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QUASI-HEREDITARY ALGEBRAS

BY

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Dedicated to the memory of Irving Reiner

In their work on highest weight categories arising in the representation theory of Lie algebras and algebraic groups, E. Cline, B. Parshall and L. Scott recently introduced the notion of a quasi-hereditary algebra (see [1] and [2]). They define a quasi-hereditary algebra recursively in terms of the existence of a particular idempotent ideal; finite-dimensional hereditary algebras are typical examples of quasi-hereditary algebras. On the other hand, they showed that every quasi-hereditary algebra has finite global dimension.

The purpose of this note is to establish the following three results. First, finite-dimensional hereditary algebras are characterized as those quasi-hereditary algebras which satisfy a certain refinement property on chains of their idempotent ideals (Theorem 1). Second, all finite-dimensional algebras of global dimension 2 are shown to be quasi-hereditary (Theorem 2). Third, the question of whether every finite-dimensional algebra of finite global dimension is quasi-hereditary is answered in the negative by providing an example of an (11-dimensional serial) algebra of global dimension 4 which is not quasi-hereditary. The same example illustrates that the class of quasi-hereditary algebras is not closed under tilting (in the sense of [4]).

In what follows, all rings are semiprimary rings. An associative ring A with 1 is called *semiprimary* if its Jacobson radical N is nilpotent and A/N is semisimple artinian. Recall that an ideal I of A is idempotent if and only if $I = AeA$ for an idempotent e of A; in particular, I is a minimal (non-zero) idempotent ideal provided that e is primitive. An ideal J of A is said to be a *heredity* ideal of A if $J^2 = J$, $JNJ = 0$ and J, considered as a right A-module J_A, is projective. In fact, this also implies that the left A-module ${}_AJ$ is projective (see [2] or [3]). A semiprimary ring A is called *quasi-hereditary* if there is a chain

$$0 = J_0 \subset J_1 \subset \cdots \subset J_{t-1} \subset J_t \subset \cdots \subset J_m = A$$

of ideals of A such that, for any $1 \le t \le m$, J_t/J_{t-1} is a heredity ideal of A/J_{t-1}. Such a chain of idempotent ideals is called a *heredity chain*. Let us

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remark that A is quasi-hereditary if and only if A^{op} is quasi-hereditary (see [2] or [3]).

THEOREM 1. *Let A be a semiprimary ring. Then A is hereditary if and only if every chain of idempotent ideals of A can be refined to a heredity chain.*

Proof. First, let A be a hereditary semiprimary ring. Note that, for any idempotent ideal, the ring A/I is again hereditary. Thus, it is sufficient to show that any non-zero idempotent ideal I of A contains a non-zero heredity ideal. To this end, consider $J = AeA$ with a primitive idempotent e in I. Since A is hereditary, J_A is projective. Moreover, since the left multiplication by any element $x \in eNe$ defines a non-invertible map from the indecomposable module eA onto xA and xA is projective, necessarily $x = 0$. It follows that $JNJ = AeNeA = 0$.

Conversely, assume that every chain of idempotent ideals of A can be refined to a heredity chain. We want to show that A is hereditary. We shall proceed by induction; thus assume that A/I is hereditary for every non-zero idempotent ideal I of A.

Let us start by choosing a primitive idempotent f in A such that the Loewy length $L(fA)$ of the module fA is maximal. Observe that $I = AfA$ is a heredity ideal. Indeed, AfA is a minimal idempotent ideal of A. Thus $fNf = 0$ and I_A is a projective A-module. We claim that $Nf = 0$. For, if $Nf \neq 0$ then there is a primitive idempotent e such that $eNf \neq 0$ and thus $eNf = eAf$. Furthermore, $I = eI \oplus (1 - e)I$, and thus $(eI)_A$ is a projective A-module. Since $eI = eAfA$ is generated by fA, it is a non-zero direct sum of copies of fA and therefore $L(eI) = L(fA)$. On the other hand, $eI = eAfA = eNfA \subseteq eN$, and consequently

$$L(eI) \leq (eN) = L(eA) - 1$$

which contradicts the maximality of $L(fA)$. Hence, $Nf = 0$.

Now, write $1 = \sum_{i=1}^{n} e_i$ with primitive orthogonal idempotents e_i. Assume that $f = e_1$ is equivalent to e_i if and only if $1 \leq i \leq k$. Thus $e_j A e_i = 0$ for all $i \leq k < j$. If $k = n$, then A is simple artinian and thus hereditary. If $k < n$, we show that $(AfN)_A$ is a projective A-module. For then, in conjunction with the fact that $Nf = 0$ and that $B = A/AfA$ is hereditary, we can conclude that A is hereditary, as required.

In order to prove that $(AfN)_A$ is projective, consider the A-module $X_A = AfN/(AfN \cap AgA)$, where g is a primitive idempotent satisfying $Ng \subseteq I$ $(= AfA)$. Such an idempotent $g = e_j$ for some $j > k$ exists, since $B = A/I$ is hereditary. Clearly, X is isomorphic to the ideal

$$(AfN + AgA)/AgA$$

of the ring $C = A/AgA$ which is, by induction, hereditary. Consequently, X is a projective C-module. Since X is annihilated by $A(f+g)A$, it is a projective D-module for $D = A/A(f+g)A$. But, in view of $Ng \subseteq I$, every projective D-module is a projective B-module, and in view of $Nf = 0$, every projective B-module is a projective A-module. Hence, X_A is a projective A-module. As a result, the canonical A-homomorphism $(AfN)_A \to X_A$ splits and thus $(AfN)_A$ is isomorphic to the direct sum of the projective A-module X_A and $(AfN \cap AgA)_A$. However, the latter A-module is also projective. This follows from the fact that

$$(*) \qquad AfN \cap AgA = \bigoplus_{i=1}^{k} e_i AgA,$$

which is trivially a direct summand of the projective A-module

$$AgA = \bigoplus_{i=1}^{n} e_i AgA.$$

In order to verify $(*)$, notice that, on the one hand,

$$e_i AgA \subseteq AgA \quad \text{and} \quad e_i AgA \subseteq (Ae_iA)(AgA) = AfNgA \subseteq AfN,$$

and that, on the other hand,

$$AfN \cap AgA \subseteq \left(\bigoplus_{i=1}^{k} e_i A \right) \cap AgA = \bigoplus_{i=1}^{k} e_i AgA.$$

This concludes the proof of Theorem 1.

THEOREM 2. *A semiprimary ring of global dimension 2 is quasi-hereditary.*

The proof is based on the following proposition.

PROPOSITION. *Let A be a semiprimary ring of global dimension at most 2. Let e be a primitive idempotent of A such that the Loewy length $L(eA)$ is minimal. Then AeA is a heredity ideal.*

Proof. Write $J = AeA$. Clearly $J^2 = J$. First, we are going to show that $eNe = 0$; this implies immediately that $JNJ = 0$. Assume that there is a non-zero element $x \in eNe$. The left multiplication by x yields an endomorphism of eA with a non-zero kernel $K \subseteq eN$. Since $\operatorname{gl\,dim} A = 2$, kernels of maps between projective modules are projective. Thus K is projective and therefore the Loewy length $L(K) \geq L(eA)$. But

$$L(K) \leq L(eN) = L(eA) - 1,$$

in contradiction to the preceding inequality.

Second, in order to show that J_A is projective, consider the minimal projective cover p: $P \to J_A$. We claim that p is an isomorphism. Otherwise, there exists a finitely generated direct summand P' of P such that the restriction of p to P' is not a monomorphism. Now, P and therefore P' is a direct sum of copies of eA. We may assume that $P' = P'' \oplus \bar{P}$ with $\bar{P} \simeq eA$ and such that the restriction of p to P'' is a monomorphism. Denoting by X the image of P'' under p, consider the commutative diagram of A-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'' & \longrightarrow & P' & \longrightarrow & \bar{P} & \longrightarrow & 0 \\ & & {\scriptstyle p''}\downarrow & & {\scriptstyle p'}\downarrow & & {\scriptstyle \bar{p}}\downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & A & \longrightarrow & A/X & \longrightarrow & 0, \end{array}$$

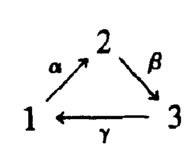
where the horizontal maps are the canonical inclusions and projections, and the vertical ones are induced by p and the embedding of J in A. Observe that p' is not a monomorphism and $\bar{p} \neq 0$. Forming the kernels, we obtain the exact sequence

$$0 = \operatorname{Ker} p'' \to \operatorname{Ker} p' \to \operatorname{Ker} \bar{p}.$$

Now, since p' is a map between projective modules, $\operatorname{Ker} p'$ is a non-zero projective module, and thus its Loewy length $L(\operatorname{Ker} p') \geq L(eA)$. But this contradicts the inequality $L(\operatorname{Ker} p') \leq L(\operatorname{Ker} \bar{p}) < L(eA)$. The proof of the proposition is completed.

Proof of Theorem 2. Choose a primitive idempotent e of A such that the Loewy length $L(eA)$ is minimal. Since, by the proposition, $J = AeA$ is a heredity ideal, it follows that $\operatorname{gl\,dim} A/J \leq \operatorname{gl\,dim} A$ (see [2] or [3]). Therefore, by induction, A/J is quasi-hereditary, and thus A is quasi-hereditary.

Example. Let A be the path algebra (over a field) of the diagram



modulo the ideal generated by $\gamma\alpha\beta$ and $\alpha\beta\gamma\alpha$. Denote by e_i the idempotent corresponding to the vertex i, and let $P_i = e_i A$. Thus, A is a serial algebra, and the lengths of the indecomposable projective A-modules are as follows: P_1 and P_2 are of length 4 and P_3 is of length 3. One can check easily that gl dim $A = 4$ and that A has no (non-zero) heredity ideals; for, e_1 Rad A $e_1 \neq 0$, e_2 Rad A $e_2 \neq 0$ and $(Ae_3A)_A$ is not projective.

Let $S_2 = P_2/\text{Rad}\, P_2$ be the simple A-module of projective dimension 1. Since the modules P_1 and P_2 are both projective and injective, it follows that $T = P_1 \oplus P_2 \oplus S_2$ is a tilting module [4]. Let $B = \text{End}\, T_A$. Then B is a quasi-hereditary algebra. Indeed, if J_1 is the set of all endomorphisms of T which factor through S_2 and J_2 the set of all those which factor through $S_2 \oplus P_2$, then $0 \subset J_1 \subset J_2 \subset A$ is a heredity chain. This may be verified without difficulty. In fact, one can see easily that B is the path algebra (over a field) of the diagram

$$1 \underset{\delta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 3$$

modulo the ideal generated by $\alpha\beta$, $\alpha\delta$, $\gamma\delta$ and $\beta\gamma - \delta\alpha$.

Appendix: Heredity ideals

For the convenience of the reader, we wish to collect here various results of [1] and [2]. In doing so, we aim to minimize assumptions and, at the same time, to strengthen conclusions (cf. examples at the end). Furthermore, we include some assertions from the general ring theory, and provide proofs which are direct and elementary.

In what follows, unless specified otherwise, A is an arbitrary associative ring with 1, J and ideal and $B = A/J$. The B-modules will always be viewed as those A-modules which are annihilated by J.

Let us remark that, under the assumption that A is semiprimary, every A-module X has finite Loewy length which will always be denoted by $L(X)$. Note that for semiprimary rings, $L(A_A)$ and $L({}_AA)$ are equal and will be simply denoted by $L(A)$. Always, $L(X) \leq L(A)$.

STATEMENT 1. *Let J_A be projective. If X is a B-module, then*

$$\text{proj dim } X_A \leq 1 + \text{proj dim } X_B.$$

In particular, proj dim $B_A \leq 1$.

Proof. Proceed by induction. The exact sequence $0 \to J_A \to A_A \to B_A \to 0$ shows that

$$\text{proj dim } B_A \leq 1.$$

Therefore, for any projective B-module X_B, proj dim $X_A \leq 1$. Now, assume that

$$\text{proj dim } X_B = d > 0,$$

and consider an exact sequence $0 \to X'_B \to P_B \to X_B \to 0$ with a projective B-module P_B. Thus proj dim $X'_B = d - 1$. An application of $\text{Hom}_A(-, Y_A)$ yields the exact sequence

$$\text{Ext}_A^{d+1}(X'_A, Y_A) \to \text{Ext}_A^{d+2}(X_A, Y_A) \to \text{Ext}_A^{d+2}(P_A, Y_A).$$

Since $d \geq 1$ and proj dim $P_A \leq 1$, the last term is zero. By induction,

$$\text{proj dim } X'_A \leq d,$$

and consequently the first term is zero also. This yields

$$\text{Ext}_A^{d+2}(X_A, Y_A) = 0$$

for any Y_A, and thus proj dim $X_A \leq d + 1$.

STATEMENT 2. *$J^2 = J$ if and only if $\text{Hom}_A(J_A, X_A) = 0$ for any B-module X. If J is projective, then $J^2 = J$ if and only if $\text{Hom}_A(J_A, B_A) = 0$.*

Proof. First, assume that $J^2 = J$ and let ϕ: $J_A \to X_A$ be a homomorphism. Then $\phi(J) = \phi(J^2) \subseteq XJ = 0$, and thus $\phi = 0$. Conversely, let $\text{Hom}_A(J_A, X_A) = 0$ for any B-module X. Write $Y_A = J/J^2$. Since $YJ = 0$, Y can be viewed as a B-module. Hence, $\text{Hom}_A(J_A, Y_A) = 0$, and the canonical epimorphism $J_A \to Y_A$ shows that $Y = 0$.

Finally, assume that J_A is projective and that $\text{Hom}_A(J_A, B_A) = 0$. Given a B-module X, let F_B be a free B-module with an epimorphism π: $F_B \to X_B$. Since J_A is projective, any map ϕ: $J_A \to X_A$ lifts to a map ϕ': $J_A \to F_A$ with $\phi = \pi\phi'$. But $\text{Hom}_A(J_A, F_A) = 0$, because F_A is a direct sum of copies of B_A.

STATEMENT 3. *Let $J^2 = J$ and J_A be projective. If X, Y are B-modules, then*

$$\text{Ext}^i_B(X_B, Y_B) \simeq \text{Ext}^i_A(X_A, Y_A) \quad \textit{for all } i \geq 0.$$

In particular, $\text{Ext}^i_A(B_A, B_A) = 0$ *for* $i \geq 1$.

Proof. Clearly,

$$\text{Hom}_B(X_B, Y_B) = \text{Hom}_A(X_A, Y_A).$$

To prove the case $i = 1$, regard $\operatorname{Ext}^1_B(X_B, Y_B)$ as a subgroup of $\operatorname{Ext}^1_A(X_A, Y_A)$. Then, given an exact sequence

$$0 \to Y_A \xrightarrow{\mu} Z_A \to X_A \to 0$$

of A-modules with an inclusion map μ, it follows, in view of $XJ = 0$, that $ZJ \subseteq \mu(Y)$. Furthermore,

$$ZJ = ZJ^2 \subseteq \mu(Y)J = 0,$$

and thus Z is a B-module. This completes the proof for $i = 1$.

Finally, let $0 \to X'_B \to P_B \to X_B \to 0$ be an exact sequence with a projective B-module P_B. An application of $\operatorname{Hom}_B(-, Y_B)$ yields, for all $i \geq 0$, the exact sequences

$$\operatorname{Ext}^i_B(P_B, Y_B) \to \operatorname{Ext}^i_B(X'_B, Y_B) \to \operatorname{Ext}^{i+1}_B(X_B, Y_B) \to \operatorname{Ext}^{i+1}_B(P_B, Y_B).$$

Here, since P_B is projective, the first and the last terms are zero for $i \geq 1$. Similarly, an application of $\operatorname{Hom}_A(-, Y_A)$ yields, for all $i \geq 0$, the exact sequences

$$\operatorname{Ext}^i_A(P_A, Y_A) \to \operatorname{Ext}^i_A(X'_A, Y_A) \to \operatorname{Ext}^{i+1}_A(X_A, Y_A) \to \operatorname{Ext}^{i+1}_A(P_A, Y_A).$$

Here, according to Statement 1, proj dim $P_A \leq 1$, and thus the last term is zero for all $i \geq 1$. Moreover, $\operatorname{Ext}^1_A(P_A, Y_A) \simeq \operatorname{Ext}^1_B(P_B, Y_B) = 0$, and therefore the first term is zero also for all $i \geq 1$. By induction, we may assume

$$\operatorname{Ext}^i_B(X'_B, Y_B) \simeq \operatorname{Ext}^i_A(X'_A, Y_A)$$

and

$$\operatorname{Ext}^{i+1}_B(X_B, Y_B) \simeq \operatorname{Ext}^{i+1}_A(X_A, Y_A),$$

as required.

Statement 4. *Let $J^2 = J$ and J_A be projective. Then*

$$\operatorname{gl\,dim} B \leq \operatorname{gl\,dim} A.$$

Proof. Let $\operatorname{gl\,dim} A = d < \infty$. If X, Y are B-modules, then, in view of Statement 3, $\operatorname{Ext}^{d+1}_B(X_B, Y_B) \simeq \operatorname{Ext}^{d+1}_A(X_A, Y_A) = 0$. Hence, $\operatorname{gl\,dim} B \leq d$.

Statement 5. *Let A be a semiprimary ring with radical N. Let $JNJ = 0$ and J_A be projective. Then*

$$\operatorname{gl\,dim} A \leq \operatorname{gl\,dim} B + 2.$$

Proof. Let $\operatorname{gl\,dim} B = d < \infty$. Given an A-module X, first calculate the projective dimension of the A-module XJ. Let $\pi\colon P \to XJ$ be a minimal projective cover. For every $x \in X$, let $J_x = J_A$, and let $P' = \bigoplus_{x \in X} J_x$; finally, define $\pi'\colon P' \to XJ$ by sending $y \in J_x$ to xy. Since P' is projective and π' is surjective, it follows that P can be identified with a direct summand of P', and thus

$$\ker \pi \subseteq \operatorname{rad} P = PN \subseteq P'N = \bigoplus_{x \in X} J_x N.$$

Consequently, $(\ker \pi)J = 0$, and thus $\ker \pi$ is a B-module. Therefore, by Statement 1, $\operatorname{proj\,dim} (\ker \pi)_A \leq d + 1$. Hence, $\operatorname{proj\,dim} (XJ)_A \leq d + 2$.

On the other hand, X/XJ is a B-module, and thus, making use of Statement 1 again, $\operatorname{proj\,dim} (X/XJ)_A \leq d + 1$. Now, since X is an extension of the A-modules XJ by X/XJ both of projective dimension $\leq d + 2$,

$$\operatorname{proj\,dim} X_A \leq d + 2.$$

STATEMENT 6. *If e is an idempotent of A, then $(AeA)^2 = AeA$. Conversely, if A is semiprimary and J is an ideal of A such that $J^2 = J$, then $J = AeA$ for an idempotent e of A.*

Proof. The first assertion is trivial. So assume that A is semiprimary with radical N and that $J^2 = J$. Any ideal of A/N is generated by an idempotent, and any idempotent of A/N is of the form $\bar{e} = e + N$ with an idempotent e in A. Thus $J + N = AeA + N$ for some idempotent e of A. Now, $J^2 = J$ implies $(J + N)^i = J + N^i$ for all $i \geq 1$; similarly, $(AeA + N)^i = AeA + N^i$ for all $i \geq 1$. But for large i, $N^i = 0$, and therefore $J = AeA$.

STATEMENT 7. *Let e be an idempotent of a ring A. If the right module $(AeA)_A$ or the left module ${}_A(AeA)$ is projective, then the multiplication map*

$$\mu\colon Ae \bigotimes_{eAe} eA \to AeA$$

is bijective. Conversely, assume that A is semiprimary with radical N and that $eNe = 0$. Then, if μ is bijective, both modules $(AeA)_A$ and ${}_A(AeA)$ are projective.

Proof. For any A-module X_A, consider the multiplication map

$$\mu_X\colon X \bigotimes_A Ae \bigotimes_{eAe} eA \to X.$$

The map μ_X is bijective for $X_A = eA$, and therefore for all direct summands of direct sums of the module eA. Now, if $(AeA)_A$ is projective, there is a surjective map of the form $\oplus eA \to AeA$, where the direct sum is indexed by all elements of A. Since this epimorphism splits, it follows that μ_{AeA} is bijective. But this means that μ is bijective. The same argument applies in the case that ${}_A(AeA)$ is projective.

Now, assume that A is semiprimary with radical N and that $eNe = 0$. Since $eNe = 0$, the ring eAe is simple artinian, and thus any eAe-module is projective. Since $(Ae)_{eAe}$ and $(eA)_A$ are projective, $(Ae \otimes_{eAe} eA)_A$ is projective also. Thus, the bijectivity of μ implies that $(AeA)_A$ is projective. Similarly, it implies that ${}_A(AeA)$ is projective.

STATEMENT 8. *Let A be a semiprimary ring with radical N and $JNJ = 0$. Then $L(A) \leq 2L(B) + 1$.*

Proof. Since A_A is an extension of J_A by B_A, we have

$$L(A) \leq L(J_A) + L(B).$$

Moreover, J_A is an extension of $(JN)_A$ by $(J/JN)_A$. Since $JNJ = 0$, JN is a B-module and thus $L((JN)_A) \leq L(B)$. On the other hand, J/JN is semisimple and thus $L((J/JN)_A) \leq 1$. Consequently,

$$L(J_A) \leq L((JN)_A) + L((J/JN)_A) \leq L(B) + 1.$$

Now, making a subsequent use of Statements 7, 5 and 8, one can easily derive the following statement for quasi-hereditary rings.

STATEMENT 9. *Let A be a semiprimary quasi-hereditary ring with a heredity chain*

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_t \subseteq \cdots \subseteq J_m = A.$$

Then A^{op} is quasi-hereditary also, and

$$0 = J_0^{\mathrm{op}} \subseteq J_1^{\mathrm{op}} \subseteq \cdots \subseteq J_t^{\mathrm{op}} \subseteq \cdots \subseteq J_m^{\mathrm{op}} = A^{\mathrm{op}}$$

is a heredity chain.

Moreover, $\operatorname{gl\,dim} A \leq 2m - 2$ *and* $L(A) \leq 2^m - 1$.

The following assertion can also be easily obtained.

STATEMENT 10. *Let A be a semiprimary quasi-hereditary ring with a heredity chain*

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_t \subseteq \cdots \subseteq J_m = A.$$

Then, for each $1 \le t \le m$, $J_t = Ae_tA$ *for an idempotent* e_t *of* A *and*

$$0 = e_tJ_0e_t \subseteq e_tJ_1e_t \subseteq \cdots \subseteq e_tJ_te_t = e_tAe_t$$

is a heredity chain of e_tAe_t.

Proof. First, $J_t = Ae_tA$ for some e_t, $e_t^2 = e_t$, by Statement 6. Clearly,

$$e_tJ_te_t = e_tAe_tAe_t \supseteq e_tAe_t,$$

and thus $e_tJ_te_t = e_tAe_t$. Now, if we show that $e_tJ_1e_t$ is a heredity ideal of e_tAe_t, then the statement follows easily by induction. Since $J_1 = Ae_1A$ for an idempotent e_1 and $e_1 \in Ae_tA$,

$$e_tJ_1e_t \supseteq (e_tJ_1e_t)^2 = e_tAe_1Ae_tAe_1Ae_t \supseteq e_tAe_1Ae_t = e_tJ_1e_t.$$

Furthermore, if N is the radical of A, then e_tNe_t is the radical of e_tAe_t and

$$(e_tJ_1e_t)(e_tNe_t)(e_tJ_1e_t) \subseteq e_tJ_1NJ_1e_t = 0.$$

Finally, the A-module $(J_1)_A$ can be written as the finite direct sum of projective A-modules

$$J_1 = \bigoplus_p a_pA \oplus P,$$

where $P \subseteq (1 - e_t)A$, $a_pA \simeq eA$ with $e^2 = 1$, $ee_1 = e_1e = e$ and $a_p = e_ta_p = a_pe_1$, i.e.,

$$a_p \in e_tAe_1 \subseteq e_tAe_t.$$

Hence,

$$e_tJ_1e_t = \bigoplus_p a_p(e_tAe_t)$$

is a decomposition into a direct sum of projective e_tAe_t-modules

$$a_p(e_tAe_t) \simeq eAe_t = (e_tee_t)(e_tAe_t).$$

Examples. *Now we exhibit some examples (mostly path algebras over a field) to illustrate the necessity of some assumptions in the previous statements and the optimality of some bounds.*

(i) The assumption that J_A is projective cannot be omitted in Statements 1, 2 and 5.

Let A be the path algebra of the diagram

$$\circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ$$

modulo the ideal generated by $\alpha\beta$ and $\beta\alpha$. Let J be a one-dimensional ideal. Then, $\operatorname{Hom}_A(J_A, B_A) = 0$, $J^2 = 0 \neq J$, $JNJ = 0$,

$$\operatorname{gl\,dim} A = \infty \quad \text{and} \quad \operatorname{gl\,dim} B = 1.$$

(ii) The assumption $J^2 = J$ cannot be omitted in Statements 3 and 4.

Let A be the path algebra of $\circ \to \circ \to \circ$ and J the unique one-dimensional ideal. Then J_A is projective, $J^2 = 0 \neq J$,

$$\operatorname{gl\,dim} A = 1 \quad \text{and} \quad \operatorname{gl\,dim} B = 2.$$

(iii) The assumption that J_A is projective cannot be omitted in Statements 3 and 4.

Let A be the (basic) Auslander algebra of the algebra in (ii), i.e., the endomorphism algebra of the direct sum of all six indecomposable modules over the algebra in (ii). Let J be the set of all endomorphisms which factor through the largest indecomposable (projective and injective) module. Then J is an idempotent ideal,

$$\operatorname{gl\,dim} A = 2 \quad \text{and} \quad \operatorname{gl\,dim} B = 4.$$

(iv) The assumption $JNJ = 0$ cannot be omitted in Statement 5.

Let A be the path algebra of the diagram

$$1 \xrightarrow{\alpha} 2 \circlearrowright \beta$$

modulo the ideal generated by β^2. Let J be the (idempotent) ideal AeA, where e is the idempotent corresponding to the vertex 2. Then J_A is projective, $\operatorname{gl\,dim} A = \infty$ and $\operatorname{gl\,dim} B = 0$.

(v) The assumption $eNe = 0$ cannot be omitted in Statement 7.

Let A be the path algebra of the diagram

$$1 \xrightarrow{\alpha} 2 \circlearrowright \beta$$

modulo the ideal generated by $\alpha\beta$ and β^2. Let e be the idempotent corresponding to the vertex 2. Then the multiplication map

$$Ae \underset{eAe}{\otimes} eA \to AeA$$

is bijective, ${}_A(AeA)$ is projective and $(AeA)_A$ is not projective.

(vi) The bounds on global dimensions in Statements 5 and 9 are best possible.

Let A be the path algebra of the diagram

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \cdots m-1 \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} m$$

modulo the ideal generated by all $\alpha_t\alpha_{t+1}$, $\beta_{t+1}\beta_t$, $\beta_t\alpha_t - \alpha_{t+1}\beta_{t+1}$ for $1 \leq t \leq m-2$ and $\beta_{m-1}\alpha_{m-1}$. Let e_t be the idempotent corresponding to the vertex t, $1 \leq t \leq m$. Put $J_t = A(e_m + e_{m-1} + \cdots + e_{m-t+1})A$. Then $0 = J_0 \subset J_1 \subset \cdots \subset J_t \subset \cdots \subset J_m = A$ is a heredity chain of the quasi-hereditary algebra A and $\operatorname{gl\,dim} A = 2m - 2$. Let us remark that $L(A) = 3$.

(vii) The bounds on Loewy lengths in Statements 8 and 9 are best possible. Consider the complete oriented graph (without loops) on m vertices; denote the arrow from the vertex r to the vertex s by α_{rs}, $1 \leq r, s \leq m$, $r \neq s$. Let A be the path algebra of this graph modulo the ideal I generated by all products $\alpha_{t_1t_2}\alpha_{t_2t_3} \cdots \alpha_{t_kt_{k+1}}$, where $t_1 = t_{k+1}$ and all $t_i < t_1$ for $2 \leq i \leq k$. Let e_t be the idempotent corresponding to the vertex t, $1 \leq t \leq m$. Let

$$J_t = A(e_m + e_{m-1} + \cdots + e_{m-t+1})A.$$

Then

$$0 = J_0 \subset J_1 \subset \cdots \subset J_t \subset \cdots \subset J_m = A$$

is a heredity chain of the quasi-hereditary algebra A. Consider the element

$$a = \alpha_{t_1t_2}\alpha_{t_2t_3} \cdots \alpha_{t_kt_{k+1}} \cdots \alpha_{t_{2^m-2}t_{2^m-1}} \quad \text{modulo } I,$$

where $t_k = r + 1$ such that $k = 2^r(2s - 1)$. It is not difficult to see that a is a non-zero element of the socle of A, and to deduce, by induction, that $L(A) = 2^m - 1$. Let us remark that $\operatorname{gl\,dim} A = 2$.

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