Lie Algebras

Arising in Representation Theory

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One of the reasons for the introduction of the Hall algebras for finitary algebras in [R1, R2, R3] was the following: Let A be a finite dimensional algebra which is hereditary, say of Dynkin type Δ. Let $\mathbf{g}$ be the simple complex Lie algebra of type Δ, with triangular decomposition $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$. The degenerate Hall algebra $\mathcal{H}(A)_1$ of A is the free abelian group on the set of isomorphism classes of A-modules of finite length. The Grothendieck group $K(A\text{-mod})$ of all A-modules of finite length modulo split exact sequences may be identified with the free abelian group on the set of isomorphism classes of indecomposable A-modules of finite length, thus with a subgroup of $\mathcal{H}(A)_1$. Now, with respect to the degenerate Hall multiplication, the subgroup $K(A\text{-mod})$ becomes a Lie subalgebra of $\mathcal{H}(A)_1$, so that $K(A\text{-mod})$ is isomorphic to the Chevalley $\mathbb{Z}$-form of $\mathbf{n}_+$, and $\mathcal{H}(A)_1$ to the corresponding Kostant $\mathbb{Z}$-form of the universal enveloping algebra $U(\mathbf{n}_+)$.

With $U(\mathbf{n}_+)$ also $\mathcal{H}(A)_1$ is a bialgebra. The papers mentioned before have concentrated on the definition of a multiplication using the evaluation of certain polynomials at 1. This approach was first presented at the Antwerp Conference in 1987, and the discussion there helped to direct the further investigations. In particular, M. van den Bergh proposed to consider instead of the Hall polynomials the Euler characteristic of corresponding varieties. This idea was developed in detail by Schofield [Sc] and Riedtmann [Rm], and also Lusztig's presentation [L] of the Hall algebras proceeds in this way. Schofield has considered the complete coalgebra structure. The aim of this short note is to point out the nature of the comultiplication of $\mathcal{H}(A)_1$. Of course, these considerations also may be used in the Euler characteristic approach.

1. The Comultiplication

An additive category (not necessarily with finite sums) will be called a ***Krull–Schmidt category*** provided any object can be written as a finite direct sum of objects with local endomorphism rings. Let $\mathcal{A}$ be a Krull–Schmidt category. For any object M in $\mathcal{A}$, we denote by $[M]$ its isomorphism class, and we assume that the isomorphism classes of objects in $\mathcal{A}$ form a set. Given M in $\mathcal{A}$, we denote by $d(M)$ the number of indecomposable direct summands in any direct decomposition of M, this is an invariant of M, according to the Krull–Schmidt theorem.

We denote by $C(\mathcal{A})$ the free abelian group with basis $(u_{[M]})_{[M]}$ indexed by the set of isomorphism classes $[M]$ of objects M in $\mathcal{A}$, with the following comultiplication:

$$\Delta(u_{[M]}) = \sum_{D(M)} u_{[M_1]} \otimes u_{[M_2]}$$

where $D(M)$ is the set of pairs $([M_1],[M_2])$ such that $[M_1 \oplus M_2] = [M]$, and with the counit

$$\epsilon(u_{[0]}) = 1, \quad \epsilon(u_{[M]}) = 0, \text{ for } [M] \neq [0].$$

Let $C(\mathcal{A})_{(n)}$ be the free abelian group with basis $(u_{[M]})_{[M]}$ indexed by the set of isomorphism classes $[M]$ of objects M with $d(M) = n$. Thus $C(\mathcal{A}) = \bigoplus_{n\geq 0} C(\mathcal{A})_{(n)}$.

Proposition 1. $C(\mathcal{A})$ is a strictly graded cocommutative $\mathbb{Z}$–coalgebra.

Recall that a graded $\mathbb{Z}$–coalgebra $C = \bigoplus_{n\geq 0} C_{(n)}$ is called *strictly graded* provided $C_{(0)} = \mathbb{Z}$, and $C_{(1)}$ is the set of primitive elements of C (an element $x \in C$ being called *primitive* in case $\Delta x = x \otimes 1 + 1 \otimes x$.)

Proof. Let $\Delta_2(u_{[M]}) = \sum u_{[M_1]} \otimes u_{[M_2]} \otimes u_{[M_3]}$, where the sum is taken over all triples $([M_1],[M_2],[M_3])$ such that $[M_1 \oplus M_2 \oplus M_3] = [M]$. Since

$$(1 \otimes \Delta)\Delta(u_{[M]}) = \Delta_2(u_{[M]}) = (\Delta \otimes 1)\Delta(u_{[M]}),$$

Δ is coassociative. If we apply $1 \otimes \epsilon$ to $\Delta(u_{[M]}) = \sum_{D(M)} u_{[M_1]} \otimes u_{[M_2]}$, then only $u_{[M]} \otimes 1$ remains, similarly, if we apply $\epsilon \otimes 1$ to $\Delta(u_{[M]})$, then only $1 \otimes u_{[M]}$ remains. Thus ϵ is a counit. This shows that $C = C(\mathcal{A})$ is a $\mathbb{Z}$–coalgebra, and, of course, it is cocommutative. Also,

$$\Delta(C_{(n)}) \subseteq \bigoplus_{0\leq i\leq n} C_{(i)} \oplus C_{(n-i)},$$

and, for $n \geq 1$, $\epsilon(C_{(n)}) = 0$, thus we deal with a coalgebra grading. It remains to be seen that any primitive element belongs to $C_{(1)}$. Consider an element x, say $x = \sum_{i=1}^{n} x_i u_{[M_i]}$ with non-zero integers x_i, and pairwise different isomorphism classes $[M_i]$. We can assume that $d(M_1) \le d(M_2) \le \cdots \le d(M_n)$. We remark that $C \otimes C$ is the free abelian group with basis $u_{[M']} \otimes u_{[M'']}$, where both $[M']$ and $[M'']$ run through the isomorphism classes of objects in $\mathcal{A}$. If x is primitive, then $[M_1] \neq [0]$, since otherwise $\Delta(x_1 u_{[0]}) = x_1 u_{[0]} \otimes u_{[0]}$ shows that x_1 is the coefficient of $\Delta(x)$ at $u_{[0]} \otimes u_{[0]}$, whereas the coefficient of $x \otimes u_{[0]} + u_{[0]} \otimes x$ at $u_{[0]} \otimes u_{[0]}$ is $2x_1$. Since the elements in $C_{(1)}$ are primitive, we may assume that $d(M_1) \ge 2$. Take a direct decomposition $[M_1' \oplus M_1''] = [M_1]$, with an indecomposable object M_1', then x_1 is the coefficient at $u_{[M_1']} \otimes u_{[M_1'']}$ for $\Delta(x)$. On the other hand, the coefficient of $x \otimes u_{[0]} + u_{[0]} \otimes x$ at $u_{[M_1']} \otimes u_{[M_1'']}$ is zero. This shows that any primitive element is in $C_{(1)}$. Thus $C = \bigoplus C_{(n)}$ is strictly graded.

Remark. In case $\mathcal{A}$ is a length category, there is a different grading on $C(\mathcal{A})$, which is of interest in representation theory, namely let $C(\mathcal{A})_n$ be the free abelian group with basis $(u_{[M]})_{[M]}$ indexed by the set of isomorphism classes $[M]$ of objects M of length n. Then $\Delta(C_n) \subseteq \bigoplus_{0 \le i \le n} C_i \oplus C_{n-i}$, and, for $n \ge 1$, $\epsilon(C_n) = 0$, whereas $C_0 = C_{(0)} = \mathbb{Z}$. Thus, we deal with a coalgebra grading, however, $C(\mathcal{A}) = \bigoplus C(\mathcal{A})_n$ is not strictly graded.

Example. Let $\mathcal{A} = \mathcal{A}(k)$ be the category of finite dimensional vector spaces over the field k, or, more generally, a Krull Schmidt category which contains (up to isomorphism) just one indecomposable object X, and all its finite direct sums nX, with $n \in \mathbb{N}_0$. Let $u_n = u_{[nX]}$. Then $\Delta(u_n) = \sum_{i=0}^{n} u_i \otimes u_{n-i}$, and, for $n \ge 1$, $\epsilon(u_n) = 0$ (this is Example 2.2 in [A]).

We should remark that the coalgebra $C(\mathcal{A})$ only depends on the free commutative semigroup $S(\mathcal{A})$ of all isomorphism classes of objects in $\mathcal{A}$ with multiplication $\oplus$. Given any free commutative semigroup S, the free abelian group $C(S)$ with basis S is a coalgebra with respect to the following operations:

$$\Delta(d) = \sum_{d'd''=d} d' \otimes d'', \quad \epsilon(1) = 1, \quad \epsilon(d) = 0, \text{ for } d \neq 1,$$

and we have $C(\mathcal{A}) = C(S(\mathcal{A}))$.

2. Bialgebras

Let $C(\mathcal{A})$ be endowed with an associative and unitary multiplication $\circ$. Let $c^Y_{ZX} \in \mathbb{Z}$ be the structure constants, for $X, Y, Z \in \mathcal{A}$; thus

$$u_{[Z]} \circ u_{[X]} = \sum_{[M]} c^M_{ZX} u_{[M]}.$$

Proposition 2. *$(C(\mathcal{A}), \circ)$ is a bialgebra if and only if $u_{[0]}$ is the unit element, and the following condition is satisfied for all objects $X, Z, M_1, M_2 \in \mathcal{A}$*

$$c^{M_1 \oplus M_2}_{ZX} = \sum c^{M_1}_{Z_1 X_1} c^{M_2}_{Z_2 X_2},$$

where the sum on the right ranges over all pairs $([Z_1], [Z_2]) \in D(Z)$, and all pairs $([X_1], [X_2]) \in D(X)$.

Proof: We have

$$\begin{aligned}\Delta(u_{[Z]} \circ u_{[X]}) &= \Delta(\sum_{[M]} c^M_{ZX} u_{[M]}) = \sum_{[M]} c^M_{ZX} \Delta(u_{[M]}) \\ &= \sum_{[M]} c^M_{ZX} \sum_{D(M)} u_{[M_1]} \otimes u_{[M_2]},\end{aligned}$$

thus the coefficient at $u_{[M_1]} \otimes u_{[M_2]}$ in $\Delta(u_{[Z]} \circ u_{[X]})$ is just $c^{M_1 \oplus M_2}_{ZX}$.

On the other hand, since

$$\begin{aligned}\Delta(u_{[Z]}) \circ \Delta(u_{[X]}) &= (\sum_{D(Z)} u_{[Z_1]} \otimes u_{[Z_2]})(\sum_{D(X)} u_{[X_1]} \otimes u_{[X_2]}) \\ &= \sum_{D(Z), D(X)} u_{[Z_1]} u_{[X_1]} \otimes u_{[Z_2]} u_{[X_2]} \\ &= \sum_{D(Z), D(X)} \sum_{[M_1]} \sum_{[M_2]} c^{M_1}_{Z_1 X_1} c^{M_2}_{Z_2 X_2} u_{[M_1]} \otimes u_{[M_2]},\end{aligned}$$

the coefficient at $u_{[M_1]} \otimes u_{[M_2]}$ in $\Delta(u_{[Z]}) \circ \Delta(u_{[X]})$ is $\sum c^{M_1}_{Z_1 X_1} c^{M_2}_{Z_2 X_2}$. Of course, Δ is a ring homomorphism if and only if the coefficients of $\Delta(u_{[Z]} \circ u_{[X]})$ and $\Delta(u_{[Z]}) \circ \Delta(u_{[X]})$ at any $u_{[M_1]} \otimes u_{[M_2]}$ coincide.

We denote by $K(\mathcal{A}) = C(\mathcal{A})_{(1)}$ the subgroup of $C(\mathcal{A})$ generated by the elements $u_{[M]}$, where M is an indecomposable object of $\mathcal{A}$. Since $K(\mathcal{A})$ is the free abelian group on the set of isomorphism classes of indecomposable objects in $\mathcal{A}$, we may consider it as the Grothendieck group of $\mathcal{A}$ with respect to split exact sequences.

Corollary. *Assume that $(C(\mathcal{A}), \circ)$ is a bialgebra. Then $K(\mathcal{A})$ is a Lie subalgebra (with respect to the Lie bracket $[z,x] = z \circ x - x \circ z$.)*

Proof: We have to show that for indecomposable objects $X, Z \in \mathcal{A}$, the commutator $[u_{[Z]}, u_{[X]}]$ is a linear combination of elements $u_{[M]}$, with M indecomposable. Clearly, it is a linear combination of elements $u_{[M]}$, with M non-zero, thus consider an object $M = M_1 \oplus M_2$, with both M_1, M_2 non-zero, and let us calculate c^M_{ZX} and c^M_{XZ}. Since X and Z are indecomposable, the sets $D(Z), D(X)$ both have just two elements, namely $([Z],[0]),([0],[Z])$, and $([X],[0]),([0],[X])$, respectively. Since the objects M_1, M_2 are non-zero, we have $c^{M_1}_{00} = 0 = c^{M_2}_{00}$, thus

$$c^M_{ZX} = c^{M_1}_{Z0}c^{M_2}_{0X} + c^{M_1}_{0X}c^{M_2}_{Z0} = c^M_{XZ}.$$

Consequently, the coefficient of $[u_{[Z]}, u_{[X]}]$ at $u_{[M]}$ is 0.

3. Hall algebras

Let A be a k-algebra, and X, Y, Z A-modules of finite length. We denote by $\mathcal{M}^Y_{ZX}$ the set of submodules U of Y which are isomorphic to X such that M/U is isomorphic to Z. Also, let $\mathcal{A} = A$-mod be the category of A-modules of finite length. In order to introduce a multiplication on $C(A\text{-mod})$, we will work with $\mathcal{M}^Y_{ZX}$.

Let us assume that k is a finite field, say with q elements. Let X, Y, Z be A-modules of finite length, and define

$$c^Y_{ZX} = |\mathcal{M}^Y_{ZX}|.$$

Proposition 3. *Let X, Z, M_1, M_2 be A-modules. Then $q-1$ divides*

$$c^{M_1 \oplus M_2}_{ZX} - \sum c^{M_1}_{Z_1X_1}c^{M_2}_{Z_2X_2}.$$

where the sum on the right ranges over all pairs $([Z_1],[Z_2]) \in D(Z)$, and all pairs $([X_1],[X_2]) \in D(X)$.

For the proof of Proposition 3, we need the following Lemma (see [R1] and [Sc]): Given a direct sum $M = M_1 \oplus M_2$, we may describe its endomorphisms by 2×2-matrices. For an endomorphisms f of M_1, let $*f$ be the action of the matrix $\begin{bmatrix} f & 0 \\ 0 & 1 \end{bmatrix}$ on $M_1 \oplus M_2$. Note that any element $\alpha \in k$ yields an endomorphism of any A-module, using multiplication. We denote by k^* the set of non-zero elements of k.

Lemma. *Let U be a submodule of $M = M_1 \oplus M_2$. The following assertions are equivalent:*

(i) $U = (U \cap M_1) \oplus (U \cap M_2)$,

(ii) For all $\alpha \in k$ *we have* $U * \alpha \subseteq U$,

(iii) There exists $\alpha \in k^*$ *with* $U * (1 + \alpha) \subseteq U$.

Proof: We only have to show that (iii) implies (i): Let $u \in U$. Write $u = (m_1, m_2)$ with $m_i \in M_i$. By assumption, $u' = u * (1 + \alpha)$ belongs to U, thus also $u' - u = m_1\alpha$. Since α is invertible, we see that m_1, and then also m_2 belong to U.

Proof of Proposition 3. Consider $\mathcal{M}^M_{ZX}$ for $M = M_1 \oplus M_2$, and we fix this decomposition. We have defined above an operation $*$ of k^* on $\mathcal{M}^{M_1 \oplus M_2}_{ZX}$, and we want to use it now. Let $\mathcal{M}'$ be the subset of $\mathcal{M}^M_{ZX}$, consisting of all submodules U of M which satisfy the equivalent conditions of Lemma, thus $\mathcal{M}'$ consists of fix points of the action $*$, and the elements of $\mathcal{M}^M_{ZX} \setminus \mathcal{M}'$ have trivial stabilizers. It follows that the k^*-orbits in $\mathcal{M}^M_{ZX} \setminus \mathcal{M}'$ are of length $q - 1$. Consequently

$$|\mathcal{M}^M_{ZX}| \equiv |\mathcal{M}'| \pmod{q-1}.$$

On the other hand, we may identify $\mathcal{M}'$ with the disjoint union of the products $\mathcal{M}^{M_1}_{Z_1X_1} \times \mathcal{M}^{M_2}_{Z_2X_2}$, the union being indexed by the pairs in $D(Z) \times D(X)$. For, given a submodule $U = (U \cap M_1) \oplus (U \cap M_2)$, isomorphic to X, and with $(M_1 \oplus M_2)/U$ isomorphic to Z, let $U_1 = U \cap M_1$, and $U_2 = U \cap M_2$. Then U_1 belongs to some $\mathcal{M}^{M_1}_{Z_1X_1}$, and U_2 to some $\mathcal{M}^{M_2}_{Z_2X_2}$, where $[Z_1 \oplus Z_2] = [Z]$, and $[X_1 \oplus X_2] = [X]$. Thus $|\mathcal{M}'|$ is just the sum of the cardinalities of the various $\mathcal{M}^{M_1}_{Z_1X_1} \times \mathcal{M}^{M_2}_{Z_2X_2}$. This completes the proof.

Given an extension field E of k, and a k-algebra A, we may consider the E-algebra $A^E = A \otimes E$. A field extension E of k will be said to be *conservative* for the k-algebra A provided for any indecomposable A-module M of finite length, the algebra $(\operatorname{End} M / \operatorname{rad} \operatorname{End} M)^E$ is a field. Given a representation-finite k-algebra A, there are infinitely many finite field extensions of k which are conservative. Given a k-algebra A with infinitely many finite field extensions which are conservative, we say that A has *Hall polynomials* provided for all A-modules X, Y, Z of finite length, there exists a polynomial $\varphi^Y_{ZX} \in \mathbb{Z}[T]$, such that for any conservative field extension E of k, we have

$$\varphi^Y_{ZX}(|E|) = |\mathcal{M}^{Y^E}_{Z^E X^E}|.$$

For representation-directed algebras, the existence of Hall polynomials has been shown in [R1]. Of course, there also is the classical example: any local uniserial algebra has Hall polynomials [H,M]. One may conjecture that any representation-finite algebra has Hall polynomials.

Assume that the algebra A has Hall–polynomials. Then the *degenerate Hall–algebra* $\mathcal{H}(A)_1$ is defined as the coalgebra $C(A\text{–mod})$ with the multiplication

$$u_{[N_1]}u_{[N_2]} = \sum_{[M]} \varphi^M_{N_1N_2}(1)u_{[M]},$$

where N_1, N_2 are arbitrary A–modules of finite length.

Theorem. *$\mathcal{H}(A)_1$ is a bialgebra, and the subgroup $K(A\text{–mod})$ is a Lie–subalgebra.*

As a $\mathbf{Q}$–bialgebra, $\mathcal{H}(A)_1 \otimes \mathbf{Q}$ is isomorphic to the universal enveloping algebra of $K(A\text{–mod}) \otimes \mathbf{Q}$.

Proof: Let E be a conservative field extension of k, and assume that $|E| = q^n$. Then, according to Proposition 3, $q^n - 1$ divides $\varphi^{M_1\oplus M_2}_{ZX}(q^n) - \sum_{D(Z),D(X)} \varphi^{M_1}_{Z_1X_1}(q^n)\varphi^{M_2}_{Z_2X_2}(q^n)$, and therefore, $T-1$ divides $\varphi^{M_1\oplus M_2}_{ZX}(T) - \sum_{D(Z),D(X)} \varphi^{M_1}_{Z_1X_1}(T)\varphi^{M_2}_{Z_2X_2}(T)$, (see [R1]), thus

$$\varphi^{M_1\oplus M_2}_{ZX}(1) = \sum_{D(Z),D(X)} \varphi^{M_1}_{Z_1X_1}(1)\varphi^{M_2}_{Z_2X_2}(1).$$

It follows that we may apply Proposition 2 and its Corollary, thus $\mathcal{H}(A)_1$ is a bialgebra and $K(A\text{–mod})$ is a Lie subalgebra.

The remaining assertion follows from general Hopf algebra theory: Any graded coalgebra $C = \bigoplus_{n\geq 0} C_n$ with $C_0 = \mathbf{Z}$ is irreducible (there is just one group–like element). But a bialgebra over a field which is irreducible as a coalgebra is always a Hopf algebra ([Sw], Theorem 9.2.2). And an irreducible cocommutative Hopf algebra H over a field of characteristic zero with $P(H)$ the set of primitive elements is just the universal enveloping algebra $U(P(H))$ of the Lie algebra $P(H)$ ([Sw], Theorem 13.0.1). We apply this to the $\mathbf{Q}$–bialgebra $H = \mathcal{H}(A)_1 \otimes \mathbf{Q}$, its set of primitive elements being $P(H) = C(A\text{–mod})_{(1)} \otimes \mathbf{Q} = K(A\text{–mod}) \otimes \mathbf{Q}$.

4. Subbialgebras

Let C be a bialgebra, and let C' be a subalgebra of C generated by a subset S such that $\Delta(S) \subseteq C' \otimes C'$, then C' is a subbialgebra. Indeed, assume that we deal with elements $x, y \in C'$ so that $\Delta(x), \Delta(y) \in C' \otimes C'$. Then also $\Delta(x+y), \Delta(xy)$ belong to $C' \otimes C'$, since Δ is a ring homomorphism. Of course, with C also C' is irreducible. Also, if C is cocommutative, then also C' is cocommutative.

Assume we have endowed $C(\mathcal{A})$ with a multiplication so that it is a bialgebra. Let $\mathcal{B}$ be a full subcategory of $\mathcal{A}$ which is closed under direct summands, let $C(\mathcal{A};\mathcal{B})$ be the subalgebra of $C(\mathcal{A})$ generated by the elements u_X, with $X \in \mathcal{B}$. Then $C(\mathcal{A};\mathcal{B})$ is a subbialgebra.

We may apply these considerations to the category $\mathcal{A} = A$–mod of finite length A-modules, where A is some algebra, and to the full subcategory $\mathcal{B}$ either of all simple, or of all semisimple modules. In the Hall algebra case, we will obtain the corresponding composition algebra, and the corresponding Loewy algebra, respectively (see [R4],[R5]).

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