

## THE HOCHSCHILD COCYCLE CORRESPONDING TO A LONG EXACT SEQUENCE

Dedicated to Hiroyuki Tachikawa on his 60th birthday

By

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1. Let  $k$  be a field, and  $A$  an associative  $k$ -algebra with 1. Let  $M, N$  be right  $A$ -modules. We denote by  $H^t$  the Hochschild cohomology of  $A$ . It is well-known that there is a natural isomorphism

$$\eta_{MN} : \text{Ext}_A^t(M, N) \longrightarrow H^t(A, \text{Hom}_k(M, N))$$

see Cartan-Eilenberg [CE], Corollary IX. 4.4. For  $t \geq 1$ , the elements of  $\text{Ext}_A^t(M, N)$  may be considered as equivalence classes of long exact sequences, see Mac Lane [M], chapter III. Let

$$E = (0 \longleftarrow M \xleftarrow{g_0} Y_1 \xleftarrow{g_1} Y_2 \longleftarrow \cdots \longleftarrow Y_t \xleftarrow{g_t} N \longleftarrow 0)$$

be an exact sequence. We want to derive a recipe for obtaining a corresponding cocycle  $A^{\otimes(t+2)} \rightarrow \text{Hom}_k(M, N)$ .

For  $0 \leq i \leq t+1$ , let  $Z_i$  be right  $A$ -modules, and for  $0 \leq i \leq t$ , let  $\beta_i : Z_i \rightarrow Z_{i+1}$  be  $k$ -linear maps. With  $\beta = (\beta_0, \dots, \beta_t)$  we associate a map

$$\Omega_\beta : A^{\otimes(t+2)} \longrightarrow \text{Hom}_k(Z_0, Z_{t+1})$$

defined by

$$(a_0, \dots, a_{t+1}) \Omega_\beta = \bar{a}_0 \beta_0 \bar{a}_1 \beta_1 \cdots \bar{a}_t \beta_t \bar{a}_{t+1},$$

for  $a_0, \dots, a_{t+1} \in A$ , where  $\bar{a}_i$  denotes the scalar multiplication by  $a_i$  (on  $Z_i$ ); note that all maps will be written on the right of the argument, thus the composition of  $\beta_0 : Z_0 \rightarrow Z_1$ , and  $\beta_1 : Z_1 \rightarrow Z_2$  is denoted by  $\beta_0 \beta_1$ .

Given the exact sequence  $E$  exhibited above, it clearly splits as a sequence of  $k$ -spaces, thus there are  $k$ -linear maps

$$M \xrightarrow{\gamma_0} Y_1 \xrightarrow{\gamma_1} Y_2 \longrightarrow \cdots \longrightarrow Y_t \xrightarrow{\gamma_t} N$$

such that

$$\gamma_{i-1} \gamma_i = 0, \quad g_{i-1} \gamma_{i-1} + \gamma_i g_i = 1_{Y_i}, \quad \text{for } 1 \leq i \leq t,$$

and

$$\gamma_0 g_0 = 1_M, \quad g_t \gamma_t = 1_N,$$

(see section 2).

**THEOREM.** *The map  $\Omega_\gamma: A^{\otimes(t+2)} \rightarrow \text{Hom}_k(M, N)$  is a cocycle, and the cohomology classes  $[\Omega_\gamma]$  and  $\eta([E])$  in  $H^t(A, \text{Hom}_k(M, N))$  are equal up to sign.*

One reason for our interest in this problem is the following: Consider the case  $t=2$ . Given any bimodule  ${}_A T_A$ , the elements of  $H^2(A, T)$  index the various "Hochschild extensions"  $\tilde{A}$  of  $A$  by  $T$  (here,  $\tilde{A}$  is a  $k$ -algebra with a square zero ideal  $I$  such that  $\tilde{A}/I=A$ , and such that  $I$ , as an  $A$ - $A$ -bimodule, is isomorphic to  $T$ ; note that the multiplication of  $\tilde{A}$  can be recovered from  $A$  and  $T$  using the corresponding 2-cocycle, see [H] or [CE], XIV. 2). There is a recursive construction for quasi-hereditary algebras due to Parshall and Scott ([PS], Theorem 4.6) which uses Hochschild extensions of quasi-hereditary algebras  $A$  by bimodules of the form  $\text{Hom}_k(M, N)$ , so we have to deal with 2-cocycles  $A^{\otimes(t)} \rightarrow \text{Hom}_k(M, N)$ . Our presentation of such 2-cocycles using long exact sequences should help to understand these algebras. Also, we remark that the Hochschild cohomology groups with values in  $\text{Hom}_k(DA, A)$ , where  $DA = \text{Hom}_k(A, k)$ , play a prominent role in Tachikawa's discussion of the Nakayama conjecture [T].

2. *The splitting for  $E$  over  $k$ .* In order to work with the sequence  $E$ , it will be convenient to use the notation:  $Y_{-1}=0, Y_0=M, Y_{t+1}=N, Y_{t+2}=0$ , and to deal also with the zero maps  $g_{-1}: Y_0 \rightarrow Y_{-1}, \gamma_{-1}: Y_{-1} \rightarrow Y_0, g_{t+1}: Y_{t+2} \rightarrow Y_{t+1}, \gamma_{t+1}: Y_{t+1} \rightarrow Y_{t+2}$ ; so that the conditions mentioned above can be rewritten in the form

$$\gamma_{i-1} \gamma_i = 0, \quad g_{i-1} \gamma_{i-1} + \gamma_i g_i = 1_{Y_i}, \quad \text{for } 0 \leq i \leq t+1.$$

Let  $X_i$  be the image of  $g_i$ , thus we have short exact sequences

$$0 \longleftarrow X_{i-1} \xleftarrow{h_{i-1}} Y_i \xleftarrow{f_i} X_i \longleftarrow 0$$

for  $1 \leq i \leq t$ , with  $g_0 = h_0, g_i = h_i f_i$  for  $1 \leq i \leq t-1$ , and  $g_t = f_t$ . These sequences split over  $k$ , thus we obtain  $k$ -linear maps  $\varphi_i: Y_i \rightarrow X_i, \eta_{i-1}: X_{i-1} \rightarrow Y_i$  such that  $\eta_{i-1} \varphi_i = 0, f_i \varphi_i = 1_{X_i}, \eta_{i-1} h_{i-1} = 1_{X_{i-1}}$  and  $h_{i-1} \eta_{i-1} + \varphi_i f_i = 1_{Y_i}$  for all  $i$ . Now, take  $\gamma_i = \varphi_i \eta_i: Y_i \rightarrow Y_{i+1}$ , in this way we obtain a splitting of  $E$  over  $k$ .

3. *Preparation for the proof.* Let  $A^e = A^{\circ p} \otimes_k A$  be the enveloping algebra

of  $A$ , where  $A^{op}$  is the opposite algebra of  $A$ . The  $A$ - $A$ -bimodules are just the (right)  $A^e$ -modules, in particular,  $A$  itself is in a canonical way an  $A^e$ -module. For  $n \geq 0$ , let  $S_n = A^{\otimes(n+2)}$ , and let  $\nabla_n : S_{n+1} \rightarrow S_n$  be defined by

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+2}) \nabla_n = \sum_{i=0}^{n+1} (-1)^i a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_{n+2}.$$

Also, let  $\nabla_{-1} : S_0 \rightarrow A$  be defined by

$$(a_0 \otimes a_1) \nabla_{-1} = a_0 a_1.$$

The  $S_n$  are  $A$ - $A$ -bimodules, or, equivalently  $A^e$ -modules, the scalar multiplication of  $a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \in S_n$  by  $a \otimes a' \in A^{op} \otimes_k A = A^e$  yields  $(aa_0) \otimes a_1 \otimes \cdots \otimes (a_{n+1}a')$ .

Note that for all  $n \geq -1$ , the maps  $\nabla_i$  are  $A^e$ -linear, in fact

$$A \xleftarrow{\nabla_{-1}} S_0 \xleftarrow{\nabla_0} S_1 \xleftarrow{\quad} \cdots$$

is a projective resolution of  $A$  as a right  $A^e$ -module, it is called the *standard resolution* of  $A$ , see [CE], IX. 6. We can use this resolution in order to calculate  $H^i(A, \text{Hom}_k(M, N)) = \text{Ext}_{A^e}^i(A, \text{Hom}_k(M, N))$ .

4. Besides  $\gamma = (\gamma_0, \dots, \gamma_t)$ , we also will need for  $0 \leq r \leq t$ , the sequences  $\gamma(r) = (\gamma_0, \dots, \gamma_r)$ , so that  $\gamma(0) = (\gamma_0)$ ,  $\gamma(t) = \gamma$ . According to section 1, there is defined  $\Omega_{\gamma(r)} : S_r \rightarrow \text{Hom}_k(Y_0, Y_{r+1})$ . In addition, by abuse of language, we also define  $\Omega_{\gamma(-1)} : A \rightarrow \text{Hom}_k(Y_0, Y_0)$  by  $a \Omega_{\gamma(-1)} = \bar{a}$ , for  $a \in A$ .

LEMMA. For  $0 \leq r \leq t$ , we have  $\nabla_{r-1} \Omega_{\gamma(r-1)} = (-1)^r \Omega_{\gamma(r)} \text{Hom}(1, g_r)$ .

PROOF. We introduce the following notation: let  $\sigma_i = \gamma_i \bar{a}_i$ ,  $\tau_i = \bar{a}_i \gamma_i : Y_i \rightarrow Y_{i+1}$  for  $0 \leq i \leq t-1$ , and let  $\sigma_{ij} = \sigma_i \sigma_{i+1} \cdots \sigma_j$ ,  $\tau_{ij} = \tau_i \tau_{i+1} \cdots \tau_j$  for  $0 \leq i \leq j \leq t-1$ ; by abuse of language, let  $\sigma_{i+1, i} = 1_{Y_i}$ , and  $\tau_{i+1, i} = 1_{Y_{i+1}}$ . Recall that

$$(a_{-1} \otimes \cdots \otimes a_r) \nabla_{r-1} = \sum_{i=0}^r (-1)^i a_{-1} \otimes \cdots \otimes (a_{i-1} a_i) \otimes \cdots \otimes a_r,$$

thus

$$\begin{aligned} (a_{-1} \otimes \cdots \otimes a_r) \nabla_{r-1} \Omega_{\gamma(r-1)} &= \sum_{i=0}^r (-1)^i \bar{a}_{-1} \sigma_{0, i-1} \tau_{i, r-1} \bar{a}_r \\ &= \sum_{i=0}^r (-1)^i \bar{a}_{-1} \sigma_{0, i-1} (g_{i-1} \gamma_{i-1} + \gamma_i g_i) \tau_{i, r-1} \bar{a}_r, \end{aligned}$$

where we have inserted  $1_{Y_i} = g_{i-1} \gamma_{i-1} + \gamma_i g_i$ . Note that for  $0 \leq i \leq r-1$ , we have

$$\begin{aligned} \sigma_{0, i-1} \gamma_i g_i \tau_{i, r-1} &= \sigma_{0, i-1} \gamma_i g_i \bar{a}_i \gamma_i \tau_{i+1, r-1} \\ &= \sigma_{0, i-1} \gamma_i \bar{a}_i g_i \gamma_i \tau_{i+1, r-1} \\ &= \sigma_{0, i} g_i \gamma_i \tau_{i+1, r-1}, \end{aligned}$$

since  $g_i$  is  $A$ -linear. As a consequence, the last term of the summand with index  $i$  and the first term of the summand with index  $i+1$  are equal up to sign, so they cancel. In addition, the first term of the summand with index  $i=0$  involves  $g_{-1}=0$ , thus vanishes. It remains

$$\begin{aligned} (a_{-1} \otimes \cdots \otimes a_r) \nabla_{r-1} \Omega_{\gamma(r-1)} &= (-1)^r \bar{a}_{-1} \sigma_{0, r-1} \check{\gamma}_r g_r \bar{a}_r \\ &= (-1)^r \bar{a}_{-1} \sigma_{0, r} g_r \\ &= (-1)^r (a_{-1} \otimes \cdots \otimes a_r) \Omega_{\gamma(r)} - g_r \\ &= (-1)^r (a_{-1} \otimes \cdots \otimes a_r) \Omega_{\gamma(r)} \text{Hom}(1, g_r). \end{aligned}$$

This finishes the proof.

5. *An injective coresolution of the  $A$ - $A$ -bimodule  $\text{Hom}_k(M, N)$ .* We choose a projective resolution

$$0 \longleftarrow M \xleftarrow{p_{-1}} P_0 \xleftarrow{p_0} P_1 \longleftarrow \cdots$$

of the  $A$ -module  $M$ , and an injective coresolution

$$0 \longrightarrow N \xrightarrow{q^{-1}} Q^0 \xrightarrow{q^0} Q^1 \longrightarrow \cdots$$

of the  $A$ -module  $N$ . For  $t \geq 0$ , let  $L^t = \bigoplus_{i=0}^t \text{Hom}_k(P_i, Q^{t-i})$ , this is an  $A$ - $A$ -bimodule, or, equivalently a right  $A^e$ -module. For  $t \geq 0$ , define an  $A^e$ -linear map  $\Delta^t: L^t \rightarrow L^{t+1}$  by

$$(\varphi_0, \dots, \varphi_t) \Delta^t = (\varphi_0 q^t, (-1)^{t+1} p_0 \varphi_0 + \varphi_1 q^{t-1}, \dots, (-1)^{t+1} p_{t-1} \varphi_{t-1} + \varphi_t q^0, (-1)^{t+1} p_t \varphi_t),$$

where  $\varphi_i \in \text{Hom}_k(P_i, Q^{t-i})$ , and define  $\Delta^{-1}: \text{Hom}_k(M, N) \rightarrow L^0$  by  $\Delta^{-1} = \text{Hom}(p_{-1}, q^{-1})$ .

We obtain a sequence

$$0 \longrightarrow \text{Hom}_k(M, N) \xrightarrow{\Delta^{-1}} L^0 \xrightarrow{\Delta^0} L^1 \longrightarrow \cdots,$$

which is an injective coresolution, see [CE], IX, Cor. 2.7a.

In order to relate the given sequence  $E$  with the injective coresolution  $Q' = (Q', q')$ , we define  $u_{-1} = 1_N$ , and, inductively, we find  $u_i: Y_{t-i} \rightarrow Q^i$  such that  $g_{t-i} u_i = u_{i-1} q^{i-1}$ , for  $0 \leq i \leq t$ .

We are going to reformulate the previous lemma using the maps  $\Delta^i$  and  $u_i$ . For  $0 \leq r \leq t-1$ , let

$$\Omega'_r: S_r \longrightarrow L^{t-r-1}$$

be defined by

$$(a_0 \otimes \cdots \otimes a_{r+1}) \Omega'_r = (p_{-1} \cdot (a_0 \otimes \cdots \otimes a_{r+1}) (\Omega_{\gamma(r)} \cdot u_{t-r-1}), 0, \dots, 0),$$

and similarly, let

$$\Omega'_{-1}: A \longrightarrow L^t$$

be defined by

$$(a)\Omega'_{-1}=(p_{-1}\bar{a}u_t, 0, \dots, 0).$$

PROPOSITION. For  $0 \leq r \leq t-1$ , we have  $\nabla_{r-1}\Omega'_{r-1}=(-1)^r\Omega'_r\Delta^{t-r-1}$ . For  $r=t$ , we have  $\nabla_{t-1}\Omega'_{r-1}=(-1)^t\Omega'_r\Delta^{-1}$ .

PROOF. For  $0 \leq r \leq t$ , and  $a_0, \dots, a_{r+1} \in A$ , we have

$$\begin{aligned} (a_0 \otimes \dots \otimes a_{r+1})\nabla_{r-1}\Omega'_{r-1} &= (p_{-1}(a_0 \otimes \dots \otimes a_{r+1})\nabla_{r-1}\Omega_{\gamma(r-1)}u_{t-r}, 0, \dots, 0) \\ &= (-1)^r(p_{-1}(a_0 \otimes \dots \otimes a_{r+1})\Omega_{\gamma(r)}g_r u_{t-r}, 0', \dots, 0) \\ &= (-1)^r(p_{-1}(a_0 \otimes \dots \otimes a_{r+1})\Omega_{\gamma(r)}u_{t-r-1}q^{t-r-1}, 0, \dots, 0), \end{aligned}$$

using the definition of  $\Omega'_{r-1}$ , the lemma, and the defining condition for  $u_{t-r}$ . On the other hand, for  $0 \leq r \leq t-1$ , we have

$$\begin{aligned} (a_0 \otimes \dots \otimes a_{r+1})\Omega'_r\Delta^{t-r-1} &= (p_{-1}(a_0 \otimes \dots \otimes a_{r+1})\Omega_r u_{t-r-1}, 0, \dots, 0)\Delta^{t-r-1} \\ &= (p_{-1}(a_0 \otimes \dots \otimes a_{r+1})\Omega_r u_{t-r-1}q^{t-r-1}, 0, \dots, 0) \end{aligned}$$

using the definitions of  $\Omega'_r$ ,  $\Delta^{t-r-1}$ , and the fact that  $p_0 p_{-1} = 0$ . Similarly, for  $r=t$ , we have

$$\begin{aligned} (a_0 \otimes \dots \otimes a_{t+1})\Omega'_t\Delta^{-1} &= p_{-1}(a_0 \otimes \dots \otimes a_{t+1})\Omega_t q^{-1} \\ &= p_{-1}(a_0 \otimes \dots \otimes a_{t+1})\Omega_{\gamma(t)}u_{-1}q^{-1}, \end{aligned}$$

since  $\Omega_\gamma = \Omega_{\gamma(t)}$  and  $u_{-1} = 1$ .

6. *Some homological algebra.* We will need some basic result of homological algebra which we want to review. We have chosen already a projective resolution of  $M$ , and an injective coresolution of  $N$ . In order to calculate  $\text{Ext}^t(M, N)$  we may use one of these sequences, or else the double complex  $\text{Hom}_A(P_i, Q^j)$ . So let  $R^t = \bigoplus_{i=0}^t \text{Hom}_A(P_i, Q^{t-i})$ , this is a subset of  $L^t = \bigoplus_{i=0}^t \text{Hom}_k(P_i, Q^{t-i})$ , and let  $\delta^t: R^t \rightarrow R^{t+1}$  be the restriction of  $\Delta^t$  to  $R^t$ , similarly, let  $\delta^{-1}: \text{Hom}_A(M, N) \rightarrow L^0$  be the restriction of  $\Delta^{-1} = \text{Hom}(p_{-1}, q^{-1})$  to  $\text{Hom}_A(M, N)$ . So we obtain a complex

$$R^\cdot = (R^0 \xrightarrow{\delta^0} R^1 \xrightarrow{\delta^1} R^2 \longrightarrow \dots),$$

which we want to compare with the complexes

$$\text{Hom}_A(P_\cdot, N) \text{ and } \text{Hom}_A(M, Q^\cdot).$$

Note that there are maps

$$\text{Hom}(1, q^{-1}): \text{Hom}_A(P, N) \longrightarrow R^*,$$

$$\text{Hom}(p_{-1}, 1): \text{Hom}_A(M, Q^*) \longrightarrow R^*,$$

and they are quasi-isomorphisms: they induce isomorphisms when passing to the cohomology ([B], §5.2).

Consider now the given exact sequence  $E$ . Its equivalence class  $[E]$  in  $\text{Ext}_A^t(M, N) = H^t(\text{Hom}_A(P, N))$  is given by the cocycle  $u_t: M \rightarrow Q_t$ . Under the map  $\text{Hom}(p_{-1}, 1): \text{Hom}_A(M, Q^*) \rightarrow R^*$ , the cocycle  $u_t$  is mapped onto the cocycle  $(p_{-1}u_t, 0, \dots, 0) \in \bigoplus_{i=0}^t \text{Hom}_A(P_i, Q^{t-i}) = R^t$ .

7. *Proof of the theorem.* We apply the previous considerations to the ring  $A^e$  (instead of  $A$ ), and the  $A^e$ -modules  $A$  and  $\text{Hom}_k(M, N)$ . For  $A$ , we use the standard resolution  $S := (S, \nabla)$ , for  $\text{Hom}_k(M, N)$ , we use the injective coresolution  $L := (L, \Delta)$ . We form  $C^t = \bigoplus_{i=0}^t \text{Hom}_{A^e}(S_i, L^{t-i})$ , with differential  $D^t: C^t \rightarrow C^{t+1}$  given by

$$\begin{aligned} (\Phi_0, \dots, \Phi_t) D^t = & (\Phi_0 \Delta^t, (-1)^{t+1} \nabla_0 \Phi_0 + \Phi_1 \Delta^{t-1}, \dots, \\ & (-1)^{t+1} \nabla_{t-1} \Phi_{t-1} + \Phi_t \Delta^0, (-1)^{t+1} \nabla_t \Phi_t), \end{aligned}$$

for  $\Phi_i \in \text{Hom}_{A^e}(S_i, L^{t-i})$ . The maps

$$\text{Hom}(1, \Delta^{-1}): \text{Hom}_{A^e}(S, \text{Hom}_k(M, N)) \longrightarrow C^*$$

and

$$\text{Hom}(\nabla_{-1}, 1): \text{Hom}_{A^e}(A, L^*) \longrightarrow C^*$$

are quasi-isomorphisms. Clearly, we have an isomorphism

$$\rho: \text{Hom}_{A^e}(A, L^*) \longrightarrow R^*,$$

since for  $A$ -modules  $X, Y$ , the bimodule maps  $\Sigma: A \rightarrow \text{Hom}_k(X, Y)$  correspond bijectively to the elements of  $\text{Hom}_A(X, Y)$ , with  $(\Sigma)\rho = (1)\Sigma$ .

It remains to chase elements via the various quasi-isomorphisms

$$\text{Hom}_{A^e}(S, \text{Hom}_k(M, N)) \xrightarrow{\text{Hom}(1, \Delta^{-1})} C^* \xleftarrow{\text{Hom}(\nabla_{-1}, 1)} \text{Hom}_{A^e}(A, L^*),$$

and

$$\text{Hom}_A(M, Q^*) \xrightarrow{\text{Hom}(p_{-1}, 1)} R^* \cong \text{Hom}_{A^e}(A, L^*).$$

The last map  $\text{Hom}(p_{-1}, 1)$  sends the cocycle  $u_t$  onto the element  $(p_{-1}u_t, 0, \dots, 0) \in R^t$ , thus to  $\mathcal{Q}'_1$  in  $\text{Hom}_{A^e}(A, L^*)$ . So it remains to consider the elements

$$\Omega_r \Delta^{-1} = (\Omega_r) \text{Hom}(1, \Delta^{-1}) \quad \text{and} \quad \nabla_{-1} \Omega'_{-1} = (\Omega'_{-1}) \text{Hom}(\nabla_{-1}, 1)$$

in  $C^t$ . Let  $\varepsilon_{2i} = (-1)^i$ , and  $\varepsilon_{2i+1} = (-1)^{i+1}$ , thus  $\varepsilon_j = (-1)^{t+j+1} \varepsilon_{j-1}$ , for all  $j$ . Let  $\Phi_i = \varepsilon_i \Omega'_i$  for  $0 \leq i \leq t-1$ , and  $(\Psi_0, \dots, \Psi_t) := (\Phi_0, \dots, \Phi_{t-1}) D^{t-1}$ . Then

$$\Psi_0 = \Phi_0 \Delta^{t-1} = \varepsilon_0 \Omega'_0 \Delta^{t-1} = \nabla_{-1} \Omega'_{-1}$$

$$\Psi_t = \varepsilon_t \nabla_{t-1} \Phi_{t-1} = \varepsilon_t (-1)^t \Omega_r \Delta^{-1},$$

whereas, for  $1 \leq r \leq t-1$ ,

$$\begin{aligned} \Psi_r &= (-1)^t \nabla_{r-1} \Phi_{r-1} + \Phi_r \Delta^{t-1-r} \\ &= (-1)^t \varepsilon_{r-1} \nabla_{r-1} \Omega'_{r-1} + \varepsilon_r \Omega'_r \Delta^{t-1-r} \\ &= (-1)^t \varepsilon_{r-1} (-1)^r \Omega'_r \Delta^{t-r-1} + (-1)^{t+r+1} \varepsilon_{r-1} \Omega'_r \Delta^{t-1-r} = 0, \end{aligned}$$

always using the proposition. This shows that

$$(\nabla_{-1} \Omega'_{-1}, 0, \dots, 0, (-1)^t \varepsilon_t \Omega_r \Delta^{-1}) = (\Phi_0, \dots, \Phi_{t-1}) D^{t-1}$$

is a coboundary in  $C^t$ , thus  $\nabla_{-1} \Omega'_{-1}$  and  $(-1)^{t+1} \varepsilon_t \Omega_r \Delta^{-1}$  yield the same cohomology class in  $H^t(C^t)$ .

Let us summarize: the composition of  $H^t(\text{Hom}(p_{-1}, 1))$ ,  $H^t(p^{-1})$ ,  $H^t(\text{Hom}(\nabla_{-1}, 1))$  and  $H^t(\text{Hom}(1, \Delta^{-1}))^{-1}$  yields a natural isomorphism

$$\eta_{MN}: \text{Ext}_A^t(M, N) \longrightarrow H^t(A, \text{Hom}_k(M, N))$$

and  $\eta_{MN}([E]) = (-1)^{t+1} \varepsilon_t [\Omega_r]$ , thus  $\eta_{MN}([E])$  and  $[\Omega_r]$  are equal up to sign. This completes the proof.

REMARK. As the proof shows, the precise relation (under the given identification of  $H^t(A, \text{Hom}_k(M, N))$  and  $\text{Ext}_A^t(M, N)$ ) is

$$\eta_{MN}([E]) = (-1)^{t+1} [\Omega_r],$$

where  $i$  is the largest integer with  $2i \leq t$  (for  $t=2i$ , we have the sign  $(-1)^{t+1} \varepsilon_{2i} = (-1)^{t+1+i} = (-1)^{i+1}$ , for  $t=2i+1$ , we have  $(-1)^{t+1} \varepsilon_{2i+1} = (-1)^{t+1} (-1)^{t+i} = (-1)^{i+1}$ ).

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