COMPOSITIO MATHEMATICA

VLASTIMIL DLAB
CLAUS MICHAEL RINGEL

A construction for quasi-hereditary algebras

Compositio Mathematica, tome 70, nº 2 (1989), p. 155-175.
<http://www.numdam.org/item?id=CM_1989__70_2_155_0>

Compositio Mathematica **70**: 155–175, 1989.

A construction for quasi-hereditary algebras

VLASTIMIL DLAB[1] & CLAUS MICHAEL RINGEL[2]
[1]*Department of Mathematics, Carleton University, Ottawa K1S 5B6, Canada;*
[2]*Fakultät für Mathematik, Universität Bielefeld, D-4800 Bielefeld, F.R.G.*

Received 6 May 1988; accepted 15 September 1988

Introduction

Two different algebraic approaches have been introduced in order to deal with highest weight categories arising in representation theory (for semisimple complex Lie algebras [BGG] or semisimple algebraic groups) and with the categories of perverse sheaves over suitable spaces [BBD]. One approach starts with the axiomatization of highest weight categories in papers by Cline, Parshall and Scott [S], [CPS], [PS], where it is shown that the highest weight categories with a finite number of weights are just the module categories over finite dimensional algebras which are quasi-hereditary. The other approach is based on descriptions of the categories of perverse sheaves by Mebkhout [Me] and MacPherson and Vilonen [MV]; recently, Mirollo and Vilonen [MiV] have shown that these categories are again equivalent to module categories over certain finite dimensional algebras. The aim of our paper is to exhibit more explicitly the algebras $A(\gamma)$ studied by Mirollo and Vilonen, and to formulate the precise relationship between this construction and the quasi-hereditary algebras introduced by Cline, Parshall and Scott. In particular, we obtain in this way a construction for all quasi-hereditary algebras. In contrast to the "not so trivial extension" method oulined in [PS], one avoids in this way the use of Hochschild extensions.

Let us outline the construction. Let k be a perfect field, let C, D be finite dimensional k-algebras, assume that C is quasi-hereditary and D is semisimple. Let ${}_CS_D$ and ${}_DT_C$ be bimodules such that ${}_CS$ and T_C have good filtrations with respect to some heredity chain of C. Let γ: ${}_CS_D \otimes {}_DT_C \to {}_CC_C$ be a bimodule map with image in the radical of C. Then an algebra $A(\gamma)$ is defined, which again is quasi-hereditary. We obtain all quasi-hereditary algebras by iterating this procedure, starting with C the zero ring.

1. The rings $A(\gamma)$

Let C, D be rings (associative, with 1), ${}_CS_D$, ${}_DT_C$ bimodules, and $\gamma\colon {}_CS_D \otimes {}_DT_C \to {}_CC_C$ a bimodule homomorphism. These are the data we will work with. In particular, starting from these data, we are going to define a ring $A(\gamma)$.

The direct sum of two abelian groups M_1, M_2 will be denoted by $M_1 + M_2$, in order to make terms which involve both the direct sum and the tensor product symbol more readable. We denote by $C \times D$ the product of the rings C and D, and we consider $S + T$ as a $C \times D$–$C \times D$-bimodule (the left action of C on T and of D on S being zero, and similar conditions hold on the right). Denote by $\mathscr{T}(S, T)$ the tensor algebra of the $C \times D$–$C \times D$-bimodule $S + T$, thus as an additive group

$$\mathscr{T}(S,T) = C + D + S + T + S \underset{D}{\otimes} T + T \underset{C}{\otimes} S + S \underset{D}{\otimes} T \underset{C}{\otimes} S + T \underset{C}{\otimes} S \underset{D}{\otimes} T + \cdots,$$

with multiplication induced by forming tensor products. Let $\mathscr{R}(\gamma)$ be the ideal of $\mathscr{T}(S, T)$ generated by all elements of the form $s \otimes t - \gamma(s \otimes t)$, with $s \in S$, $t \in T$. Then, by definition, $A(\gamma) = \mathscr{T}(S, T)/\mathscr{R}(\gamma)$. We denote by e_C the image of the unit element of C in $A(\gamma)$, and by e_D the image of the unit element of D in $A(\gamma)$. Note that e_C, e_D are orthogonal idempotents in $A(\gamma)$ with $1 = e_C + e_D$.

We want to investigate properties of $A(\gamma)$. Before we do this, let us insert a description of the category of $A(\gamma)$-modules. Let $\mathscr{C}(\gamma)$ be the following category: an object of $\mathscr{C}(\gamma)$ is of the form $(X_C, Y_D, \varphi, \psi)$, where $\varphi\colon X_C \otimes {}_CS_D \to Y_D$, $\psi\colon Y_D \otimes {}_DT_C \to X_C$ such that $\psi(\varphi \otimes 1_T) = 1_X \otimes \gamma$; the maps $(X, Y, \varphi, \psi) \to (X', Y', \varphi', \psi')$ are of the form (ξ, η), where $\xi\colon X_C \to X'_C$, $\eta\colon Y_D \to Y'_D$ such that $\varphi'(\xi \otimes 1_S) = \eta\varphi$ and $\psi'(\eta \otimes 1_T) = \xi\psi$, and the composition of the maps is componentwise. In case both C and D are k-algebras for some field k, the object $(X_C, Y_D, \varphi, \psi)$ in $\mathscr{C}(\gamma)$ is said to be finite dimensional provided both X_C and Y_D are finite dimensional over k.

PROPOSITION 1: *The category of (right) $A(\gamma)$-modules is equivalent to $\mathscr{C}(\gamma)$. In case both C and D are k-algebras over some field k, the finite dimensional $A(\gamma)$-modules correspond to the finite dimensional objects in $\mathscr{C}(\gamma)$, under such an equivalence.*

Proof: This can be easily verified. For the convenience of the reader, we outline the construction of the relevant functors. Given an object $(X_C, Y_D, \varphi, \psi)$

in $\mathscr{C}(\gamma)$, then $X + Y$ is canonically a right $\mathscr{T}(S, T)$-module, and the condition $\psi(\varphi \otimes 1_T) = 1_X \otimes \gamma$ implies that the $\mathscr{T}(S, T)$-module $X + Y$ is annihilated by $\mathscr{R}(\gamma)$, thus it is an $A(\gamma)$-module. Conversely, given a right $A(\gamma)$-module M, then $M = Me_C + Me_D$, and Me_C may be considered as a right C-module, Me_D as a right D-module, and the operation of $A(\gamma)$ on M gives, in addition, maps φ: $Me_C \otimes {}_CS_D \to Me_D$, ψ: $Me_D \otimes {}_DT_C \to Me_C$, which satisfy $\psi(\varphi \otimes 1_T) = 1_{Me_C} \otimes \gamma$.

REMARK: The objects in $\mathscr{C}(\gamma)$ may be exhibited also in an alternative way: Instead of specifying a map ψ: $Y_D \otimes {}_DT_C \to X_C$, one may consider the adjoint map $\bar{\psi}$: $Y_D \to \mathrm{Hom}_C({}_DT_C, X_C)$. Note that γ induces a natural transformation γ^*: $F \to G$, where $F = - \otimes {}_CS_D$ and $G = \mathrm{Hom}_C({}_DT_C, -)$ are considered as functors from the category of C-modules to the category of D-modules, namely $\gamma^*_X = \overline{1_X \otimes \gamma}$, for any C-module X. The condition $\psi(\varphi \otimes 1_T) = 1_X \otimes \gamma$ translates to the condition $\bar{\psi}\varphi = \gamma^*_X$, thus the commutation of the triangle

$$\begin{array}{ccc} F(X) & \xrightarrow{\gamma^*_X} & G(X) \\ & {\scriptstyle\varphi}\searrow \quad \nearrow{\scriptstyle\bar{\psi}} & \\ & Y & \end{array}$$

This is the form of the objects considered by Mirollo and Vilonen in [MiV]. They start with a right exact functor F, a left exact functor G, and a natural transformation η: $F \to G$. It has been used in [MiV] that under their assumptions, any right exact functor F is a tensor product functor, any left functor G is a Hom functor. But also, any natural transformation η: $F \to G$, where $F = - \otimes {}_CS_D$ and $G = \mathrm{Hom}_C({}_DT_C, -)$, is induced by a bimodule homomorphism ${}_CS_D \to \mathrm{Hom}_C({}_DT_C, {}_CC_C)$, namely by η_X, where $X = C_C$ (note that this η_X is not only a map of right D-modules, but also commutes with the left action by C, using the naturality condition). However, the bimodule homomorphisms ${}_CS_D \to \mathrm{Hom}_C({}_DT_C, {}_CC_C)$ correspond bijectively to the bimodule homomorphism ${}_CS_D \otimes {}_DT_C \to {}_CC_C$, to the case considered above is the general case.

PROPOSITION 2: *The subgroup*

$$C + D + S + T + T \underset{C}{\otimes} S$$

of $\mathscr{T}(S, T)$ *is a direct complement of* $\mathscr{R}(\gamma)$.

Proof: Let $\mathcal{T}_0 = C \times D$, and $\mathcal{T}_{n+1} = \mathcal{T}_n \otimes_{C\times D} (S + T)$, for $n \in \mathbb{N}_0$. Thus $\mathcal{T} = \mathcal{T}(S, T) = \oplus_{n\geqslant 0} \mathcal{T}_n$. By induction on n, one easily shows that $\mathcal{T}_n$ is contained in $C + D + S + T + T \otimes_C S + \mathcal{R}(\gamma)$. On the other hand, let $u \in \mathcal{R}(\gamma)$, say $u = \Sigma\, x_j(s_j \otimes t_j - \gamma(s_j \otimes t_j))y_j \in C + D + S + T + T \otimes_C S$, with $s_j \in S$, $t_j \in T$, and $x_j, y_j \in \mathcal{T}$. We can assume $x_j \in \mathcal{T}_{n_j}, y_j \in \mathcal{T}_{m_j}$ for some $n_j, m_j \in \mathbb{N}_0$. For any i, let $I(i)$ be the set of all j with $n_j + m_j = i$. Then $v_i := \Sigma_{j\in I(i)}\, x_j(s_j \otimes t_j)y_j \in \mathcal{T}_{i+2}$, and $w_i := \Sigma_{j\in I(i)}\, x_j\gamma(s_j \otimes t_j)y_j \in \mathcal{T}_i$. Note that $v_i = 0$ implies $w_i = 0$, since w_i is the image of v_i under the linear map $1 \otimes \gamma \otimes 1 : \mathcal{T} \otimes_{C\times D} (S \otimes_D T) \otimes_{C\times D} \mathcal{T} \to \mathcal{T} \otimes_{C\times D} \mathcal{T}$. Now, if $u = \Sigma_i\,(v_i + w_i)$ is non-zero, then choose n maximal with $v_n \neq 0$. Then $u - v_n$ belongs to $\otimes_{i\leqslant n+1} \mathcal{T}_i$, whereas v_n is non-zero in $\mathcal{T}_{n+2}$. However, we also assume that u belongs to $\mathcal{T}_0 + \mathcal{T}_1 + T \otimes_C S$. It follows that $n = 0$ and that v_n belongs both to $S \otimes T$ and $T \otimes S$. But these additive subgroups of $\mathcal{T}_2$ intersect trivially, thus $u = 0$.

COROLLARY 1: *Let k be a field. If C, D are finite dimensional k-algebras and S, T are finite dimensional over k, with k operating centrally on them, then $A(\gamma)$ is a finite dimensional k-algebra.*

Note that this corollary is essentially due to Mirollo–Vilonen. In [MiV], they have shown that under the given assumptions, $\mathcal{C}(\gamma)$ is equivalent to the module category over a finite dimensional k-algebra A. This algebra is not specified further, but by Morita theory, A has to be Morita equivalent to our $A(\gamma)$.

COROLLARY 2: *The canonical projection $\mathcal{T}(S, T) \to A(\gamma)$ induces the following identifications:*

$$e_C A(\gamma) e_C = C, \quad e_D A(\gamma) e_D = D + T \underset{C}{\otimes} S, \quad e_c A(\gamma) e_D = S,$$

$$e_D A(\gamma) e_C = T.$$

REMARK: The ring structure of $D' := e_D A(\gamma) e_D = D + T \otimes_C S$ is given by the following multiplication:

$$(d, t \otimes s)\,(d', t' \otimes s') + (dd', dt' \otimes s' + t \otimes sd' + t\gamma(s \otimes t') \otimes s'),$$

for $d, d' \in D$; $t, t' \in T$, and $s, s' \in S$. The right $e_D A(\gamma) e_D$-module structure on $e_C A(\gamma) e_D = S$ is given by

$$s \cdot (d, t \otimes s') = sd + \gamma(s \otimes t)s',$$

for $s, s' \in S$; $d \in D$ and $t \in T$; similarly, the left $e_D A(\gamma) e_D$-module structure on $e_D A(\gamma) e_C = T$ is given by

$$(d, t \otimes s)t' = dt' + t\gamma(s \otimes t'),$$

for $d \in D$; $t, t' \in T$ and $s \in S$. Finally, the multiplication yields a map

$$e_D A(\gamma) e_C \underset{C}{\otimes} e_C A(\gamma) e_D \to e_D A(\gamma) e_D$$

which is just the inclusion $T \otimes_C S \to D + T \otimes_C S$, and a map

$$e_C A(\gamma) e_D \underset{D'}{\otimes} e_D A(\gamma) e_C \to e_C A(\gamma) e_C$$

which is induced by γ: $S \otimes_D T \to C$. Note that these data form "pre-equivalence data" in the sense of [B] p. 61. Of course, one may obtain a different proof of proposition 2 by defining first the multiplication on $D' = D + T \otimes_C S$, then a right D'-module structure on S and a left D'-module structure on T as above, and verifying the various associativity conditions in order to be sure to deal with "preequivalence data". Then $A(\gamma)$ may be defined as the matrix ring

$$\begin{bmatrix} C & S \\ T & D' \end{bmatrix}.$$

Observe that in the ring $e_D A(\gamma) e_D = D + T \otimes_C S$, the subgroup $e_D A(\gamma) e_C A(\gamma) e_D = T \otimes_C S$ is an ideal, that this ideal is complemented by the subring D, and that the multiplication map

$$e_D A(\gamma) e_C \underset{e_C A(\gamma) e_C}{\otimes} e_C A(\gamma) e_D \to e_D A(\gamma) e_C A(\gamma) e_D$$

is bijective. These properties in fact yield a characterization of the construction, as we will show in the next proposition.

In general, given a ring A and an idempotent e, the multiplication map

$$(1 - e)Ae \underset{eAe}{\otimes} eA(1 - e) \to (1 - e)AeA(1 - e)$$

is bijective if and only if the multiplication map

$$Ae \underset{eAe}{\otimes} eA \to AeA$$

is bijective. For, the multiplication map $Ae \otimes_{AeA} eA \to AeA$ is the direct sum of the four multiplication maps $e_1 Ae \otimes_{eAe} eAe_2 \to e_1 AeAe_2$, where $e_1, e_2 \in \{e, 1 - e\}$, and, for trivial reasons, three of the four are always bijective, namely those when e_1 or e_2 is equal to e.

PROPOSITION 3: *Let A be a ring, let e be an idempotent of A. Assume that the multiplication map $Ae \otimes_{eAe} eA \to AeA$ is bijective and that there is a subring D of $(1 - e)A(1 - e)$ such that $(1 - e)A(1 - e) = (1 - e)AeA(1 - e) + D$. Let $C = eAe$, $S = eA(1 - e)$, $T = (1 - e)Ae$, and $\gamma\colon S \otimes_D T \to C$ the multiplication map. Then A is isomorphic to $A(\gamma)$.*

Proof: There is an obvious ring surjection $\mathscr{T}(S, T) \to A$ which maps $\mathscr{R}(\gamma)$ to zero. Thus we obtain a surjective map $A(\gamma) \to A$. The kernel will be a subset of $T \otimes_C S \subseteq A(\gamma)$. However, since the multiplication map $(1 - e)Ae \otimes_{eAe} eA(1 - e) \to (1 - e)AeA(1 - e)$ is bijective, the kernel of $A(\gamma) \to A$ is zero. Thus A is isomorphic to $A(\gamma)$.

2. Morita equivalence

The structure of $A(\gamma)$ strongly depends on the bimodule map γ. Assume that there are given additional bimodules ${}_CS'_D$ and ${}_DT'_C$ and a bimodule map $\gamma'\colon {}_CS'_D \otimes {}_DT'_C \to {}_CC_C$. Then we denote by $\gamma \perp \gamma'$ the bimodule map ${}_C(S + S')_D \otimes {}_D(T + T')_C \to {}_CC_C$ with $\gamma = \gamma \perp \gamma' | S \otimes T$, $\gamma' = \gamma \perp \gamma' | S' \otimes T'$, $0 = \gamma \perp \gamma' | S \otimes T'$, and $0 = \gamma \perp \gamma' | S' \otimes T$. If ${}_DM_C$ is a bimodule, let ${}_C\tilde{M}_D = \mathrm{Hom}_C({}_DM_C, {}_CC_C)$ and $\varepsilon_M\colon {}_C\tilde{M}_D \otimes {}_DM_C \to {}_CC_C$ the evaluation map ($\varepsilon(\varphi \otimes m) = \varphi(m)$).

PROPOSITION 4: *Let ${}_DP_C$ be a bimodule with P_C finitely generated projective. Then $A(\gamma)$ and $A(\gamma \perp \varepsilon_P)$ are Morita equivalent algebras.*

Proof: We show that the categories $\mathscr{C}(\gamma)$ and $\mathscr{C}(\gamma \perp \varepsilon_P)$ are equivalent. Let $\iota_S\colon {}_CS_D \to {}_C(S + \tilde{P})_D$ be the inclusion map, $\pi_T\colon {}_D(T + P)_C \to {}_DT_C$ the canonical projection. For any C-module X_C, we obtain the following commutative diagram

$$\begin{array}{ccc}
X_C \otimes {}_CS_D & \xrightarrow{\ \gamma_X^*\ } & \mathrm{Hom}_C({}_DT_C, X_C) \\
\Big\downarrow {\scriptstyle 1 \otimes \iota_S} & & \Big\downarrow {\scriptstyle \mathrm{Hom}(\pi_T, 1)} \\
X_C \otimes {}_C(S + \tilde{P})_D & \xrightarrow{(\gamma \perp \varepsilon_P)_X^*} & \mathrm{Hom}_C({}_D(T + P)_C, X_C).
\end{array}$$

Note that the bottom map can be written in the form

$$X_C \otimes {}_C S_D + X_C \otimes {}_C \tilde{P}_D \xrightarrow{\begin{bmatrix} \gamma_X^* & 0 \\ 0 & (\varepsilon_P)_X^* \end{bmatrix}} \mathrm{Hom}_C({}_D T_C, X_C) + \mathrm{Hom}_C({}_D P_C, X_C),$$

and, since P_C is finitely generated projective, $(\varepsilon_P)_X^*$ is bijective, for all X_C. It follows that $1 \otimes \iota_S$ and $\mathrm{Hom}\,(\pi_T, 1)$ induce isomorphisms $\mathrm{Ker}\,\gamma_X^* \to \mathrm{Ker}\,(\gamma \perp \varepsilon)_X^*$ and $\mathrm{Cok}\,\gamma_X^* \to \mathrm{Cok}\,(\gamma \perp \varepsilon_P)_X^*$. So we can apply proposition 1.2 of the MacPherson–Vilonen paper [MV].

REMARK: Observe that there exists an idempotent e in $A(\gamma \perp \varepsilon_P)$ such that $eA(\gamma \perp \varepsilon_P)e$ is isomorphic to $A(\gamma)$ (so that $eA(\gamma \perp \varepsilon)_{A(\gamma \perp \varepsilon)}$ with $\varepsilon = \varepsilon_P$ is a progenerator). Such an idempotent e may be constructed as follows: Let $E = \mathrm{End}\,P_C$. Since P_C is finitely generated projective, there is a bimodule isomorphism ${}_E P_C \otimes {}_C \tilde{P}_E \to {}_E E_E$, defined by $p \otimes \alpha \mapsto (p' \mapsto p\alpha(p'))$, for $p \in P$ and $\alpha \in \tilde{P}$, see [B], p. 68. In particular, there is a finite set of elements $p_i \in P$, $\alpha_i \in \tilde{P}$ such that $p = \Sigma_i p_i \alpha_i(p)$ for all $p \in P$, namely, let $f = \Sigma\, p_i \otimes \alpha_i$ be the element in $P \otimes \tilde{P}$ which is mapped to 1_E. Since ${}_D P_C$ is a D–C-bimodule, and $E = \mathrm{End}\,P_C$, the D–D-submodule of ${}_D P_C \otimes {}_C \tilde{P}_D$ generated by f is isomorphic to ${}_D D_D$. We consider f as an element of $(T + P) \otimes_C (S + \tilde{P}) \subseteq A(\gamma \perp \varepsilon)$. It is an idempotent and $e_D f = f = f e_D$. Let $e = 1 - f$. Then $e = (e_D - f) + e_C$, where $e_D - f$ and e_C are orthogonal idempotents. If we identify $\mathscr{C}(\gamma \perp \varepsilon_P)$ with the category of $A(\gamma \perp \varepsilon_P)$-modules, and $\mathscr{C}(\gamma)$ with the category of $A(\gamma)$-modules, then we obtain an equivalence $\mathscr{C}(\gamma \perp \varepsilon_P) \to \mathscr{C}(\gamma)$ by multiplying with the idempotent e.

COROLLARY 1: *Let ${}_D P_C$ be a bimodule with P_C finitely generated projective. Then $A(\varepsilon_P)$ is Morita equivalent to $C \times D$.*

The map γ: ${}_C S_D \otimes {}_D T_C \to {}_C C_C$ will be said to be *non-degenerate* provided $\gamma(s \otimes t) = 0$ for all $t \in T$ implies $s = 0$, and $\gamma(s \otimes t) = 0$ and all $s \in S$ implies $t = 0$.

COROLLARY 2: *Let C be semisimple artinian and T_C finitely generated and assume γ is non-degenerate. Then $A(\gamma)$ is Morita equivalent to $C \times D$.*

Proof: Since C is semisimple artinian, T_C is also projective. Since γ is non-degenerate, we can identify ${}_C S_D$ with ${}_C \tilde{T}_D$ so that $\gamma = \varepsilon_T$. Corollary 1 shows that $A(\gamma)$ is Morita equivalent to $C \times D$.

3. Semiprimary rings

Recall that a ring A is called *semiprimary* provided there exists a nilpotent ideal N such that A/N is semisimple artinian. Clearly, if such an ideal N exists, it is uniquely determined and is called the radical of A; we will denote it by $N(A)$. In particular, any finite dimensional algebra over a field k is a semiprimary ring.

We assume that both C and D are semiprimary. As before, there is given a bimodule map $\gamma\colon {}_CS_D \otimes {}_DT_C \to {}_CC_C$. We denote by S' the set of all elements $s \in S$ satisfying $\gamma(s \otimes t) \in N(C)$ for all $t \in T$. Similarly, we denote by T' the set of all elements $t \in T$ satisfying $\gamma(s \otimes t) \in N(C)$ for all $s \in S$. Note that S' is a C–D-submodule of S with $N(C)S \subseteq S'$, and T' is a C–D-submodule of T with $TN(C) \subseteq T'$. The kernel of the canonical map

$$T \underset{C}{\otimes} S \to (T/T') \underset{C}{\otimes} (S/S')$$

will be denoted by U. Let $\bar{C} = C/N(C)$. Since S/S' is annihilated by $N(C)$ from the left, and T/T' is annihilated by $N(C)$ from the right, we may consider S/S' as a left $\bar{C}$-module and T/T' as a right $\bar{C}$-module, and γ induces a bimodule map

$$\bar{\gamma}\colon {}_{\bar{C}}(S/S') \underset{D}{\otimes} (T/T')_{\bar{C}} \to {}_{\bar{C}}\bar{C}_{\bar{C}}.$$

PROPOSITION 5: *The subset $I := N(C) + S' + T' + U$ of $A(\gamma)$ is a nilpotent ideal, and $A(\gamma)/I = A(\bar{\gamma})$.*

Proof: The canonical maps yield an exact sequence

$$T \underset{C}{\otimes} S' + T' \underset{C}{\otimes} S \to T \underset{C}{\otimes} S \to (T/T') \underset{C}{\otimes} (S/S') \to 0,$$

thus U is generated by the image of $T \otimes_C S'$ and $T' \otimes_C S$ in $T \otimes_C S$. It follows that $UT \subseteq T'$, since for $t \in T, s' \in S'$, and for $t' \in T', s \in S$, we have

$$(t \otimes s') \cdot T \subseteq T \cdot N(C) \subseteq T', (t' \otimes s) \cdot T \subseteq T'C \subseteq T',$$

and similarly, $SU \subseteq S'$. As a consequence, I is an ideal of $A(\gamma)$. Also, $A(\gamma)/I = A(\bar{\gamma})$. It remains to show that I is nilpotent. However, any element of I^m is a sum of monomials $x_1x_2 \ldots x_m$ with x_i in $N(C)$, $N(D)$, S', T', TS' or ST'. Since there exists n with $N(C)^n = 0 = N(D)^n$, it follows easily that $I^m = 0$ for large m. This completes the proof.

COROLLARY 1: *Assume $(T/T')_C$ is finitely generated. Then $A(\gamma)$ is semi-primary.*

Proof: Clearly, $\bar{\gamma}$ is non-degenerate, thus $A(\bar{\gamma})$ is Morita equivalent to $D \times \bar{C}$, by corollary 2 to proposition 4. In particular, $A(\bar{\gamma})$ is semiprimary. Since I is nilpotent, also $A(\gamma)$ is semiprimary.

COROLLARY 2: *Assume the image of γ is contained in $N(C)$. Then $N(A(\gamma)) = N(C) + N(D) + S + T + T \otimes_C S$, and $A(\gamma)/N(A(\gamma)) = C/N(C) \times D/N(D)$.*

Proof: Since the image of γ is contained in $N(C)$, we have $S' = S$, $T' = T$, thus $U = T \otimes_C S$. Also, $A(\bar{\gamma}) = \bar{C} \times D$, and the radical of $A(\gamma)$ is $0 \times N(D)$.

Recall that a semiprimary ring A is said to be *basic* provided $A/N(A)$ is a product of division rings. Any semiprimary ring is Morita equivalent to a uniquely determined basic semiprimary ring.

COROLLARY 3: *If C, D are basic and the image of γ is contained in $N(C)$, also $A(\gamma)$ is basic.*

REMARK: It is not difficult to see that all the conditions are also necessary in order to have $A(\gamma)$ basic.

Now assume that both C and D are finite dimensional k-algebras and that the bimodules ${}_CS_D$ and ${}_DT_C$ are finite dimensional over k, with k operating centrally on them. As we have seen, for any γ: ${}_CS_D \otimes {}_DT_C \to {}_CC_C$, the ring $A(\gamma)$ is a finite dimensional k-algebra. We consider now the special case $D = k$.

PROPOSITION 6: *Let $D = k$. Then $\gamma = \gamma' \perp \varepsilon_P$, where P_C is (finitely generated) projective, and the image of γ' is contained in $N(C)$. In particular, $A(\gamma')$ is the basic algebra Morita equivalent to $A(\gamma)$.*

Proof: In case the image γ is contained in $N(C)$, let $\gamma' = \gamma$ and $P = 0$. So assume the image of γ is not contained in $N(C)$. Since the image of γ is a C–C-subbimodule, it has to contain a primitive idempotent e of C. Thus, let $s_i \in S$, $t_i \in T$ with $\gamma(\Sigma\, s_i \otimes t_i) = e$. Without loss of generality, we can assume $s_i = es_i$, $t_i = t_ie$ for all i. For some i, we must have $\gamma(s_i \otimes t_i) \notin N(C)$, thus $\gamma(s_i \otimes t_i) \in eCe \backslash N(eCe)$. But eCe is a local ring, thus there is some ece with $e = \gamma(s_i \otimes t_i)ece = \gamma(s_i \otimes t_iece)$. This shows that there is $s = es \in S$ and $t = te \in T$ such that $\gamma(s \otimes t) = e$.

Note that the canonical map $Ce \to Cs$, given by $ce \mapsto ces$ is bijective: it is surjective, since $s = es$, and if $xs = 0$, then $0 = \gamma(xs \otimes t) = x\gamma(s \otimes t) = xe$, thus it is also injective. Similarly, the canonical map $eC \to tC$ is bijective. It follows that tC is a projective right C-module and that we may identify Cs with $\widetilde{tC}$ such that $\gamma | Cs \otimes_k tC$ is equal to ε_{tC}.

Let S' be the set of all $s' \in S$ with $\gamma(s' \otimes t) = 0$, and T' the set set of all $t' \in T$ such that $\gamma(s \otimes t') = 0$. We claim

$$S = S' + Cs \quad \text{and} \quad T = T' + tC.$$

For, if $c \in C$ and $cs \in S'$, then $0 = \gamma(cs \otimes t) = c\gamma(s \otimes t) = ce$, thus $cs = 0$, and so $S' \cap Cs = 0$. On the other hand, given $u \in S$, then $u - \gamma(u \otimes t))s$ belongs to S', since

$$\begin{aligned} \gamma((u - \gamma(u \otimes t)s) \otimes t) &= \gamma(u \otimes t) - \gamma(\gamma(u \otimes t)s \otimes t) \\ &= \gamma(u \otimes te) - \gamma(u \otimes t)\gamma(s \otimes t) \\ &= \gamma(u \otimes t)e - \gamma(u \otimes t)e = 0. \end{aligned}$$

thus $u \in S' + Cs$. The dual arguments give the second assertion.

Let γ' be the restriction of γ to $S' \otimes_k T'$. Since $\gamma | S' \otimes_k tC$ and $\gamma | Cs \otimes_k T'$ both are zero, we see that $\gamma = \gamma' \perp \varepsilon_{tC}$. The proof of the proposition can be completed by using induction: the process of splitting off bimodule maps must stop since we deal with finite dimensional modules.

Note that $A(\gamma')$ is basic by corollary 2 to proposition 5, and is Morita equivalent to $A(\gamma)$ by proposition 4.

4. Quasi-hereditary algebras

We recall the relevant definitions. The rings considered will usually be assumed to be semiprimary. An ideal J of A is said to be a *heredity* ideal of A, if $J^2 = J$, $JN(A)J = 0$, and J, considered as right A-module, is projective. The (semiprimary) ring A is called *quasi-hereditary* if there exists a chain $\mathscr{J} = (J_i)_i$ of ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_m = A$$

of A such that, for any $1 \leqslant t \leqslant m$, the ideal J_t/J_{t-1} is a heredity ideal of A/J_{t-1}. Such a chain of ideals is called a *heredity chain*.

Let A be quasi-hereditary with heredity chain $\mathscr{J} = (J_i)_{0 \leqslant i \leqslant m}$. Given an A-module X_A the chain of submodules

$$0 = XJ_0 \subseteq XJ_1 \subseteq \cdots \subseteq XJ_m = X$$

will be called the *$\mathscr{J}$-filtration* of X_A. We say that the $\mathscr{J}$-filtration of X_A is *good*, provided XJ_i/XJ_{i-1} is a projective A/J_{i-1}-module, for $0 \leqslant i \leqslant m$, and similarly for left modules.

THEOREM 1: *Let A be a semi-primary ring, and e an idempotent of A, let $C = eAe$. The following conditions are equivalent:*

(*i*) *There exists a heredity chain for A containing AeA.*

(*ii*) *Both rings C and A/AeA are quasi-hereditary, the multiplication map*

$$Ae \otimes_C eA \to AeA$$

is bijective, and there exists a heredity chain $\mathscr{I}$ of C such that the $\mathscr{I}$-filtrations of $(Ae)_C$ and ${}_C(eA)$ are good.

(*iii*) *Both rings C and A/AeA are quasi-hereditary, the multiplication map*

$$(1 - e)Ae \otimes_C eA(1 - e) \to (1 - e)AeA(1 - e)$$

is bijective, and there exists a heredity chain $\mathscr{I}$ of C such that the $\mathscr{I}$-filtrations of $((1 - e)Ae)_C$ and ${}_C(eA(1 - e))$ are good.

The proof of the theorem requires some preparation. Note that an ideal J of A satisfies $J^2 = J$ if and only if there exists an idempotent e of A with $J = AeA$.

PROPOSITION 7: *Let e be an idempotent in a quasi-hereditary ring A such that AeA belongs to a heredity chain. Then the multiplication map $Ae \otimes_{eAe} eA \to AeA$ is bijective.*

Proof: In case AeA is a heredity ideal, the result is known, see the appendix of [DR]. We proceed by induction on t, where

$$0 = J_0 \subset J_1 \subset \cdots \subset J_t = AeA \subset \cdots \subset J_m = A$$

is a heredity chain of A.

Let $J = J_{t-1}$. Let $\bar{A} = A/J$, and denote by $\bar{e}$ the image of e in $\bar{A}$. Let $e = \Sigma_{i=1}^{s} e_i$ with orthogonal primitive idempotents e_i. We can assume that $e_1, \ldots, e_s$ are ordered in such a way that $e_i \in J$ if and only if $i \leqslant s'$. Let $f = \Sigma_{i=1}^{s'} e_i$. Then $J = AfA$ and $f = ef = fe$, thus $fAf \subseteq eAe$.

We claim that the following sequence

$$Af \underset{fAf}{\otimes} fA \xrightarrow{\varphi} Ae \underset{eAe}{\otimes} eA \xrightarrow{\varphi} \bar{A}\bar{e} \underset{\bar{e}\bar{A}\bar{e}}{\otimes} \bar{e}\bar{A} \longrightarrow 0$$

with φ induced by inclusion maps, and ψ induced by the canonical surjections, is exact. For the proof, we proceed as follows. The canonical exact sequence

$$0 \to AfAe \to Ae \to Ae/AfAe \to 0$$

of right eAe-modulus is tensored on the right with ${}_{eAe}eA$, thus we obtain

$$AfAe \underset{AeA}{\otimes} eA \xrightarrow{\varphi_1} Ae \underset{eAe}{\otimes} eA \xrightarrow{\psi_1} (Ae/AfAe) \underset{eAe}{\otimes} eA \to 0.$$

We tensor the canonical exact sequence

$$0 \to eAfA \to eA \to eA/eAfA \to 0$$

of left eAe-modules with $AfAe_{eAe}$ and with $(Ae/AfAe)_{eAe}$ and obtain

$$AfAe \underset{eAe}{\otimes} eAfA \xrightarrow{\varphi_2} AfAe \underset{eAe}{\otimes} eA \to AfAe \underset{eAe}{\otimes} (eA/eAfA) \to 0$$

and

$$(Ae/AfAe) \underset{eAe}{\otimes} eAfA \to (Ae/AfAe) \underset{eAe}{\otimes} eA$$

$$\xrightarrow{\psi_0} (Ae/AfAe) \underset{eAe}{\otimes} (eA/eAfA) \to 0.$$

Since both $AfAe \otimes_{eAe} (eA/eAfA)$ and $(Ae/AfAe) \otimes_{eAe} eAfA$ are zero, we see that φ_2 is surjective, and ψ_0 is bijective. Note, that $(Ae/AfAe) \otimes_{eAe} (eA/eAfA)$ may be identified with $\bar{A}\bar{e} \otimes_{\bar{e}\bar{A}\bar{e}} \bar{e}\bar{A}$, so that $\psi = \psi_0\psi_1$. Also, there is a canonical map

$$Af \underset{fAf}{\otimes} fA \xrightarrow{\phi_3} AfAe \underset{eAe}{\otimes} eAfA$$

induced by the inclusion maps, and one easily checks that φ_3 is surjective. Since $\varphi = \varphi_1\varphi_2\varphi_3$, it follows that φ maps onto the kernel of ψ.

There is the following commutative diagram

$$\begin{array}{ccccccccc} & & Af \otimes_{fAf} fA & \xrightarrow{\varphi} & Ae \otimes_{eAe} eA & \xrightarrow{\psi} & \bar{A}\bar{e} \otimes_{\bar{e}\bar{A}\bar{e}} \bar{e}\bar{A} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \bar{\mu}} & & \\ 0 & \longrightarrow & J_{t-1} & \longrightarrow & J_t & \longrightarrow & J_t/J_{t-1} & \longrightarrow & 0 \end{array}$$

where the vertical maps are the multiplication maps, and the lower exact sequence is the canonical one. By definition, J_t/J_{t-1} is a heredity ideal of $\bar{A}$, thus $\bar{\mu}$ is bijective. By induction, $\bar{\mu}$ is bijective. It follows that φ is injective and that μ is bijective. This completes the proof.

LEMMA 1: *Let A be a semiprimary ring, J a heredity ideal of A, and $e \in A$ an idempotent with $J \subseteq AeA$. Then eJe is a heredity ideal in eAe and the right eAe-module Je_{eAe} and the left eAe-module ${}_{eAe}eJ$ both are projective.*

Proof: Since $J^2 = J$ and $J \subseteq AeA$, there is an idempotent f in A with $J = AfA$ and $f = efe$. Therefore $(eJe)^2 = eAfAeAfAe = eAfAe = eJe$. Of course, $N(eAe) = eN(A)e$, thus, $eJeN(eAe)eJe \subseteq JN(A)J = 0$. As a right A-module, $J = AfA$ is an epimorphic image of some direct sum $\oplus fA$, and, since J_A is projective, it follows that J_A is isomorphic to a direct summand of $\oplus fA$. Thus Je_{eAe} is isomorphic to a direct summand of $\oplus fAe$, and since f is an idempotent in AeA, we know that fAe_{eAe}, and therefore Je_{eAe} is projective. Similarly, since ${}_AJ$ is projective (see [PS] or also [DR]), we also have ${}_{eAe}eJ$ projective.

LEMMA 2: *Let C be any ring, f an idempotent in C, and M a right C-module. Assume that $(MfC)_C$ is projective. Then the multiplication map μ: $Mf \otimes_{fCf} fC \to MfC$ is bijective.*

Proof: Since μ is a surjective map of right C-modules, it splits. Thus, there is a C-submodule U of $Mf \otimes_{fCf} fC$ such that the restriction of μ to U is bijective. Multiply U, $Mf \otimes_{fCf} fC$ and MfC from the right by f. Since the map $Mf \otimes_{fCf} fCf \to MfCf = Mf$ induced by μ is bijective, the same is true for the inclusion map $Uf \to Mf \otimes_{fCf} fCf$. Thus $Uf = Mf \otimes_{fCf} fCf$. But the C-module $Mf \otimes_{fCf} fC$ is generated by $Mf \otimes_{fCf} fCf$, thus $Mf \otimes_{fCf} fC = U$.

PROPOSITION 8: *Let A be a semiprimary ring. Let e be an idempotent of A, let $C = eAe$, and assume that the multiplication map $Ae \otimes_C eA \to AeA$ is* *bijective. Let J be an ideal with $J \subseteq AeA$. The following conditions are equivalent:*

(*i*) *J is a heredity ideal of A.*

(*ii*) *eJe is a heredity ideal of C, the C-modules $(Je)_C$ and ${}_C(eJ)$ are projective, and the multiplication map $Je \otimes_C eJ \to J$ is bijective.*

(*iii*) *eJe is a heredity ideal of C, the C-modules $((1-e)Je)_C$ and ${}_C(eJ(1-e))$ are projective, and the multiplication map $(1-e)Je \otimes_C eJ(1-e) \to (1-e)J(1-e)$ is bijective.*

Proof: If J is a heredity ideal of A, then clearly eJe is a heredity ideal of C, thus all conditions include the assumption that eJe is a heredity ideal of C. Let f be an idempotent of C with $eJe = CfC$. Thus, $fe = ef = e$, and $J = AfA$. Let $D = fAf$. There is the following commutative diagram

$$\begin{array}{ccc} Af \underset{D}{\otimes} fAe \underset{C}{\otimes} eAf \underset{D}{\otimes} fA & \xrightarrow{\mu_1 \otimes \mu_2} & AfAe \underset{C}{\otimes} eAfA \\ \downarrow{\scriptstyle 1 \otimes \mu_4 \otimes 1} & & \downarrow{\scriptstyle \mu_5} \\ Af \underset{D}{\otimes} fA & \xrightarrow{\mu_3} & AfA \end{array}$$

where all the maps μ_i are multiplication maps. Since we assume that the multiplication map $Ae \otimes_C eA \to AeA$ is bijective, the map μ_4: $fAe \otimes_C eAf \to fAf$ is bijective, thus also $1 \otimes \mu_4 \otimes 1$ is bijective.

(i) $\Rightarrow$ (ii): Assume that J is a heredity ideal. According to lemma 1, we know that $(Je)_C$ is projective. Dually, also ${}_C(eJ)$ is projective. Since the multiplication map μ_3: $Af \otimes_D fA \to AfA$ is bijective, we see that also μ_1, μ_2 are bijective. Thus we conclude that μ_5: $Je \otimes_C eJ \to J$ is bijective.

(ii) $\Rightarrow$ (iii): We only have to observe that $(Je)_C = (eJe)_C \oplus ((1-e)Je)_C$, and ${}_C(eJ) = {}_C(eJe) \oplus {}_C(eJ(1-e))$.

(iii) $\Rightarrow$ (i): Since $J = AfA$, we have $J^2 = J$ and $JN(A)J = AfN(A)fA = AfN(C)fA = 0$. It remains to be seen that the multiplication map μ_3 is bijective. Lemma 2 applied to $M = A$ asserts that the map μ_1 is bijective, since $(Je)_C$ is projective. Dually, also μ_2 is bijective. By assumption, μ_5 is bijective, thus μ_3 is bijective. This completes the proof.

LEMMA 3: *Let C be a ring, f an idempotent in C. Let M_C and ${}_CN$ be C-modules. Assume $(MfC)_C$ and ${}_C(CfN)$ are projective C-modules. Then there is an exact sequence*

$$\mathrm{Tor}_1^C(M/MfC, N/CfN) \xrightarrow{\eta} MfC \underset{C}{\otimes} CfN \xrightarrow{\nu} M \underset{C}{\otimes} N$$

$$\xrightarrow{\pi} (M/MfC) \underset{C}{\otimes} (N/CfN) \to 0,$$

where ν is induced by the inclusion maps, and π is induced by the projection maps.

Proof: Let $\bar{M}_C = M/MfC$, and ${}_C\bar{N} = N/CfN$. The canonical sequence

$$0 \to (MfC)_C \to M_C \to \bar{M}_C \to 0$$

gives the long exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^C(M, \bar{N}) \xrightarrow{\alpha} \mathrm{Tor}_1^C(\bar{M}, \bar{N}) \longrightarrow MfC \underset{C}{\otimes} \bar{N} \longrightarrow M \underset{C}{\otimes} \bar{N}$$

$$\xrightarrow{\gamma} \bar{M} \underset{C}{\otimes} \bar{N} \longrightarrow 0,$$

where we use that $(MfC)_C$ is projective. Since $f\bar{N} = 0$, we see that $MfC \otimes_C \bar{N} = 0$. Also, we obtain the sequence

$$0 \longrightarrow MfC \underset{C}{\otimes} CfN \xrightarrow{\beta} M \underset{C}{\otimes} CfN \longrightarrow \bar{M} \underset{C}{\otimes} CfN \longrightarrow 0,$$

which is exact, since ${}_C(CfN)$ is projective. Here, $\bar{M} \otimes_C CfN = 0$, since $\bar{M}f = 0$. As a consequence, the maps α, β, γ all are bijective. The canonical exact sequence

$$0 \to {}_C(CfN) \to {}_CN \to {}_C\bar{N} \to 0$$

yields the upper row of the following commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{Tor}_1^C(M, \bar{N}) & \xrightarrow{\delta} & M \underset{C}{\otimes} CfN & \longrightarrow & M \underset{C}{\otimes} N & \longrightarrow & M \underset{C}{\otimes} \bar{N} & \longrightarrow & 0 \\
\downarrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} & & \| & & \downarrow{\scriptstyle\gamma} & & \\
\mathrm{Tor}_1^C(\bar{M}, \bar{N}) & \xrightarrow{\beta^{-1}\delta\alpha^{-1}} & MfC \otimes CfN & \xrightarrow{\nu} & M \otimes N & \xrightarrow{\pi} & \bar{M} \otimes \bar{N} & \longrightarrow & 0.
\end{array}$$

Since α, β, γ are bijective, and the upper row is exact, also the lower one is exact.

LEMMA 4: *Let J be a heredity ideal in A, let $B = A/J$. If X_B, ${}_BY$ are B-modules, we may consider them as A-modules, and we have* $\mathrm{Tor}_1^B(X, Y) \simeq \mathrm{Tor}_1^A(X, Y)$.

Proof: Write $X_A = A_A^n/X'$ for some submodule X' of A_A^n and some n. Since $XJ = 0$, it follows that $J^n \subseteq X'$, and $X = B^n/X''$, where $X'' = J^n/X'$.

We have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & J^n & = & J^n & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X'_A & \longrightarrow & A^n_A & \longrightarrow & X_A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X''_A & \longrightarrow & B^n_A & \longrightarrow & X_A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Tensoring with ${}_AY$ gives the following commutative diagram, with all tensor products being over A:

$$
\begin{array}{ccccccc}
J^n \otimes Y & = & J^n \otimes Y & & & & \\
\downarrow & & \downarrow & & & & \\
X' \otimes Y & \xrightarrow{\alpha} & A^n \otimes Y & \longrightarrow & X \otimes Y & \longrightarrow & 0 \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \| & & \\
X'' \otimes Y & \xrightarrow{\beta} & B^n \otimes Y & \longrightarrow & X \otimes Y & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}
$$

with exact rows and columns. Since $JY = 0$, and $J^2 = J$, we see that $J^n \otimes_A Y = 0$, thus γ, δ are isomorphisms. But the kernel of α is $\mathrm{Tor}_1^A(X, Y)$, the kernel of β is $\mathrm{Tor}_1^B(X, Y)$. This completes the proof.

LEMMA 5: *Let A be quasi-hereditary, with heredity chain $\mathscr{J}$. Assume that the $\mathscr{J}$-filtrations of X_A, ${}_AY$ are good. Then* $\mathrm{Tor}_1^A(X, Y) = 0$.

Proof: Let $\mathscr{J} = (J_i)_{0 \leqslant i \leqslant m}$. The proof is by induction on m. Let $B = A/J_1$. By induction, we have $\mathrm{Tor}_1^B(X/XJ_1, Y/J_1Y) = 0$, thus $\mathrm{Tor}_1^A(X/XJ_1, Y/J_1Y) = 0$ by lemma 4. Since $(XJ_1)_A$ is projective, also $\mathrm{Tor}_1^A(XJ_1, Y/J_1Y) = 0$, thus $\mathrm{Tor}_1^A(X, Y/J_1Y) = 0$ by the long exact Tor-sequence. Also, ${}_A(J_1Y)$ is projective, thus $\mathrm{Tor}_1^A(X, J_1Y) = 0$ and therefore $\mathrm{Tor}_1^A(X, Y) = 0$, again using a long exact Tor-sequence.

Proof of the theorem: Let $\mathscr{J} = (J_i)_i$ be a chain of idempotent ideals of A, say

$$0 = J_0 \subset J_1 \subset \cdots \subset J_m = A$$

and assume that $J_t = AeA$ for some t. Note that for $0 \leqslant i \leqslant t$, we have

$$AeJ_ie = AeAJ_ie = J_tJ_ie = J_ie.$$

(i) $\Rightarrow$ (ii): We assume that $\mathscr{J}$ is a heredity chain. Clearly, $A/AeA = A/J_t$ is quasi-hereditary. Also, $C = AeA$ is quasi-hereditary, with heredity chain $\mathscr{I} = (eJ_ie)_{0 \leqslant i \leqslant t}$, see [DR]. According to Proposition 7, the multiplication map $Ae \otimes_C eA \to AeA$ is bijective. It remains to be shown that the $\mathscr{I}$-filtrations of $(Ae)_C$ and ${}_C(eA)$ are good. We deal with $(Ae)_C$, the other case follows from dual considerations. Let $1 \leqslant i \leqslant t$, we have to show that $AeJ_ie/AeJ_{i-1}e$ is a projective right $C/eJ_{i-1}e$-module. We apply Proposition 8 to the ring $\bar{A} = A/J_{i-1}$, the idempotent $\bar{e} = e + J_{i-1}$, and the ideal $\bar{J} = J_i/J_{i-1}$. Since $\bar{A}\bar{e}\bar{A}$ belongs to a heredity chain of $\bar{A}$, the assumption concerning the multiplication map is satisfied. Let $\bar{C} = \bar{e}\bar{A}\bar{e}$. Since $\bar{J}$ is a heredity ideal of $\bar{A}$, it follows that $(\bar{J}\bar{e})_{\bar{C}}$ is a projective $\bar{C}$-module. However, $\bar{C}$ can be identified with $C/eJ_{i-1}e$, and $\bar{J}\bar{e}$ can be identified with $J_ie/J_{i-1}e = AeJ_ie/\ AeJ_{i-1}e$. It follows that $AeJ_ie/AeJ_{i-1}e$ is a projective $C/eJ_{i-1}e$-module.

(ii) $\Leftrightarrow$ (iii): Let $e_1 = e$, $e_2 = 1 - e$. There are the direct decompositions of C-modules $(Ae)_C = (e_1Ae)_C \oplus (e_2Ae)_C$ and ${}_C(eA) = {}_C(eAe_1) \oplus {}_C(eAe_2)$. The multiplication map μ: $Ae \otimes_C eA \to eAe$ is the direct sum of the four multiplication maps

$$\mu_{ij}\colon e_iAe \underset{C}{\otimes} eAe_j \to e_iAeAe_j,$$

$1 \leqslant i, j \leqslant 2$. But $\mu_{11}, \mu_{12}, \mu_{21}$ are always bijective. Thus μ is bijective if and only if μ_{22} is bijective. Also, given a heredity chain $\mathscr{I}$ of C, the $\mathscr{I}$-filtration of C_C is always good. Thus the $\mathscr{I}$-filtration of $(Ae)_C$ is good if and only if the $\mathscr{I}$-filtration of $((1 - e)Ae)_C$ is good. A similar argument for ${}_C(eA)$ and ${}_C(eA(1 - e))$ completes the proof.

(ii) $\Rightarrow$ (i): Let $\mathscr{I} = (I_i)_i$ be a heredity chain for C, say

$$0 = I_0 \subset I_1 \subset \cdots \subset I_t = C.$$

Let $J_i = AI_iA$, for $0 \leqslant i \leqslant t$, thus $J_t = eAe$. Also note that $eJ_ie = I_i$ for all $0 \leqslant i \leqslant t$. We want to apply Proposition 8 to the ideal $J = J_1$. Since the

$\mathscr{I}$-filtration of Ae is good, we know that $(AeI_1)_C$ is a projective C-module. However, $AeI_1 = AeJ_1e = J_1e$, thus $(J_1e)_C$ is a projective C-module. Similarly, ${}_C(eJ_1)$ is a projective C-module. Since the $\mathscr{I}$-filtrations of $(Ae/Je)_C$ and ${}_C(eA/eJ)$ are good, we have $\mathrm{Tor}_1^C(Ae/Je, eA/eJ) = 0$ by lemma 5. We can apply lemma 3 to $M = Ae$ and $N = eA$, since $AefC = Je$ is a projective right C-module, and $CfeA = eJ$ is a projective left C-module. There is the following commutative diagram of canonical maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Je \otimes_C eJ & \xrightarrow{\nu} & Ae \otimes_C eA & \xrightarrow{\pi} & (Ae/Je) \otimes_C (eA/eJ) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \bar{\mu}} & & \\ 0 & \longrightarrow & J & \longrightarrow & AeA & \longrightarrow & AeA/J & \longrightarrow & 0 \end{array}$$

(with ν induced by the inclusion maps, π by the projection maps, and all maps μ', μ, $\bar{\mu}$ being multiplication maps). Both rows are exact, the first one according to lemma 3. Now μ is bijective by assumption, thus μ' is injective. But clearly μ' is also surjective, thus μ' is bijective too. Thus all conditions of (ii) in proposition 8 are satisfied, therefore J is a heredity ideal. It remains to be shown that $\bar{A} = A/J$ and $\bar{e} = e + J$ again satisfy the conditions (ii) of the theorem, so that we can use induction. Let $\bar{C} = \bar{e}\bar{A}\bar{e}$. Clearly, $\bar{A}/\bar{A}\bar{e}\bar{A} \simeq A/AeA$, and $\bar{C} \simeq C/I_1$, so both rings are quasi-hereditary. The ring $\bar{C}$ has the heredity chain $\bar{\mathscr{I}} = (I_i/I_1)_{1 \leqslant i \leqslant t}$ and one easily checks that the $\bar{\mathscr{I}}$-filtrations both of $(\bar{A}\bar{e})_{\bar{C}}$ and of ${}_{\bar{C}}(\bar{e}\bar{A})$ are good. Finally, the multiplication map $\bar{A}\bar{e} \otimes_C \bar{e}\bar{A} \to \bar{A}\bar{e}\bar{A}$ is just the map $\bar{\mu}$ in the diagram above, and therefore bijective. This completes the proof of the theorem.

In the special case when C is semisimple, the conditions (ii) and (iii) of theorem 1 are easier to formulate.

COROLLARY: *Let A be a semisimple ring, e an idempotent of A, and assume that $C = eAe$ is semisimple. Then the following conditions are equivalent:*

(*i*) *There exists a heredity chain containing AeA.*

(*ii*) *A/AeA is quasi-hereditary, and the multiplication map $Ae \otimes_C eA \to$ AeA is bijective.*

(*iii*) *A/AeA is quasi-hereditary, and the multiplication map $(1 - e)Ae \otimes_C eA(1 - e) \to (1 - e)AeA(1 - e)$ is bijective.*

REMARK: The 'not so trivial extension' method outlined by Parshall and Scott in [PS] can be based on this corollary: if $\mathscr{J} = (J_i)_{0 \leqslant i \leqslant m}$ is a heredity chain for A, and $J_1 = AeA$ for some idempotent e of A, then $C = eAe$ is semisimple. Also, we can assume that e is chosen in such a way that we have, in addition, $eA(1 - e) \subseteq N(A)$. In this case, the

the multiplication map

$$eA(1-e) \underset{\mathbb{Z}}{\otimes} (1-e)Ae \to eAe = C$$

is zero, in particular, the ideal $U = (1-e)AeA(1-e)$ of $\tilde{D} = (1-e)A(1-e)$ satisfies $U^2 = 0$. It follows that A is uniquely determined by $C, D := A/AeA$, the C–D-bimodule $M = eA(1-e)$, the D–C-bimodule $N = (1-e)Ae$, and the 'Hochschild extension'

$$0 \to N \underset{C}{\otimes} M \to \tilde{D} \to D \to 0.$$

5. The inductive construction of quasi-hereditary algebras

THEOREM 2: *Let C, D be quasi-hereditary rings, let ${}_CS_D$, ${}_DT_C$ be bimodules, and γ: ${}_CS_D \otimes {}_DT_C \to {}_CC_C$ a bimodule homomorphism. Assume that there exists a heredity chain $\mathscr{J}$ of C such that the $\mathscr{J}$-filtrations both of ${}_CS$ and of T_C are good. Then $A(\gamma)$ is quasi-hereditary.*

Proof: Let $e = e_C$. Then ${}_CS_D = eA(1-e)$, ${}_DT_C = (1-e)Ae$. The assertion is just the implication (iii) $\Rightarrow$ (i) of theorem 1.

We consider now the converse problem of writing a given quasi-hereditary ring in the form $A(\gamma)$.

PROPOSITION 9: *Let A be a quasi-hereditary ring, let e be an idempotent of A such that AeA belongs to a hereditary chain of A. Assume that there exists a subring D of $(1-e)A(1-e)$ such that $D + (1-e)AeA(1-e) = (1-e)A(1-e)$. Let $C = AeA$, $S = eA(1-e)$, $T = (1-e)Ae$, and γ: $S \otimes_D T \to C$ the multiplication map. Then $A = A(\gamma)$.*

Proof: This is a direct consequence of propositions 7 and 3.

As a consequence, we obtain the following result which gives the inductive procedure for constructing quasi-hereditary rings. Here, given a semiprimary ring A, we denote by $s(A)$ the number of isomorphism classes of simple right A-modules.

THEOREM 3: *Let k be a field. Let A be a non-zero quasi-hereditary finite dimensional k-algebra with a heredity chain $\mathscr{J} = (J_i)_{0 \leqslant i \leqslant m}$. Assume $D := A/J_{m-1}$ is a separable k-algebra. Then there exists a quasi-hereditary k-algebra C with $s(C) < s(A)$, with a heredity chain $\mathscr{I} = (I_i)_{0 \leqslant i \leqslant m-1}$, bimodules ${}_CS_D$, ${}_DT_C$, such that the $\mathscr{I}$-filtrations of ${}_CS$ and T_C are good, and a bimodule* *homomorphism* γ: ${}_CS_D \otimes {}_DT_C \to {}_CC_C$ *with image contained in* $N(C)$, *such that* $A = A(\gamma)$.

Proof: Choose an idempotent e of A such that $J_{m-1} = AeA$ and such that, moreover, $eA(1-e) \subseteq N(A)$. Note that

$$(1-e)A(1-e)/(1-e)AeA(1-e) \simeq A/AeA,$$

thus, since A/AeA is assumed to be separable, there exists a subring $D \subseteq (1-e)AeA(1-e)$ such that $D + (1-e)AeA(1-e) = (1-e)A(1-e)$. Let $C = eAe$, $S = eA(1-e)$, $T = (1-e)Ae$, and γ: $S \otimes_D T \to C$ be the multiplication map. Then $A = A(\gamma)$ by proposition 9. The assumption $eA(1-e) \subseteq N(A)$ implies that the image of γ is contained in $N(C)$. Of course, $s(A(\gamma)) = s(C) + s(D)$, thus $s(C) < s(A)$. Let $\mathscr{I} = (I_i)_{0 \leqslant i \leqslant m-1}$ with $I_i = eJ_ie$, this is a heredity chain by [DR], and the $\mathscr{I}$-filtrations of ${}_CS$ and T_C are good, by (the proof of) the theorem in section 4.

Corollary: *Let k be a perfect field. Let A be a non-zero quasi-hereditary finite dimensional k-algebra. Then there exists a semisimple k-algebra D, a quasi-hereditary k-algebra C, with* $s(C) < s(D)$, *and a bimodule homorphism* γ: ${}_CS_D \otimes {}_DT_C \to {}_CC_C$ *such that* $A = A(\gamma)$.

Proof: Let $\mathscr{J} = (J_i)_{0 \leqslant i \leqslant m}$ be a heredity chain of A. Always, A/J_{m-1} is semisimple. Since k is perfect, A/J_m is even separable. So we apply theorem 3.

6. Examples

Let C, D be quasi-hereditary rings, and γ: ${}_CS_D \otimes {}_DT_C \to {}_CC_C$ a bimodule homomorphism. Theorem 2 asserts that $A(\gamma)$ is quasi-hereditary provided there exists a heredity chain $\mathscr{I}$ for C such that the $\mathscr{I}$-filtrations both of ${}_CS$ and T_C are good. We want to give two examples which show what may happen in general. We consider quasi-hereditary algebras C with $s(C) = 2$ and D will be a division ring. The simple right C-modules will be denoted by $E(1)$, $E(2)$. The projective cover of $E(i)$ will be denoted by $P(i)$. The simple left C-modules will be denoted by $E^*(i)$, with $E^*(i) \otimes_C E(i) \neq 0$.

Example 1: Let C be serial, with $P(1)$ of length 3, and $P(2)$ of length 2. Let T_C be the indecomposable right C-module of length 2 with top $E(1)$, and ${}_CS$ the indecomposable left C-module of length 2 with top $E^*(2)$. The endomorphism rings of T_C and ${}_CS$ are isomorphic division rings (always, we assume that endomorphisms act on the opposite side as the scalars), say $D = \text{End}\,(T_C) = \text{End}\,({}_CS)$. Note that the D–C-bimodule $\text{Hom}\,({}_CS_D,$

${}_CC_C$) can be identified with ${}_DT_C$, let γ: ${}_CS \otimes_D T_C \to {}_CC_C$ be adjoint to the identity map ${}_DT_C \to \operatorname{Hom}({}_CS_D, {}_CC_C)$. One may check without difficulties that $A = A(\gamma)$ is again serial, with simple right modules $E(1)$, $E(2)$, $E(3)$, (where $E(1)$, $E(2)$ are the given C-modules). If $P_A(i)$ denotes the projective cover of $E(i)$, then $P_A(i)$ has length 4, 3, 4 for $i = 1, 2, 3$, respectively. It follows that *gl. dim.* $A = 4$, but A is not quasi-hereditary.

EXAMPLE 2: Let C again be serial, with $P(1)$ of length 2, and $P(2)$ of length 1. (Thus, C is Morita equivalent to the ring of upper triangular 2×2–matrices over some division ring). Let T_C be the simple injective right C-module, ${}_CS$ the simple injective left C-module (thus, $T_C = E(1)$, and ${}_CS = E^*(2)$), and $D = \operatorname{End}(T_C) = \operatorname{End}({}_CS)$. Let γ: ${}_CS \otimes_D T_C \to {}_CC_C$ be the zero map. Then $A = A(\gamma)$ is again serial with all indecomposable projective A-modules of length 2. Consequently, A is self-injective with $N(A)^2 = 0$. In particular, *gl. dim.* $A = \infty$.

Acknowledgements

The authors are endebted to L. Scott for helpful discussions concerning the presentation of the results. Also, he informed us that proposition 7 has been obtained independently by B. Parshall, see [P]. The authors are endebted to the referee for pointing out the proof of proposition 4 given here.

References

[B] Bass, H., *Algebraic K-Theory*, Benjamin, New York (1968).
[BBD] Beilinson, A., Bernstein, J. and Deligne, P., Faisceaux pervers, *Asterisque* 100 (1983).
[BGG] Bernstein, J., Gelfand, I. and Gelfand, S., Category of g-modules, *Funct. Anal. Appl.* 10 (1976), 87–92.
[CPS] Cline, E., Parshall, B. and Scott, L., Finite dimensional algebras and highest weight categories (To appear), *J. Reine Angew. Math.*
[DR] Dlab, V. and Ringel, C.M., Quasi-hereditary algebras (To appear), *Ill. J. Math.*
[MV] MacPherson, R. and Vilonen, K., Elementary construction of perverse sheaves, *Inv. Math.* 84 (1986), 403–435.
[Me] Mebkhout, Z., Une èquivalence des catègories. Une autre èquivalence des catègories, *Comp. Math.* 51 (1984), 51–88.
[MiV] Mirollo, R. and Vilonen, K., Bernstein–Gelfand–Gelfand reciprocity on perverse sheaves. *Ann. Scient. Éc. Norm. Sup.* 4e série 20 (1987), 311–324.
[P] Parshall, B. J., Finite dimensional algebras and algebraic groups (To appear).
[PS] Parshall, B.J. and Scott, L.L., Derived categories, quasi-hereditary algebras, and algebraic groups (To appear).
[S] Scott, L.L., Simulating algebraic geometry with algebra I: Derived categories and Morita theory, *Proc. Symp. Pure Math., Amer. Math. Soc., Providence* 47 (1987), part 1, 271–282.