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A bypass of an arrow is sectional

By

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Given a vertex y in a quiver, we denote by y^+ the set of vertices z with an arrow $y \to z$, and by y^- the set of vertices x with an arrow $x \to y$. Let $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ be a translation quiver, thus (Γ_0, Γ_1) is a (locally finite) quiver without multiple arrows, and $\tau: \Gamma_0' \to \Gamma_0$ is an injective map, where Γ_0' is a subset of Γ_0, such that for any $z \in \Gamma_0'$ we have $z^- = (\tau z)^+$. A vertex of Γ which does not belong to Γ_0' is said to be *projective*, one which does not belong to $\tau(\Gamma_0')$ is said to be *injective*. Recall that a path $y_0 \to y_1 \to \cdots \to y_n$ in Γ is said to be *sectional* provided for every $0 < i < n$, we have $\tau y_{i+1} \neq y_{i-1}$. It is called *cyclic* if $y_0 = y_n$ and $n \geqq 1$. We consider the following conditions:

(NC) There is no cyclic path.

(PQ) If $x_0 \to p$ is an arrow, with p projective, and $x_0 \to x_1 \to \cdots \to x_n = q$ is a sectional path, with q injective, then $n \geqq 1$, and $p = x_1$.

If $x \to z$ is an arrow in a quiver without cyclic paths, any path $x = y_0 \to y_1 \to \cdots \to y_n = z$ of length $n \geqq 2$ will be called a *bypass* for $x \to z$.

If $x \to z$ is an arrow in a translation quiver any sectional path $x = y_0 \to y_1 \to \cdots \to y_n = z$ of length $n \geqq 2$ will be called a *sectional bypass* for $x \to z$, provided we have in addition $y_1 \neq y_n$, $y_0 \neq y_{n-1}$.

Proposition 1. *Assume the conditions* (NC) *and* (PQ) *are satisfied. Then any bypass of an arrow is sectional.*

Proof. Let $x \to z$ be an arrow, and $x = y_0 \to y_1 \to \cdots \to y_n = z$ a bypass, and assume it is not sectional.

Consider first the case when z is projective. We have $y_1 \neq z$, since otherwise we would have a cyclic path. Take r maximal with $0 < r < n$, such that the path $y_0 \to y_1 \to \cdots \to y_r$ is sectional. The condition (PQ) asserts that none of the vertices y_i, with $0 \leqq i \leqq r$ can be injective, since $y_1 \neq z$. Therefore, we can form the vertices $\tau^- y_i$, and we do this for $0 \leqq i \leqq r-1$. We obtain a path $z \to \tau^- y_0 \to \tau^- y_1 \to \cdots \to \tau^- y_{r-1} = y_{r+1}$ of length $r \geqq 1$, which we can compose with the given path from y_{r+1} to $y_n = z$ in order to obtain a cyclic path, in contradiction to (NC).

Assume now that z is not projective. We have $x \neq y_{n-1}$, since otherwise we would have a cyclic path. Take s minimal with $0 < s < n$, such that the path $y_s \to y_{s+1} \to \cdots \to y_n$ is sectional, therefore $\tau y_{s+1} = y_{s-1}$.

Consider the case where one of the vertices y_t with $s + 1 < t < n$ is projective, and take t maximal with this property. We can form τy_i for $t + 1 \leqq i \leqq n$, and we obtain a path $\tau y_{t+1} \to \cdots \to \tau y_n \to x$. If we compose this path with the given path $x = y_0 \to y_1 \to \cdots \to y_t$, then we have a bypass for the arrow $\tau y_{t+1} \to y_t$. On the one hand, this bypass is not sectional, since it passes through $y_{s-1} \to y_s \to y_{s+1}$, on the other hand, it ends in the projective vertex y_t. But we have seen already that this is impossible.

It follows that none of the vertices y_i, with $s + 1 \leqq i < n$ is projective, thus we can form τy_i, for these i, and we obtain a path $\tau y_{t+1} \to \cdots \to \tau y_n \to x$ of length $n - t \geqq 1$. We compose this with the given path from $x = y_0$ to $y_{t-1} = \tau y_{t+1}$ and obtain in this way a cyclic path, in contradiction to (NC). This completes the proof.

Recall that a function $f: \Gamma_0 \to \mathbb{N}_1$ is said to be subadditive, provided $f(\tau z) + f(z) \geqq \sum_{y \in z^-} f(y)$, for every non-projective z. The following conditions will be of interest:

($P_{\leqq}$) If $y \to p$ is an arrow, and p is projective, then $f(y) \leqq f(p)$.
($P_{<}$) If $y \to p$ is an arrow, and p is projective, then $f(y) < f(p)$.
($Q_{\geqq}$) If $q \to y$ is an arrow, and q is injective, then $f(q) \geqq f(y)$.
($Q_{>}$) If $q \to y$ is an arrow, and q is injective, then $f(q) > f(y)$.
(A) If $x \to y$ is an arrow, then $f(x) \neq f(y)$.

Of course, under the condition (A), the conditions ($P_{<}$) and ($P_{\leqq}$) coincide, and similarly also ($Q_{>}$) and ($Q_{\geqq}$).

Lemma. *Assume there exists a subadditive function $f: \Gamma_0 \to \mathbb{N}_1$ which satisfies the conditions* ($P_{\leqq}$) *and* ($Q_{>}$). *Then the condition* (PQ) *holds.*

Proof. Let $x_0 \to p$ be an arrow, with p projective, and $x_0 \to x_1 \to \cdots \to x_n = q$ a sectional path, with q injective. If $n = 0$, then we deal with an arrow $q \to p$. However the condition ($P_{\leqq}$) asserts $f(q) \leqq f(p)$, whereas the condition ($Q_{>}$) yields $f(q) > f(p)$. Thus, we must have $n \geqq 1$. Assume we have $p \neq x_1$. We can assume that none of the vertices x_i with $0 \leqq i < n$ is injective. Denote $y_0 = p$, and, $y_i = \tau^- x_{i-1}$, for $1 \leqq i \leqq n$. Then, for $0 \leqq i < n$, the set x_i^+ contains the vertices y_i and x_{i+1}, and they are always different, thus the subadditivity gives $f(x_i) + f(y_{i+1}) \geqq f(y_i) + f(x_{i+1})$ for these i. We rewrite this as $f(x_i) - f(x_{i+1}) \geqq f(y_i) - f(y_{i+1})$, add up, and obtain $f(x_0) + f(y_n) \geqq f(y_0) + f(x_n)$. But y_0 is projective, thus $f(x_0) \leqq f(y_0)$, and x_n is injective, thus $f(x_n) > f(y_n)$. So we obtain a contradiction.

Note that the condition (PQ) is selfdual: if it is satisfied in Γ, then also in the opposite of Γ. Thus (PQ) also follows from the conditions ($P_{<}$) and ($Q_{\geqq}$).

Examples. First of all, the conditions (NC), ($P_{\leqq}$), ($Q_{\geqq}$) are not sufficient to enforce that bypasses of arrows are sectional. Take the translation quiver with vertices x, y, a, b, c and arrows $x \to y, x \to a, a \to b, b \to c, c \to y$, with $\tau c = a$, and $f(b) = 2$, whereas $f(z) = 1$ for the remaining vertices. Then $x \to y$ has a bypass which is not sectional.

Second, the translation quiver $\mathbb{Z}\Delta$, where Δ has three vertices a, b, c and arrows $a \to b, a \to c, b \to c$. Then there is a sectional path $(0, a) \to (0, c) \to (1, b)$, and the non-sectional path $(0, a) \to (0, b) \to (1, a) \to (1, b)$. We see that even in a stable translation quiver without cyclic paths, a bypass of a sectional path of length two does not have to be sectional.

We consider now translation quivers which may have cyclic paths. The following is a special case of considerations in [1].

Proposition 2. *Let Γ be a translation quiver, and assume there exists a bounded subadditive function f which satisfies the conditions* (A), ($P_<$) *and* ($Q_>$). *Then no arrow has a sectional bypass.*

Proof. Assume $y_0 \to y_1 \to \cdots \to y_n$ is a sectional bypass to the arrow $y_0 \to y_n$. We consider the case $f(y_0) < f(y_n)$, the remaining case $f(y_0) > f(y_n)$ follows by duality.

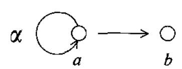
Because of $f(y_0) < f(y_n)$, the vertex y_0 cannot be injective, thus we can form $y_{n+1} = \tau^- y_0$. There are arrows $y_1 \to y_{n+1}$ and $y_n \to y_{n+1}$. By definition we have $y_0 \neq y_{n-1}$, thus the path $y_1 \to y_2 \to \cdots \to y_n \to y_{n+1}$ is sectional. Now $y_1 \neq y_n$ by the definition of a sectional bypass, and $y_2 \neq y_{n+1}$, since the original path was sectional. Therefore $y_1 \to y_2 \to \cdots \to y_n \to y_{n+1}$ is a sectional bypass to the arrow $y_1 \to y_{n+1}$. Also, $y_1 \neq y_n$, therefore $f(y_0) + f(y_{n+1}) \geqq f(y_1) + f(y_n)$, thus $f(y_{n+1}) - f(y_1) \geqq f(y_n) - f(y_0) > 0$. Inductively, we obtain in this way an infinite sequence of vertices y_i, with $i \in \mathbb{N}_0$, such that for all i we have $f(y_{n+i}) - f(y_i) \geqq f(y_n) - f(y_0)$. It follows that f cannot be bounded.

Application. The Auslander-Reiten quiver $\Gamma(\Lambda)$ of an Artin algebra Λ (see e.g. [3]) has as vertices the isomorphism classes of the indecomposable modules, there is an arrow $[X] \to [Y]$ provided there exists an irreducible map, and τ is the Auslander-Reiten translation. Of course, the length function is subadditive, and satisfies conditions ($P_<$), ($Q_>$) and (A). Thus, if $\mathscr{C}$ is a component of an Auslander-Reiten quiver which has no cycles, then any bypass of an arrow in $\mathscr{C}$ is sectional. This can be used for many components, since according to Zhang [4], a component without projective or injective vertices which is not a tube has no cyclic path.

If Λ is representation-finite (i.e. $\Gamma(\Lambda)$ is finite), Proposition 2 implies that an irreducible map does not allow a sectional bypass.

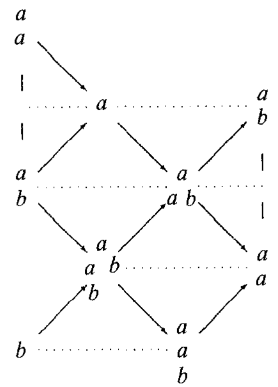
Corollary. *Let Λ be a representation-directed algebra (i.e. $\Gamma(\Lambda)$ is finite and satisfies* (NC)). *If $\alpha: X \to Y$ is an irreducible map between indecomposable Λ-modules, then α has no bypass.*

Example. Let us comment on the definition of a sectional bypass. Consider the following algebra given as quiver with relation by:



with $\alpha^2 = 0$.

We denote the indecomposable modules by their Loewy-series. Then the Auslander-Reiten quiver is given as follows, where the horizontal dotted lines indicate the Auslander-Reiten translation, while identification is along the vertical dashed lines.



We obtain a sectional path

$$b \longrightarrow \begin{smallmatrix} & a & \\ a & & b \\ & b & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & a & \\ a & & b \end{smallmatrix} \longrightarrow \begin{smallmatrix} a \\ b \end{smallmatrix} \longrightarrow \begin{smallmatrix} & a & \\ a & & b \\ & b & \end{smallmatrix},$$

(the first map is the inclusion map of a radical summand, and the second map is surjective). Since we require $y_1 \neq y_n$, this is not a sectional bypass to the first arrow.

We say that a cyclic path $y_0 \to y_1 \to \cdots \to y_n = y_0$ is a *sectional cycle* if it is sectional and $\tau y_1 \neq y_{n-1}$. The last example shows that one has to be careful when speaking about sectional cycles. The last three arrows form a sectional path which is cyclic, but it is not a sectional cycle. So the result in [2] should be formulated that the Auslander-Reiten quiver of a representation-finite algebra does not contain a sectional cycle.

References

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