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Algebras Whose Auslander–Reiten Quivers Have Large Regular Components

WILLIAM CRAWLEY-BOEVEY* AND CLAUS MICHAEL RINGEL

Fakultät für Mathematik, Universität Bielefeld, 4800 Bielefeld, Germany

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For any infinite valued quiver Δ with only finitely many points of valency more than two and satisfying appropriate necessary conditions, we construct an algebra whose Auslander–Reiten quiver has a regular component of shape $\mathbb{Z}\Delta$. © 1992 Academic Press, Inc.

A theorem of Zhang [9] states that a regular component of the Auslander–Reiten quiver of an artin algebra is either a tube $\mathbb{Z}\mathbb{A}_\infty/n$ or is of the form $\mathbb{Z}\Delta$ with Δ a valued quiver. Any finite wild quiver Δ with at least three points actually does occur [8], but up to now the only infinite Δ known to occur were the quivers $\mathbb{A}_\infty$, $\mathbb{A}_\infty^\infty$, $\mathbb{B}_\infty$, $\mathbb{C}_\infty$, and $\mathbb{D}_\infty$, and since rather a lot of examples had been considered, one was tempted to suppose that these were the only possibilities. This is not so. Before stating our result we recall the valued versions of some standard definitions [5].

A *valued quiver* $Q = (Q_0, Q_1, a)$ consists of a quiver (Q_0, Q_1) with no loops or multiple edges, where Q_0 is the set of vertices and Q_1 the set of arrows, and function $a\colon Q_1 \to \mathbb{N}_1 \times \mathbb{N}_1$, where $\mathbb{N}_1 = \{1, 2, ...\}$. If $\alpha\colon x \to y$ is an arrow one writes (a_α, a'_α) for $a(\alpha)$. A *valued translation quiver* $(\Gamma_0, \Gamma_1, \tau, a)$ consists of a valued quiver (Γ_0, Γ_1, a) and an injective function $\tau\colon \Gamma'_0 \to \Gamma_0$ defined on a subset Γ'_0 of Γ_0, such that for all $x \in \Gamma'_0$ and $y \in \Gamma_0$, there is an arrow $\alpha\colon \tau x \to y$ if and only if there is an arrow $\beta\colon y \to x$, and if these exist then $a_\alpha = a'_\beta$ and $a'_\alpha = a_\beta$.

The *Auslander–Reiten quiver* Γ_A of an artin algebra A is the valued translation quiver $(\Gamma_0, \Gamma_1, \tau, a)$ with Γ_0 the set of isomorphism classes $[M]$ of indecomposable finitely generated A-modules M, an arrow $\alpha\colon [M] \to [N]$ if there is an irreducible map $M \to N$, with τ defined by $\tau[M] = [D\,\mathrm{Tr}\, M]$ on the vertices $[M]$ with M non-projective, and with a_α (respectively a'_α) equal to the length of $\mathrm{Irr}(M, N)$ as an $\mathrm{End}(N)$-module (respectively as an $\mathrm{End}(M)$-module), where $\mathrm{Irr}(M, N) = \mathrm{Rad}(M, N)/\mathrm{Rad}^2(M, N)$ is the bimodule of irreducible maps.

* Current address: Mathematical Institute, Oxford University, 24–29 St. Giles, Oxford OX1 3LB, England.

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If $\Delta = (\Delta_0, \Delta_1, a)$ is a valued quiver then $\mathbb{Z}\Delta = (\Gamma_0, \Gamma_1, \tau, a)$ is the valued translation quiver defined by $\Gamma_0 = \mathbb{Z} \times \Delta_0$, $\tau(k, q) = (k+1, q)$ for all $(k, q) \in \Gamma_0$, and for each arrow $\alpha: x \to y$ in Δ_1 and each $k \in \mathbb{Z}$ there are two arrows

$$(k, \alpha): (k, x) \to (k, y) \qquad \text{and} \qquad \sigma(k, \alpha): (k+1, y) \to (k, x)$$

in $\mathbb{Z}\Delta$, with $a_{(k,\alpha)} = a_\alpha = a'_{\sigma(k,\alpha)}$ and $a'_{(k,\alpha)} = a'_\alpha = a_{\sigma(k,\alpha)}$.

For $\mathbb{Z}\Delta$ to occur as a connected component of an Auslander–Reiten quiver Γ_A, it is necessary that Δ be locally finite, connected and have no oriented cycles. Moreover, the valuation must be *symmetrizable*, that is, there must be a function $d: \Delta_0 \to \mathbb{N}_1$ such that $a_\alpha d(y) = d(x)\, a'_\alpha$ for all arrows $\alpha: x \to y$ in Δ_1. If $x \in \Delta_0$, the *valency* of x is the sum

$$\sum_\alpha a_\alpha + \sum_\beta a'_\beta,$$

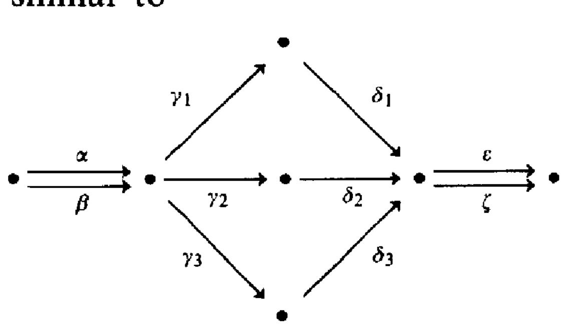
where α ranges over all arrows starting at x and β over all arrows terminating at x.

THEOREM. *Let Δ be an infinite, locally finite, connected, valued quiver with no oriented cycles and symmetrizable valuation. If Δ has only finitely many vertices of valency ≥ 3 then there is a finite dimensional algebra over some field k with a regular component of shape $\mathbb{Z}\Delta$ in its Auslander–Reiten quiver.*

This is proved in Section 6.

In our construction the field k is a prime field $\mathbb{F}_p$ or $\mathbb{Q}$. However, if the valuation of Δ is symmetric, that is if $a_\alpha = a'_\alpha$ for all arrows α, then k can be taken to be any field.

It is perhaps worthwhile to describe the evolution of the construction we use. In discussions between S. Brenner, M. C. R. Butler and the first author, algebras similar to

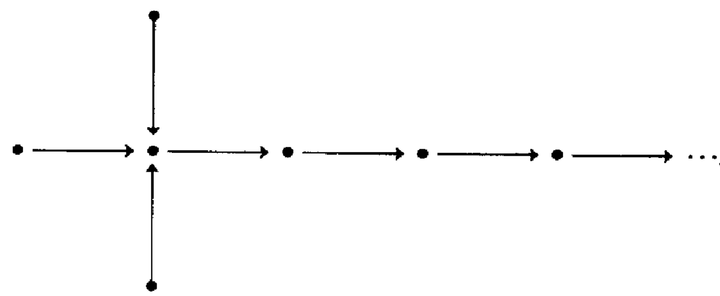


$$\delta_1\gamma_1 = \delta_2\gamma_2 = \delta_3\gamma_3$$

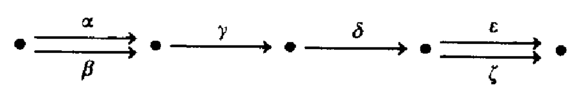
were considered. It is easy to see that there is an Auslander–Reiten sequence

$$0 \to \begin{smallmatrix}1\\00110\\1\end{smallmatrix} \to \begin{smallmatrix}1\\00000\\0\end{smallmatrix} \oplus \begin{smallmatrix}0\\00100\\0\end{smallmatrix} \oplus \begin{smallmatrix}0\\00000\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\01110\\1\end{smallmatrix} \to \begin{smallmatrix}1\\01100\\1\end{smallmatrix} \to 0 \qquad (*)$$

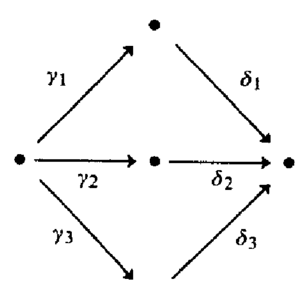
and since this cannot be in the preprojective or preinjective component, and the algebra is not tilted, there was some hope that it was in a regular component of type $\mathbb{Z}Q$, with Q the quiver



Unfortunately, no way has been found to verify this. We modify this algebra in two ways. The first way is to add the relations $\gamma_i \alpha = 0 = \varepsilon \delta_i$ $(1 \leqslant i \leqslant 3)$, in which case many of the modules in the component containing the Auslander–Reiten sequence $(*)$ behave like modules for the domestic special biserial algebra

$A =$  $\gamma\alpha = 0 = \varepsilon\delta,$

and this fact makes it possible to prove that the component does now have shape $\mathbb{Z}Q$. We formalize this in Section 1 with the notion of "Kronecker biextensions": our algebra is a Kronecker biextension of the algebra

$B =$  $\delta_1\gamma_1 = \delta_2\gamma_2 = \delta_3\gamma_3$

In order to realize the full range of $\mathbb{Z}\Delta$ mentioned above, we make our second modification, replacing B with a tilted QF-3 algebra. As preliminaries we need a little more information about slices for tilted algebras, which is included in Section 2, and about QF-3 algebras, which is given in Section 3.

We deal with algebras A (associative, with 1) which are finite dimensional over a field k, and by an A-module mean a finite dimensional left A-module. We denote by A-mod the category of A-modules, and write maps on the left. By a module class we mean a full subcategory of A-mod closed under direct sums, summands, and isomorphisms. If M is an A-module we denote by $P_A(M)$ or $P(M)$ the projective cover of M, and by $I_A(M)$ or $I(M)$ the injective envelope of M. If S is a simple module then $[M:S]$ denotes the multiplicity of S in M, and M is said to be sincere if $[M:S]\neq 0$ for all S. The modules N orthogonal to M are those with $\operatorname{Hom}_A(M,N)=\operatorname{Hom}_A(N,M)=0$. We denote by D the duality $\operatorname{Hom}_k(-,k)$, by τ_A and τ_A^- the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr}D$, and by ν_A the Nakayama functor $D\operatorname{Hom}_A(-,A)$. Other unexplained notation can be found in [7].

If D is a division algebra and Q is a quiver we denote by DQ the path algebra of Q over D, so $DQ=D\otimes_k kQ$. If $\alpha_i\colon x_{i-1}\to x_i$ $(1\leqslant i\leqslant n)$ are arrows in Q, we adopt the convention that the path from x_0 to x_n is the product $\alpha_n\alpha_{n-1}\cdots\alpha_1$, so that DQ-modules correspond to representations of Q by means of left D-vector spaces.

1. Kronecker Biextensions

(1.1) Let B be a finite dimensional k-algebra, and X an indecomposable projective-injective B-module. We suppose that soc X is projective. $X/\operatorname{rad}X$ is injective and X has length $\geqslant 3$.

Let $E=\operatorname{End}_B(X)^{\mathrm{op}}$, so that X is an B-E-bimodule. Since soc X and rad X are fully invariant submodules of X, the simple modules

$$S=\operatorname{soc}X \qquad \text{and} \qquad T=X/\operatorname{rad}X$$

are naturally $B-E$-bimodules. Now $\nu_B(S)\cong X$ and $\nu_B(X)\cong T$ so

$$\operatorname{End}_B(S)\cong\operatorname{End}_B(X)\cong\operatorname{End}_B(T).$$

In particular, E is a division ring.

Definition. The *Kronecker biextension* C of B with respect to X is the algebra

$$C=\begin{pmatrix} E & \operatorname{Hom}_E(X\oplus S,E) & E \\ 0 & B & T\oplus X \\ 0 & 0 & E\end{pmatrix}$$

defined using the evaluation map

$$\operatorname{Hom}_E(X\oplus S,E)\otimes_B(T\oplus X)\to E, \qquad f\otimes(t,x)\mapsto f(x,0).$$

Any B-module is naturally a C-module, and there are two additional simple C-modules: a simple projective S' given by the first column, and a simple injective T' whose projective cover is given by the third column. One can immediately write down a number of exact sequences relating various C-modules,

$$0 \to T \oplus P(T) \to P(T') \to T' \to 0 \qquad 0 \to S' \to I(S') \to S \oplus I(S) \to 0$$

$$0 \to S'^2 \to P(S) \to S \to 0 \qquad 0 \to T \to I(T) \to T'^2 \to 0$$

$$0 \to S' \to P(T) \to X \to 0 \qquad 0 \to X \to I(S) \to T' \to 0$$

(1.2) Let $M = \bigoplus_{i=1}^{5} M_i$, where

$$M_1 = P(T'), \quad M_2 = P(T), \quad M_3 = \text{rad } P(T), \quad M_4 = P(S), \quad M_5 = P(S') = S',$$

and set $A = \text{End}_C(M)^{\text{op}}$ so that M is a $C-A$-bimodule. Let Q be the quiver

$$Q = 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \xrightarrow{\gamma} 3 \xrightarrow{\delta} 4 \underset{\zeta}{\overset{\varepsilon}{\rightrightarrows}} 5.$$

LEMMA. $A \cong EQ/\langle \gamma\alpha, \varepsilon\delta \rangle$.

Proof. Each M_i has endomorphism ring E^{op}, for example M_3 has simple socle S', and $[M_3 : S'] = 1$, which implies that any non-zero endomorphism of M_3 is an automorphism, and hence that $\text{End}_C(M_3) \cong \text{End}_C(S') \cong E^{\text{op}}$.

We define maps $\alpha, ..., \zeta$ between the M_i in the scheme

$$M_1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} M_2 \xleftarrow{\gamma} M_3 \xleftarrow{\delta} M_4 \underset{\zeta}{\overset{\varepsilon}{\leftleftarrows}} M_5. \qquad (*)$$

Namely, α is the projection of $P(T)$ onto the summand T of rad $P(T') \cong T \oplus P(T)$; β is the inclusion of $P(T)$ as the other summand of rad $P(T')$; γ is the inclusion of rad $P(T)$ into $P(T)$; δ is the natural map $P(S) \to$ rad $P(T)$ coming from the inclusion of S as the socle of rad X; ε is the inclusion of S' as the summand of rad $P(S) \cong S'^2$ which is killed by δ; and ζ is the inclusion of S' as a complementary summand of rad $P(S)$.

Now the maps $\alpha, ..., \zeta$ commute with the action of E^{op} on M_i, and in fact any map between the M_i is an E^{op}-linear combination of products of these maps. Passing to the opposite this gives a surjection $EQ \to \text{End}_C(M)^{\text{op}}$ which sends the trivial path e_i at vertex i in Q to the projection of M onto M_i. One can now check that the kernel is generated by $\gamma\alpha$ and $\varepsilon\delta$.

(1.3) LEMMA. *M is projective as a right A-module.*

Proof. Equivalently $\operatorname{Hom}_C(P, M)$ is a projective right A-module for each indecomposable projective C-module P. If $P \cong P(T')$, $P(T)$, $P(S)$, or $P(S')$, then P is a summand of M, so that $\operatorname{Hom}_C(P, M)$ is a summand of $\operatorname{Hom}_C(M, M) = A^{\mathrm{op}}$, and the assertion follows. Otherwise $\operatorname{Hom}_C(P, M_4) = \operatorname{Hom}_C(P, M_5) = 0$ and any map from P to M_1 (respectively M_2) factors through the mono β (respectively γ), so that as a right A-module $\operatorname{Hom}_C(P, M)$ can be displayed as the representation of Q^{op} given by

$$\operatorname{Hom}_C(P, M_1) \underset{\beta \text{ iso}}{\overset{\alpha = 0}{\leftrightarrows}} \operatorname{Hom}_C(P, M_2) \xleftarrow{\gamma \text{ iso}} \operatorname{Hom}_C(P, M_3) \longleftarrow 0 \leftrightarrows 0,$$

and this is projective.

(1.4) We denote the projective, injective, and simple left A-modules corresponding to vertex i by $P(i) = Ae_i$, $I(i)$, $S(i)$.

LEMMA. *$M \otimes_A P(i) \cong M_i$, $M \otimes_A S(1) \cong T'$, $M \otimes_A S(2) \cong T$, $M \otimes_A S(3) \cong \operatorname{rad} X/\operatorname{soc} X$, $M \otimes_A S(4) \cong S$, and $M \otimes_A S(5) \cong S'$.*

Proof. The first isomorphism is clear and the other isomorphisms follow from the presentations of the simple C-modules.

(1.5) LEMMA. *There is a natural transformation $\phi\colon M \otimes_A \nu_A(-) \to \nu_C(M \otimes_A -)$ of functors A-mod $\to$ C-mod. If P is a projective A-module with no summand $P(3)$ then $\phi_P\colon M \otimes_A \nu_A(P) \to \nu_C(M \otimes_A P)$ is an isomorphism.*

Proof. Recall that $\nu_A(Z) = D \operatorname{Hom}_A(Z, A)$, and $\nu_C(M \otimes_A Z) \cong D \operatorname{Hom}_C(M \otimes_A Z, C)$. We define $\phi_Z\colon M \otimes_A \nu_A(Z) \to \nu_C(M \otimes_A Z)$ by sending $m \otimes f$ with $m \in M$ and $f\colon \operatorname{Hom}_A(Z, A) \to k$ to the map

$$\phi_Z(m \otimes f)\colon \operatorname{Hom}_C(M \otimes_A Z, C) \to k$$

which sends $g \in \operatorname{Hom}_C(M \otimes_A Z, C)$ to $f(g_m)$, where $g_m \in \operatorname{Hom}_A(Z, A)$ is defined by letting $g_m(z) \in A = \operatorname{Hom}_C(M, M)^{\mathrm{op}}$ be the map which sends $n \in M$ to $g(n \otimes z)\, m \in M$. Tedious calculation shows that ϕ_Z is well defined and that ϕ is a natural transformation.

We show next that if $P = P(1) \oplus P(2) \oplus P(4) \oplus P(5)$ then ϕ_P is epi. Let $e = e_1 + e_2 + e_4 + e_5$ so that $P = Ae$. The dual of ϕ_P is the map

$$D\phi_P\colon \operatorname{Hom}_C(Me, C) \to D(M \otimes_A D(eA)) \cong \operatorname{Hom}_A(M, eA),$$

which after identifying eA with $\operatorname{Hom}_C(Me, M)$ sends $g \in \operatorname{Hom}_C(Me, C)$ to the map taking $m \in M$ to the map taking $x \in Me$ to $g(x)\, m \in M$. Now

$$Me = P(T') \oplus P(T) \oplus P(S) \oplus P(S') = Cf$$

for some idempotent $f \in C$, and

$$D(fC) \cong \nu_C(Cf) \cong I(T') \oplus I(T) \oplus I(S) \oplus I(S')$$

so $D(fC)/\text{rad}\, D(fC)$ is a direct sum of copies of T' and S, and hence $D(fC)/\text{rad}\, D(fC)$ is cogenerated by $D(fC)$. Thus, as right C-modules, $\text{soc}\, fC$ is generated by fC, and hence $\text{Hom}_C(fC, U) \neq 0$ for all non-zero submodules U of fC. Taking $U = uC$ it follows that $uCf = \text{Hom}_C(fC, uC) \neq 0$ for all non-zero $u \in fC$. Now $fC \cong \text{Hom}_C(Cf, C) \cong \text{Hom}_C(Me, C)$, and it follows that $g(Me)\, Me \neq 0$ for all non-zero $g \in \text{Hom}_C(Me, C)$. Thus $D\phi_P$ is mono, so ϕ_P is epi.

To complete the proof of the lemma it suffices to observe that for $i = 1, 2, 4, 5$ the modules $M \otimes_A I(i) \cong M \otimes_A \nu_A(P(i))$ have the same multiplicities of each simple as the injective C-modules $I(T')$, $I(T)$, $I(S)$, and $I(S')$, respectively. This is possible since these injectives have a known structure and $M \otimes_A -$ is exact.

(1.6) Let $S(3)^{\perp}$ be the module class in A-mod defined by

$$S(3)^{\perp} = \langle U \mid \text{Hom}_A(U, S(3)) = \text{Ext}^1_A(U, S(3)) = 0 \rangle.$$

LEMMA. *If U, V are A-modules and $U \in S(3)^{\perp}$ then the functor $M \otimes_A -$ induces an isomorphism* $\text{Hom}_A(U, V) \to \text{Hom}_C(M \otimes_A U, M \otimes_A V)$ *and a monomorphism* $\text{Ext}^1_A(U, V) \to \text{Ext}^1_C(M \otimes_A U, M \otimes_A V)$.

Proof. Suppose first that U is projective, so the condition $U \in S(3)^{\perp}$ is precisely that U has no summand $P(3)$, and hence $M \otimes_A U$ is projective. If P is a projective C-module, then $\text{Hom}_C(P, M \otimes_A V) \cong \text{Hom}_C(P, M) \otimes_A V$ as follows from reducing to the case $P = C$, and therefore

$$\begin{aligned} &\text{Hom}_C(M \otimes_A U, M \otimes_A V) \\ &\quad \cong \text{Hom}_C(M \otimes_A U, M) \otimes_A V \\ &\quad \cong \text{Hom}_A(U, \text{Hom}_C(M, M)) \otimes_A V \\ &\quad = \text{Hom}_A(U, A) \otimes_A V \end{aligned}$$

since $\text{End}_C(M) = A^{\text{op}}$. Since U is projective this reduces to $\text{Hom}_A(U, V)$, as required.

In the general case U has a projective presentation $P_1 \to P_0 \to U \to 0$ with $P_0, P_1 \in S(3)^{\perp}$. Tensoring with M gives a projective presentation $M \otimes_A P_1 \to M \otimes_A P_0 \to M \otimes_A U \to 0$ of $M \otimes_A U$. Now in the morphism

$$\begin{array}{ccccc} 0 \longrightarrow & \text{Hom}_A(P_0, V) & \longrightarrow & \text{Hom}_A(P_1, V) \\ & \downarrow & & \downarrow \\ 0 \longrightarrow & \text{Hom}_C(M \otimes_A P_0, M \otimes_A V) & \longrightarrow & \text{Hom}_C(M \otimes_A P_1, M \otimes_A V) \end{array}$$

of complexes, the vertical maps are isomorphisms, so the induced maps $\mathrm{Hom}_A(U, V) \to \mathrm{Hom}_C(M \otimes_A U, M \otimes_A V)$ and $\mathrm{Ext}^1_A(U, V) \to \mathrm{Ext}^1_C(M \otimes_A U, M \otimes_A V)$ on homology are an isomorphism and a monomorphism, respectively.

(1.7) LEMMA. *If $\xi: 0 \to \tau_A U \to G \to U \to 0$ is an Auslander–Reiten sequence in A-mod and $U \in S(3)^{\perp}$ then $M \otimes_A \xi: 0 \to M \otimes_A \tau_A U \to M \otimes_A G \to M \otimes_A U \to 0$ is an Auslander–Reiten sequence in C-mod.*

Proof. U has local endomorphism ring, and by (1.6) so also does $M \otimes_A U$, so $M \otimes_A U$ is indecomposable. If $P_1 \to P_0 \to U \to 0$ is a minimal projective presentation of U then $M \otimes_A P_1 \to M \otimes_A P_0 \to M \otimes_A U \to 0$ is a projective presentation of $M \otimes_A U$, and it is in fact minimal, as follows using the categorical formulation of minimality and (1.6). The commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & M \otimes_A \tau_A(U) & \longrightarrow & M \otimes_A \nu_A(P_1) & \longrightarrow & M \otimes_A \nu_A(P_0) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \tau_C(M \otimes_A U) & \longrightarrow & \nu_C(M \otimes_A P_1) & \longrightarrow & \nu_C(M \otimes_A P_0) \end{array}$$

has exact rows, and the second two vertical maps are isomorphisms by (1.5), so $M \otimes_A \tau_A U \cong \tau_C(M \otimes_A U)$. By [1], the $\mathrm{End}_A(U)$-module $\mathrm{Ext}^1_A(U, \tau_A U)$ has a simple socle generated by ξ, and by (1.6) there is an embedding

$$\mathrm{Ext}^1_A(U, \tau_A U) \hookrightarrow \mathrm{Ext}^1_C(M \otimes_A U, M \otimes_A \tau_A U).$$

After identifying $\mathrm{End}_A(U)$ with $\mathrm{End}_C(M \otimes_A U)$, this is an $\mathrm{End}_A(U)$-module map. Now $\mathrm{Ext}^1_C(M \otimes_A U, M \otimes_A \tau_A U)$ has a simple socle, which must therefore be generated by the image $M \otimes_A \xi$ of ξ. Thus $M \otimes_A \xi$ is an Auslander–Reiten sequence.

(1.8) We now come to the main result of this section.

THEOREM. *There are C-modules, denoted by $X[p, q]$ ($p, q \geq 0$), with the following properties.*

(1) $X[1, 1] \cong X$, $X[1, 0] \cong X/\mathrm{soc}\, X$, $X[0, 1] \cong \mathrm{rad}\, X$ *and* $X[0, 0] \cong \mathrm{rad}\, X/\mathrm{soc}\, X$.

(2) *For $p > 0$ there are Auslander–Reiten sequences*

$$0 \to X[p-1, q+1] \to X[p-1, q] \oplus X[p, q+1] \to X[p, q] \to 0.$$

(3) $[X[p, q]: T'] = \max(0, p-1)$ *and* $[X[p, q]: S'] = \max(0, q-1)$.

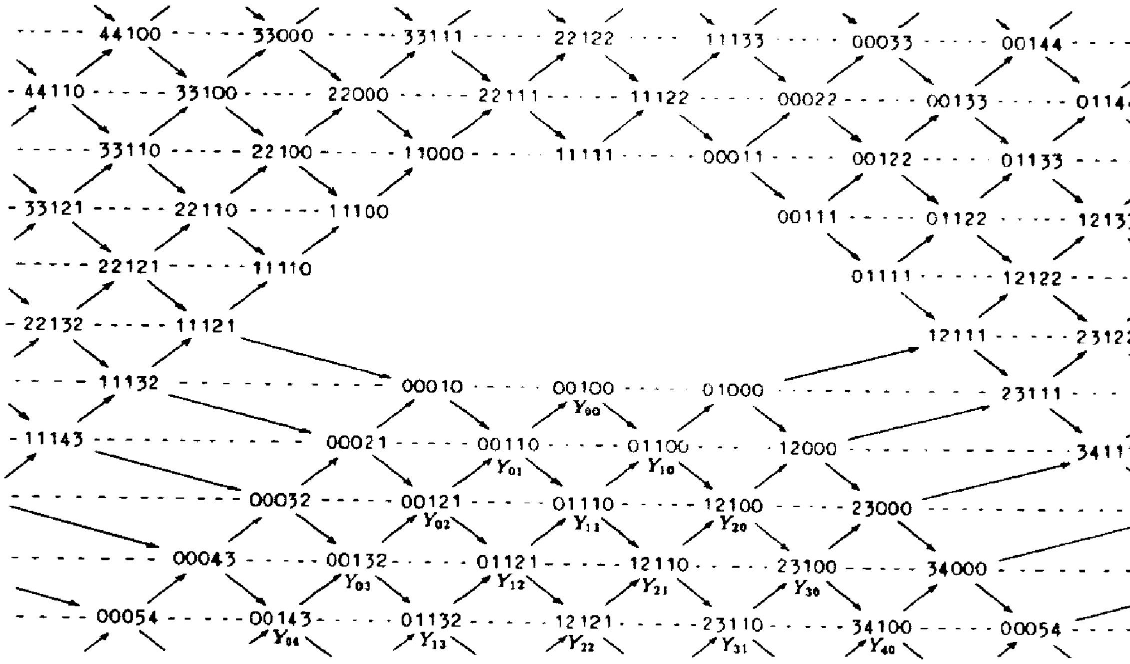


FIGURE 1

Remark. The modules $X[p, q]$ with $(p, q) \neq (0, 0)$ are indecomposable since they occur as end terms of Auslander–Reiten sequences, but $X[0, 0]$ may well decompose—indeed we shall see later that this is the whole point.

Proof. The algebra $A \cong EQ/\langle \gamma\alpha, \varepsilon\delta \rangle$ has an Auslander–Reiten component displayed in Fig. 1, where the numbers indicate the multiplicities of the $S(i)$. This can be seen by applying the results of [3] to the string algebra $kQ/\langle \gamma\alpha, \varepsilon\delta \rangle$, and then using the tensor product functor $E \otimes_k -$, or it can be checked directly. Set $X[p, q] = M \otimes_A Y_{pq}$, where the Y_{pq} are indicated in the diagram. Now (1) is clear, (2) follows from (1.7) and the fact that $Y_{pq} \in S(3)^{\perp}$ for $p > 0$, and (3) follows from (1.4).

(1.9) We shall also need the

PROPOSITION. *If* $\xi: 0 \to L \to M \to N \to 0$ *is an Auslander–Reiten sequence in* B-mod *and* $[N : S] = [L : T] = 0$, *then* ξ *is an Auslander–Reiten sequence in* C-mod.

The algebra C can be obtained from B by taking a one-point extension $\begin{pmatrix} B & T \oplus X \\ 0 & E \end{pmatrix}$ followed by a one-point coextension. The proposition thus follows by using both parts of the next lemma. Let G and H be k-algebras, let V be a $G-H$-bimodule and let R be the matrix algebra $\begin{pmatrix} G & V \\ 0 & H \end{pmatrix}$. Clearly G-modules and H-modules can be regarded as R-modules.

LEMMA. (1) *If* $g: 0 \to L \to M \to N \to 0$ *is an Auslander–Reiten sequence in* G-mod *and* $\mathrm{Hom}_G(V, L) = 0$, *then* g *is an Auslander–Reiten sequence in* R-mod.

(2) *If* $h: 0 \to L \to M \to N \to 0$ *is an Auslander–Reiten sequence in* H-mod *and* $\mathrm{Hom}_H(N, DV) = 0$, *then* h *is an Auslander–Reiten sequence in* R-mod.

Proof. The pair (G-mod, H-mod) is a torsion theory in R-mod, say with idempotent functor t, and if $X \in G$-mod and $Y \in H$-mod then

$$\mathrm{Ext}^1_R(Y, X) \cong \mathrm{Hom}_G(V \otimes_H Y, X) \cong \mathrm{Hom}_H(Y, \mathrm{Hom}_G(V, X)).$$

To prove (1) it suffices to prove that the map $\theta: L \to M$ is still a source map in R-mod. Let $\phi: L \to M'$ be a map in R-mod. If $t\phi: L \to tM'$ is split mono, then tM' has a summand isomorphic to L, and since $\mathrm{Ext}^1_R(Y, L) = 0$ for all $Y \in H$-mod, this summand splits off as a summand of M', so that ϕ is split mono. Otherwise, $t\phi$, and hence ϕ itself, factors through θ, as required. The proof of (2) is dual.

(1.10) We need to iterate the construction of Kronecker biextensions.

DEFINITION. If B is a f.d. algebra with orthogonal projective-injectives $X_1, \ldots, X_n$, each having projective socle, injective top and length $\geqslant 3$, then the *multiple Kronecker biextension* C of B with respect to $X_1, \ldots, X_n$ is defined by setting $B_0 = B$, letting B_i be the Kronecker biextension of B_{i-1} with respect to X_i for $1 \leqslant i \leqslant n$, and taking $C = B_n$.

This makes sense because the orthogonality of the X_i ensures that when regarded as a B_{i-1}-module, X_i is still a projective-injective with projective socle and injective top. Let $S_i = \mathrm{soc}\, X_i$ and $T_i = X_i/\mathrm{rad}\, X_i$, and denote by T_i' and S_i' the two simple modules introduced by the Kronecker biextension with respect to X_i. Now C is a Kronecker biextension of B_{n-1} with respect to X_n so there are C-modules $X_n[p, q]$ given by (1.8). In fact, up to isomorphism, C is unchanged if the X_i are permuted, so any result for X_n applies equally for the other X_i. Thus we obtain

THEOREM. *There are C-modules* $X_i[p, q]$ ($1 \leqslant i \leqslant n$, $p, q \geqslant 0$) *satisfying*

(1) $X_i[1, 1] \cong X_i$, $X_i[1, 0] \cong X_i/\mathrm{soc}\, X_i$, $X_i[0, 1] \cong \mathrm{rad}\, X_i$ *and* $X_i[0, 0] \cong \mathrm{rad}\, X_i/\mathrm{soc}\, X_i$.

(2) *For* $p > 0$ *there are Auslander–Reiten sequences*

$$0 \to X_i[p-1, q+1] \to X_i[p-1, q] \oplus X_i[p, q+1] \to X_i[p, q] \to 0.$$

(3) $[X_i[p, q] : T_j'] = \delta_{ij} \max(0, p-1)$ *and* $[X_i[p, q] : S_j'] = \delta_{ij} \max(0, q-1)$.

And by induction from (1.9).

PROPOSITION. *If $\xi: 0 \to L \to M \to N \to 0$ is an Auslander–Reiten sequence in B-*mod *and $[N : S_i] = [L : T_i] = 0$ for all i, then ξ is an Auslander–Reiten sequence in C-*mod.

2. PARTIAL SLICE MODULES

(2.1) Let B be a finite dimensional algebra. If X and Y are B-modules, the notation $X \preccurlyeq Y$ means that there are indecomposable modules $Z_1, ..., Z_n$ and non-zero maps

$$X \to Z_1 \to Z_2 \to \cdots \to Z_n \to Y.$$

Note that this relation is only transitive when restricted to indecomposable modules, which is the case considered in [7]. Allowing X and Y to be arbitrary, however, simplifies our notation. Note in particular that for all X one has $0 \not\preccurlyeq X$ and $X \not\preccurlyeq 0$.

We define another relation by $X \diamond Y$ if there is an indecomposable module Z with $X \preccurlyeq \tau_B Z$ and $Z \preccurlyeq Y$. (Loosely, $X \diamond Y$ when there is a mesh between X and Y.)

DEFINITION. A sincere module X is a *partial slice module* if $X \not\diamond X$.

LEMMA. *A partial slice module is faithful.*

Proof. The same argument as [7, 2.4.7′].

(2.2) LEMMA. *If X is a partial slice module and M is a module, then*

$$X \diamond M \Leftrightarrow X \preccurlyeq \tau_B M$$

$$M \diamond X \Leftrightarrow \tau_B^- M \preccurlyeq X.$$

Proof. We prove the first relation. The second is dual. If $X \preccurlyeq \tau_B M$ then M has an indecomposable summand Z with $X \preccurlyeq \tau_B Z$ and $Z \preccurlyeq M$ so $X \diamond M$. Conversely, if $X \diamond M$ there is some indecomposable Z with $X \preccurlyeq \tau_B Z$ and $Z \preccurlyeq M$. Taking a path $Z = Z_0 \to Z_1 \to \cdots \to Z_t$ with Z_t an indecomposable summand of M, if there is a non-zero map $Z_k \to P$ with P indecomposable projective, then

$$X \preccurlyeq \tau_B Z \qquad \text{and} \qquad Z \preccurlyeq Z_k \preccurlyeq P \preccurlyeq X$$

since X is sincere, but this contradicts $X \not\diamond X$. Thus

$$\overline{\mathrm{Hom}}_B(\tau_B Z_k, \tau_B Z_{k+1}) \cong \underline{\mathrm{Hom}}_B(Z_k, Z_{k+1}) \cong \mathrm{Hom}_B(Z_k, Z_{k+1}) \neq 0$$

so $\mathrm{Hom}_B(\tau_B Z_k, \tau_B Z_{k+1}) \neq 0$, and hence $X \preccurlyeq \tau_B Z \preccurlyeq \tau_B M$, as required.

(2.3) If X is a module we define module classes via

$$S(\to X) = \langle M \text{ indecomposable} \mid M \preccurlyeq X \text{ and } M \not\diamond X \rangle$$

$$S(X \to) = \langle M \text{ indecomposable} \mid X \preccurlyeq M \text{ and } X \not\diamond M \rangle.$$

Recall [8] that a *slice* is a module class $\mathscr{S}$ satisfying (α) $\mathscr{S}$ contains a sincere module; (β) $\mathscr{S}$ is convex, that is, if S_0, S_1, M are indecomposable, $S_0, S_1 \in \mathscr{S}$ and $S_0 \preccurlyeq M \preccurlyeq S_1$ then $M \in \mathscr{S}$; (γ) if M is indecomposable and not projective then at most one of M and $\tau_B M$ belongs to $\mathscr{S}$; and (δ) if M, S are indecomposable, M not projective, $S \in \mathscr{S}$, and if there is an irreducible map $S \to M$, then M or $\tau_B M$ belongs to $\mathscr{S}$.

PROPOSITION. *If X is a partial slice module then $S(\to X)$ and $S(X\to)$ are slices.*

Proof. We prove that $S(X\to)$ is a slice, the proof for $S(\to X)$ is dual. Each indecomposable summand of X lies in $S(X\to)$, so the sincere module X belongs to $S(X\to)$, which is (α). Now (β) and (γ) are clear. For (δ), if $X \not\diamond M$ then $M \in S(X\to)$ since $X \preccurlyeq M$. On the other hand, if $X \diamond M$, then $X \preccurlyeq \tau_B M$ by Lemma (2.2), and so $\tau_B M \in S(X\to)$ since if $X \diamond \tau_B M$ then also $X \diamond S$.

Remark. The arguments above are adaptations of the proof given in [7, Addendum to 4.2] that $S(X\to)$ and $S(\to X)$ are slices when X is a sincere directing module. In fact, it is shown there that if moreover $Y \in S(X\to)$ is indecomposable, then $S(\to Y)$ is a slice. The proposition generalizes that fact since $Y \oplus X$ is a partial slice module and $S(\to Y) = S(\to Y \oplus X)$.

COROLLARY. *A sincere module is a partial slice module if and only if it belongs to some slice.*

3. *QF*-3 ALGEBRAS

Recall that a finite dimensional algebra B is said to be QF-3 provided that it has a faithful projective-injective module. An equivalent condition is that B has a faithful module X which is isomorphic to a summand of any other faithful module. The module X, called the *minimal faithful module*, is unique up to isomorphism, and is in fact just the direct sum of one copy of each indecomposable projective-injective.

(3.1) LEMMA. *If B is a tilted algebra and X is a sincere projective-injective, then X is faithful. In particular B is QF-3.*

Proof. Since B is tilted, it has a slice $\mathscr{S}$. Now any indecomposable projective-injective must occur in any slice, so the indecomposable summands of X, and hence X itself, belongs to $\mathscr{S}$. Thus X is a partial slice module, so faithful by Lemma (2.1).

Note that the non-tilted QF-3 algebra given by quiver with relations

$$\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} \bullet \qquad \gamma\beta = \beta\alpha = 0.$$

has a sincere projective-injective module which is not faithful.

(3.2) An algebra B is said to be *socle-projective* if $\mathrm{soc}_B\, B$ is projective, or equivalently, if the socle of any projective module is projective. For QF-3 algebras this notion is symmetric.

PROPOSITION. *If B is a QF-3 algebra with minimal faithful module X, then the following conditions are equivalent*

(1) *B is socle-projective.*

(1*) *B^{op} is socle-projective, or what is the same, $I/\mathrm{rad}\, I$ is injective for all injective B-modules I.*

(2) $\mathrm{End}_B(X)$ *is semisimple.*

Proof. (1)$\Rightarrow$(2). The Nakayama functor induces an equivalence ν_B: B-proj $\to$ B-inj, $X \cong \nu_B(\mathrm{soc}\, X)$ and $\mathrm{soc}\, X$ is projective, so $\mathrm{End}_B(X) \cong \mathrm{End}_B(\mathrm{soc}\, X)$ is semisimple.

(2)$\Rightarrow$(1). Since X is faithful, B embeds in a direct sum of copies of X, so it suffices to prove that $\mathrm{soc}\, X$ is projective. Suppose for a contradiction that S is a non-projective simple submodule of X. Again $P(S)$ embeds in a direct sum of copies of X, and since $P(S)$ is not simple, it follows that $X/\mathrm{soc}\, X$ has S as a composition factor. This gives a non-zero map $X/\mathrm{soc}\, X \to I(S) \subsetneq X$. However, since $\mathrm{End}_B(X)$ is semisimple, the nilpotent ideal $\{\phi \in \mathrm{End}_B(X) \mid \phi(\mathrm{soc}\, X) = 0\}$ is zero.

(1)$\Leftrightarrow$(1*). The algebra B^{op} is QF-3 with minimal faithful module DX, and $\mathrm{End}_{B^{\mathrm{op}}}(DX) \cong \mathrm{End}_B(X)^{\mathrm{op}}$, so statement (2) is self dual.

(3.3) PROPOSITION. *Let B be a socle-projective, connected, QF-3 algebra, and let X be an indecomposable projective-injective module. If $\mathrm{soc}\, X$ or $X/\mathrm{rad}\, X$ belongs to a slice, or if X has length $\leqslant 2$, then B is hereditary.*

Proof. If $X/\mathrm{rad}\, X$ belongs to a slice $\mathscr{S}$, then any submodule Y of X is projective, for if $Y \neq 0$, X then $X \preccurlyeq X/Y \preccurlyeq X/\mathrm{rad}\, X$ and since slices are convex, $X/Y \in \mathscr{S}$. Thus X/Y has projective dimension $\leqslant 1$, and so Y is projective. The same assertion holds if X has length $\leqslant 2$ since $\mathrm{soc}_B\, B$ is projective.

Continuing with these two cases, since X has simple socle, all non-zero submodules of X are indecomposable projectives. We show that these projectives form a connected component of B. Thus let Y be a submodule of X and let P be another indecomposable projective.

If there is a non-zero map $\alpha: P \to Y$, then $\mathrm{Im}(\alpha)$ is projective so that α must be an embedding. Thus P embeds in X.

Suppose, on the other hand, that there is a non-zero map $\beta: Y \to P$. Let S be a simple submodule of $\mathrm{Im}(\beta)$. Since B is socle-projective the module S is projective, and since $\beta^{-1}(S)$ is indecomposable projective and maps onto S, it follows that $\beta^{-1}(S) \cong S$. Thus $X \cong I(S)$ so there is a non-zero map $P \to X$. As before, this is an embedding.

Thus the submodules of X do form a connected component of B. Since B is connected, these are all indecomposable projectives, and therefore every submodule of an indecomposable projective is projective. Thus B is hereditary.

The case when soc X is in a slice is dual. This time one must use the fact that B is QF-3 to ensure that B^{op} is socle-projective.

(3.4) *Remark.* Much more can be deduced in (3.3), since the only connected hereditary algebras with a non-zero projective-injective module are of type $\mathbb{A}_n$ with linear orientation.

4. A Combinatorial Lemma

In this section we give a simple but rather useful combinatorial lemma. Recall that if Γ is a valued translation quiver, then a function $f: \Gamma_0 \to \mathbb{N}$ is said to be *subadditive* if

$$f(x) + f(\tau^{-1}x) \geqslant \sum_{\alpha: x \to y} a_\alpha f(y)$$

for all non-projective $x \in \Gamma_0$, where the summation is over all arrows $\alpha: x \to y$ starting at x. We say that f *respects injectives* if

$$f(x) \geqslant \sum_{\alpha: x \to y} a_\alpha f(y)$$

for all injective vertices x.

(4.1) Definition. A point $x \in \Gamma_0$ is *successor-monotone* if there is a sequence of points and arrows

$$x \xrightarrow{\phi_1} x_1 \xrightarrow{\phi_2} x_2 \xrightarrow{\phi_3} \cdots$$

in Γ, with strict inequalities $f(x) < f(x_1) < f(x_2) < \cdots$.

LEMMA. *Let Γ be a valued translation quiver and let f be a subadditive function which respects injectives.*

(1) *A successor-monotone point $x \in \Gamma_0$ is not injective, and* $f(\tau^{-1}x) > 0$.

(2) *Any successor of a successor-monotone point is successor-monotone.*

Proof. (1) If x is injective then since f respects injectives,

$$f(x) \geqslant a_{\phi_1} f(x_1) \geqslant f(x_1),$$

contrary to the assumption. The second assertion follows in the same way from the subadditivity of f.

(2) It suffices to prove that if x is successor-monotone, then any immediate successor y of x is successor-monotone. If $y = x_1$, the assertion is clear, so suppose that $y \neq x_1$. The x_i are successor-monotone, so not injective and therefore there is a sequence of arrows

$$y \to \tau^{-1}x \to \tau^{-1}x_1 \to \tau^{-1}x_2 \to \cdots.$$

By subadditivity

$$f(x) + f(\tau^{-1}x) \geqslant \sum_{\alpha: x \to z} a_\alpha f(z) \geqslant f(x_1) + f(y),$$

so $f(\tau^{-1}x) - f(y) \geqslant f(x_1) - f(x) > 0$. Similarly, for $i > 0$,

$$f(x_i) + f(\tau^{-1}x_i) \geqslant \sum_{\beta: x_i \to w} a_\beta f(w) \geqslant f(x_{i+1}) + f(\tau^{-1}x_{i-1}),$$

where $x_0 = x$. Thus $f(\tau^{-1}x_i) - f(\tau^{-1}x_{i-1}) \geqslant f(x_{i+1}) - f(x_i) > 0$, and so

$$f(y) < f(\tau^{-1}x) < f(\tau^{-1}x_1) < f(\tau^{-1}x_2) < \cdots$$

as required.

5. Construction of a Component

In this section B is a non-hereditary, socle-projective, connected, tilted, QF-3 algebra and $\mathscr{S}$ is a slice in B-mod.

(5.1) Let $X_1, ..., X_n$ be the indecomposable projective-injective modules, so that $X = X_1 \oplus \cdots \oplus X_n$ is the minimal faithful module.

By Proposition (3.2) the X_i are orthogonal, have projective socles S_i and injective tops T_i, and by Proposition (3.3) the X_i have length ≥ 3. Thus one can form the multiple Kronecker biextension C of B with respect to $X_1, ..., X_n$. Each Kronecker biextension introduces two new simple modules, and we denote by S_i' and T_i' the two arising from the biextension by X_i.

In what follows we shall need to work with both algebras B and C, so some care is required. We point out here that the notation related to tilting theory: $\preccurlyeq$, $S(X\rightarrow)$, etc. will only be used for the algebra B.

(5.2) As in Lemma (2.1) the module X is a partial slice module which occurs in every slice, so in addition to the slice $\mathscr{S}$ there are two canonical slices $S(\rightarrow X)$ and $S(X\rightarrow)$. Now if $S \in \mathscr{S}$ is indecomposable, then there are non-negative integers $p = p_S$ and $q = q_S$ uniquely determined by requiring that $0 \neq \tau_B^p S \in S(\rightarrow X)$ and $0 \neq \tau_B^{-q} S \in S(X\rightarrow)$. Of course if $S = X_i$, then $p = q = 0$. On the other hand,

LEMMA. *If $S \in \mathscr{S}$ is indecomposable and not an X_i then $p_S > 0$ or $q_S > 0$.*

Proof. Otherwise $S \in S(X\rightarrow)$ and $S \in S(\rightarrow X)$, so $X_i \preccurlyeq S \preccurlyeq X_j$ for some i, j. Since slices are convex and the endomorphism ring of a slice module is hereditary, there are non-zero maps $\alpha: X_i \rightarrow S$ and $\beta: S \rightarrow X_i$ with $\beta\alpha \neq 0$. Now the X_k are orthogonal so $i = j$, and $\mathrm{End}_B(X_i)$ is a division ring, so α is a split monomorphism. Since S is indecomposable, $S \cong X_i$, contrary to the assumption.

(5.3) Recall that the valued quiver $\Delta(\mathscr{S})$ of $\mathscr{S}$ is the full subquiver of the Auslander–Reiten quiver Γ_B on the vertices $[S]$ with S an indecomposable module belonging to $\mathscr{S}$.

For $1 \leq i \leq n$, let Δ^i be the valued quiver

$$z_1^i \xrightarrow{(1,1)} z_2^i \xrightarrow{(1,1)} z_3^i \xrightarrow{(1,1)} \cdots$$

and let Δ be the connected union of $\Delta(\mathscr{S})$ and the Δ^i, in which the point z_1^i is identified with the point $[X_i]$ in $\Delta(\mathscr{S})$.

Recall that $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$ and $\tau(k, d) = (k+1, d)$ so that $(k, d) = \tau^k(0, d)$. To each point x in $\mathbb{Z}\Delta$ we associate a C-module $M(x)$ by setting

$$M(k, z_j^i) = \tau_C^k X_i[j, 1] \qquad \text{for} \quad k \in \mathbb{Z},\ 1 \leq i \leq n,\ j \geq 1,$$

$$M(k, [S]) = \tau_C^k S \qquad \text{for} \quad k \in \mathbb{Z},\ [S] \in (\Delta(\mathscr{S}))_0,$$

where the modules $X_i[p, q]$ are as in Theorem (1.10). The identification $z_1^i = [X_i]$ makes sense here since $X_i[1, 1] \cong X_i$.

(5.4) We define $2n$ points r_i, s_i $(1 \leqslant i \leqslant n)$ in $\mathbb{Z}\Delta$. The module X_i is in $\mathcal{S}$, and we have an exact sequence

$$0 \to \operatorname{rad} X_i \to \operatorname{rad} X_i/\operatorname{soc} X_i \oplus X_i \to X_i/\operatorname{soc} X_i \to 0 \qquad (*)$$

which is an Auslander–Reiten sequence in both B-mod and C-mod, for it is the sequence terminating at $X_i[1, 0]$. It follows from the definition of a slice that exactly one of rad X_i and X_i/soc X_i is in $\mathcal{S}$. In the first case there is a point [rad X_i] in $\Delta(\mathcal{S})$, and we set

$$r_i = (0, [\operatorname{rad} X_i]) \qquad \text{and} \qquad s_i = (-1, [\operatorname{rad} X_i]);$$

in the second case there is a point $[X_i/\operatorname{soc} X_i]$ in $\Delta(\mathcal{S})$ and we set

$$r_i = (1, [X_i/\operatorname{soc} X_i]) \qquad \text{and} \qquad s_i = (0, [X_i/\operatorname{soc} X_i]).$$

Since $(*)$ is an Auslander–Reiten sequence in C-mod, in both cases we have $M(r_i) \cong \operatorname{rad} X_i$ and $M(s_i) \cong X_i/\operatorname{soc} X_i$.

(5.5) We divide $\mathbb{Z}\Delta$ into three regions. The *predecessor region* $\mathcal{R}_{\text{pred}}$ consists of the points x with a path to some τr_i, the *successor region* $\mathcal{R}_{\text{succ}}$ consists of the points x which are successors of some $\tau^{-1} s_i$, and the *middle region* $\mathcal{R}_{\text{mid}}$ consists of the remaining points.

Since each τr_i is of the form (k, d) with $k = 1$ or 2 and each $\tau^{-1} s_j$ is of the form (k, d) with $k = -1$ or -2. It follows that there is no path from any $\tau^{-1} s_j$ to τr_i. Thus

LEMMA. *$(\mathbb{Z}\Delta)_0$ is the disjoint union $\mathcal{R}_{pred} \cup \mathcal{R}_{mid} \cup \mathcal{R}_{succ}$. The set $\mathcal{R}_{pred}$ is closed under predecessors in $\mathbb{Z}\Delta$, $\mathcal{R}_{succ}$ is closed under successors, and $\mathcal{R}_{mid}$ is convex.*

(5.6) The middle region $\mathcal{R}_{\text{mid}}$ can be determined explicitly.

LEMMA. (1) *$(k, z_j^i) \in \mathcal{R}_{mid}$ if and only if $-1 \leqslant k \leqslant j$, and if this holds then $M(k, z_j^i) \cong X_i[j-k, 1+k]$.*

(2) *If $S \in \mathcal{S}$ is indecomposable and not an X_i, then $(k, [S]) \in \mathcal{R}_{mid}$ if and only if $-q_S \leqslant k \leqslant p_S$, and if this holds then $M(k, [S]) \cong \tau_B^k S$.*

Proof. (1) The first assertion is clear. Now $\tau_C X_i[p, q] \cong X_i[p-1, q+1]$ for $p \geqslant 1$ by Theorem (1.10), and the second assertion follows.

(2) Let $\mathcal{C}$ be the component in B-mod containing $\mathcal{S}$. By tilting theory one knows that if $M, N \in \mathcal{C}$ are indecomposable, then $M \preccurlyeq N$ if and only if there is a path in $\Gamma\mathcal{C}$ from $[M]$ to $[N]$. By [7, 4.2.4] there is an

embedding $\iota: \Gamma\mathscr{C} \hookrightarrow \mathbb{Z}\Delta(\mathscr{S})$. Let H be the set of points in $\mathbb{Z}\Delta(\mathscr{S})$ such that there is a path from $\iota[S_0]$ to x and from x to $\iota[S_1]$ with $S_0 \in S(\to X)$ and $S_1 \in S(X\to)$ indecomposable. Since $S(\to X)$ and $S(X\to)$ give rise to complete sectional subgraphs of $\Gamma\mathscr{C}$, it follows that every point in H is in the image of ι.

Let $p = p_S$ and $q = q_S$, the smallest integers such that $\tau_B^p S \preccurlyeq X_i$ and $X_j \preccurlyeq \tau_B^{-q} S$ for some i, j. Since S is not an X_k one actually has that

$$\tau_B^p S \preccurlyeq \operatorname{rad} X_i \qquad \text{and} \qquad X_j/\operatorname{soc} X_j \preccurlyeq \tau_B^{-q} S,$$

and since these paths in B-mod give rise to paths in $\Gamma\mathscr{C}$ and hence in $\mathbb{Z}\Delta$, it follows that

$$(p+1, [S]) \in \mathscr{R}_{\text{pred}} \qquad \text{and} \qquad (-q-1, [S]) \in \mathscr{R}_{\text{succ}}.$$

On the other hand, $\tau_B^{p-1} S \not\preccurlyeq \operatorname{rad} X_i$ and $X_i/\operatorname{soc} X_i \not\preccurlyeq \tau_B^{-(q-1)} S$ for all i, so there are no paths between these points in $\Gamma\mathscr{C}$, and hence also in $\mathbb{Z}\Delta(\mathscr{S})$ by the remark above. Thus $(p, [S])$ and $(-q, [S]) \in \mathscr{R}_{\text{mid}}$. This proves the first assertion.

For $-q \leqslant j \leqslant p-1$, consider the Auslander–Reiten sequence

$$0 \to \tau_B^{j+1} S \to E \to \tau_B^j S \to 0 \tag{$*$}$$

in B-mod. Since $\tau_B^j S \not\preccurlyeq X_i$ one has $[\tau_B^j S : S_i] = 0$ for all i. Similarly $[\tau_B^{j+1} S : T_i] = 0$, so by Proposition (1.10) the sequence $(*)$ is an Auslander–Reiten sequence in C-mod. It follows that $M(k, [S]) \cong \tau_B^k S$ for $-q \leqslant k \leqslant p$.

(5.7) Now that $\mathscr{R}_{\text{mid}}$ is known, many other facts can be determined.

LEMMA. (1) *The modules $M(x)$ with $x \in \mathscr{R}_{mid}$ are not projective, injective, summands of the radical of a projective or summands of $I/\operatorname{soc} I$ with I injective.*

(2) *$\mathscr{R}_{mid}$ contains at least two points in each τ-orbit in $\mathbb{Z}\Delta$.*

(3) *If $z, \tau^{-1}z \in \mathscr{R}_{mid}$ there is an Auslander–Reiten sequence of the form*

$$\xi_z: 0 \to M(z) \to \bigoplus_{\alpha: z \to w} a_\alpha M(w) \to M(\tau^{-1}z) \to 0,$$

where the summation is over all arrows $\alpha: z \to w$ starting at w.

(4) *If there is an arrow $x \to y$ in $\mathbb{Z}\Delta$ between two points in $\mathscr{R}_{mid}$, then there is an irreducible map $M(x) \to M(y)$.*

Proof. (1) We divide into cases and use Lemma (5.6). These assertions are clearly true for the modules $X_i[p, q]$ (p or $q \neq 0$). For $(k, [S])$ with $S \in \mathcal{S}$, observe that the modules $\tau_B^k S$ do not involve any S_i' or T_i', so there are only two possibilities

(a) $\tau_B^k S$ is the simple summand T_i of rad $P(T_i')$.

(b) $\tau_B^k S$ is the simple summand S_i of $I(S_i')/\text{soc}\, I(S_i')$.

In case (a), $\tau_B^k S$ is an injective B-module, so $k = -q_S$ and therefore T_i belongs to the slice $S(X\rightarrow)$, which is impossible by Lemma (3.3). Case (b) is excluded similarly.

(2) This follows from (5.2) and (5.6).

(3) In (5.6) we have observed that the Auslander–Reiten sequence starting at $M(z)$ can be obtained from (1.10), so it has the required form.

(4) Since $x \in \mathcal{R}_{\text{mid}}$, so also is $\tau^{-1}x$ or τx. In the first case the assertion follows from ξ_x; in the second case $\tau y \in \mathcal{R}_{\text{mid}}$ and the assertion follows from $\xi_{\tau y}$.

(5.8) LEMMA. (1) *If* $x \in \mathcal{R}_{succ}$ *then* $M(x)$ *is not injective and* $[M(x) : T_i'] \neq 0$ *for some* i.

(2) *If* $x \in \mathcal{R}_{pred}$ *then* $M(x)$ *is not projective and* $[M(x) : S_i'] \neq 0$ *for some* i.

Proof. (1) Consider the Auslander–Reiten quiver Γ_C of C, and let f_i be the function $(\Gamma_C)_0 \rightarrow \mathbb{N}$ sending an indecomposable C-module M to $[M : T_i']$. Clearly this is subadditive (even additive) and respects injectives. By Theorem (1.10) the module $X_i/\text{soc}\, X_i \cong X_i[1, 0]$ is successor-monotone.

Let $y \in \mathcal{R}_{\text{mid}}$ be a point with $\tau^{-1}y \in \mathcal{R}_{\text{succ}}$, so $\tau^{-1}y$ is a successor of some $\tau^{-1}s_i$ and hence y is a successor of s_i. By Lemma (5.6), $s_i \in \mathcal{R}_{\text{mid}}$, so Lemma (5.7) shows that $M(y)$ is a successor of $M(s_i) \cong X_i/\text{soc}\, X_i$ in Γ_C. Applying Lemma (4.1) with the function f_i it follows that $M(y)$ is not injective and $[\tau_C^{-1}M(y) : T_i'] \neq 0$. Now observe that $\tau_C^{-1}M(y)$ is also a successor of $X_i/\text{soc}\, X_i$, so $\tau_C^{-1}M(y)$ is not injective and $[\tau_C^{-2}M(y) : T_i'] \neq 0$. Repeating in this way one finds for all $m > 0$ that $M(\tau^{-m}y) \cong \tau_C^{-m}M(y)$ is not injective and $[M(\tau^{-m}y) : T_i'] \neq 0$.

Varying y and m, the points $\tau^{-m}y$ range over all elements of $\mathcal{R}_{\text{succ}}$.

(2) This is dual, using the opposite of Lemma (4.1).

(5.9) THEOREM. *The modules* $M(x)$ *with* $x \in (\mathbb{Z}\Delta)_0$ *form a connected component of the Auslander–Reiten quiver of* C, *and this component has shape* $\mathbb{Z}\Delta$.

Proof. We begin by showing that if $x \in \mathscr{R}_{\text{mid}} \cup \mathscr{R}_{\text{succ}}$ then there is an Auslander–Reiten sequence of the form

$$\xi_x : 0 \to M(x) \to \bigoplus_{\alpha: x \to y} a_\alpha M(y) \to M(\tau^{-1}x) \to 0.$$

Suppose first that $\tau x \in \mathscr{R}_{\text{mid}} \cup \mathscr{R}_{\text{succ}}$ and there is an Auslander–Reiten sequence of form $\xi_{\tau x}$,

$$\xi_{\tau x} : 0 \to M(\tau x) \to \bigoplus_{\alpha: \tau x \to z} a_\alpha M(z) \to M(x) \to 0.$$

Now the z which arise also lie in $\mathscr{R}_{\text{mid}} \cup \mathscr{R}_{\text{succ}}$ so by Lemmas (5.7) and (5.8) the modules $M(x)$ and $M(z)$ are not injective. Thus by [2, Proposition 2.3] there is an Auslander–Reiten sequence of the form

$$0 \to \tau_C^- M(\tau x) \to P \oplus \bigoplus_{\alpha: \tau x \to z} a_\alpha \tau_C^- M(z) \to \tau_C^- M(x) \to 0$$

with P projective. If P is non-zero, then $M(x) \cong \tau_C^- M(\tau x)$ is a summand of rad P. This is impossible, for either $x \in \mathscr{R}_{\text{mid}}$ in which case Lemma (5.7) applies, or $x \in \mathscr{R}_{\text{succ}}$, so $[M(x) : T_i'] \neq 0$ for some i by Lemma (5.8). Now this sequence can be rewritten as ξ_x.

This inductive step implies our assertion since every $x \in \mathscr{R}_{\text{mid}} \cup \mathscr{R}_{\text{succ}}$ is of the form $\tau^{-m}y$ $(m \geqslant 0)$ with $y, \tau^{-1}y \in \mathscr{R}_{\text{mid}}$, and by Lemma (5.7) there is an Auslander–Reiten sequence of form ξ_y.

Dually, whenever $\tau^{-1}x \in \mathscr{R}_{\text{pred}} \cup \mathscr{R}_{\text{mid}}$ there is an Auslander–Reiten sequence ξ_x. Thus there is an Auslander–Reiten sequence of the form ξ_x for all $x \in (\mathbb{Z}\Delta)_0$.

Since B is connected, so is $\Delta(\mathscr{S})$, and hence so also is $\mathbb{Z}\Delta$. Thus the modules $M(x)$ form a connected component of Γ_C.

Finally suppose that $M(x) \cong M(y)$ for $x, y \in (\mathbb{Z}\Delta)_0$. We show that $x = y$. By applying τ^m for suitable $m \in \mathbb{Z}$ and possibly exchanging the two points we may assume that $x \in \mathscr{R}_{\text{mid}}$, $\tau x \in \mathscr{R}_{\text{pred}}$ and $y \notin \mathscr{R}_{\text{pred}}$. Thus $M(x)$ is either an $X_i[0, q]$ $(q \geqslant 1)$ or it belongs to $S(\to X)$. In both cases $[M(x) : T_j'] = 0$ for all j, so $y \in \mathscr{R}_{\text{mid}}$ by Lemma (5.8). Now the modules $M(z)$ for $z \in \mathscr{R}_{\text{mid}}$ are known, and different z give different $M(z)$, so $x = y$ as required. Thus the component has shape $\mathbb{Z}\Delta$.

6. Realizing Any Component

We prove the theorem stated in the introduction. First we need some lemmas.

(6.1) LEMMA. *If Δ is a connected, locally finite, valued quiver, and almost all vertices $x \in \Delta_0$ have valency $\leqslant 2$, then Δ can be written as the connected union of a finite valued quiver H and quivers G_i $(1 \leqslant i \leqslant n)$ of the form*

$$g_{i0} \overset{(1,1)}{\text{———}} g_{i1} \overset{(1,1)}{\text{———}} g_{i2} \overset{(1,1)}{\text{———}} \cdots$$

in which the orientation is unspecified, and g_{i0} is identified with some point h_i in H.

Proof. Since Δ is connected, we can define the distance $d(x, y) \in \mathbb{N}$ between two points $x, y \in \Delta_0$ as the length of the shortest walk from x to y. Choose any point $p \in \Delta_0$, and $m \in \mathbb{N}$ such that $d(p, x) \leqslant m$ for all points x with valency $\geqslant 3$. As a first approximation, let H to be the full valued subquiver with $H_0 = \{q \mid d(p, q) \leqslant m\}$. This is finite since Δ is locally finite. One readily sees that any point $x \notin H_0$ lies in a full valued subquiver of Δ of one of the forms

$$\text{(a)}\quad q \text{———} \bullet \overset{(1,1)}{\text{———}} \bullet \overset{(1,1)}{\text{———}} \bullet \overset{(1,1)}{\text{———}} \cdots$$

$$\text{(b)}\quad q \text{———} \bullet \overset{(1,1)}{\text{———}} \bullet \overset{(1,1)}{\text{———}} \cdots \overset{(1,1)}{\text{———}} \bullet \text{———} q',$$

$$\text{(c)}\quad q \text{———} \bullet \overset{(1,1)}{\text{———}} \cdots \overset{(1,1)}{\text{———}} \bullet \text{———} \bullet$$

in which $q, q' \in H_0$, the other points are not in H_0 and have no arrows connected to them other than those marked, the orientations of the arrows are unspecified and the valuations of unmarked arrows are unspecified. Since Δ is locally finite, there are only finitely many configurations of these forms. Now by enlarging m we can eliminate cases (b) and (c) and ensure that the first arrow in case (a) has valuation (1, 1).

(6.2) LEMMA. *If F is a finite valued quiver without oriented cycles and with symmetrizable valuation and if k is a prime field then there is an hereditary k-algebra with A with $\Delta(A\text{-inj}) \cong F$.*

Proof. One can choose extensions k_n/k of degree n for all $n \geqslant 1$, in such a way that if $n \mid m$ then $k_n \subseteq k_m$. If the function $d\colon F_0 \to \mathbb{N}_1$ symmetrizes the valuation, let A be the tensor algebra of the species with quiver F, in which the vertex x is assigned the field $k_{d(x)}$ and an arrow $\alpha\colon x \to y$ in F with $m = a_\alpha d(y) = d(x)\, a'_\alpha$ is assigned the field k_m regarded as a $k_{d(x)} - k_{d(y)}$-bimodule. See [4].

(6.3) LEMMA. *Let A be an hereditary algebra and let B be the iterated one-point extension of A with respect to the indecomposable injective A-modules $I_1, \ldots, I_n$ with projective socle. If X_i denotes the indecomposable* *projective B-module with radical I_i, then B is a socle-projective, tilted, QF-3 algebra with minimal faithful module $X_1 \oplus \cdots \oplus X_n$,*

$$\mathscr{S} = A\text{-inj} \vee \langle X_1, ..., X_n \rangle$$

is a slice, and $\Delta(\mathscr{S})$ is obtained from $\Delta(A\text{-inj})$ by attaching new vertices $[X_i]$ and arrows $[I_i] \to [X_i]$ with valuation $(1, 1)$.

Proof. Since the I_i are orthogonal, recall that B is the matrix algebra

$$B = \begin{pmatrix} A & I_1 & \cdots & I_n \\ 0 & E_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & E_n \end{pmatrix},$$

where $E_i = \mathrm{End}_A(I_i)^{\mathrm{op}}$. The X_i are injective by [7, 2.5.5]. The fact that $\mathscr{S}$ is a slice, and the form of $\Delta(\mathscr{S})$, follows from the usual iterative construction of the component of B-mod containing $\mathscr{S}$. Since $X = X_1 \oplus \cdots \oplus X_n$ is sincere, Lemma (3.1) shows that B is QF-3 and X is the minimal faithful module. The fact that B is socle-projective is obvious.

(6.4) *Proof of the Theorem.* If Δ has underlying valued graph $\mathbb{A}_\infty$ the assertion is known—one can take any wild hereditary algebra [6]—so suppose otherwise. Write Δ as the connected union of valued quivers H and G_i $(1 \leqslant i \leqslant n)$ as in Lemma (6.1). Since Δ is infinite, $n \geqslant 1$.

If $f: \Delta_0 \to \mathbb{Z}$ is a function with the property

$$f(x) - f(y) \in \{0, 1\} \text{ whenever there is an arrow } \alpha: x \to y \text{ in } \Delta, \qquad (*)$$

then $\mathbb{Z}\Delta \cong \mathbb{Z}\Delta(f)$, where $\Delta(f)$ is the full valued subquiver of $\mathbb{Z}\Delta$ on the points

$$\Delta(f)_0 = \{(-f(d), d) \mid d \in \Delta_0\}.$$

The classical case is a reflection: if $z \in \Delta_0$ is a source and $f(x) = \delta_{zx}$ then $(-1, z)$ is a sink in $\Delta(f)$.

Using this construction we reduce to the case when the sources in Δ are precisely the vertices g_{i1} $(1 \leqslant i \leqslant n)$. First we consider $\mathbb{Z}H$ and define a sequence of functions $f_k: H_0 \to \mathbb{Z}$, setting $f_0(h) = 0$ for all $h \in H_0$, and iteratively, if there is a source $(-f_k(z), z) \neq (0, h_1)$ in $H(f_k)$, we set $f_{k+1}(x) = f_k(x) + \delta_{zx}$. These functions satisfy $(*)$ and $f_k(h_1) = 0$. Since H is finite, $\sum_{x \in H_0} f_k(x)$ is bounded, but as this sum increases in each step, the sequence must terminate, say with $f = f_j$. Since H, and hence the $H(f_k)$ have no oriented cycles this means that $(0, h_1)$ is the unique source in $H(f)$. We extend f to Δ by setting

$$f(g_{i1}) = \begin{cases} f(h_i) - 1 & (\text{if } g_{i0} \to g_{i1}) \\ f(h_i) & (\text{if } g_{i0} \leftarrow g_{i1}) \end{cases}$$

and

$$f(g_{i,j+1}) = \begin{cases} f(g_{ij}) & (\text{if } g_{ij} \to g_{i,j+1}) \\ f(g_{ij}) + 1 & (\text{if } g_{ij} \leftarrow g_{i,j+1}) \end{cases}$$

for $j \geqslant 1$. Now the g_{i1} are the sources in $\Delta(f)$ and $\mathbb{Z}\Delta \cong \mathbb{Z}\Delta(f)$.

Let A be an hereditary algebra with $\Delta(A\text{-inj})$ isomorphic to the full valued subquiver F of Δ on $H_0 \cup \{g_{11}, ..., g_{n1}\}$. Such an algebra exists by Lemma (6.2). Let B be the iterated one-point extension constructed in Lemma (6.3). If $\mathscr{S}$ is the corresponding slice, then $\Delta(\mathscr{S})$ is isomorphic to the full valued subquiver of Δ on the points $H_0 \cup \{g_{ij} \mid 1 \leqslant i \leqslant n, j = 1, 2\}$. Since Δ is connected so is F, and therefore B is connected.

If B is hereditary then the indecomposable injective A-modules with projective socle must be projective. As remarked in (3.4), the only connected hereditary algebras with a non-zero projective-injective are of type $\mathbb{A}_n$ with linear orientation. In this case Δ has graph $\mathbb{A}_\infty$, which has been excluded.

Thus B is a non-hereditary, socle-projective, connected, tilted, QF-3 algebra with slice $\mathscr{S}$. Applying Theorem (5.9), the corresponding multiple Kronecker biextension C has a component of shape $\mathbb{Z}\Delta$, as required.

References

1. M. Auslander and I. Reiten, Representation theory of artin algebras, III, Almost split sequences, *Comm. Algebra* **3** (1975), 239–294.
2. M. Auslander and I. Reiten, Representation theory of artin algebras, V, Methods for computing almost split sequences and irreducible morphisms, *Comm. Algebra* **5** (1977), 519–554.
3. M. C. R. Butler and C. M. Ringel, Auslander–Reiten sequences with few middle terms and applications to string algebras, *Comm. Algebra* **15** (1987), 145–179.
4. V. Dlab and C. M. Ringel, Indecomposable representations of graphs and algebras, *Mem. Amer. Math. Soc.* **173** (1976).
5. D. Happel, U. Preiser, and C. M. Ringel, Vinberg's characterization of Dynkin diagrams using subadditive functions with applications to DTr-periodic modules, *in* "Representation Theory II," Proceedings Ottawa 1979, Lecture Notes in Mathematics, Vol. 832, Springer-Verlag, Berlin, 1980.
6. C. M. Ringel, Finite dimensional hereditary algebras of wild representation type, *Math. Z.* **161** (1978), 235–255.
7. C. M. Ringel, Tame algebras and integral quadratic forms, *in* "Lecture Notes in Mathematics," Vol. 1099, Springer-Verlag, Berlin, 1984.
8. C. M. Ringel, The regular components of the Auslander–Reiten quiver of a tilted algebra, *Chinese Ann. Math. Ser. B* **9** (1988), 1–18.
9. Y. Zhang, The structure of stable components, *Can. J. Math.*, to appear.