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Towers of Semi-simple Algebras

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This is an extension of the work of Goodman–de la Harpe–Jones on pairs of multi-matrix algebras and the corresponding index. In particular, it is shown that the basic invariant of a pair of finite-dimensional semi-simple algebras is a bimodule together with an integral n-tuple, and that the entire theory is equivalent to the theory of finite-dimensional hereditary algebras whose square of the radical equals zero.

The present note provides an algebraic approach to the theory of pairs of semi-simple algebras, the Jones fundamental construction of a tower, and its index. It underlines the fundamental role of a bimodule and a vector space which are attached to every tower, and which provide the ingredients for the corresponding weighted valued graph. The Jones index is expressed in terms of the largest real part of the eigenvalues of a Coxeter transformation associated with this graph. In this way, the "mysteries" of a "discrete" nature of the set of all possible values for this index are clarified.

Chains of finite-dimensional semi-simple algebras

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_t \subseteq \cdots$$

and their inductive limits have been the subject of a number of studies in the theory of C^*-algebras, in particular in the language of AF-algebras (see, e.g., [Br, C, D, E, G, J1]). The methods and results are well recorded in the recent book of Goodman, de la Harpe, and Jones [GHJ]. A significant

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part of the theory is purely algebraic, including the study of multi-matrix algebras and their Bratteli diagrams, the Jones fundamental construction of the towers, and the corresponding index.

In this brief account, we wish to provide a general approach to these concepts and clarify the links to existing algebraic theories. This will anable us to consider general pairs and towers of finite-dimensional semi-simple algebras; the respective invariants are a bimodule (together with a vector space), its (weighted) valued graph, the corresponding (weighted) preprojective component, and the largest real part of the eigenvalues of the corresponding Coxeter transformation. Throughout the paper, k is some fixed field. We may formulate the main results as follows (for explanations concerning the terminology see Section 1).

THEOREM 1. *There is a one-to-one correspondence between the pairs $A \subseteq B$ of finite-dimensional semi-simple k-algebras and the pairs $({}_FM_G, X_F)$ of finite-dimensional F–G-bimodules and finite-dimensional F-vector spaces over basic semi-simple k-algebras F and G. Given $({}_FM_G, X_F)$, the corresponding pair is*

$$\mathrm{End}(X_F) \subseteq \mathrm{End}(X_F \otimes {}_FM_G).$$

An explicit description of this correspondence is given in Proposition 1. The following theorem underlines the central role of these bimodules (note that in terms of the representation theory of associative algebras, we deal with k-species) for the towers defined by the fundamental construction of V.F.R. Jones [J1, J2]. In particular, one recognizes the concept of a bimodule over basic semi-simple k-algebras as an important link in the full understanding of the interrelations between the existing theories of the pairs $A \subseteq B$ on the one hand and those of the associated matrix algebra $\mathscr{A}(A, B) = \left(\begin{smallmatrix} A & B \\ 0 & B \end{smallmatrix}\right)$ on the other.

THEOREM 2. *Given a pair $A \subseteq B$, let $C = \mathrm{End}(B_A)$, so that there is a pair $B \subseteq C$. If $({}_FM_G, X_F)$ defines $A \subseteq B$, then*

$$(\mathrm{Hom}_G({}_FM_G, {}_GG_G), X_F \otimes {}_FM_G)$$

defines $B \subseteq C$.

Thus, in particular, the valued graph of $B \subseteq C$ is obtained from the valued graph of $A \subseteq B$ by reversing the arrows. A successive application of Theorem 2 leads to the tower defined by $A \subseteq B$, whose valued graph is just that of the preprojective component of the Auslander–Reiten graph of the hereditary algebra $\mathscr{A}(A, B)$ (see Remark in Section 2).

THEOREM 3. *Let $A \subseteq B$ be a (connected) pair of finite-dimensional semi-simple k-algebras and c the Coxeter transformation defined by its valued graph. If the graph is Dynkin, then the Jones index of $A \subseteq B$ is*

$$[B : A] = 2(r+1),$$

where r is the largest value of the real parts of the eigenvalues of c. Otherwise,

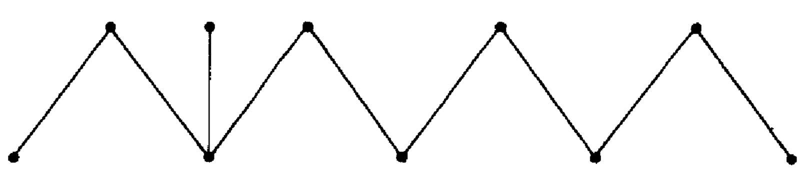
$$[B : A] = 2 + \lambda + \lambda^{-1},$$

where λ is the largest real eigenvalue of c. Thus, $[B : A] = 4$ if and only if the graph of $A \subseteq B$ is Euclidean (extended Dynkin) and $[B : A] > 4.026$ in all other cases.

This explains the rather "discrete" nature of the set of all possible values for the Jones index [J3]. In particular, we get the well-known values $4\cos^2(\pi/n)$ for $n \geqslant 3$ in the case of a Dynkin graph ($\neq A_1$) when $r = \cos(2\pi/n)$. Also, if the graph is neither Dynkin nor Euclidean, then the values of the Jones index are always greater than or equal to $\rho_0 = 4.02641794918695985 99\ldots$, the largest real root of the polynomial

$$x^5 - 9x^4 + 27x^3 - 31x^2 + 12x - 1$$

corresponding to the (Bratteli) graphs



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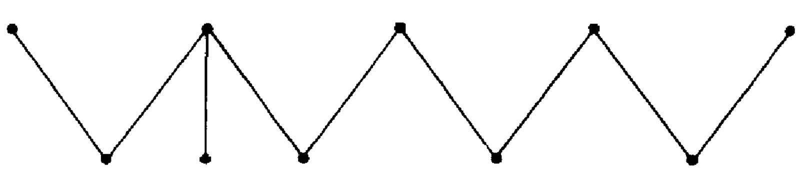


(The corresponding eigenvalue of the Coxeter transformation is the largest root of the irreducible polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.)$$

Let us point out that the previous theorem answers a question of V. F. R. Jones in [J2] by clarifying the link between small values of $[B : A]$ and the finite representation type of $\mathscr{A}(A, B)$. Let us formulate this relationship explicitly.

THEOREM 4. *Let $A \subseteq B$ be a (connected) pair of finite-dimensional semi-simple k-algebras, let $[B:A]$ be its Jones-index, and let $\mathscr{A}(A, B) = \left(\begin{smallmatrix} A & B \\ 0 & B \end{smallmatrix}\right)$. Then*

(i) $[B:A] < 4$ *if and only if $\mathscr{A}(A, B)$ is of finite representation type, and, in this case,*

$$[B:A] = 4\cos^2\frac{\pi}{n} \qquad \textit{for some} \quad n \geqslant 3.$$

(ii) $[B:A] = 4$ *if and only if $\mathscr{A}(A, B)$ is of tame representation type.*

(iii) $[B:A] > 4$ *if and only if $\mathscr{A}(A, B)$ is of wild representation type, and, in this case, $[B:A] \geqslant \rho_0$.*

1. PAIRS AND THEIR GRAPHS

Throughout the paper, $A \subseteq B$ denotes a k-pair, i.e., a pair of finite-dimensional semi-simple k-algebras with unital embedding. Two such pairs $A \subseteq B$ and $A' \subseteq B'$ are said to be equivalent if there is a k-algebra isomorphism $\varphi: B \to B'$ such that $\varphi(A) = A'$. Recall that semi-simple k-algebras are finite products of full matrix rings over division k-algebras; those which are products of division k-algebras are called basic.

We shall associate every k-pair $A \subseteq B$ with the hereditary k-algebra of 2×2 matrices

$$\mathscr{A}(A, B) = \begin{pmatrix} A & B \\ 0 & B \end{pmatrix} = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \middle| a \in A; b, c \in B \right\}.$$

LEMMA 1. *The k-pairs $A \subseteq B$ and $A' \subseteq B'$ are equivalent if and only if $\mathscr{A}(A, B)$ and $\mathscr{A}(A', B')$ are isomorphic k-algebras.*

Proof. If φ is an equivalence of $A \subseteq B$ and (A', B'), then Φ given by

$$\Phi\left(\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} \varphi(a) & \varphi(c) \\ 0 & \varphi(b) \end{pmatrix}$$

defines obviously a k-algebra isomorphism of $\mathscr{A}(A, B)$ and $\mathscr{A}(A', B')$.

Conversely, if Ψ is an isomorphism of $\mathscr{A}(A, B)$ and $\mathscr{A}(A', B')$, then Ψ induces an isomorphism of their maximal semi-simple quotients and thus isomorphisms

$$\Psi_1 : A \to A' \qquad \text{and} \qquad \Psi_2 : B \to B'.$$

In view of the fact that the restriction Ψ_3 of Ψ to the radicals of the algebras is a bimodule isomorphism and thus $\Psi_3(acb) = \Psi_1(a) \Psi_3(c) \Psi_2(b)$, we infer that the homomorphism $\Psi: B \to B$ defined by

$$\Psi(b) = \Psi_3(1_B) \Psi_2(b) \Psi_3(1_B)^{-1}$$

is an isomorphism such that $\Psi(a) = \Psi_1(a)$ for all $a \in A$. This completes the proof of Lemma 1.

Now, $\mathscr{A}(A, B)$ is Morita equivalent (i.e., the respective module categories are equivalent) to a k-algebra of 2×2 matrices

$$\begin{pmatrix} F & M \\ 0 & G \end{pmatrix} = \left\{ \begin{pmatrix} f & m \\ 0 & g \end{pmatrix} \middle| f \in F, g \in G, m \in M \right\},$$

where ${}_FM_G$ is an F–G-bimodule (with k acting centrally) and F and G are basic semi-simple k-algebras. This provides a clue to the proofs of Theorems 1 and 2 on the one hand and, following the techniques of the theory of representations, to the definition of a (weighted) valued (bipartite) graph of a pair on the other.

Recall that an (oriented, symmetrizable) valued graph is defined by a pair of non-negative integral $n \times n$ matrices ($U = (u_{ij})$, $V = (v_{ij})$) such that there is a positive integral invertible diagonal $n \times n$ matrix D satisfying $UD = DV$. The index set $\{1, 2, ..., n\}$ is the set of vertices of the graph, and if $u_{ij} \neq 0$ (and thus $v_{ij} \neq 0$), we say that there is an arrow from i to j with valuation (u_{ij}, v_{ij}).

PROPOSITION 1. *Let $A \subseteq B$ be a k-pair and*

$$A = \prod_{i=1}^{a} \mathrm{Mat}(x_i, F_i), \qquad B = \prod_{j=1}^{b} \mathrm{Mat}(y_j, G_j).$$

Then there are (finite-dimensional) F_i-spaces X_i and F_i–G_j-bimodules ${}_{F_i}({}_iM_j)_{G_j}$ such that

$$A \simeq \mathrm{End}\, X_F \qquad \text{and} \qquad B \simeq \mathrm{End}(X_F \otimes {}_FM_G),$$

where $X_F = \bigoplus_{i=1}^{a} X_i$ and ${}_FM_G = \bigoplus_{i=1}^{a} \bigoplus_{j=1}^{b} {}_iM_j$ with canonical operations by the basic k-algebras

$$F = F_1 \times F_2 \times \cdots \times F_a \qquad \text{and} \qquad G = G_1 \times G_2 \times \cdots \times G_b.$$

Furthermore,

$$\mathrm{End}\, B_A \simeq \mathrm{End}(X_F \otimes {}_FM_G \otimes {}_GM_F^*),$$

where ${}_GM_F^* = \mathrm{Hom}_G({}_FM_G, {}_GG_G)$.

ADDENDUM. *Write* $\dim({}_iM_j)_{G_j} = u_{ij}$ *and* $\dim_{F_i}({}_iM_j) = v_{ij}$; *then*

$$\dim(X_i)_{F_i} = x_i \qquad \textit{and} \qquad \sum_{i=1}^{a} x_i u_{ij} = y_j.$$

Furthermore, End $B_A = \prod_{l=1}^{a} \mathrm{Mat}(z_l, F_l)$, *where*

$$z_l = \sum_{i=1}^{a} x_i \sum_{j=1}^{b} u_{ij} v_{lj}.$$

Proof. The k-algebra $\mathscr{A}(A, B)$ assigned to the pair $A \subseteq B$ is Morita equivalent to an algebra $\mathscr{A} = \left(\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right)$ with basic semi-simple k-algebras F and G. Now $F = F_1 \times F_2 \times \cdots \times F_a$ and $G = G_1 \times G_2 \times \cdots \times G_b$ with division algebras F_i $(1 \leqslant i \leqslant a)$ and G_j $(1 \leqslant j \leqslant b)$. Accordingly, the F–G-bimodule ${}_FM_G$ is a (unique) direct sum of the F_i–G_j-bimodules ${}_iM_j = F_i M G_j$. Consequently, $A \simeq \mathrm{End}\, X_F$ where $X = \bigoplus_{i=1}^{a} X_i$ with F_i-spaces X_i of dimension x_i $(1 \leqslant i \leqslant a)$ and $B \simeq \mathrm{End}(X_F \otimes {}_FM_G)$. Indeed, $\mathscr{A}(A, B)$ is the endomorphism algebra of the direct sum of the (projective right) $\mathscr{A}(A, B)$-modules consisting of x_i copies of $\left(\begin{smallmatrix} F_i & {}_iM \\ 0 & G \end{smallmatrix}\right)$, where ${}_iM$ is the F_i–G-bimodule $\bigoplus_{j=1}^{b} {}_iM_j$, for each $1 \leqslant i \leqslant a$.

Observe that

$$B \simeq \mathrm{End}\left(\bigoplus_{j=1}^{b} X \otimes {}_iM_j\right) \qquad \text{with} \quad M_j = \bigoplus_{i=1}^{a} {}_iM_j.$$

Furthermore,

$$\begin{aligned}
\mathrm{End}\, B_A &= \mathrm{End}[\mathrm{End}({}_AX_F \otimes {}_FM_G)]_A \\
&\simeq \mathrm{End}[\mathrm{Hom}_F({}_AX_F, \mathrm{Hom}_G({}_FM_G, {}_AX_F \otimes {}_FM_G))]_A \\
&\simeq \mathrm{Hom}_A[\mathrm{Hom}_F({}_AX_F, X_F \otimes {}_FM_G \otimes {}_GM_F^*), \\
&\qquad\qquad \mathrm{Hom}_F(X_F, {}_AX_F \otimes {}_FM_G \otimes {}_GM_F^*)] \\
&\simeq \mathrm{Hom}_F[\mathrm{Hom}_F({}_AX_F, X_F \otimes {}_FM_G \otimes {}_GM_F^*) \otimes {}_AX_F, \\
&\qquad\qquad X_F \otimes {}_FM_G \otimes {}_GM_F^*] \\
&\simeq \mathrm{End}(X_F \otimes {}_FM_G \otimes {}_GM_F^*)_F,
\end{aligned}$$

as required.

In order to complete the proof, we note that $\dim(G_j M^* F_i)_{F_i} = v_{ij}$ and $\dim_{G_j}(G_j M^* F_i) = u_{ij}$. In this way, we get the two formulae for y_j and z_l of the addendum, respectively.

Now, both Theorems 1 and 2 follow easily from Proposition 1; indeed, one takes into account only the fact that the k-algebras

$$\begin{pmatrix} \operatorname{End} X_F & \operatorname{End}(X_F \otimes {}_F M_G) \\ 0 & \operatorname{End}(X_F \otimes {}_F M_G) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$$

are Morita equivalent.

In order to clarify the relationship between various k-pairs, we shall call two pairs $A \subseteq B$ and $A' \subseteq B'$ Morita equivalent if the respective k-algebras $\mathscr{A}(A, B)$ and $\mathscr{A}(A', B')$ are Morita equivalent.

PROPOSITION 2. *Two pairs $A \subseteq B$ and $A' \subseteq B'$ are Morita equivalent if and only if there is a finite-dimensional bimodule ${}_F M_G$ and there are finite-dimensional vector spaces X_F and ${}_F Y$ over basic semi-simple k-algebras F and G such that A is Morita equivalent to A' via $X \otimes {}_F Y$ ($A \simeq \operatorname{End} X_F$ and $A' \simeq \operatorname{End} Y_F^*$ with $Y_F^* = \operatorname{Hom}_F({}_F Y, {}_F F_F)$) and B is Morita equivalent to B' via $(X_F \otimes {}_F M_G) \otimes ({}_G M_F^* \otimes {}_F Y)$($B \simeq \operatorname{End}(X_F \otimes {}_F M_G)$, $B' \simeq \operatorname{End}(Y_F^* \otimes {}_F M_G)$) where ${}_G M_F^* = \operatorname{Hom}_G({}_F M_G, {}_G G_G)$.*

Proof. Referring to Proposition 1, we have

$$A = \operatorname{End}(X_F), \qquad B = \operatorname{End}(X_F \otimes {}_F M_G)$$

and

$$A' = \operatorname{End}(X'_F), \qquad B' = \operatorname{End}(X'_F \otimes {}_F M_G).$$

Thus, writing ${}_F Y = \operatorname{Hom}_F(X'_F, {}_F F_F)$, the equivalences of the module categories of the semi-simple algebras A, F, A' are given by

$$Z_A \leftrightarrow Z_A \otimes {}_A X_F \leftrightarrow Z_A \otimes {}_A X_F \otimes {}_F Y_{A'}.$$

Moreover, the corresponding equivalences of the module categories of B, G, B' are

$$\begin{aligned} Z_B &\leftrightarrow Z_B \otimes {}_B({}_A X_F \otimes {}_F M_G)_G \\ &\leftrightarrow Z_B \otimes {}_B({}_A X_F \otimes {}_F M_G)_G \otimes {}_G({}_G M_F^* \otimes {}_F Y_{A'})_{B'}. \end{aligned}$$

Now, we are prepared to introduce the definition of the weighted valued graph of a k-pair.

DEFINITION. Let $A \subseteq B$ be a k-pair as in Proposition 1. Then its (bipartite) valued graph is defined by the $(a+b) \times (a+b)$ matrices

$$\left(\begin{pmatrix} 0_{a \times a} & U_0 \\ 0_{b \times a} & 0_{b \times b} \end{pmatrix}, \begin{pmatrix} 0_{a \times a} & V_0 \\ 0_{b \times a} & 0_{b \times b} \end{pmatrix} \right),$$

where $U_0=(u_{ij})$, $V_0=(v_{ij})$ are $a\times b$ matrices exhibiting the dimensions u_{ij}, v_{ij} given by the pair. Moreover, the weighting of the graph is achieved by attaching the dimensions x_i and y_j to the vertices $1\leqslant i\leqslant a$ and $a+1\leqslant a+j\leqslant a+b$, respectively.

Note that y_j's are determined uniquely by the x_i's. Observe also that in the case of the pairs of multi-matrix algebras (i.e., the pairs for which all $F_i=G_j=k$) this notion coincides with that of the weighted Bratteli diagrams [Br]. However, in contrast to the case of multi-matrix algebras where the weighted Bratteli diagrams provide a full characterization of the pairs, in general, one has to consider the respective bimodules ${}_iM_j$ (and not only their dimensions, i.e., the valuation of the graph) in order to characterize the k-pairs fully. As a simple illustration of the situation, consider the central embedding

$$c\mapsto\begin{pmatrix} c & 0\\ 0 & c\end{pmatrix}$$

of the $\mathbb{R}$-algebra $\mathbb{C}$ into $\mathrm{Mat}(2,\mathbb{C})$ and the non-central one

$$c=a+bi\mapsto\begin{pmatrix} a & -b\\ b & a\end{pmatrix}.$$

The respective $\mathbb{R}$-pairs are clearly non-equivalent. However, the weighted valued graph is

$$\underset{(1)}{\circ}\xrightarrow{(2,2)}\underset{(2)}{\circ}$$

in either case. An easy calculation shows that the $\mathbb{C}$–$\mathbb{C}$-bimodule is ${}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}\oplus{}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$ with canonical operations in the first case, and ${}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}\oplus{}_{\mathbb{C}}\mathbb{C}_{\bar{\mathbb{C}}}$ with the bimodule operation on ${}_{\mathbb{C}}\mathbb{C}_{\bar{\mathbb{C}}}$ given by $c_1\cdot a\cdot c_2=c_1a\overline{c_2}$ in the second case.

2. Towers of Pairs and Their Indices

Given a k-pair $A\subseteq B$, the fundamental construction of V. F. R. Jones [J1] defines the tower of finite-dimensional semi-simple k-algebras

$$A_0\subseteq A_1\subseteq A_2\subseteq\cdots\subseteq A_p\subseteq A_{p+1}\subseteq\cdots,$$

where $A_0=A$, $A_1=B$, and $A_{p+1}=\mathrm{End}(A_p)_{A_{p-1}}$ for $p\geqslant 1$.

Let ${}_FM_G$ be the F-G-bimodule associated with the k-pair $A_0=A\subseteq B=A_1$ as in Proposition 1. Then, in view of Theorem 2, the bimodule associated with the k-pair $A_p\subseteq A_{p+1}$ is either ${}_FM_G$ if p is even or

${}_GM_F^* = \mathrm{Hom}_G({}_FM_G, {}_GG_G)$ if p is odd. Consequently, the valued graph of the k-pair $A_p \subseteq A_{p+1}$ is the valued graph given by $(\left(\begin{smallmatrix} 0 & U_0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & V_0 \\ 0 & 0 \end{smallmatrix}\right))$ as for $A \subseteq B$ if p is even, and by $(\left(\begin{smallmatrix} 0 & 0 \\ V_0^{tr} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ U_0^{tr} & 0 \end{smallmatrix}\right))$ if p is odd. The weighting $(\mathbf{x}, \mathbf{y})$ with $\mathbf{x} = (x_i)_{1 \leqslant i \leqslant a}$ and $\mathbf{y} = \mathbf{x}U_0 = (y_j)_{1 \leqslant j \leqslant b}$ of the graph as described in Proposition 1 determines the weighting of the graph of the entire tower by successive application of the k-linear transformation of the real $(a+b)$-dimensional space $\mathbb{R}^{(a+b)}$ given by

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, \mathbf{y}) \begin{pmatrix} U_0 V_0^{tr} & 0 \\ 0 & V_0^{tr} U_0 \end{pmatrix}.$$

Let us attach to the k-pair $A \subseteq B$ the *tower transformation* t_{AB} of the a-dimensional real space $\mathbb{R}^{(a)}$ defined by

$$t_{AB}(\mathbf{x}) = \mathbf{x} U_0 V_0^{tr}$$

and recall [GHJ] that the *Jones index* of $A \subseteq B$ *is*

$$[B : A] = \limsup_{p \to \infty} \sqrt[p]{|t_{AB}^p(\mathbf{x})|},$$

where the norm $|\cdots|$ denotes the sum of the (positive) coordinates of $t_{AB}^p(\mathbf{x})$.

Thus, the Perron–Frobenius theory yields that $[B : A]$ is the largest real eigenvalue of t_{AB} (and equal to $\lim_{p \to \infty} (|t_{AB}^{p+1}(\mathbf{x})|/|t_{AB}^p(\mathbf{x}))$.

Now, recall the concept of a Coxeter transformation defined by a valued graph (U, V). First, for each vertex $1 \leqslant q \leqslant n$, one defines an involution s_q of the n-dimensional real space $\mathbb{R}^{(n)}$ by

$$s_q(\mathbf{z}) = \mathbf{z}',$$

where $z'_r = z_r$ for $1 \leqslant r \leqslant n, r \neq q$, and $z'_q = -z_q + \sum_{r=1}^n z_r u_{rq}$. Then, a product $s_{q_1} s_{q_2} \cdots s_{q_n}$ of all these involutions (taken in some order) defines a Coxeter transformation [Bo].

We attach, to a given k-pair $A \subseteq B$, the Coxeter transformation c_{AB} given by the product

$$c_{AB} = s_1 s_2 \cdots s_a s_{a+1} \cdots s_{a+b}$$

of involutions s_i, $1 \leqslant i \leqslant a$, and s_{a+j}, $1 \leqslant j \leqslant b$. Thus c_{AB} is the transformation of the real space

$$\mathbb{R}^{(a+b)} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathbb{R}^{(a)}, \mathbf{y} \in \mathbb{R}^{(b)}\}$$

given by the matrix

$$\begin{pmatrix} -I & -U_0 \\ V_0^{tr} & V_0^{tr}U_0 - I \end{pmatrix} = \begin{pmatrix} -I & 0 \\ V_0^{tr} & I \end{pmatrix}\begin{pmatrix} I & U_0 \\ 0 & -I \end{pmatrix}.$$

Now, the eigenvalues of c_{AB} and $t_{A,B}$ are closely related as follows (cf. [A, DR1, SS, V, Z]).

LEMMA 2. *If $\lambda \neq 1$ is an eigenvalue of c_{AB}, then $\lambda + 2 + \lambda^{-1}$ is an eigenvalue of t_{AB}, and all eigenvalues of t_{AB} are obtained this way.*

Proof. It is easy to verify the following identities for $\lambda = \rho^2$, $(\rho^2+1)/\rho = \kappa$, $\lambda \neq 0, -1$:

$$\det\left[\begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I \end{pmatrix} - \begin{pmatrix} -I & 0 \\ V_0^{tr} & I \end{pmatrix}\begin{pmatrix} I & U_0 \\ 0 & -I \end{pmatrix}\right]$$

$$= \det\left[\begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I \end{pmatrix} + \begin{pmatrix} I & 0 \\ -V_0^{tr} & I \end{pmatrix}\begin{pmatrix} I & U_0 \\ 0 & I \end{pmatrix}\right] \det\left[\begin{pmatrix} I & -U_0 \\ 0 & I \end{pmatrix}\right]$$

$$= \det\left[\begin{pmatrix} \lambda I & -\lambda U_0 \\ 0 & \lambda I \end{pmatrix} + \begin{pmatrix} I & 0 \\ -V_0^{tr} & I \end{pmatrix}\right]$$

$$= \frac{\rho^{a+b}}{\kappa^{a+b}} \det\begin{pmatrix} \kappa I & -\rho U_0 \\ -\frac{1}{\rho} V_0^{tr} & \kappa I \end{pmatrix} \cdot \det\begin{pmatrix} \kappa I & \rho U_0 \\ 0 & \kappa I \end{pmatrix}$$

$$= \frac{\rho^{a+b}}{\kappa^{a+b}} \det\begin{pmatrix} \kappa^2 I & 0 \\ 0 & \kappa^2 I - V_0^{tr} U_0 \end{pmatrix}$$

$$= \left(\frac{\lambda}{1+\lambda}\right)^{a+b} \det\left[\begin{pmatrix} \frac{(\lambda+1)^2}{\lambda} I & 0 \\ 0 & \frac{(\lambda+1)^2}{\lambda} I \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & V_0^{tr} U_0 \end{pmatrix}\right].$$

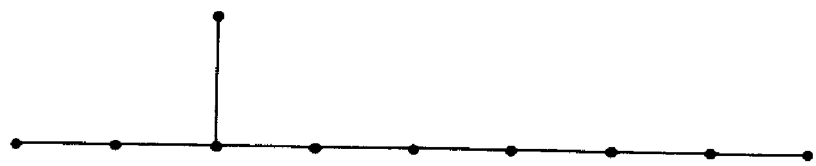
Taking into account that the non-zero eigenvalues of $V_0^{tr} U_0$ and of $U_0 V_0^{tr}$ coincide, the lemma follows.

Now, we also get a proof of Theorem 3 easily. Indeed, the largest real part of the eigenvalues of c_{AB} leads to the largest (real) eigenvalue of t_{AB}.

In the case of a Dynkin graph,

$$\lambda = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n},$$

and thus $2+\lambda+1/\lambda=2(\cos(2\pi/n)+1)=4\cos^2(\pi/n)$. Furthermore, Xi [X] has shown that the Coxeter transformations of the graph



attain the least possible value of a maximal (real) eigenvalue λ_0 greater than 1. In fact,

$$\lambda_0 = 1.17628081825991750 65...$$

is the largest real root of the irreducible polynomial

$$x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$$

(cf. [Z]). Thus the corresponding minimal value

$$\rho_0 = 4.02641794918695985 99...$$

of an index greater than 4 is attained for the valued graph with

$$U_0 = V_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 4 follows from the basic properties of the eigenvalues of Coxeter transformations and the theorem characterizing the representation type of a hereditary tensor algebra [DR2].

Remark (cf. [DR2]). In this connection, let us point out that the valued graph of a tower of semi-simple algebras defined by a k-pair $A \subseteq B$ also represents the so-called preprojective component of the algebra $\mathscr{A}(A, B)$. Moreover, while the weighting of the graph obtained from the weighting $(\mathbf{x}, \mathbf{y})$ of the k-pair $A \subseteq B$ is determined by the successive applications of the tower transformation t_{AB}, we note that $(\mathbf{x}, \mathbf{y})$ can be interpreted as the so-called dimension type of the direct sum of x_i indecomposable projective left $\mathscr{A}(A, B)$-modules defined at the vertex i for all $1 \leqslant i \leqslant a$ and the values $c_{AB}^p(\mathbf{x}, \mathbf{y})$, $p=1, 2, ...$, obtained by the successive application of the Coxeter transformation as the dimension types of the corresponding direct sums of preprojective indecomposable representations. Of course, in the case of a Dynkin graph, the preprojective component is finite, since the positive values of $c^{p}_{AB}(\mathbf{x},\mathbf{y})$ will terminate after a finite number of steps.

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