T M_2$ be a map. Denote by $\pi_i : M_i \to M_i/\eta_T M_i$ the canonical projections. Since $\operatorname{Ext}^1(M_1, \eta_T M_2) = 0$, the map $\pi_1 g : M_1 \to M_2/\eta_T M_2$ can be lifted to M_2, thus there is $g' : M_1 \to$

M_2 such that $g'\pi_2 = \pi_1 g$. But this means that $\phi(g') = g$, thus our functor is also full. This completes the proof.

5. Quasi-hereditary algebras with many projective–injective modules

In this section we consider quasi-hereditary algebras such that the projective cover of any costandard module is injective.

Lemma 5.1. *Let A be a quasi-hereditary algebra. The following conditions are equivalent:*
(i) The projective cover of any costandard module is injective.
(ii) The projective cover of T is injective.
(iii) The projective cover of $D(A_A)$ is injective, and $\operatorname{top}\nabla(\lambda)$ *belongs to* $\operatorname{add}\operatorname{top} D(A_A)$, *for all $\lambda \in \Lambda$.*
(iv) Every module in $\mathcal{F}(\nabla)$ is divisible, and $\mathcal{F}(\nabla)$ is closed under projective covers.

Proof:(i) implies (iv): If the projective cover of any costandard module is injective, the same is true for the projective cover $P(M)$ of any module M in $\mathcal{F}(\nabla)$. Thus any module M in $\mathcal{F}(\nabla)$ is generated by an injective module. Also, $P(M)$ as an injective module again belongs to $\mathcal{F}(\nabla)$.

(iv) implies (iii): The module $D(A_A)$ belongs to $\mathcal{F}(\nabla)$, thus also its projective cover $P(D(A_A))$. Also, as a module in $\mathcal{F}(\nabla)$, we know that $P(D(A_A))$ is divisible. But a projective divisible module is injective. Also, since $\nabla(\lambda)$ is divisible, we know that $\operatorname{top}\nabla(\lambda)$ is in $\operatorname{add}\operatorname{top} D(A_A)$.

(iii) implies (ii): Since $T \in \mathcal{F}(\nabla)$, we know that every composition factor of $\operatorname{top} T$ belongs to some $\operatorname{top}\nabla(\lambda)$, thus to $\operatorname{add}\operatorname{top} D(A_A)$. Thus, the projective cover of T belongs to $\operatorname{add} P(D(A_A))$, and therefore is injective.

(ii) implies (i): For every $\lambda \in \Lambda$, we know that $T(\lambda)$ maps onto $\nabla(\lambda)$, thus the projective cover $P(\nabla(\lambda))$ is a direct summand of the projective cover $P(T)$. This completes the proof.

Dually, we may consider the case where the injective envelope of any standard module is projective. (Recall that the *dominant dimension* $\operatorname{dd} M$ of a module M (as introduced by Tachikawa) is greater or equal to 1 if and only if its injective hull is projective.)

Theorem 4. *Let A be a quasi-hereditary algebra, and assume the projective cover of any costandard module is injective. Let $B = A/\eta_T A$. Then $\mathcal{H}(T) = B$–mod.*

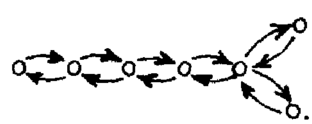
Proof: By Lemma 5.1, we know that the projective cover of T is injective, thus belongs to $\operatorname{add} T$. It follows that $\eta_T M = \eta_{P(T)} M$, for any module M, and therefore $\operatorname{Hom}(T, M) = 0$ if and only if $\operatorname{Hom}(P(T), M) = 0$.

However, the modules M with $\mathrm{Hom}(P(T), M) = 0$ are just the $A/\eta_{P(T)}A$–modules, thus the B–modules.

6. The preprojective algebra of type A_n

The preprojective algebra of a finite graph has been introduced by Gelfand and Ponomarev [GP] in order to study the preprojective representations of a finite quiver without oriented cycles. A general account which covers the more general situation of a valued graph (thus dealing with the preprojective representations of a finite species) is [DR1].

It seems to be convenient to start with the following rather fancy definition of a graph (possibly with loops and multiple edges): a *graph* G is a quiver with a fixpointfree involution σ on the set of arrows such that for any arrow $\alpha : x \to y$, the arrow $\sigma(\alpha)$ points from y to x. (The usual definition will replace the two arrows α and $\sigma(\alpha)$ by a single edge between x and y.) Important graphs for representation theory are the Dynkin diagrams $\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$, and the Euclidean diagrams $\tilde{\mathsf{A}}_n, \tilde{\mathsf{D}}_n, \tilde{\mathsf{E}}_6, \tilde{\mathsf{E}}_7, \tilde{\mathsf{E}}_8$; note that according to our convention we have to draw the graph $\mathbf{D}_7$ as follows:



Given a Dynkin diagram of the form $\mathsf{A}_n, \mathsf{D}_m$, or E_6, where $n \geq 2$, and $m \geq 5$ is odd, we denote by ν the unique automorphism of order precisely 2. For the remaining Dynkin diagrams, we denote by ν the identity automorphism.

The *preprojective algebra* $\mathcal{P}(G)$ of the graph G is the factor ring of of the path algebra kG (here, G is considered as a quiver) modulo the ideal $(\rho_x | x \in G_0)$ generated by the elements $\rho_x = \sum_{t(\alpha)=x} \sigma(\alpha) \cdot \alpha$ (where $t(\alpha)$ denotes the terminal vertex of the arrow α, and G_0 is the set of vertices of G). [Note that we may consider any graph as a (stable) polarized translation quiver, as defined in [R1], using the identity map on G_0 as translation, and then $\mathcal{P}(G)$ is just the mesh algebra.]

The following result is due to Riedtmann [Rm] and Gelfand–Ponomarev [GP], see also [Ro]. Several proofs are available, we may refer also to [G], [DR1], and [R1].

Proposition 6.1. *The preprojective algebra $\mathcal{P}(G)$ is finite dimensional if and only if G is the disjoint union of Dynkin diagrams $\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$.*

Proposition 6.2. *Let G be a disjoint union of Dynkin diagrams $\mathsf{A}_n, \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$. Then $\mathcal{P}(G)$ is a self-injective algebra with Nakayama functor given by ν.*

Proof: Let e_x be the primitive idempotent of $\mathcal{P}(G)$ corresponding to the vertex x of G. We consider the indecomposable projective $\mathcal{P}(G)$–module $\mathcal{P}(G)e_x$. It follows from the hammock considerations in [RV] that the socle of $\mathcal{P}(G)e_x$ is simple and not annihilated by $e_{\nu(x)}$. Consequently, $\mathcal{P}(G)$ is self–injective, and ν is its Nakayama permutation.

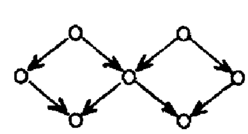
We denote by Ω the Heller functor: $\Omega(M)$ is the kernel of a projective cover $P(M) \to M$, and we recall from [G] that for any self–injective algebra B with Nakayama functor ν, we have $\tau = \Omega^2\nu = \nu\Omega^2$.

We are going to consider the case $G = \mathsf{A}_n$ in more detail.

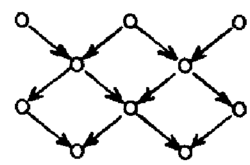
Proposition 6.3. *Let G be a Dynkin diagram of type A_n. The algebra $B = \mathcal{P}(G)$ is representation–finite in case $n \leq 4$. For $n = 5$, the algebra is of tubular type E_8.*

The proof will use the universal cover $\tilde{B}$ of B, as introduced by Bongartz and Gabriel [BG]. Note that $\tilde{B}$ is an infinite dimensional algebra without 1, but with sufficiently many idempotents. We may construct $\tilde{B}$ as the mesh algebra of the translation quiver $\mathbf{Z}\mathsf{A}_n$. We recall that a graph G is said to be a *tree* provided it is connected, but is no longer connected when a pair $\alpha, \sigma(\alpha)$ is deleted. Of course, for a tree, there are no loops, and σ is determined by the underlying quiver. Given a tree G, there exists a stable translation quiver $\mathbf{Z}G$ without oriented cycles such that $\mathbf{Z}G/\tau \cong G$, and $\mathbf{Z}G$ is unique up to isomorphism. The translation quivers $\mathbf{Z}G$ have been introduced by Riedtmann [Rm], those of the form $\mathbf{Z}\mathsf{A}_n$ already have appeared in [GR].

Proof of Proposition 6.3. The consideration of representation–finite algebras is by now standard, so we only deal with the case $n = 5$ (and here, the arguments are similar to those used in [HR2]). There exists a convex subquiver of $\mathbf{Z}\mathsf{A}_5$ of the form



and the corresponding factor algebra C_0 of $\tilde{B}$ is tame concealed of type $\tilde{\mathsf{D}}_6$, thus of tubular type (4,2,2). The convex subquiver



of $\mathbf{Z}\mathsf{A}_5$ (with the induced relations) yields a tubular extension C of C_0 of extension type (6,3,2), thus C is a tubular algebra of tubular type E_8.

Note that the algebra C_∞ obtained from C by removing all sinks is a tame concealed algebra of type $\tilde{\mathbb{E}}_7$, and the opposite algebra C^o of C is a tubular extension of C_∞. This completes the proof.

Let us add the following remark:

Lemma 6.4. *Let G be a Dynkin diagram of type $\mathbb{A}_n$. For $B = \mathcal{P}(G)$, we have*

$$\Omega^3 E(x) \cong E(\nu(x)) \cong \tau^3 E(x).$$

Proof: It is easy to verify that $\Omega \operatorname{rad} P(x) \cong P(x)/\operatorname{soc} P(x)$, and this implies that $\Omega^3 E(x) \cong E(\nu(x))$. The second isomorphism is a direct consequence: $\tau^3 E(x) \cong \nu^3 \Omega^6 E(x) \cong E(\nu(x))$.

In case $n \leq 5$, any indecomposable non-projective $\mathcal{P}(\mathbb{A}_n)$-module M satisfies

$$\Omega^3 M \cong \nu(M) \cong \tau^3 M,$$

and it seems that this is true for any n.

7. The Auslander algebra of $k[T]/\langle T^n \rangle$

Let $R_n = k[T]/\langle T^n \rangle$, this is a representation finite algebra, with indecomposable modules $M(i)$, $1 \leq i \leq n$, where $M(i)$ is of length $n - i + 1$. Let $A_n = \operatorname{End}(\bigoplus_i M(i))$ be its Auslander–algebra with the corresponding indexing of the simple A_n–modules, thus the indecomposable projective A_n–modules embed as follows into each other

$$P(1) \supset P(2) \supset \cdots \supset P(n-1) \supset P(n).$$

Note that A_n is quasi-hereditary in a unique way, with weight set the canonically ordered set $\{1, 2, \ldots, n\}$.

Lemma 7.1. *The standard modules are the serial modules with socle $E(1)$. The class $\mathcal{F}(\Delta)$ of all Δ–filtered modules is just the set of all torsionless modules, and also the set of all modules with socle generated by $P(1)$.*

Of course, dually, the costandard modules are the serial ones with top $E(1)$, and $\mathcal{F}(\nabla)$ is the set of all divisible modules, and also the set of all modules generated by $P(1)$. The modules $T(i)$ are the indecomposable modules with top and socle isomorphic to $E(1)$.

If we set $P(n+1) = 0$, then we have exact sequences

$$0 \to P(i+1) \to P(i) \to \Delta(i) \to 0$$

for all $1 \leq i \leq n$, therefore all standard modules have projective dimension at most 1. Dually, the costandard modules have injective dimension at most 1. This shows that we can apply Theorem 3, thus the categories $\mathcal{F}(\Delta)/\langle T\rangle$ and $\mathcal{H}(T)$ are equivalent. Also, since all costandard modules are generated by $P(1)$, and $P(1)$ is projective–injective, we see that the projective cover of any costandard module is injective. As a consequence, Theorem 4 can be applied, and it yields an equivalence of $\mathcal{H}(T)$ and B_n-mod, where $B_n = A_n/\eta_T A_n$. It remains to calculate B_n. It is easy to see that B_n is just the preprojective algebra of type $\mathbb{A}_{n-1}$, thus it follows:

Proposition 7.2. *Let $A = A_n$. The category $\mathcal{F}(\Delta)$ is finite for $n \leq 5$, and of tubular type E_8 for $n = 6$.*

Remark: It is known that the module category A_n–mod itself is finite only for $n \leq 3$, and that it is of tubular type E_7 for $n = 4$.

The following pages exhibit the Auslander–Reiten quivers of $\mathcal{F}(\Delta)$ for $3 \leq n \leq 5$, the vertical dashed lines have to be identified in order to form some kind of cylinder (for $n = 3, 5$) or Möbius strip (for $n = 4$). We use the following conventions: Let $\tilde{A}_n$ be the universal cover of the algebra A_n, note that the Galois group is just $\mathbb{Z}$. Let $3 \leq n \leq 6$. In this case, for any indecomposable module M in $\mathcal{F}(\Delta)$, there exists an $\tilde{A}_n$-module $\tilde{M}$ (unique up to shift by the Galois group) with push–down M. Always, the tables present the support and the Jordan–Hölder multiplicities of the modules $\tilde{M}$. For a better identification of the support of different modules $\tilde{M}$ inside the quiver of $\tilde{A}_n$, one vertex is encircled. As our presentation has shown, for the indecomposable modules M in $\mathcal{F}(\Delta)$ which are not relative injective, it is sufficient to know the factor module $M/\eta_T M = M/\eta_{P(1)}M$. Let $\mathcal{P}(1)$ be the set of indecomposable projective $\tilde{A}_n$-modules with push–down of the form $P(1)$. In our tables, the composition factors belonging to $\eta_{\mathcal{P}(1)}\tilde{M}$ are given by crosses, the remaining ones by a digit.

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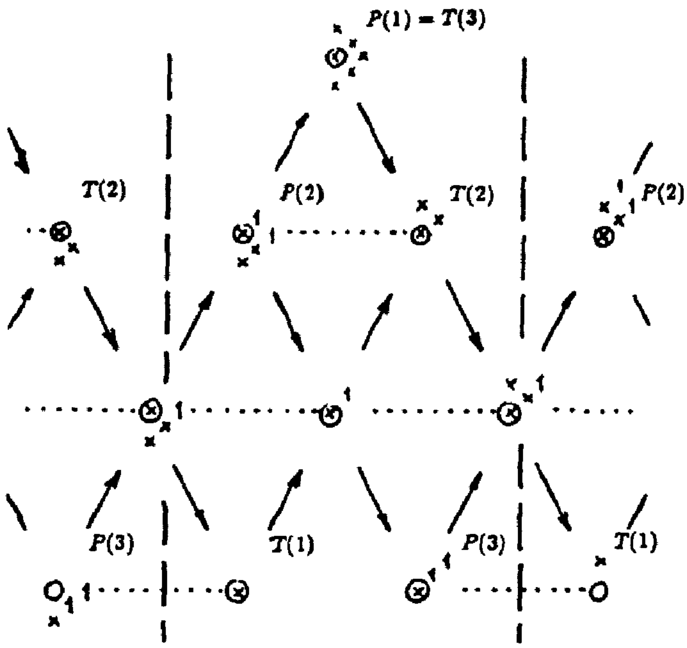

Table 1: The Auslander–Reiten quiver of $\mathcal{F}(\Delta)$ for A_3

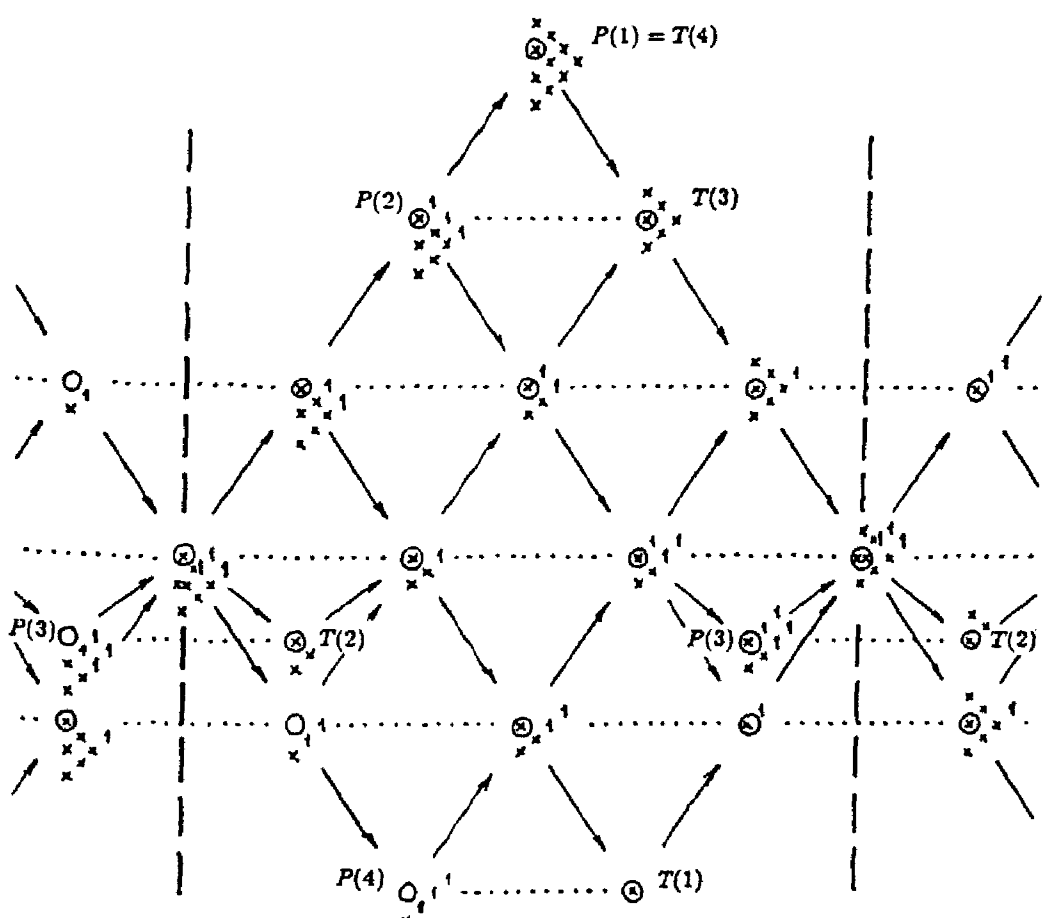

Table 2: The Auslander–Reiten quiver of $\mathcal{F}(\Delta)$ for A_4

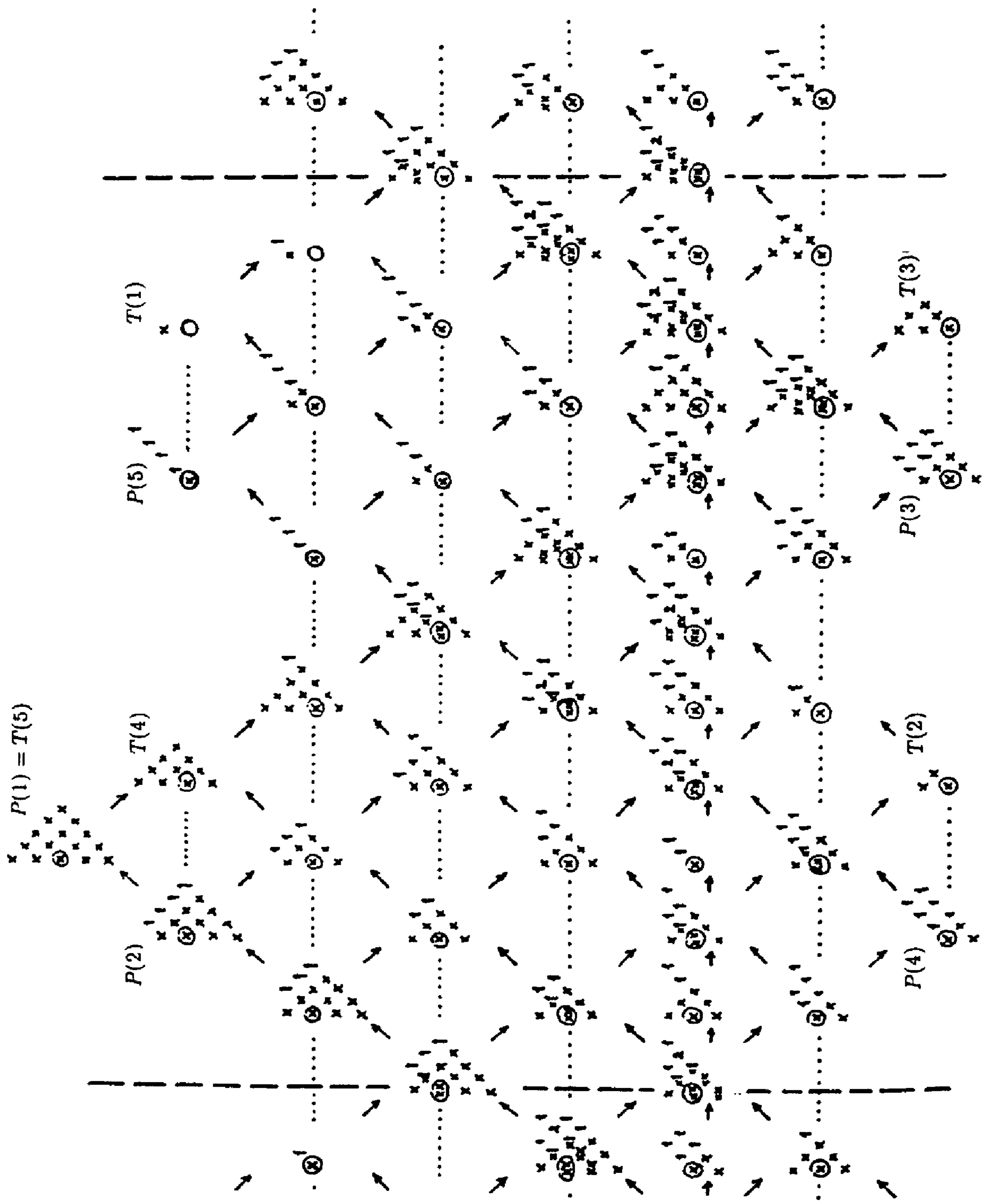

Table 3: The Auslander–Reiten quiver of $\mathcal{F}(\Delta)$ for A_5

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