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THE INTERSECTION PROPERTY OF AMALGAMATIONS

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Let M denote a class of monomorphisms in a category, with the following properties:

(i) If $fg \in M$ and $f \in M$ then $g \in M$.

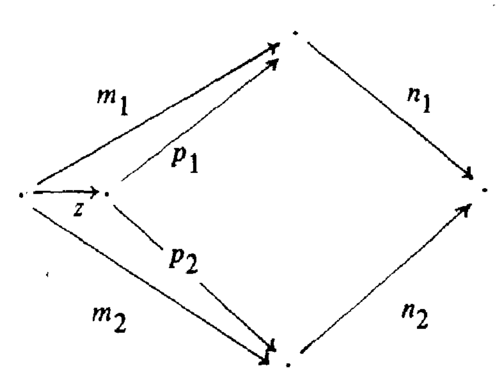
(ii) The intersection of two M's is an M.

(iii) The pushout of an M by an M exists and is an M.

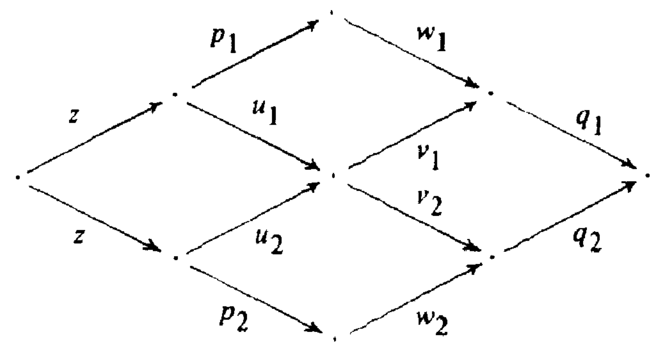
The first two properties are enjoyed by the class of *all* monomorphisms, by the class of regular monomorphisms, and again by the distinguished class of monomorphisms in an Isbell bicategory structure. The third is the so-called amalgamation property. There has been some interest of late in categories with the amalgamation property, and in particular in such categories wherein the pushout of two M's is also a pullback (= intersection). We give a necessary and sufficient condition for the latter to be the case.

Proposition. *In the diagram below, let $m_i \in M$, let n_1, n_2 be the pushout of m_1, m_2, let p_1, p_2 be the pullback of n_1, n_2, and let z be the unique map rendering the diagram commutative. Then z is an epimorphism.*

(1)



Proof. The $n_i \in M$ by (iii), the $p_i \in M$ by (ii) and (i), and $z \in M$ by (i). Form the diagram

(2) 

in which every diamond is a pushout. Then the exterior of (2) coincides with that of (1), so we may assume that $q_i w_i = n_i$. Because z and the p_i are M's, every map in (2) is an M by (iii) (and the pushouts do exist). Since

$$q_1 v_1 u_1 = q_1 w_1 p_1 = n_1 p_1 = n_2 p_2 = q_2 v_2 u_2 = q_1 v_1 u_2,$$

and since q_1 and v_1 are both monomorphisms, we have $u_1 = u_2$. Since u_1, u_2 is the cokernel-pair of z, it follows that z is an epimorphism.

Corollary. *Let the category admit intersections of M's. Then every pushout of two M's is also a pullback if and only if every M that is an epimorphism is in fact an isomorphism.*

Proof. The "if" part follows from the proposition. For the "only if" part let m be an epimorphism in M. Taking $m_1 = m_2 = m$ in diagram (1), we have $n_1 = n_2 = 1$, whence $p_1 = p_2 = 1$, whence $z = m$; and, by assumption, z is an isomorphism.

Remark. The condition in the corollary is automatic if M is the extremal monomorphisms or the regular monomorphisms. If M is the monomorphism, it says that every bimorphism (= epimorphic monomorphism) is an isomorphism. Suppose (i) holds in the stronger form:

If $fg \in$ M, then $g \in$ M. Then, for a fixed m_1, the z in (1) will always be an isomorphism if and only if m_1 is an extremal monomorphism.

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