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On Modular Representations of A_4

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1. INTRODUCTION

Let k be a field of characteristic 2. The representation theory of the alternating group A_4 over k has an essentially twofold character depending on whether k does or does not contain the cubic root of unity. The case when x^2+x+1 is reducible over k is reflected in the case when k is algebraically closed, which has been treated in detail by several authors (see S. B. Conlon [2], E. Kern [9]). The case when x^2+x+1 is irreducible over k has been considered explicitly only recently (see U. Schoenwaelder [10]). Of course, there is a well-known general procedure (see D. G. Higman [8]) to find the represenations of A_4 in both cases. This is due to the fact that A_4 contains the Klein 4-group V_4 as a normal subgroup and the representations of V_4 are well-understood (see V. A. Bašev [1], A. Heller, I. Reiner [7]). It is this procedure that has been used by U. Schoenwaelder in the case where x^2+x+1 is irreducible over k. An inherent difficulty of this method appears in the explicit listing and full understanding of certain representations, viz. those belonging to the one-parameter family of kA_4-modules which is induced from the one-parameter family of indecomposable kV_4-modules. Whereas in the case of V_4 these representations are indexed by the irreducible polynomials over k, the corresponding index set appears to be much more involved in the case A_4: One has to consider an action of the cyclic group of order 3 on the polynomials and determine the corresponding orbits. However, there is an alternative approach presented in our paper, which shows that also in the case of A_4, the one-parameter family of representations may be indexed again by the set of all irreducible

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polynomials over k, without any further modification. In fact, we offer an explicit description of all (finite-dimensional) representations of A_4 in this case, by translating the problem to the respective problem of describing all representations of a (tame) hereditary k-algebra of type $\tilde{\mathbb{B}}_3$. It should be pointed out that the description of the representation of A_4 over an algebraically closed field k can be reduced by the same process to the (tame) hereditary k-algebra of type $\tilde{\mathbb{A}}_5$.

2. STATEMENTS

Let $A_4 = \langle h, g \mid h^2 = g^3 = (hg)^3 = 1 \rangle$ be a (fixed) presentation of A_4. Then, as a kA_4-module,

$$kA_4 = \varepsilon_1 kA_4 \oplus \varepsilon_2 kA_4 \qquad \text{with} \quad \varepsilon_1 = g + g^2, \varepsilon_2 = 1 + g + g^2.$$

Thus, all non-projective-injective indecomposable representation are $kA_4/\mathrm{Soc}(kA_4)$-modules, and can be described as modules over the algebra

$$A = \left\{ \begin{bmatrix} y & o & u & w \\ & x & o & v \\ & & x & o \\ & & & \bar{y} \end{bmatrix} \mid x \in k;\ y, u, v, w, \in K = k(\omega) \right\},$$

where $\omega \neq 1$, $\omega^3 = 1$, and $^-: K \to K$ is the involution induced by mapping ω to ω^2. This is due to the fact that we have

PROPOSITION. $kA_4/\mathrm{Soc}(kA_4) \simeq A$.

Hence, the Loewy structure of the two indecomposable projective-injective kA_4 – modules is

$$\begin{array}{ccc} & \langle a_1, a_2 \rangle & \\ \langle a_3 \rangle & \langle a_4 \rangle & \langle a_5, a_6 \rangle \\ & \langle a_7, a_8 \rangle & \end{array} \quad \text{and} \quad \begin{array}{c} \langle b_1 \rangle \\ \langle b_2, b_3 \rangle \\ \langle b_4 \rangle \end{array}$$

here

$$\{a_1 = g + g^2, a_2 = 1 + g^2, a_3 = (g + g^2)(1 + h)(1 + g + g^2),$$
$$a_4 = (1 + g^2)(1 + h)(1 + g + g^2), a_5 = (g + g^2)(1 + h)(g + g^2)$$
$$a_6 = (g + g^2)(1 + h)(1 + g), a_7 = (g + g^2)(1 + h)(1 + g + g^2)(1 + h),$$
$$a_8 = (g + g^2)(1 + h)(1 + g)(1 + h)\}$$

and

$$\{b_1 = 1 + g + g^2,\ b_2 = (1 + g + g^2)(1 + h),\ b_3 = (1 + g + g^2)(1 + h)\, g,$$
$$b_4 = (1 + g + g^2)(1 + h)(1 + g + g^2)\}$$

are k-bases for these modules. With respect to these bases, the two representations have the forms

(i)

$$h \mapsto \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline & & 1 & 0 & & & 1 & 0 \\ & & 0 & 1 & & & 0 & 1 \\ \hline & & & & 1 & 0 & 1 & 0 \\ & & & & 0 & 1 & 0 & 1 \\ \hline & & & & & & 1 & 0 \\ & & & & & & 0 & 1 \end{array}\right], \qquad g \mapsto \left[\begin{array}{cc|cc|cc|cc} 0 & 1 & & & & & & \\ 1 & 1 & & & & & & \\ \hline & & 1 & 0 & & & & \\ & & 0 & 1 & & & & \\ \hline & & & & 1 & 1 & & \\ & & & & 1 & 0 & & \\ \hline & & & & & & 0 & 1 \\ & & & & & & 1 & 1 \end{array}\right];$$

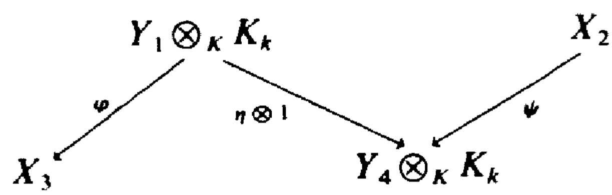
(ii)

$$h \mapsto \left[\begin{array}{c|cc|c} 1 & 1 & 0 & \\ \hline & 1 & 0 & 0 \\ & 0 & 1 & 1 \\ \hline & & & 1 \end{array}\right], \qquad g \mapsto \left[\begin{array}{c|cc|c} 1 & & & \\ \hline & 1 & 1 & \\ & 1 & 0 & \\ \hline & & & 1 \end{array}\right].$$

Moreover, since $\operatorname{rad}^2(A) = 0$, there is—up to the two simple injective B-modules—a bijection between the indecomposable modules over A and the indecomposable modules over the "separated" algebra (see, e.g., [3])

$$B = \begin{bmatrix} K & 0 & K & K \\ & k & 0 & K \\ & & k & 0 \\ & & & K \end{bmatrix}.$$

The latter is a (tame) hereditary k-algebra of the type $\tilde{\mathbb{B}}_3$ (see V. Dlab and C. M. Ringel [4]) and hence, denoting by e_i, $1 \leqslant i \leqslant 4$, the obvious diagonal idempotents of the algebra B, every A-module M can be described in terms of the K-spaces $Y_1 = Me_1$, $Y_4 = Me_4$, the k-spaces $X_2 = Me_2$, $X_3 = Me_3$, two k-linear maps φ, ψ, and a K-linear map η:



Call the quadruple

$$\begin{pmatrix} & y_1 = \dim Y_1 & & x_2 = \dim X_2 \\ x_3 = \dim X_3 & & y_4 = \dim Y_4 & \end{pmatrix}$$

the dimension type of M, or of the corresponding representation of A_4. It follows that every module can be, up to a choice of generators in Y_1, X_2, X_3, Y_4, uniquely described by a triple of matrices U, W, V over k of dimensions $2y_1 \times x_3$, $2y_1 \times 2y_4$, $x_2 \times 2y_4$, respectively. Summarizing, it turns out that every non-projective-injective indecomposable representation of A_4 of dimension type $({}_{x_3}\,{}^{y_1}\,{}_{y_4}\,{}^{x_2})$ is given by a pair of $(2y_1 + x_2 + x_3 + 2y_4) \times (2y_1 + x_2 + x_3 + 2y_4)$ matrices over k, partitioned into the 4×4 corresponding blocks

$$h \mapsto \begin{bmatrix} E & & U & W \\ & E & & V \\ & & E & \\ & & & E \end{bmatrix}, \qquad g \mapsto \begin{bmatrix} E' & & & \\ & E & & \\ & & E & \\ & & & E'' \end{bmatrix},$$

where E are the identity matrices and

$$E' = \begin{bmatrix} \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix} \end{bmatrix}, \qquad E'' = \begin{bmatrix} \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} \end{bmatrix}.$$

In view of the latter, these representations will be presented simply by specifying the three matrices

$$U, W, V;$$

all required types, in an appropriate notation, will be introduced at the end of this section. With these provisions, we are able to formulate the main result.

THEOREM. *Let* $A_4 = \langle h, g \mid h^2 = g^3 = (hg)^3 = 1 \rangle$ *and let* k *be a field of characteristic* 2 *not containing the cubic roots of unity. Furthermore, let* $K = k(\omega)$ *with* $\omega \neq 1, \omega^3 = 1$. *Referring to the preceding mode of description and the notation below, the following is a complete list of indecomposable representations of* A_4 *over* k:

(1) *the two projective-injective representations of dimensions* 8 *and* 4, *repectively, presented in* (i) *and* (ii);

(2) *for each* $d \geqslant 0$, *the "preprojective" representations*

$P(3d)$ *of dimension*	$6d+1 = 2d+d+(d+1)+2d$	U_-EV_0;
$P(3d+1)$	$6d+3 = 2d+(d+1)+d+2(d+1)$	$U_0W_-V_0$;
$P(3d+2)$	$6d+5 = 2(d+1)+d+(d+1)+2(d+1)$	U_0EV_-;
$Q(3d)$	$12d+2 = 4d+2d+2d+2(2d+1)$	$U_0W_-V_-$;
$Q(3d+1)$	$12d+6 = 2(2d+1)+2d+2(d+1)+2(2d+1)$	U_-EV_-;
$Q(3d+2)$	$12d+10 = 2(2d+1)+2(d+1)+2(d+1)+4(d+1)$	$U_-W_-V_0$;

(3) *for each* $d \geqslant 0$, *the "preinjective" representations*

$S(3d)$ *of dimension*	$6d+1 = 2d+(d+1)+d+2d \ (d \neq 0)$	U_0EV_+;
$S(3d+1)$	$6d+3 = 2(d+1)+d+(d+1)+2d$	$U_0W_+V_0$;
$S(3d+2)$	$6d+5 = 2(d+1)+(d+1)+d+2(d+1)$	U_+EV_0;
$T(3d)$	$12d+2 = 2(2d+1)+2d+2d+4d \ (d \neq 0)$	$U_+W_+V_0$;
$T(3d+1)$	$12d+6 = 2(2d+1)+2(d+1)+2d+2(2d+1)$	U_+EV_+;
$T(3d+2)$	$12d+10 = 4(d+1)+2(d+1)+2(d+1)+2(2d+1)$	$U_0W_+V_+$;

(4) *for each* $d \geqslant 0$, *the "regular nonhomogeneous" representations*

$R_1^1(d)$ *of dimension*	$12d+4 = 2(2d+1)+2d+2(d+1)+4d$	$U^{10}EV$;
$R_1^3(d)$	$12d+8 = 4(d+1)+2d+2(d+1)+2(2d+1)$	U^0EV^0;
$R_1^2(d)$	$12d+12 = 4(d+1)+2(d+1)+2(d+1)+4(d+1)$	U^0E^0V;
$R_2^2(d)$	$12d+4 = 4d+2(d+1)+2d+2(2d+1)$	$UE^{01}V$;
$R_2^1(d)$	$12d+8 = 2(2d+1)+2(d+1)+2(d+1)+2(2d+1)$	U^1E^1V;
$R_2^3(d)$	$12d+12 = 4(d+1)+2(d+1)+2(d+1)+4(d+1)$	UE^1V^0;
$R_3^3(d)$	$12d+4 = 2(2d+1)+2d+2d+2(2d+1)$	U^0EV^0;
$R_3^2(d)$	$12d+8 = 2(2d+1)+2(d+1)+2d+4(d+1)$	U^0E^0V;
$R_3^1(d)$	$12d+12 = 4(d+1)+2(d+1)+2(d+1)+4(d+1)$	${}^0U^1EV$;

(5) *for each* $d \geqslant 1$ *and each irreducible* $f(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} + x^n$ *of* $k[x], f(x) \neq 1 + x + x^2$, *the "homogeneous" representations*

$$H_f(d) \text{ of dimension } \quad 6nd = 2nd + nd + nd + 2nd \quad U_fEV_0;$$

and for each $d \geqslant 1$

$$K_\infty(d) \text{ of dimension } \quad 6d = 2d + d + d + 2d \quad U_\infty EV_0.$$

Here, for each natural m, we consider the following types of matrices:

$$U = \left[\begin{array}{c|c|c} \begin{matrix}1&0\\0&1\end{matrix} & & \\ \hline & \ddots & \\ \hline & & \begin{matrix}1&0\\0&1\end{matrix} \end{array}\right] \quad \text{if size } 2m \times (m+1),$$

$$U_0 = \left[\begin{array}{c|c|c} \begin{matrix}1\\0\end{matrix} & & \\ \hline & \ddots & \\ \hline & & \begin{matrix}1\\0\end{matrix} \end{array}\right] \quad \text{of size } 2m \times m,$$

$$U_+ = \left[\begin{array}{c|c|c} \begin{matrix}1\\0\\0\\1\end{matrix} & & \\ \hline & \ddots & \\ \hline & & \begin{matrix}1\\0\\0\\1\end{matrix} \end{array}\right] \quad \text{of size } 2(m+1) \times m,$$

$$U = \left[\begin{array}{c|c|c} \begin{matrix}1&0\\0&1\\0&1\\0&0\end{matrix} & & \\ \hline & \ddots & \\ \hline & & \begin{matrix}1&0\\0&1\\0&1\\0&0\end{matrix} \end{array}\right] \quad \text{of size } 4m \times 2m,$$

$$U_\infty = \begin{bmatrix} 0 & & & \\ 1 & & & \\ 1 & & & \\ 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & 1 & \\ & & 1 & 0 \\ & & 0 & 1 \end{bmatrix} \qquad \text{of size } 2m \times m,$$

$$U_f = \begin{bmatrix} U_f^\vee & & & \\ E^\vee & U_f^\vee & & \\ & \ddots & \ddots & \\ & & E^\vee & U_f^\vee \end{bmatrix}, \qquad \text{of size } 2m \times m \text{ for } m = nd,$$

where

$$U_f^\vee = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & 1 & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \\ & & & 1 \\ & & & 1 \\ c_0 & c_1 & & c_{n-1} \end{bmatrix} \qquad \text{and} \qquad E^\vee = \begin{bmatrix} & & 0 \\ & & 1 \\ & & \\ & & \\ & & \end{bmatrix}$$

for $f(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + x^n \in k[x]$,

$V_- = V_+^{\mathrm{tr}}$, $V_0 = U_0^{\mathrm{tr}}$, $V_+ = U_-^{\mathrm{tr}}$,

$$V = \begin{bmatrix} 0 & 1 & 1 & 0 & & & & & \\ 0 & 0 & 0 & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & 1 & 1 & 0 \\ & & & & & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{of size } 2m \times 4m,$$

$$W = \left[\begin{array}{cccc:cc:cccc}
1 & 0 & 0 & 1 & & & & & & \\
0 & 1 & 1 & 1 & & & & & & \\
\hdashline
 & & & & \ddots & & & & & \\
\hdashline
 & & & & & & 1 & 0 & 0 & 1 \\
 & & & & & & 0 & 1 & 1 & 1
\end{array}\right] \quad \text{of size } 2m \times 2(m+1),$$

$W_0 = E$ (identity) of size $2m \times 2m$, and $W_+ = W_-^{\mathrm{tr}}$. In addition, for an $r \times s$ matrix Z, we define the $(r+2) \times s$ matrix

$$^{0}Z = \left[\begin{array}{c} \\ \hdashline \\ Z \\ \\ \end{array}\right],$$

the $(r+2) \times (s+2)$ matrix

$$Z^{1} = \left[\begin{array}{c:cc} & & \\ Z & & \\ & & \\ \hdashline & 1 & 0 \\ & 0 & 1 \end{array}\right],$$

the $(r+4) \times (s+2)$ matrix $^{0}Z^{1} = {}^{0}(Z^{1}) = ({}^{0}Z)^{1}$, the $r \times (s+2)$ matrix

$$Z^{0} = \left[\begin{array}{c:c} Z & \quad \end{array}\right],$$

the $(r+2) \times (s+2)$ matrix

$$^{1}Z = \left[\begin{array}{cc:c} 1 & 0 & \\ 0 & 1 & \\ \hdashline & & \\ & & Z \\ & & \end{array}\right],$$

and the $(r+2) \times (s+4)$ matrix $^{1}Z^{0} = ({}^{1}Z)^{0} = {}^{1}(Z^{0})$.

3. Methods and Proofs

Let us start with the following.

Proof of proposition. Consider the homomorphism $\mu\colon kA_4 \to A$ given by

$$\mu(h) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ & 1 & 0 & 1 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \quad \text{and} \quad \mu(g) = \begin{bmatrix} \omega & & & \\ & 1 & & \\ & & 1 & \\ & & & \omega^2 \end{bmatrix}.$$

Then, for $a, b \in k$,

$$\mu[(a+bg)(1+h)(1+g+g^2)] = \begin{bmatrix} 0 & 0 & a+b\omega & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix},$$

$$\mu[(1+g+g^2)(1+h)(a+bg^2)] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & a+b\omega \\ & & 0 & 0 \\ & & & 0 \end{bmatrix},$$

and

$$\mu[(a+bg)[(1+h)(g+g^2)+(1+g+g^2)(1+h)]] = \begin{bmatrix} 0 & 0 & 0 & a+b\omega \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix},$$

and thus μ is surjective. Clearly, $\operatorname{rad}^2(A) = 0$. Since

$$\mu[\operatorname{rad}^2(kA_4)] = \{\mu[\operatorname{rad}(kA_4)]\}^2 \subseteq \operatorname{rad}^2 A = 0,$$
$$\operatorname{Ker} \mu \supseteq \operatorname{rad}^2(kA_4) \supseteq \operatorname{Soc}(kA_4).$$

Furthermore,

$$\operatorname{Top}(kA_4) = kA_4/\operatorname{rad}(kA_4) \simeq kC_3 = k \times K,$$

and therefore

$$\dim_k \operatorname{Soc}(kA_4) = \dim_k \operatorname{Top}(kA_4) = 3.$$

Since $\dim_k kA_4 = 12$ and $\dim_k A = 9$, it follows that

$$\mathrm{Soc}(kA_4) = \mathrm{rad}^2(kA_4) = \mathrm{Ker}\,\mu,$$

and that μ induces an isomorphism between $kA_4/\mathrm{Soc}(kA_4)$ and A, as required.

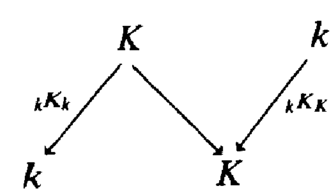
As a consequence of the proposition, we immediately get the following

COROLLARY. *The category of all finite dimensional representations of A_4 over k which do not have injective direct summands is equivalent to the category of all finite dimensional A-modules.*

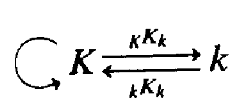
Now,

$$B = \begin{bmatrix} K & & K & K \\ & k & & K \\ & & k & \\ & & & k \end{bmatrix} = \left\{ \begin{bmatrix} y_1 & & u & w \\ & x_2 & & v \\ & & x_3 & \\ & & & y_4 \end{bmatrix} \middle| x_2, x_3 \in k;\ y_1, y_4, u, v, w \in K \right\}$$

is a hereditary k-algebra whose k-species



is the "separated" species of the k-species



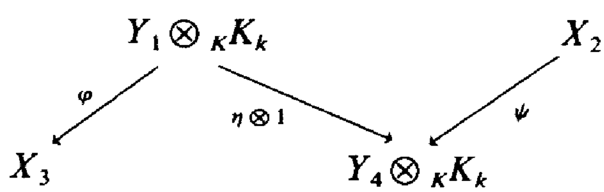
of the algebra A. Hence, there is a (full, dense) "separation" functor

$$\Phi\colon \mathrm{mod}\, B \to \mathrm{mod}\, A$$

defined by

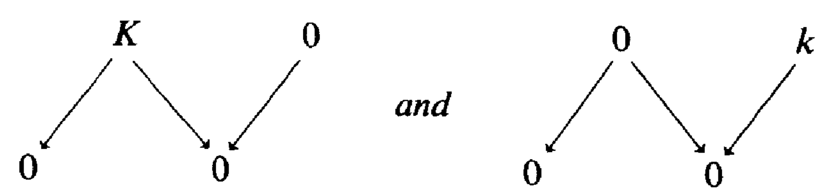
$$\Phi(M_B) = {}^{\left(\begin{smallmatrix} 0 & \eta\otimes 1 \\ 0 & 0 \end{smallmatrix}\right)} \circlearrowright (Y_1 \oplus Y_4) \otimes {}_KK_k \underset{\left(\begin{smallmatrix} \bar{0} & \psi \\ 0 & 0 \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 0 & \varphi \\ 0 & 0 \end{smallmatrix}\right)}{\rightleftarrows}} X_2 \oplus X_3$$

where

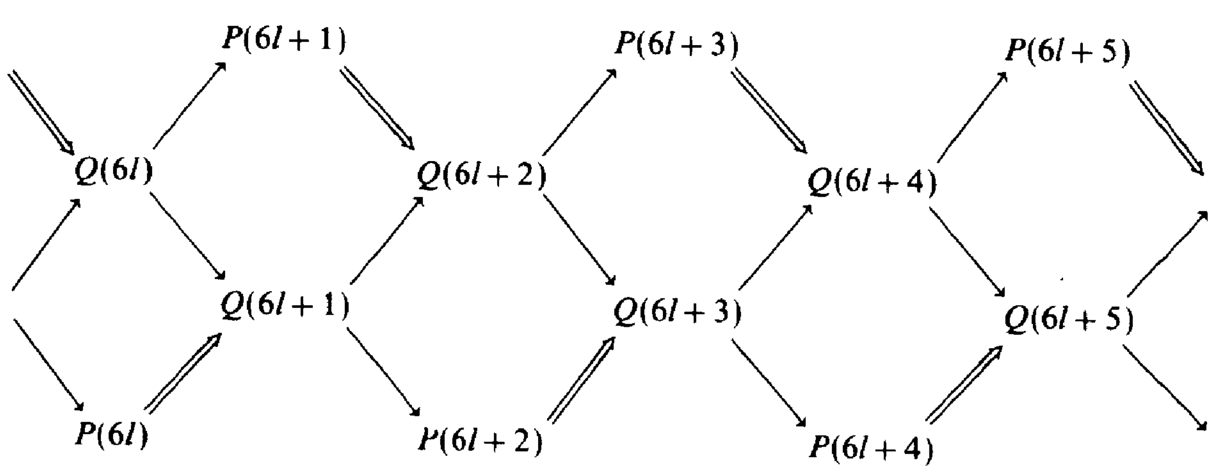
$M_B =$ 

which induces a representation equivalence between mod A and the full subcategory of mod B of all B-modules M_B with monic $\varphi^*\colon Y_1 \to X_3 \otimes_k K_K$ and ψ. Hence, we may formulate the following

COROLLARY. *There is a bijection between the set of all isomorphism classes of indecomposable A-modules, i.e., of non-injective indecomposable representations of A_4 over k, and the set of all isomorphism classes of indecomposable B-modules with the exception of the two classes represented by*



Now, we are in position to apply the results and method of V. Dlab and C. M. Ringel [4, 5]. First of all, we compute the dimension types of the indecomposable preprojective and preinjective representations and describe the respective components of the Auslander–Reiten graph of A: for $l \geq 0$,



where

$$\dim P(3d)=\begin{pmatrix} & d & & d \\ d+1 & & d & \end{pmatrix},$$

$$\dim P(3d+1)=\begin{pmatrix} & d & & d+1 \\ d & & d+1 & \end{pmatrix},$$

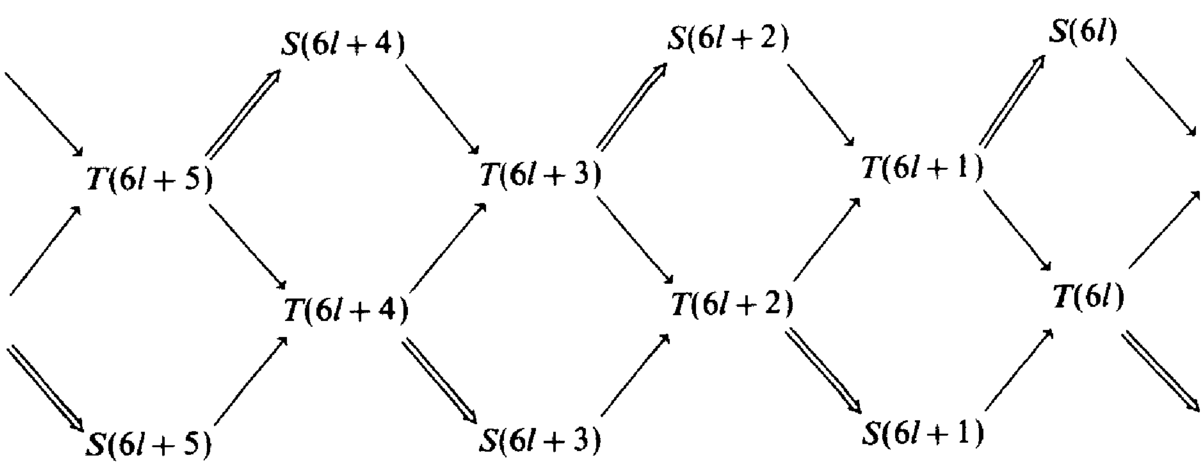
$$\dim P(3d+2)=\begin{pmatrix} & d+1 & & d \\ d+1 & & d+1 & \end{pmatrix},$$

$$\dim Q(3d)=\begin{pmatrix} & 2d & & 2d \\ 2d & & 2d+1 & \end{pmatrix},$$

$$\dim Q(3d+1)=\begin{pmatrix} & 2d+1 & & 2d \\ 2d+2 & & 2d+1 & \end{pmatrix},$$

$$\dim Q(3d+2)=\begin{pmatrix} & 2d+1 & & 2d+2 \\ 2d+2 & & 2d+2 & \end{pmatrix};$$

and, for $l \geqslant 0$,



where

$$\dim S(3d)=\begin{pmatrix} & d & & d+1 \\ d & & d & \end{pmatrix},$$

$$\dim S(3d+1)=\begin{pmatrix} & d+1 & & d \\ d+1 & & d & \end{pmatrix},$$

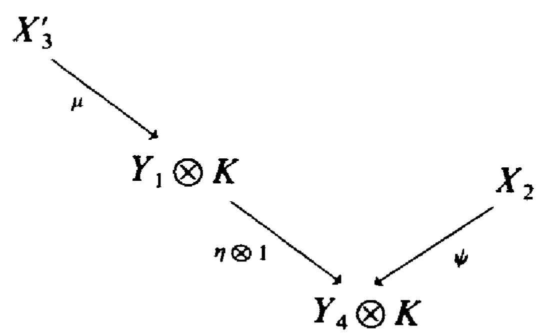
$$\dim S(3d+2)=\begin{pmatrix} & d+1 & & d+1 \\ d & & d+1 & \end{pmatrix},$$

$$\dim T(3d)=\begin{pmatrix} & 2d+1 & & 2d \\ 2d & & 2d & \end{pmatrix},$$

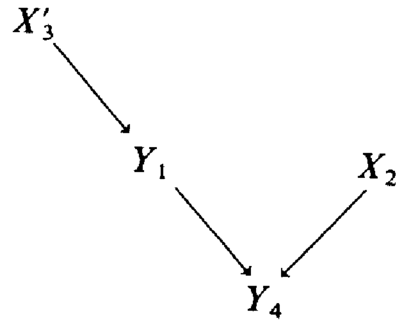
$$\dim T(3d+1)=\begin{pmatrix} & 2d+1 & & 2d+2 \\ 2d & & 2d+1 & \end{pmatrix},$$

$$\dim T(3d+2)=\begin{pmatrix} & 2d+2 & & 2d+2 \\ 2d+2 & & 2d+1 & \end{pmatrix}.$$

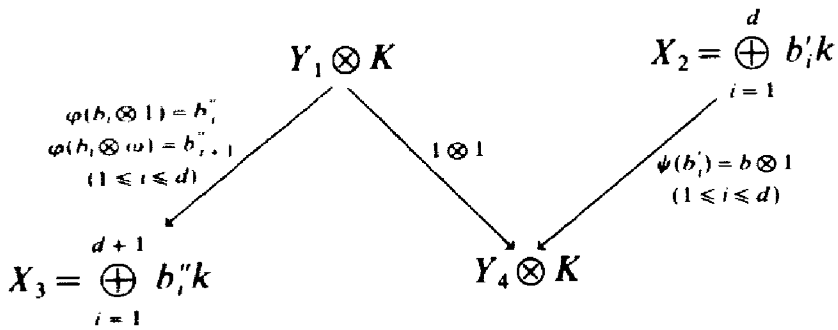
Each of the preprojective representations can be associated with a K-space Y_4 that is endowed with a structure of a k-subspace X_2 and a K-subspace Y_1 containing a k-subspace X_3'



taking $X_3'=\operatorname{Ker}\varphi$ and $\mu=\ker\varphi$. In this way, choosing bases of the spaces appropriately, we arrive at the representation presentations listed under (2) in the theorem. Let us illustrate this procedure in the case of, say, $P(3d)$. We have $\dim P(3d)=\begin{pmatrix} & d & & d \\ d+1 & & d & \end{pmatrix}$; thus, there is a unique indecomposable d-dimensional K-space Y_4 endowed with a d-dimensional k-subspace X_2 and a $(d-1)$-dimensional k-subspace X_3' (identifying Y_1 with Y_4). We claim that the vector space $Y_4=\bigoplus_{i=1}^{d} b_iK$ with $X_2=\bigoplus_{i=1}^{d} b_ik$ and $X_3'=\bigoplus_{i=1}^{d-1}(b_i\omega+b_{i+1})k$ is indecomposable. Indeed, for $d=1$, the statement is trivial, and we proceed by induction. Observe that the maximal K-subspace $\bar{Y}_4$ of the k-subspace X_2+X_3' must be compatible with any decomposition of

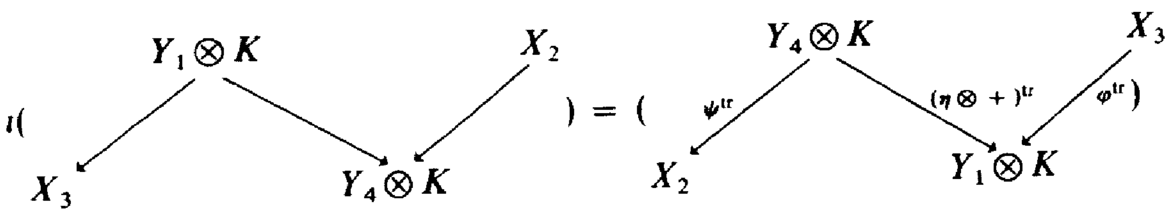


and that $\bar{Y}_4=\bigoplus_{i=1}^{d-1} b_iK$ together with $\bar{Y}_1=\bar{Y}_4$, $\bar{X}_2=\bigoplus_{i=1}^{d_1} b_ik$, $\bar{X}_3'=\bigoplus_{i=1}^{d-2}(b_i\omega+b_{i+1})k$ is indecomposable by induction. Consequently, $\bar{Y}_4$ is a direct summand of any decomposition of Y_4. However, then the one-dimensional K-complement cannot nontrivially intersect both X_2 and X'_3. It turns out that



is the presentation of the representation $P(3d)$ listed in the theorem.

To deal with the preinjective indecomposable representrations, we note that the opposite algebra A^{opp} is isomorphic to A and that this isomorphism yields a bijection ι between the (right) A-modules which may be described as



where the matrix of $\varphi^{\text{tr}}, (\eta \otimes 1)^{\text{tr}}$, and ψ^{tr} with respect to chosen bases of Y_1, X_2, X_3 and X_4 is the transpose of the respective matrix of $\varphi, \eta \otimes 1$ and ψ. Since, under ι, the preinjective indecomposable representations correspond to the preprojective ones, we get immediately the presentations (3) of the theorem.

In order to deal with the regular nonhomogeneous representations, we apply the theory described in [4] and the vector space method of constructing the explicit presentations, as described in the case of preprojective representations. Note that there are nine types of such regular indecomposable representations depending on the simple regular types in the socle and at the top. Thus, denoting by (1), (2), and (3) the simple types of dimension of types

$$\begin{pmatrix} & 1 & & 0 \\ 2 & & 0 & \end{pmatrix}, \quad \begin{pmatrix} & 0 & & 2 \\ 0 & & 1 & \end{pmatrix}, \quad \begin{pmatrix} & 1 & & 0 \\ 0 & & 1 & \end{pmatrix}, \qquad \text{respectively,}$$

the indecomposable representation $R_i^j(d)$, $1 \leqslant i, j \leqslant 3$, has the socle of type (i) and the top of type (j); the regular length of $R_i^i(d)$ is $3d+1$, that of $R_1^3(d)$, $R_2^1(d)$ and R_3^2 is $3d+2$, and that of $R_1^2(d)$, $R_2^3(d)$ and $R_3^1(d)$ is $3d+3$.

Finally, we apply the Tables of [4] and the well-known theory of homogeneous Kronecker modules. It follows that the subcategory of all homogeneous representations is equivalent to the category

$$U_\infty \times \operatorname{mod}^\circ k[x],$$

where $\operatorname{mod}^\circ k[x]$ is the full subcategory of the category of all finite dimensional modules over the polynomial algebra $k[x]$ whose modules do not contain a submodule isomorphic to $R_3^1(0)$. Indeed, in the embedding of the category $U_\infty \times \operatorname{mod} k[x]$ of all homogeneous Kronecker modules into the category of all regular representations of the algebra A, as described in [4], the module $k[x]/(1+x+x^2)\,k[x]$ maps onto $R_3^1(0)$; in fact, it is easy to check that $R_3^1(d) \approx H_{1+x+x^2}(d)$. Consequently, a suitable choice of a basis of X_3 leads to presentation (5) of the theorem.

4. Appendix

To summarize the following is the list of all indecomposable representations of A_4 of dimension δ modulo 12, which are not of the form H_f (the latter have dimension 0 or 6 modulo 12):

δ (mod 12)	Non-regular	Regular non-homogeneous	Homogeneous	Remark
0		R_1^2, R_2^3, R_3^1	H_∞	
1	$P(3d), S(3d)$			$P(0) \approx S(0)$
2	$Q(3d), T(3d)$			$Q(0) \approx T(0)$
3	$P(3d+1), S(3d+1)$			
4	$\varepsilon_2 kA_4$	R_1^1, R_2^2, R_3^3		
5	$P(3d+2), S(3d+2)$			(Projective of dimension 4)
6	$Q(3d+1), T(3d+1)$		H_∞	
7	$P(3d), S(3d)$			
8	$\varepsilon_1 kA_4$	R_1^3, R_2^1, R_3^2		
9	$P(3d+1), S(3d+1)$			(Projective of dimension 8)
10	$Q(3d+2), T(3d+2)$			
11	$P(3d+2), S(3d+2)$			

The simple homogeneous representations $H_f(n)$ are indexed by the irreducible polynomials f over k. For the field $k = \mathbb{F}_2$, the number $\alpha(n)$ of such polynomials of a given degree n has been determined by C. F. Gauss. In [6, p. 610], he proves that

$$2^n = \sum_{t \mid n} t \cdot \alpha(t).$$

Using the Möbius inversion, we get

$$\alpha(n)=\frac{1}{n}\sum_{t\mid n}\mu\left(\frac{n}{t}\right)\cdot 2^t,$$

where μ is the Möbius function: $\mu(m)=0$ if m is divisible by the square of a prime, and $\mu(p_1 p_2 \cdots p_k)=(-1)^k$ if $p_1, p_2, ..., p_k$ are distinct primes. Thus, for the dimension $\delta=6r$, the number of homogeneous indecomposable representations of A_4 over the field $\mathbb{F}_2$ which are of the form H_f is

$$\sum_{t\mid r \text{ and } t\neq 2}\alpha(t).$$

Consequently, the total numner N_δ of the indecomposable representations of A_4 over $\mathbb{F}_2$ of dimension δ is given by

δ	N_δ	for $t\geqslant 1$	δ	N_δ
1	1		$12t+1$	2
2	1		$12t+2$	2
3	2		$12t+3$	2
4	4		$12t+4$	3
5	2		$12t+5$	2
6	5			
7	2		$12t+7$	2
8	4		$12t+8$	3
9	2		$12t+9$	2
10	2		$12t+10$	2
11	2		$12t+11$	2

and for $\delta=6r$, $r\geqslant 2$,

$$N_\delta=3+\sum_{t/r}\alpha(t)$$

(comp. the computations for $\delta\leqslant 36$ and an error for $\delta\leqslant 42$ in [10])

r	δ	N_δ	r	δ	N_δ
2	12	6	14	84	1185
3	18	7	15	90	2195
4	24	9	16	96	4119
5	30	11	17	102	7715
6	36	17	18	108	14605
7	42	23	19	114	27599
8	48	39	20	120	52491
9	54	63	⋮	⋮	⋮
10	60	111	24	144	699255
11	66	191	⋮	⋮	⋮
12	72	355	50	300	22517998808031
13	78	635	⋮	⋮	⋮

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