Math. Ann. 195, 279--291 (1972)

A Class of Balanced Non-Uniserial Rings*

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Let R be a ring with unity. An R-module M is called balanced, if the natural homomorphism from R to the double centralizer of M is surjective. If every left R-module is balanced, R is said to be left balanced (or to satisfy the double centralizer condition for left modules). It is well-known that every artinian uniserial ring is both left and right balanced, and recently Jans [3] conjectured that "if R has minimum condition, then every R-module has the double centralizer condition if and only if R is a uniserial ring". This conjecture has been proved in [1] to be true for rings which are finitely generated over their centres. However, the following theorem shows that, in general, the conjecture is false.

Theorem. *Let R be a local ring with the radical W such that $W^2 = 0$, $\dim({}_{R/W}W) = 2$ and $\dim(W_{R/W}) = 1$. If R/W is commutative, then R is both left and right balanced.*

It is easy to see that rings satisfying the conditions of Theorem exist.

In Section 1, a sufficient condition for a direct sum of modules to be balanced is given; it represents a generalization of theorems of Nesbitt and Thrall [5] and Morita [4]. In Section 2, the indecomposable injective left module and the indecomposable injective right module over the rings R described in our theorem are calculated. From this, it follows that there are exactly three different types of indecomposable left R-modules (all of which are monogenic), three different types of indecomposable right R-modules and that every R-module is a direct sum of indecomposables. The latter is proved for left R-modules in Section 3, and for right R-modules in Section 4. A combination of the previous results yields the theorem; together with a few remarks, the proof of Theorem constitutes the final Section 5.

1.

The following Proposition generalizes results of Nesbitt and Thrall [5] and Morita [4]. We recall that a module M_0 is said to be a generator for a module M, if the images of all the morphisms $M_0 \to M$ generate M and that it is said to be a cogenerator for M, if the intersection of the

* This research has been supported by the National Research Council of Canada under Grant No. A-7257.

kernels of the morphisms $M \to M_0$ equals zero. Thus, in particular, if M is isomorphic to a quotient module of M_0, then M_0 generates M; and, if M is isomorphic to a submodule of M_0, then M_0 cogenerates M.

Proposition 1. *Let* $M = \left(\bigoplus_{\gamma \in \Gamma} M_\gamma\right) \oplus M_0$ *be a direct sum of* R*-modules such that* M_0 *is balanced and, for every* $\gamma \in \Gamma$, M_0 *is a generator or a cogenerator for* M_γ. *Then* M *is balanced.*

Proof. Let, for every $\gamma \in \Gamma \cup \{0\}$,

$$\pi_\gamma : M \to M_\gamma \quad \text{and} \quad \iota_\gamma : M_\gamma \to M$$

be the canonic projections and injections accompanying the direct sum M; in particular,

$$\iota_\gamma \pi_\gamma = 1_{M_\gamma} \quad \text{for each} \quad \gamma.$$

Let ψ be an element of the double centralizer of M. Consider, for every $\gamma \in \Gamma \cup \{0\}$, the morphism

$$\psi_\gamma = \iota_\gamma \psi \pi_\gamma = M_\gamma \xrightarrow{\iota_\gamma} M \xrightarrow{\psi} M \xrightarrow{\pi_\gamma} M_\gamma .$$

Clearly, if $\varphi : M_\gamma \to M_{\gamma'} (\gamma, \gamma' \in \Gamma \cup \{0\})$ is an R-homomorphism, then

$$(\psi_\gamma x)\,\varphi = \psi_{\gamma'}(x\varphi) \quad \text{for all} \quad x \in M_\gamma .$$

This follows easily from the fact that $\pi_\gamma \varphi \iota_{\gamma'}$ belongs to the centralizer and ψ to the double centralizer of M:

$$\begin{aligned} \psi_\gamma \varphi &= (\iota_\gamma \psi \pi_\gamma)\,\varphi(\iota_{\gamma'} \pi_{\gamma'}) = \iota_\gamma \psi(\pi_\gamma \varphi \iota_{\gamma'})\,\pi_{\gamma'} \\ &= \iota_\gamma(\pi_\gamma \varphi \iota_{\gamma'})\,\psi \pi_{\gamma'} = \varphi \psi_{\gamma'} . \end{aligned}$$

Thus, in particular, ψ_γ belongs to the centralizer of M_γ. Therefore, since M_0 is balanced, ψ_0 is induced by multiplication by an element $\varrho \in R$. We are going to show that also ψ_γ is induced by multiplication by the same element ϱ (for every $\gamma \in \Gamma$). Indeed, if $x \in M_0^\varphi$, where $\varphi : M_0 \to M_\gamma$ is an R-homomorphism and $x = x_0\varphi$ with $x_0 \in M_0$, then

$$\psi_\gamma x = \psi_\gamma(x_0\varphi) = (\psi_0 x_0)\,\varphi = (\varrho x_0)\,\varphi = \varrho(x_0\varphi) = \varrho x .$$

As a consequence, if M_0 generates M, then

$$\psi_\gamma x = \varrho x \quad \text{for all} \quad x \in M_\gamma .$$

Also, if $x \in M_\gamma$, then $\psi_\gamma x - \varrho x$ belongs to the kernel of every morphism $\varphi : M_\gamma \to M_0$; for,

$$(\psi_\gamma x - \varrho x)\,\varphi = (\psi_\gamma x)\,\varphi - (\varrho x)\,\varphi = \psi_0(x\varphi) - \varrho(x\varphi) = 0 .$$

And hence, if M_0 cogenerates M_γ, then

$$\psi_\gamma x - \varrho x = 0, \quad \text{i.e.} \quad \psi_\gamma x = \varrho x, \quad \text{for all} \quad x \in M_\gamma .$$

Finally, in order to complete the proof, it is sufficient to observe that, for every $\gamma \in \Gamma \cup \{0\}$,

$$(\psi m)\,\pi_\gamma = \psi_\gamma(m\pi_\gamma) \quad \text{for all} \quad m \in M;$$

this follows immediately from

$$\psi\pi_\gamma = (\psi\pi_\gamma\iota_\gamma)\,\pi_\gamma = (\pi_\gamma\iota_\gamma\psi)\,\pi_\gamma = \pi_\gamma\psi_\gamma .$$

And thus, it turns out that

$$\psi m = \sum_{\gamma \in \Gamma \cup \{0\}} (\psi m)\,\pi_\gamma\iota_\gamma = \sum_\gamma [\psi_\gamma(m\pi_\gamma)]\,\iota_\gamma$$

$$= \sum_\gamma \varrho(m\pi_\gamma)\,\iota_\gamma = \sum_\gamma (\varrho m)\,\pi_\gamma\iota_\gamma = \varrho m$$

for all $m \in M$, as required.

2.

In what follows, R will always stand for a ring described in our Theorem, i.e. R will always be a local ring with radical W such that $Q = R/W$ is commutative, $W^2 = 0$ and

$$\dim({}_Q W) = 2, \qquad \dim(W_Q) = 1 .$$

For the sake of brevity, we shall often refer to these rings as to *rings of type* (2,1).

Our first objective is to determine the indecomposable injective R-modules. This is achieved in the following

Proposition 2. *Let R be a ring of type (2, 1). Let u and v be elements of W such that $Ru + Rv = W$. Then*

(*l*) ${}_R(R/Ru)$ *is an indecomposable injective left R-module and*

(*r*) $[(R \oplus R)/D]_R$ *with* $D = \{(u\varrho, -v\varrho) \mid \varrho \in R\}$ *is an indecomposable injective right R-module.*

Proof. In order to facilitate the proof of Proposition 2, let us define a multiplication (which will be denoted by $*$) on W in such a way that the bimodule ${}_R W_R$ becomes a *bialgebra* in the following sense:

A left module ${}_R W$ with a multiplication $*$ is called a left algebra, if $(W, *)$ is a ring, and for all $\lambda \in R$, and $w_1, w_2 \in W$ we have the equality

$$(\lambda w_1) * w_2 = \lambda(w_1 * w_2) .$$

A bimodule ${}_RW_R$ with multiplication $*$ is called a bialgebra, if $({}_RW, *)$ is a left algebra and $(W_R, *)$ is a right algebra.

In order to define the multiplication, we take the element $u \in W$ and proceed as follows: Any element of W has the form $u\varrho$ with $\varrho \in R$, because W is a minimal right ideal. Moreover, it is easy to see that the morphism from R to W mapping ϱ into $u\varrho$ defines an R-isomorphism of the simple right module $(R/W)_R$ onto W_R. Now $(R/W)_R$ is not only a right R-module, but in fact a right algebra with respect to the given multiplication. And, we define $*$ in such a way that the mapping $\varrho \mapsto u\varrho$ becomes a morphism of right algebras, i.e. we define

$$(u\varrho)*(u\sigma) = u\varrho\sigma \quad \text{for all} \quad \varrho, \sigma \text{ in } R.$$

One can see immediately that the operation $*$ is well-defined and that $(W_R, *)$ is a right algebra. But W is also a left R-module and, we can show that $({}_RW, *)$ is a left algebra. For, if $w_i = u\varrho_i$, $i = 1, 2$, are two elements of W, and $\lambda \in R$, then λu can be written in the form $\lambda u = u\varrho$ for some ϱ in R, and we have

$$(\lambda w_1) * w_2 = (\lambda u\varrho_1) * (u\varrho_2) = (u\varrho\varrho_1) * (u\varrho_2) = u\varrho\varrho_1\varrho_2$$
$$= \lambda u\varrho_1\varrho_2 = \lambda((u\varrho_1) * (u\varrho_2)) = \lambda(w_1 * w_2),$$

as required. This shows that ${}_RW_R$ is with respect to the operation $*$ a bialgebra. Let us also point out that the ring $(W, *)$ is isomorphic to $Q = R/W$ (and is therefore commutative) and that u is the identity element of $(W, *)$.

(*l*) Now, let us prove that the indecomposable left R-module $M = R/U$ with $U = Ru$ is injective. We need to show that every morphism $\varphi: {}_RW \to M$ can be extended to a morphism from ${}_RR$ to M.

We can assume that the kernel $\ker\varphi$ is of length 1. Thus, $\ker\varphi = Rw$ for some non-zero w of W. Since $wR = W$,

$$u = w\varrho_0 \quad \text{for some} \quad \varrho_0 \in R.$$

Moreover, ϱ_0 must obviously be a unit. Observe that the element $v_0 = v\varrho_0^{-1}$ does not belong to the kernel of φ. For, otherwise $v\varrho_0^{-1}$ would be in $Rw = Ru\varrho_0^{-1}$, i.e. v would belong to Ru. Write

$$(v\varrho_0^{-1})\varphi = \lambda v + U \in M, \qquad \lambda \in R.$$

Now, $\lambda u = u\sigma_0$ for some $\sigma_0 \in R$ and, furthermore, this implies that $\lambda v = v\sigma_0$. Indeed, refering back to the first part of the proof,

$$v\sigma_0 = (v * u)\sigma_0 = v * (u\sigma_0) = v * (\lambda u) = (\lambda u) * v$$
$$= \lambda(u * v) = \lambda v.$$

In fact, we claim that $\varrho_0\sigma_0$ induces the morphism φ. First, if $\varkappa w \in \ker \varphi$, then we have the relation

$$(\varkappa w)\,\varrho_0\sigma_0 = \varkappa u\sigma_0 = \varkappa\lambda u \in U .$$

Second, for $v_0 = v\varrho_0^{-1}$ we have the relation

$$v_0\varrho_0\sigma_0 = v\varrho_0^{-1}\varrho_0\sigma_0 = v\sigma_0 = \lambda v .$$

Thus, summarizing,

$$w\varrho_0\sigma_0 + U = U = w\varphi \quad \text{and} \quad v_0\varrho_0\sigma_0 + U = \lambda v + U = v_0\varphi ,$$

i.e. φ can be extended to a morphism from ${}_RR$ to M, as required.

(r) The proof that the right R-module $M = (R \oplus R)/D$ with $D = \{(u\varrho, -v\varrho) \mid \varrho \in R\}$ is an indecomposable injective will be given in several steps. Let us start with a remark that $v * v$ can be expressed as a linear combination of u and v and thus we have

$$u * u = u, \quad u * v = v * u = v, \quad v * v = \alpha u + \beta v \quad \text{for some} \quad \alpha, \beta \in R .$$

First, M has necessarily a simple socle. For, assume the converse, i.e. that the socle of M has length $\geqq 2$. Then, denoting by π the canonic epimorphism $R \oplus R \to (R \oplus R)/D$, R is obviously embedded by

$$R \xrightarrow{(0,1)} R \oplus R \xrightarrow{\pi} M$$

as a direct summand. Therefore, there is a morphism $\eta : M \to R$ such that

$$R \xrightarrow{(0,1)} R \oplus R \xrightarrow{\pi} M \xrightarrow{\eta} R \;=\; 1_R .$$

Now, $\pi\eta$ has the form (μ_1, μ_2), where $\mu_i : R_R \to R_R$ can be interpreted as a left multiplication by $\mu_i \in R$. Under the morphism (μ_1, μ_2), D is mapped into 0 and thus

$$\mu_1 u - \mu_2 v = (\mu_1, \mu_2)\begin{pmatrix} u \\ -v \end{pmatrix} = 0 .$$

But, obviously $\mu_2 v = v$ and hence $\mu_1 u = v$ implying that $v \in Ru$. This contradiction shows that the socle of M must be simple. As an immediate consequence, M is indecomposable.

Second, we are going to show that every socle element of M has the form $(\lambda u + \varkappa v, 0) + D$ for some $\lambda, \varkappa \in R$. On the basis of the preceeding paragraph, we know that every element of the socle of M has the form

$$(w_1, w_2) + D \quad \text{with} \quad w_i \in W \qquad (i = 1, 2) .$$

Moreover, since $W = vR$, $w_2 = v\varrho_2$ for some $\varrho_2 \in R$ and thus

$$(w_1, w_2) + D = (w_1 + u\varrho_2, w_2 - v\varrho_2) + D = (w_1 + u\varrho_2, 0) + D .$$

Obviously, $w_1 + u\varrho_2$ belongs to W and has therefore the required form.

Third, we want to show that, for every $\varkappa \in R$,

$$(\varkappa\beta u - \varkappa v, \varkappa\alpha u) \in D.$$

Again, we shall make use of the operation $*$ and its commutativity. Take $\varrho \in R$ such that $u\varrho = \varkappa\beta u - \varkappa v$.

Then,

$$\begin{aligned} v\varrho &= (v*u)\,\varrho = v*(u\varrho) = v*(\varkappa\beta u - \varkappa v) \\ &= v*(\varkappa\beta u) - v*(\varkappa v) = (\varkappa\beta u)*v - (\varkappa v)*v \\ &= \varkappa\beta(u*v) - \varkappa(v*v) = \varkappa\beta v - \varkappa(\alpha u + \beta v) = -\varkappa\alpha u. \end{aligned}$$

Therefore, the element $(u\varrho, -v\varrho) = (\varkappa\beta u - \varkappa v, \varkappa\alpha u) \in D$.

Finally, we are ready to prove that M is injective. Again, it is sufficient to verify that every morphism $\varphi: W_R \to M$ can be extended to a morphism of R_R into M. Since φu is a socle element of M,

$$\varphi u = (\lambda u + \varkappa v, 0) + D \quad \text{for some} \quad \lambda, \varkappa \in R.$$

Consider the morphism

$$(\lambda + \varkappa\beta, \varkappa\alpha): R \to R \oplus R,$$

where the ring elements operate on R by left multiplication. Obviously

$$(\lambda + \varkappa\beta, \varkappa\alpha)\,u = (\lambda u + \varkappa\beta u, \varkappa\alpha u)$$

and thus the morphism

$$R \xrightarrow{(\lambda+\varkappa\beta,\varkappa\alpha)} R \oplus R \xrightarrow{\pi} M$$

maps the element u into

$$\begin{aligned} (\lambda u + \varkappa\beta u, \varkappa\alpha u) + D &= (\lambda u + \varkappa\beta u, \varkappa\alpha u) - (\varkappa\beta u - \varkappa v, \varkappa\alpha u) + D \\ &= (\lambda u + \varkappa v, 0) + D = \varphi u. \end{aligned}$$

This completes the proof of Proposition 2.

3.

Again, throughout this and the following sections, R denotes a ring of type (2, 1). Now, knowing the indecomposable injective R-modules, it is not difficult to derive that every R-module can be decomposed into a direct sum of indecomposable R-modules. In this section, this result will be proved for left R-modules.

Lemma 1. *Let F be a free left R-module. Let $s \neq 0$ be an element of the socle of F. Then s belongs to a monogenic submodule which is isomorphic to ${}_RR$.*

Proof. The elements of F can be represented by indexed families (r_i) with $r_i \in R$ and the restriction that all but a finite number of the r_i's to be zero. An element (r_i) belongs to the socle $\operatorname{Soc} F$ of F if and only if $r_i \in W$ for all i. Let

$$s = (w_i) \in \operatorname{Soc} F .$$

Let $u \neq 0$ be a fixed element of W. Since $uR = W$, there exists $\varrho_i \in R$ such that $w_i = u\varrho_i$; here, we take $\varrho_i = 0$ if $w_i = 0$. Now, right multiplication by ϱ_i yields a homomorphism $\varrho_i : {}_R R \to {}_R R$, and thus the family (ϱ_i) defines a homomorphism

$$\varphi : {}_R R \to F .$$

Clearly, $u\varphi = s$, and hence $s \in \operatorname{Im} \varphi$. Furthermore, since $s \neq 0$, there is a unit ϱ_{i_0} such that $w_{i_0} = u\varrho_{i_0}$; as a consequence, $\operatorname{Im} \varphi \cong {}_R R$.

Let us introduce a notation for the different types of monogenic left R-modules. Let us point out that, for a given length, all monogenic left R-modules are isomorphic. The only non-trivial case is that of length 2; here, the isomorphism follows from the fact that a monogenic module of length 2 is injective. Denote by A_i the isomorphism type of the monogenic R-module of length i $(i = 1, 2, 3)$; hence, there is the simple module $A_1 = {}_R(R/W)$, the injective module $A_2 = {}_R(R/Ru)$ of Lemma 1 and the ring itself considered as a left module $A_3 = {}_R R$.

Lemma 2. *Let M be a left R-module with submodules X and Y of type A_3 such that*

$$X + Y = M \quad \textit{and} \quad X \cap Y \quad \textit{is a simple submodule}.$$

Then M contains a submodule of type A_2.

Proof. M is obviously isomorphic to the pushout P of the following diagram

$$\begin{array}{ccc} {}_R L & \xrightarrow{\eta} & {}_R R \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota'} \\ {}_R R & \longrightarrow & P \end{array},$$

where L is a minimal left ideal of R, ι the inclusion mapping and η a monomorphism. If $x \neq 0$ is an element of L, then

$$x\eta = x\varrho \quad \text{for some} \quad \varrho \in R ,$$

because $xR = W$. Thus right multiplication by ϱ is a mapping from R into R satisfying $\iota\varrho = \eta$. But this implies, in view of the properties of a pushout, that ι' splits and that the complement is just the cokernel R/L of i. Since R/L is of type A_2, the lemma follows.

Now, we are ready to prove

Proposition 3. *Let R be a ring of type (2, 1). Then A_1, A_2 and A_3 are the only (isomorphism) types of indecomposable left R-modules and every left R-module is a direct sum of indecomposables.*

Proof. To prove our proposition, we shall show that every left R-module can be expressed as a direct sum of modules of types A_1, A_2 or A_3.

Let M be a left R-module. Take a submodule X of M which is maximal with respect to the property of being a direct sum of modules of type A_2. Since X is injective, $M = X \oplus M'$, where M' is a submodule of M which contains no submodules of type A_2.

Now, let Y be a submodule of M' which is maximal with respect to the property of being a direct sum of modules of type A_3. Let Z be a complement of the socle $\operatorname{Soc} Y$ of Y in $\operatorname{Soc} M'$. Then, Z is a direct sum of modules of type A_1 and, evidently, $Y \cap Z = 0$. We want to show that

$$Y \oplus Z = M' .$$

To this end, assume that there is an element $m \in M' \setminus (Y \oplus Z)$. Then Rm must be of type A_3, because $m \notin \operatorname{Soc} M'$ and M' contains no submodule of type A_2. The submodule $Y \cap Rm$ is non-zero; for, otherwise $Y + Rm$ would be a direct sum of modules of type A_3, contradicting the maximality of Y. Take $s \neq 0$ of $Y \cap Rm$. Since $s \in \operatorname{Soc} Y$, Lemma 1 implies that there is a submodule $N \subseteqq Y$ of type A_3 with $s \in N$. In view of Lemma 2, $N \cap Rm$ cannot be simple and therefore the length of $N \cap Rm$ is 2.

If we now assume that $\operatorname{Soc}(N + Rm)$ is of length 2, then the embedding $\operatorname{Soc}(N + Rm)$ in the injective module $A_2 \oplus A_2$ yields an isomorphism $N + Rm \cong A_2 \oplus A_2$ (because both modules are of length 4). However, since M' has no submodules of type A_2, this is impossible. Thus, $\operatorname{Soc}(N + Rm)$ has to be of length 3, and therefore

$$N + Rm = N + \operatorname{Soc}(N + Rm) .$$

But this means that

$$Rm \subseteqq Y + \operatorname{Soc} M' \subseteqq Y \oplus Z ,$$

and we get a contradiction to our hypothesis. The proof is completed.

4.

In this section, we are going to prove a decomposition theorem for right R-modules analogous to that for left R-modules derived in the preceeding Section 3. Let us denote by B_1, B_2 and B_3 the isomorphism types of indecomposable right R-modules defined as follows: B_1 is the simple module $(R/W)_R$; B_2 is the ring considered as a right module; B_3 is the injective module $(R \oplus R)/D$ described in Proposition 2. Here again, the index refers to the length of the respective module. Note however that, contrary to the previous situation, B_3 is not a monogenic module.

First, let us prove by induction the following

Lemma 3. (a) *Let M be an R-module of length $2n+1$ generated by $n+1$ monogenic submodules. Let N be a submodule of M which is a direct sum of n copies of B_2. If, furthermore, M does not contain a submodule of type B_3, then*

$$M = N + \operatorname{Soc} M .$$

(b) *The only indecomposable R-modules of length $\leqq 2n+1$ are modules of type B_1, B_2 and B_3.*

Proof. If the length of M is 3, and if M contains a monogenic submodule N of length 2, then either $\operatorname{Soc} M$ is simple – in which case the injectivity of B_3 yields an isomorphism from M onto B_3, or $\operatorname{Soc} M$ is of length $\geqq 2$; in the latter case, evidently

$$M = N + \operatorname{Soc} M .$$

This establishes the validity of both (a) and (b) for $n = 1$.

Now, assume that both assertions hold for all $m \leqq n-1$.

(a) Without loss of generality, we may assume that the $n+1$ monogenic submodules which generate M are all of length 2. We can consider M as the amalgamation of N with a monogenic module of length 2 with simple submodules identified. Thus, M is isomorphic to the pushout P of the following diagram

$$\begin{array}{ccc} W_R & \xrightarrow{\;\eta\;} & R_R \oplus R_R \oplus \cdots \oplus R_R \\ \big\downarrow{\scriptstyle \iota} & & \big\downarrow{\scriptstyle \iota'} \\ R_R & \xrightarrow{\;\eta'\;} & P \end{array} ,$$

where ι is the inclusion of W in R, η is a monomorphism and ι' corresponds to the inclusion $N \subsetneqq M$. Let us take a non-zero element $w \in W$; hence, ηw is of the form $(x_1, x_2, \ldots, x_n)$ with at least one non-zero x_i. Assume that $x_1 \neq 0$ and distinguish three cases:

(i) Let $x_i \in Rw$ for all $1 \leqq i \leqq n$. Then, we can find elements σ_i such that $x_i = \sigma_i w$ and thus the morphism

$$(\sigma_1, \sigma_2, \ldots, \sigma_n) : R_R \to R_R \oplus R_R \oplus \cdots \oplus R_R$$

representing left multiplication maps w into $(x_1, x_2, \ldots, x_n) = \eta w$. But this means that $R_R \oplus R_R \oplus \cdots \oplus R_R$ is a direct summand of P. Consequently, the complement is simple and therefore $M = N + \mathrm{Soc} M$.

(ii) Let $x_1 \notin Rw$ and $x_i \in Rx_1$ for all $1 \leqq i \leqq n$. Then, we can find elements σ_i with $x_i = \sigma_i x_1$; observe that σ_1 is a unit. Now, both $\eta'(1)$ and $\iota'(\sigma_1, \sigma_2, \ldots, \sigma_n)$ generate submodules of length 2 and the equality

$$\begin{aligned} \eta'(1)\, w = \eta' w = \iota' \eta w = \iota'(x_1, x_2, \ldots, x_n) \\ = \iota'(\sigma_1 x_1, \sigma_2 x_1, \ldots, \sigma_n x_1) = \iota'(\sigma_1, \sigma_2, \ldots, \sigma_n)\, x_1 \end{aligned}$$

shows that

$$\eta' w \in \eta'(1)\, R \cap \iota'(\sigma_1, \sigma_2, \ldots, \sigma_n)\, R .$$

Let $X = \eta'(1)\, R + \iota'(\sigma_1, \sigma_2, \ldots, \sigma_n)\, R$. Assuming that $\iota'(\sigma_1, \sigma_2, \ldots, \sigma_n)\, R$ is a direct summand of X, we would have a morphism $\eta'(1)\, R \to \iota'(\sigma_1, \sigma_2, \ldots, \sigma_n)\, R$ mapping $\eta' w$ into $\iota'(\sigma_1 x_1, \sigma_2 x_1, \ldots, \sigma_n x_1)$, and thus a morphism $R_R \to R_R \oplus R_R \oplus \cdots \oplus R_R$ mapping w into $(\sigma_1 x_1, \sigma_2 x_1, \ldots, \sigma_n x_1)$. In particular, we would have a morphism $R_R \to R_R$ mapping w into $\sigma_1 x_1 = x_1$ and since such a morphism must be induced by left multiplication we would get that $x_1 \in Rw$, contradicting our hypothesis. Thus, X has to be an indecomposable R-module of length 3 and therefore of type B_3. Since M has no submodule of type B_3, we conclude that the case (ii) cannot happen.

(iii) Let $x_1 \notin Rw$ and there is x_i such that $x_i \notin Rx_1$. We may assume that $x_2 \notin Rx_1$. Thus, $W = Rx_1 + Rx_2$ and therefore there are elements σ_1, σ_2 such that

$$w = \sigma_1 x_1 + \sigma_2 x_2 .$$

In this case, the pushout P can be considered as the quotient module of $n+1$ copies of R_R by the submodule generated by $(w, -x_1, -x_2, \ldots, -x_n)$. Under the morphism

$$(1, \sigma_1, \sigma_2, 0, \ldots, 0): R_R \oplus R_R \oplus \cdots \oplus R_R \to R_R$$

representing left multiplication, the element $(w, -x_1, -x_2, \ldots, -x_n)$ is mapped into $w - \sigma_1 x_1 - \sigma_2 x_2 = 0$ and thus the morphism factors through P. As a consequence P has a homomorphic image of type B_2. The latter splits off and we deduce that M is a direct sum of a module of type B_2 and a module M' of length $2n-1$.

Now, using the induction argument, M' is a direct sum of modules of types B_1, B_2 and B_3. However, since M has no submodules of type B_3, M' is a direct sum of monogenic modules of length 1 and 2. In particular, $\mathrm{Soc} M'$ has to be of length at least n and therefore $\mathrm{Soc} M$ has to be of length at least $n+1$. Consequently, $M = N + \mathrm{Soc} M$, as required.

The statement (a) is established.

(b) Given an indecomposable R-module M of length $\leqq 2n+1$, we deduce immediately that M has no proper submodule of type B_3; this follows from the fact that B_3 is injective. Now, take a submodule N which is maximal with respect to the property of being a direct sum of copies of B_2, and let K be a complement of $\mathrm{Soc}\,N$ in $\mathrm{Soc}\,M$. In order to verify (b), it is sufficient to show that $M = N \oplus K$, i.e. to show that every element $x \in M$ generating a submodule of length 2 belongs to $N \oplus K$. Let $M' = N + xR$. If $x \notin N$, then the length of M' is $2m+1$, where m is the number of the copies of B_2 in N. Since $m \leqq n$, we get by induction

$$M' = N + \mathrm{Soc}\,M'.$$

But this means that $x \in N + K$.

The proof of Lemma 3 is completed.

As an easy consequence of Lemma 3, we can formulate the following result parallel to Proposition 3. We may remark that it shows in conjunction with Proposition 3 that rings of type (2, 1) are rings of SLCRT, but not of SRCRT in the sense of Tachikawa [7].

Proposition 4. *Let R be a ring of type* (2, 1). *Then B_1, B_2 and B_3 are the only (isomorphism) types of indecomposable right R-modules and every right R-module is a direct sum of indecomposables.*

Proof. It is sufficient to show that every right module M can be written as a direct sum of modules of types B_1, B_2 and B_3.

Following the method of proving Proposition 3, we denote by X a submodule of M which is maximal with respect to the property of being a direct sum of modules of type B_3 and observe that $M = X \oplus M'$. In M', take a submodule Y which is a maximal direct sum of modules of type B_2, and denote by Z a complement of $\mathrm{Soc}\,Y$ in $\mathrm{Soc}\,M'$. We intend to show that

$$M = X \oplus Y \oplus Z.$$

Assume the contrary, i.e. that there is an element $m \in M' \setminus (Y \oplus Z)$ which generates a submodule of length 2. Clearly, because of maximality of Y, $Y \cap mR \neq 0$. Thus, there is a direct sum Y' of a finite number of copies of B_2 contained in Y such that

$$Y' \cap mR \neq 0.$$

Now, applying Lemma 3(a) to the module $Y' + mR$ and the submodule Y' we get readily that

$$Y' + mR = Y' + \mathrm{Soc}(Y' + mR).$$

Consequently, $m \in Y' + \mathrm{Soc}(Y' + mR) \subsetneqq Y + \mathrm{Soc}\,M' = Y \oplus Z$, a contradiction. Proposition 4 follows.

5.

Finally, making use of Proposition 1, 3 and 4, we can readily present

Proof of Theorem. First, it is easy to verify that all indecomposable R-modules are balanced. This is trivial for A_3 and B_2, as well as for the simple modules A_1 and B_1; and, it follows for A_2 and B_3, because they are injective modules over a local artinian ring, from a theorem of Fuller [2] or Tachikawa [8]. In view of Propositions 3 and 4, the fact that every R-module is balanced then follows immediately from Proposition 1 (taking for M_0 a direct summand of a maximal length).

Let us conclude this section with a few remarks. The first one concerns the existence of rings of type (2, 1) (cf. Rosenberg and Zelinsky [6]).

Remark 1. Let $F(t)$ be the field of all rational functions over a field F. Denote by R_2 the ring of all pairs $(f(t), g(t))$, where $f(t), g(t) \in F(t)$, with respect to the component-wise addition and the following multiplication

$$(f_1(t), g_1(t)) \cdot (f_2(t), g_2(t)) = (f_1(t)\, f_2(t), f_1(t^2)\, g_2(t) + g_1(t)\, f_2(t)) .$$

Then R_2 is a (2, 1)-ring, its radical $W_2 = \{(0, g(t)) | g(t) \in F(t)\}$, and thus $R_2/W_2 \cong F(t)$.

It may be also appropriate to show that a local artinian ring R with the radical W does not need to be balanced if R/W^2 is balanced.

Remark 2. Let $F(t)$ be the field of all rational functions over a field F. Denote by R_3 the ring of all triples $(f(t), g(t), h(t))$, where $f(t), g(t), h(t) \in F(t)$, with respect to the component-wise addition and the following multiplication

$$\begin{aligned}&(f_1(t), g_1(t), h_1(t)) \cdot (f_2(t), g_2(t), h_2(t))\\ &= (f_1(t) f_2(t), f_1(t^2) g_2(t) + g_1(t) f_2(t), f_1(t^4) h_2(t) + g_1(t^2) g_2(t) + h_1(t) f_2(t)).\end{aligned}$$

Then the radical

$$W_3 = \{(0, g(t), h(t)) | g(t), h(t) \in F(t)\} ,$$

$$W_3^2 = \{(0, 0, h(t)) | h(t) \in F(t)\}$$

and thus R_3/W_3^2 is a (2, 1)-ring of the type described in Remark 1. However, as one can easily see, the dimension of the left vector space W_3^2 over R_3/W_3 equals 4, and therefore R_3 is not left balanced (cf. [1]).

Added in Proof (January, 1972). A full characterization of balanced rings will appear in Lecture Notes in Mathematics (The contributions to the ring and operator year at Tulane University), Springer-Verlag. In particular, the rings described in Theorem belong to the class of exceptional rings.

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(Received April 19, 1971)