BALANCED RINGS

by

Vlastimil Dlab and Claus Michael Ringel

Department of Mathematics

Carleton University

Ottawa, Canada

The aim of these notes is to report on recent investigations in the structure of balanced rings. Here, a ring R is called left balanced, if every left R-module is balanced, i.e. has the double centralizer property *). The study of balanced modules can be traced back to C. J. NESBITT and R. M. THRALL: in [23] they showed that uniserial rings in the sense of T. NAKAYAMA [22] are left and right balanced. Later, R. M. THRALL [27] introduced the class of QF-1 rings generalizing quasi-Frobenius rings as those rings R over which all finitely generated faithful R-modules are balanced. Some progress in the study of balanced and QF-1 rings was made in the papers [20], [21] and [24] of K. MORITA and H. TACHIKAWA. In particular, they established in [21] that the property to be balanced is Morita equivalent.

A further progress in the theory of balanced rings has been done quite recently. Considering the question whether every balanced ring is uniserial,

---

*) The term "balanced ring" has been introduced by V. P. CAMILLO in [3].

D. R. FLOYD [14] proved this statement for finite dimensional commutative algebras. Later, S. E. DICKSON and K. R. FULLER [8] extended the result to artinian commutative rings and J. P. JANS [18] proved it for finite dimensional algebras over algebraically closed fields *) . In [18], J. P. JANS formally conjectured that the result holds for every artinian ring. In his paper [3], V. P. CAMILLO showed that under certain conditions (if R is commutative or left noetherian) a left balanced ring is left artinian, and that, in general, a left balanced ring is always semiperfect and its radical is nil. K. R. FULLER proved in [16] that a semiprimary ring is left balanced if and only if it is a finite direct sum of full matrix rings over local left balanced rings.

These results were extended in [9]. In addition to the fact that every left balanced ring is left artinian, some necessary conditions on the structure of local left balanced rings were also established. Making use of these structural conditions, every left balanced ring which is finitely generated over its centre was shown to be uniserial. Independently, V. P. CAMILLO and K. R. FULLER [4] obtained this result for algebras. In [10], on the other hand, JANS' conjecture was shown to be false: Local rings R with the radical W such that $Q = R/W$ is commutative, $W^2 = 0$ and $\dim_Q W \times \dim W_Q = 2$ are balanced **), and essentially these are the only non-uniserial balanced rings with a commutative radical quotient [11]. This is a special case of the complete characterization of left balanced rings in terms of exceptional rings given in [12]. One of the immediate

---

*) The crucial result that a local algebra over an algebraically closed field with zero radical square is uniserial was previously proved by D. R. FLOYD [14].

**) In a recent letter, Professor H. Tachikawa informs us that he has known of counterexamples to the conjecture, as well.

consequences is the fact that a ring is left balanced if and only if it is right balanced. Moreover, a ring is balanced if and only if it is left artinian and every finitely generated module is balanced. Thus, an artinian ring R is balanced if and only if every factor ring of R is a QF-1 ring. The paper [12] and [13] include also a characterization of balanced rings in terms of their module categories.

These notes are virtually self-contained. Besides the basic properties of Morita equivalent rings, only a theorem of N. JACOBSON concerning division rings which are finitely generated over their centres will be used at one instance (Theorem III.4.4). Section I.3 provides, in fact, a proof that quasi-Frobenius rings are QF-1 rings. However, since there was no need to introduce these classes of rings (see e.g. [7]), we restrict ourselves to the class of uniserial rings. But we note that certain results of Chapters II and III can be generalized to QF-1 rings (cf. [4] and [9]). Let us also note that the duality theory of H. TACHIKAWA [25] (closely related with Lemma I.2.5) could be used to prove Proposition III.6.1. This has been done in [13]; here, we prefer to give an elementary proof.

Some of the proofs given in the present notes have not appeared in the literature. Let us mention that our proof of Morita equivalence of the property to be balanced, which follows the idea of the original proof in [21] yields, in fact, a more general statement (Proposition III.1.3). Also, the concept of "bi-T-nilpotency" of V. P. CAMILLO [3] used previously in [9] to prove the fact that a balanced ring is artinian has been avoided; as a result, the proof has become simpler.

I. PRELIMINARIES

I.1. NOTATION AND TERMINOLOGY

Throughout these notes, R denotes an (associative) ring with unity, R^* its opposite. By an R-module we always understand a unital R-module; the symbols ${}_RM$ or M_R will be used to underline the fact that M is a left or a right R-module, respectively. We usually consider left R-modules, and, in this case, speak simply about an R-module. It should be noted that homomorphisms always act from the opposite side as the operators; in particular, every (left) R-module M defines a right C-module, where C is the endomorphism ring of the R-module M. The ring C or, more precisely $C(M)$ is called the *centralizer* of M. The *double centralizer* $D(M)$ (or simply D) is the endomorphism ring of M_C. Again, D operates from the opposite side as C, that is from the left. There is a canonic ring homomorphism from R into D; if this homomorphism is surjective, then M is called *balanced*. The ring R is called *left finitely balanced*, if every finitely generated left R-module is balanced. If every left R-module is balanced, then R is called *left balanced*.

For an element m of an R-module M, the *annihilator* $\{r \in R \mid rm = 0\}$ of m will be denoted by $\mathrm{Ann}(m)$. If the R-module M has a composition series, denote by ∂M its *length*. The *radical* $\mathrm{Rad}\, M$ of M is the intersection of all maximal submodules of M; it is the set of all non-generators of M. Here, an

element $m \in M$ is called a non-generator, if it can be ommitted from any generating set of M. The radical of the ring R is by definition $\mathrm{Rad}_R R$; it will be denoted consistently by W. For the R-module M, we always have the inclusion $WM \subseteq \mathrm{Rad}\, M$; moreover, if R/W is artinian, then $WM = \mathrm{Rad}\, M$. If $\mathrm{Rad}\, M$ is the only (proper) maximal submodule of M, then M will be called *local*. Thus, all local modules are monogenic. And, if ${}_R R$ (and, for the matter R_R) is local, then R is said to be a local ring. Note that the ring R is local if and only if the non-units of R form an ideal. If the R-module M has minimal submodules, the *socle* $\mathrm{Soc}\, M$ is defined as their union. We always have $\mathrm{Soc}\, M \subseteq \{m \in M \mid Wm = 0\}$; moreover, if R/W is artinian, then $\mathrm{Soc}\, M = \{m \in M \mid Wm = 0\}$. Considering R as a left R-module, we get the concept of the *left socle* $\mathrm{Soc}_R R$ of R. A module is said to be *uniserial*, if all its submodules form a chain with respect to inclusion. Hence, a uniserial module of finite length is local. A local ring R is called uniserial, if both ${}_R R$ and R_R are uniserial modules of finite length. An arbitrary ring is called uniserial [22], if it is a finite direct sum of full matrix rings over local uniserial rings. It is not difficult to see that a left artinian ring R is uniserial if and only if R/W^2 is uniserial.

The R-module M is called *indecomposable*, if M cannot be written as the direct sum of two proper submodules. If M is indecomposable and of finite length, then the centralizer C is a local ring. Moreover, there exists a composition series

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that $M_i \mathcal{W} \subseteq M_{i-1}$ for all i, where $\mathcal{W}$ is the radical of C (see [2], Ex. 3, pp. 26-27). Thus, $\mathcal{W}^n = 0$ and, furthermore,

$$M_1 \subseteq \mathrm{Soc}\, M_C \quad \text{and} \quad \mathrm{Rad}\, M_C \subseteq M_{n-1}.$$

Let M_1 and M_2 be two R-modules. If $\varphi : M_1 \to M_2$ is a homomorphism, we denote by $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$ the kernel and the image of φ, respectively. The module M_1 is called a *generator for* M_2, if the images of all homomorphisms $M_1 \to M_2$ generate M_2. Dually, M_1 is called a *cogenerator for* M_2, if the intersection of kernels of all homomorphisms $M_2 \to M_1$ is zero. Finally, M is a generator or a cogenerator for a class of modules, if it is a generator or a cogenerator for every module of this class.

The category of all (left) R-modules will be denoted by $\operatorname{Mod} R$. Two rings R and R' are called Morita equivalent, if $\operatorname{Mod} R$ and $\operatorname{Mod} R'$ are equivalent categories. The rings R and R' are Morita equivalent if and only if there exists a right R-module P_R which is finitely generated, projective and a generator for all right R-modules such that R' is isomorphic to the centralizer of P_R ([19] or [1]). In particular, a ring R is Morita equivalent to every full matrix ring over R.

I.2. GENERATORS AND COGENERATORS

Results of this section belong in part to the folklore of the subject. They are presented here in such a way as to be easily applicable in the next chapters.

LEMMA I.2.1. *Let* M_1 *and* M_2 *be* R*-modules such that* M_1 *is either a generator or a cogenerator for* M_2. *If* M_1 *is balanced, then* $M_1 \oplus M_2$ *is balanced, as well.*

Proof. The elements of the centralizer of $M_1 \oplus M_2$ can be written as matrices $\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$, where $\varphi_{ij} : M_i \to M_j$ with $1 \leq i, j \leq 2$ are R-homomorphisms. Let ψ be in the double centralizer of $M_1 \oplus M_2$. If $x \in M_1$ and $\psi(x, 0) = (x', y')$, then

$$\psi(x, 0) = \psi[(x, 0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] = [\psi(x, y)]\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (x', y')\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (x', 0),$$

and thus $y' = 0$.

Similarly, if $y \in M_2$ and $\psi(0, y) = (x'', y'')$, then $x'' = 0$. It is easy to see that $\psi_1 : M_1 \to M_1$ defined by $(\psi_1 x, 0) = \psi(x, 0)$ belongs to the double centralizer of M_1. Therefore, there is $\rho \in R$ with

$$\psi(x, 0) = \rho(x, 0) \quad \text{for all} \quad x \in M_1.$$

Assume now that M_1 generates M_2. Then, taking an element of the form $(0, x\varphi_{12})$, where $x \in M_1$ and $\varphi_{12} \in \mathrm{Hom}(M_1, M_2)$, we get the equality

$$\psi(0, x\varphi_{12}) = \psi[(x, 0)\begin{pmatrix} 0 & \varphi_{12} \\ 0 & 0 \end{pmatrix}] = [\psi(x, 0)]\begin{pmatrix} 0 & \varphi_{12} \\ 0 & 0 \end{pmatrix} = \rho(0, x\varphi_{12}).$$

Since the elements of the form $(x, 0)$ and of the form $(0, x\varphi_{12})$ with $x \in M_1$ and $\varphi_{12} \in \mathrm{Hom}(M_1, M_2)$ generate $M_1 \oplus M_2$ additively, it follows that ψ is induced by multiplication by ρ.

If we assume that M_1 cogenerates M_2 consider $\varphi_{21} \in \mathrm{Hom}(M_2, M_1)$ and apply $\begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}$ to $\psi(0, y) - \rho(0, y)$ with $y \in M_2$; we get

$$[\psi(0, y)-\rho(0, y)]\begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix} = \psi[(0, y)\begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}]-\rho(0, y)\begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix} =$$

$$\psi(y\varphi_{21}, 0)-\rho(y\varphi_{21}, 0) = 0.$$

Therefore, $\psi(0, y) = \rho(0, y)$. Again, since the elements $(x, 0)$ and $(0, y)$ with $x \in M_1$ and $y \in M_2$ generate $M_1 \oplus M_2$ additively, we conclude that ψ is induced by multiplication by ρ . The proof is completed.

LEMMA I.2.2. *Let* M_1 *and* M_2 *be* R*-modules such that* M_1 *is both a generator and a cogenerator for* M_2 . *If* $M_1 \oplus M_2$ *is balanced, then* M_1 *is balanced, as well.*

Proof. Let ψ_1 be an element of the double centralizer of M_1 . We are going to construct ψ_2 in the double centralizer of M_2 , such that $\psi_1 \oplus \psi_2 = \begin{matrix} \psi_1 & 0 \\ 0 & \psi_2 \end{matrix}$ belongs to the double centralizer of $M_1 \oplus M_2$.

Since M_1 is a generator for M_2 , every element of M_2 has the form $\Sigma x_i \gamma_i$ with $x_i \in M_1$ and $\gamma_i \in \mathrm{Hom}(M_1, M_2)$. Define ψ_2 as follows

$$\psi_2(\Sigma x_i \gamma_i) = \Sigma(\psi_1 x_i)\gamma_i \ .$$

Here, ψ_2 is well defined. For, if $\Sigma x_i \gamma_i = 0$ and $\varphi_{21} : M_2 \to M_1$, then the fact that $\gamma_i \varphi_{21} \in C(M_1)$ implies

$$[\psi_2(\Sigma x_i \gamma_i)]\varphi_{21} = \Sigma(\psi_1 x_i)(\gamma_i \varphi_{21}) = \Sigma \psi_1 (x_i \gamma_i \varphi_{21}) = 0 \ .$$

And, since M_1 cogenerates M_2 , it follows that $\psi_2(\Sigma x_i \gamma_i) = 0$.

Furthermore, ψ_2 belongs to the double centralizer of M_2 . This is an immediate consequence of the relation

$$[\psi_2(\Sigma x_i \gamma_i)]\varphi_{22} = \Sigma(\psi_2 x_i)\gamma_i \varphi_{22} = \Sigma \psi_2(x_i \gamma_i \varphi_{22}) = \psi_2[\Sigma x_i \gamma_i)\varphi_{22}] \ ,$$

where $\varphi_{22} \in \mathrm{Hom}(M_2, M_2)$. Also, for $x \in M_1$ and for $\Sigma x_i \gamma_i \in M_2$ with $x_i \in M_1$ and $\gamma_i \in \mathrm{Hom}(M_1, M_2)$, we have

$$(\psi_1 x)\varphi_{12} = \psi_2(x\varphi_{12}) \quad \text{and} \quad (\psi_2 \Sigma x_i y_i)\varphi_{21} = \psi_1(\Sigma x_i y_i \varphi_{21}) \ ,$$

with arbitrary $\varphi_{12} \in \mathrm{Hom}(M_1, M_2)$ and $\varphi_{21} \in \mathrm{Hom}(M_2, M_1)$. This shows that $\psi_1 \oplus \psi_2$ belongs to the double centralizer of $M_1 \oplus M_2$. Since $M_1 \oplus M_2$ is balanced, $\psi_1 \oplus \psi_2$ is induced by multiplication by an element $\rho \in R$. Therefore, ψ_1 is induced by multiplication by ρ, and M_1 is balanced, as required.

LEMMA I.2.3. *Every generator of* ${}_R M$ *is balanced.*

Proof. If M is a generator of ${}_R M$, then ${}_R R$ is an epimorphic image of the direct sum $M^{(n)}$ of n copies of M. Since ${}_R R$ is projective, we can find a complement K such that ${}_R R \oplus K \cong M^{(n)}$. Applying Lemma I.2.1, we get that $M^{(n)}$ is balanced; for, ${}_R R$ is balanced and ${}_R R$ is a generator for K. Now, $M^{(n)}$ is a direct sum of copies of M, and we can therefore apply Lemma I.2.2 to conclude that M is balanced, as well.

Under certain assumptions, a similar statement can be proved for cogenerators. In what follows, we shall need the following result:

LEMMA I.2.4. *Every cogenerator of* ${}_R M$ *which is finitely generated over its centralizer is balanced.*

Proof. Let M be a cogenerator of ${}_R M$. Then, for any element ψ of the double centralizer $\mathcal{D}$ of M and any $x \in M$, we have $\psi x \in Rx$. For, assuming $\psi x \notin Rx$, we can find an R-homomorphism $\varphi : M/Rx \to M$ with $(\psi x + Rx)\varphi \neq 0$. And, denoting by ϵ the canonic epimorphism $\epsilon : M \to M/Rx$, and observing that $\epsilon\varphi$ belongs to the centralizer C of M, we get

$$0 = \psi[x(c\varphi)] = (\psi x)(c\varphi) \neq 0 ,$$

a contradiction.

Now let $x_1, \ldots, x_n$ be elements of M which generate M as a C-Module. If we form the direct sum $M^{(n)}$ of n copies of M, then it is easy to see that every element $\psi \in D$ defines an element $\psi^{(n)}$ of the double centralizer of $M^{(n)}$

$$\psi^{(n)}(m_1, \ldots, m_n) = (\psi m_1, \ldots, \psi m_n) \quad \text{for} \quad m_i \in M .$$

But $M^{(n)}$ is again a cogenerator of ${}_RM$, and we can apply the first result of our proof for $M^{(n)}$ and the element $(x_1, \ldots, x_n) \in M^{(n)}$. In this way, we get

$$(\psi x_1, \ldots, \psi x_n) = \psi^{(n)}(x_1, \ldots, x_n) = \rho(x_1, \ldots, x_n) = (\rho x_1, \ldots, \rho x_n) ,$$

for some $\rho \in R$. But, since the element x_i generate the C-module M_C, the relations $\psi x_i = \rho x_i$ yield that ψ is induced by multiplication.

The above result can be applied, in particular, for an injective cogenerator of ${}_RM$, when the ring R is left artinian.

LEMMA I.2.5. *Let R be a left artinian ring. Then any injective cogenerator of ${}_RM$ is balanced.*

Proof. Let M be an injective cogenerator of ${}_RM$ and C its centralizer. For any left ideal L of R, denote by $r(L)$ the following C-submodule of M

$$r(L) = \{m \in M \mid Lm = 0\} .$$

If $L' \subseteq L$, then we have a C-homomorphism

$$\alpha : r(L') \to \mathrm{Hom}_R(L/L', M_C) ,$$

where, for $m \in r(L')$, the R-homomorphism $m\alpha$ maps $\lambda + L' \in L/L'$ into λm. The kernel of this morphism is just $r(L)$, and therefore $r(L')/r(L)$ can be considered as a C-submodule of $\mathrm{Hom}_R(L/L', M_C)$.

If S is a simple R-module, then $\mathrm{Hom}_R(S, M_C)$ is a simple C-module. For, $\mathrm{Hom}_R(S, M_C) \neq 0$ in view of the fact that M is a cogenerator and, given any two elements γ and γ' in $\mathrm{Hom}_R(S, M_C)$ with $\gamma \neq 0$, we see easily that γ is a monomorphism and that there is an R-endomorphism φ of M with $\gamma\varphi = \gamma'$; the latter conclusion follows from the injectivity of M.

Since ${}_RR$ is artinian, we have a composition series

$$0 = L_o \subset L_1 \subset \ldots \subset L_n = {}_RR$$

of left ideals L_i of ${}_RR$. Since L_i/L_{i-1} is a simple R-module, $\mathrm{Hom}_R(L_i/L_{i-1}, M_C)$ is a simple C-module, and thus the C-submodule $r(L_{i-1})/r(L_i)$ of $\mathrm{Hom}_R(L_i/L_{i-1}, M_C)$ is either trivial or simple. Hence

$$M_C = r(L_o) \supseteq r(L_1) \supseteq \ldots \supseteq r(L_n) = 0$$

defines a composition series of the C-module M. In particular, M_C is finitely generated and Lemma I.2.4 yields the result.

I.3. UNISERIAL RINGS ARE BALANCED

The purpose of this section is to offer a brief proof of the statement in the title.

LEMMA I.3.1. *If* R *is a local uniserial ring, then* ${}_RR$ *is injective.*

Proof. Let L be a left ideal of R. Consider an R-homomorphism $\varphi : L \to {}_RR$. If $L = Rx$, then $\partial(Rx\varphi) \leq \partial(Rx)$, and thus $x\varphi$ belongs to $Rx = xR$; consequently, we can find $r \in R$ such that $x\varphi = xr$. Then, the right multiplication by r gives a homomorphism ${}_RR \to {}_RR$ which extends φ.

LEMMA I.3.2. *Any direct sum of left balanced rings is left balanced.*

Proof. Assume $R = \bigoplus_{i=1}^{n} R_i$, where all R_i's are left balanced rings. Every R-module M has a unique decomposition as a direct sum of submodules M_i, where M_i can be considered as an R_i-module. The centralizer $C(M)$ of M is the direct sum of the rings $C(M_i)$, and because R_i map surjectively onto the double centralizers $\mathcal{D}(M_i)$, R maps surjectively onto $\mathcal{D}(M)$ (which is the direct sum of the $\mathcal{D}(M_i)$'s), as well.

PROPOSITION I.3.3. (C. J. NESBITT AND R. M. THRALL [2]) *Every uniserial ring is both left and right balanced.*

Proof. Because of Lemma I.3.2, it is sufficient to consider the case, where $\underline{\underline{R}}$ is a full matrix ring over a local uniserial ring R. If L is a principal indecomposable left ideal of $\underline{\underline{R}}$, then L is the image of ${}_RR$ under category isomorphism from $\operatorname{Mod} R$ to $\operatorname{Mod} \underline{\underline{R}}$. Therefore, Lemma I.3.1 implies that L is an injective $\underline{\underline{R}}$-module.

Assume that M is a faithful $\underline{\underline{R}}$-module. Since L is a generator and a uniserial module, there exists a monomorphism $L \to M$; furthermore, since L is injective, we have $M \cong L \oplus K$ for some $\underline{\underline{R}}$-module K. Now, $L \oplus K$ is also a

generator, and thus Lemma I.2.3 yields that M is balanced.

Finally, an arbitrary $\underline{\underline{R}}$-module can be considered as a module over some factor ring of $\underline{\underline{R}}$, which again is a full matrix ring over a local uniserial ring. Hence, $\underline{\underline{R}}$ is left balanced. And, similarly $\underline{\underline{R}}$ is right balanced. The proof is completed.

II. LOCAL RINGS

II.1. A NECESSARY LENGTH CONDITION

In this section, we shall get the first information on the structure of the local left balanced rings.

PROPOSITION II.1.1. *Let* R *be a local left finitely balanced ring with the radical* W . *Then, for each natural* n ,

$$\partial_R(W^n/W^{n+1}) \leq 2 .$$

Proof. Obviously, without loss of generality, we can suppose that $W^{n+1} = 0$ and $W^n \neq 0$. Observe that ${}_R(W^n)$ is then completely reducible.

First, assume that W^n contains a minimal left ideal U which is not a two-sided ideal. Then

$$T = \{\tau | \tau \in R \text{ and } U\tau \subseteq U\}$$

is a proper subring of R . It is easy to see that T is again a local ring with the radical W . Therefore, $Q = R/W$ can be viewed as a right vector space over T/W and we have

$$\dim Q_{(T/W)} \geq 2 \ ;$$

let $1 + W$ and $r + W$ be linearly independent elements of $Q_{(T/W)}$.

Now, consider the monogenic left R-module $X = R/U$; X is obviously faithful. For every $x \in R$, write $\bar{x} = x + U \in X$. The elements of the centralizer C of X can be lifted to endomorphisms of ${}_R R$ (that is to say, to right multiplications by elements of R) and, in this way, we get just those elements $\tau \in R$ which satisfy $U\tau \subseteq U$. Thus, the ring C is isomorphic to T/U and its radical W corresponds to W/U in this isomorphism. Consequently the C-module $Q = R/W \cong X/(W/U)$ has the same structure as $Q_{(T/W)}$; let $\epsilon : X_C \to Q_C$ be the canonic epimorphism. Also, observe that, given an arbitrary element $z \in W^n$, $\bar{z}$ belongs to $\mathrm{Soc}(X_C)$; denote by ι the inclusion of $\mathrm{Soc}(X_C)$ into X_C. Now, consider a C-endomorphism

$$\psi : X_C \xrightarrow{\epsilon} Q_C \xrightarrow{\psi'} \mathrm{Soc}(X_C) \xrightarrow{\iota} X_C$$

of X_C such that

$$(1 + W)\psi' = \bar{0} \quad \text{and} \quad (r + W)\psi' = \bar{z} \ .$$

Since X is balanced, ψ is induced by the ring multiplication, say by an element $\rho \in R$:

$$\psi(\bar{x}) = \rho\bar{x} \quad \text{for all} \quad \bar{x} \in X \ .$$

Then, $\rho \cdot \bar{1} = \bar{0}$ implies $\rho \in U$ and $\rho\bar{r} = \bar{z}$ implies $z \in \rho r + U \subseteq \subseteq U + Ur$. Hence,

$$W^n \subseteq U + Ur \ ,$$

as required.

In order to complete the proof of Proposition II.1.1, we are going to show that if all left ideals of R contained in W^n are two-sided, then $\partial_R(W^n) = 1$. For, assume the contrary and let Ru and Rv be non-zero two-sided ideals of R such that

$$Ru \subseteq W^n \ , \ Rv \subseteq W^n \quad \text{and} \quad Ru \cap Rv = 0 \ .$$

Consider the finitely generated faithful left R-module

$$Y = {}_R(R \oplus R)/D \quad \text{with} \quad D = R(u, v) \ .$$

Every endomorphism of Y can be lifted to an endomorphism of ${}_R(R \oplus R)$ and, in this way, we get just those endomorphisms of the left R-module ${}_R(R \oplus R)$ which map D into D. Let $\begin{matrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{matrix}$ be the matrix representation of such an endomorphism of ${}_R(R \oplus R)$; here α_{ij} denote endomorphisms of ${}_RR$, that is to say, right multiplications by elements of R. For $(u, v) \in D$, we get

$$(u, v) \begin{matrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{matrix} = (u\alpha_{11} + v\alpha_{21}, u\alpha_{12} + v\alpha_{22}) = (\lambda u, \lambda v)$$

for a suitable $\lambda \in R$. This yields that both α_{12} and α_{21} belong to W. For, if $\alpha_{12} \notin W$, then

$$u = (\lambda v - v\alpha_{22})\alpha_{12}^{-1} \in Rv$$

and, similarly, if $\alpha_{21} \notin W$, then

$$v = (\lambda u - u\alpha_{11})\alpha_{21}^{-1} \in Ru \ ,$$

a contradiction in either case.

Now, take a non-zero element $z \in W^n$ and define an additive homomorphism $\psi : R \oplus R \to R \oplus R$ by

$$\psi(r_1, r_2) = (zr_1, 0) \quad \text{for all} \quad (r_1, r_2) \in R \oplus R .$$

Evidently, ψ is a non-zero morphism, $\psi(D) = 0$ and

$$[\psi(r_1, r_2)] \begin{matrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{matrix} = (zr_1\alpha_{11}, zr_1\alpha_{12}) = (zr_1\alpha_{11}, 0) =$$

$$= (zr_1\alpha_{11} + zr_2\alpha_{21}, 0) = \psi(r_1, r_2) \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

because both $zr_1\alpha_{12}$ and $zr_2\alpha_{21}$ belong to $W^nW = 0$. Thus, ψ induces an element ψ^* of the double centralizer of Y. Since Y is balanced, ψ^* is induced by an element $\sigma \in R$; moreover, since ψ^* is non-zero, $\sigma \neq 0$. However, this results in an immediate contradiction; for,

$$\psi^*[(0, 1) + D] = D = \sigma[(0, 1) + D]$$

implies that $(0, \sigma) \in D$, and thus $\sigma = 0$.

The proof of Proposition II.1.1 is completed.

II.2. RINGS WITH $W^2 = 0$

Let R be a local ring with the radical W such that $W^2 = 0$. Denote the skew field R/W by Q, and considering W as a left or right vector space over Q write ${}_QW$ or W_Q, respectively. In this section, we are going to show that every such left balanced ring is either uniserial or satisfies the dimension relation

$$\dim {}_QW \times \dim W_Q = 2$$

together with some conditions binding the elements of W ; these are formulated in Propositions II.2.3 and II.2.4. In the next section, the latter rings will be termed exceptional.

LEMMA II.2.1. *Let* M *be a balanced indecomposable* R*-module of finite length and* m *an element of* M *such that* $\mathrm{Ann}(m) = 0$. *Then, denoting by* C *the centralizer of* M , $mC = M$.

Proof. Let $\mathcal{W}$ be the radical of C . Since M is of finite length, $\mathcal{W}$ is nilpotent and M_C has a non-trivial socle $\mathrm{Soc}\, M_C$ and a non-trivial radical $M\mathcal{W}$. Now, $M/(mC + M\mathcal{W})$ is a completely reducible right C-module; therefore, if we show that any C-homomorphism ψ of the form

$$M_C \xrightarrow{\epsilon} M/(mC + M\mathcal{W}) \to \mathrm{Soc}\, M_C \xrightarrow{\iota} M_C$$

(where ϵ is the canonical epimorphism and ι the embedding) is trivial, then we have

$$M = mC + M\mathcal{W} \quad .$$

But M is balanced, so $\psi x = \rho x$ for some $\rho \in R$ and any $x \in M$. It follows from $\rho m = \psi m = 0$ and $\mathrm{Ann}(m) = 0$, that $\rho = 0$; therefore ψ is trivial. The equality $M = mC + M\mathcal{W}$ yields $M = mC + (mC + M\mathcal{W})\mathcal{W} = mC + M\mathcal{W}^2$, and by induction,

$$M = mC + M\mathcal{W}^n \quad .$$

Since $\mathcal{W}$ is nilpotent, $M = mC$, as required.

LEMMA II.2.2. *Let* R *be a local left artinian ring,* W *its radical and* $Q = R/W$; *let* $W^2 = 0$. *Let* $(w_i)_{i=1}^n$ *be linearly independent elements of the vector space* W_Q . *Then*

$$I_n = R^{(n)}/D \text{ with } D = R(w_1, \ldots, w_n) ,$$

where $R^{(n)}$ *is a direct sum of* n *copies of* ${}_RR$, *is an indecomposable* R*-module of finite length. If, moreover,* I_n *is balanced, then*

$$W^{(n)}/D \subseteq R[1, 0, \ldots, 0) + D] .$$

Proof. First, there is no homomorphism of I_n onto ${}_RR$. For assuming the converse, we get a homomorphism

$$R^{(n)} \xrightarrow{\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}} {}_RR$$

such that D is mapped into 0 . Thus $\sum_{i=1}^{n} w_i r_i = 0$ and, in view of our hypothesis, all $r_i \in W$ and therefore the homomorphism cannot be surjective. In order to show that I_n is indecomposable, assume that $I_n = A \oplus B$. Then, both A and B are finitely generated and $A/\mathrm{Rad}\,A \oplus B/\mathrm{Rad}\,B$ is an n-dimensional vector space over Q . The well-known fact that elements of $\mathrm{Rad}\,A$ and $\mathrm{Rad}\,B$ are non-generators implies that

$$A = \sum_{i=1}^{p} Ra_i \quad \text{and} \quad B = \sum_{j=1}^{q} Rb_j$$

for some elements a_i and b_j, where $p + q = n$. Therefore, applying a length

argument, either A or B is a direct sum of copies of ${}_RR$, and consequently either $A = 0$ or $B = 0$, because there is no homomorphism of I_n onto ${}_RR$.

To prove the second assertion we may assume $n \geq 2$. We lift every element $\varphi \in C$ to an endomorphism of R^n and write it as a matrix (α_{ij}) where the α_{ij} are endomorphisms of ${}_RR$. First we show, that for an element φ of the radical W of C, all α_{ij} belong to W. Denote by e_i the element $(0, \ldots, 0, 1, 0, \ldots, 0) + D$ with 1 at the i-th position. Then $e_i\varphi = (\alpha_{i1}, \ldots, \alpha_{in}) + D$. If $\alpha_{ij} \notin W$ for some (i, j), then $e_i\varphi$ together with all the elements e_k, $k \neq j$, generates I_n. If $\operatorname{Ann}(e_i\varphi) \neq 0$, then a length argument shows that $R(e_i\varphi)$ is a direct summand, contradicting the fact that I_n is indecomposable. So $\operatorname{Ann}(e_i\varphi) = 0$. An application of Lemma II.2.1 leads to the equality $e_i\varphi C = I_n$, and, a fortiori, $I_nW = I_n$. Since this is impossible, we conclude that for $\varphi \in W$, all elements $\alpha_{ij} \in W$. From this it follows that $(W^{(n)}/D)\varphi = 0$ for all $\varphi \in W$; thus W^n/D is contained in the socle of I_C, where we abbreviate I_n by I. Also, according to Lemma II.2.1, $(1, 0, \ldots, 0) + D$ does not belong to IW. So for any $x \in W^n/D$, we can find a C-homomorphism ψ of the form

$$I_C \xrightarrow{\epsilon} I/IW \to \operatorname{Soc} I_C \xrightarrow{\iota} I_C$$

(ϵ the canonic epimorphism, ι the inclusion), mapping $(1, 0, \ldots, 0) + D$ onto x. But ψ is induced by left multiplication and thus there is $\rho \in R$ with $x = \rho[(1, 0, \ldots, 0) + D]$. This proves the second part of Lemma II.2.2.

Now, in order to facilitate formulations of the following Propositions II.2.3 and II.2.4, let us define, for a given ring R and an element $v \in R$, the following two subrings of R:

$$T_v = \{\tau \mid \tau \in R \text{ and } v\tau \in Rv\} \text{ and } S_v = \{\sigma \mid \sigma \in R \text{ and } \sigma v \in vR\}.$$

PROPOSITION II.2.3. *Let* R *be a local ring and* W *its radical such that*

$$W^2 = 0 \quad \textit{and} \quad \dim_Q W = 2 .$$

If R *is left finitely balanced, then* $\dim W_Q = 1$ *and, for any two linearly independent elements* u *and* v *of* ${}_QW$,

$$W = Rv + uT_v .$$

Proof. First, we shall prove that $\dim W_Q = 1$. Assume the contrary and choose $0 \neq w_1 \in W$. Since the set-theoretical union of Rw_1 and w_1R is a proper subset of W , there is an element $w_2 \in W$ which is neither in Rw_1 nor in w_1R . Consider the indecomposable R-module I_2 of Lemma II.2.2. Since I_2 is balanced ${}_R(W \oplus W)/D \subseteq R[(1, 0) + D]$. Therefore, taking $(0, w_1) + D$, there is $r_o \in R$ such that $(-r_o, w_1) \in D$. Thus, in particular, $w_1 = \lambda w_2$ for some $\lambda \in R$. But λ is necessarily a unit and thus $w_2 = \lambda^{-1} w_1$, in contradiction to $w_2 \notin Rw_1$.

Now, to complete the proof, take two linearly independent elements u and v of ${}_QW$ and verify that $W = Rv + uT_v$. To this end, consider the R-module $N = R/Rv$; let C be its centralizer. Obviously, the rings C and T_v/Rv are isomorphic, and thus $(Rv + uT_v)/Rv$ is a non-zero C-submodule of N . Therefore, there is a non-zero C-homomorphism $\psi : N \to N$ mapping N into $(Rv + uT_v)/Rv$. Since N is a balanced R-module, ψ is induced by the ring multiplication:

$$\psi n = \rho n \quad \text{for all} \quad n \in N \quad \text{with a suitable non-zero} \quad \rho \in R .$$

Consequently, $\rho R \subseteq Rv + uT_v$ and, since $\dim W_Q = 1$, $W = Rv + uT_v$, as required.

PROPOSITION II.2.4. *Let* R *be a local ring and* W *its radical such that*

$$W^2 = 0 \quad \textit{and} \quad \dim_Q W = 1 .$$

If R *is left finitely balanced, then either* $\dim W_Q = 1$ *or* $\dim W_Q = 2$ *and, for any two linearly independent elements* u *and* v *of* W_Q ,

$$W = vR + S_v u .$$

Proof. First, we are going to show that $\dim W_Q \leq 2$. Assuming the contrary, choose three linearly independent elements w_1, w_2, w_3 in W_Q , and consider the indecomposable R-module I_3 of Lemma II.2.2. Since I_3 is balanced, ${}_R(W \oplus W \oplus W)/D \subseteq R[(1, 0, 0) + D]$. Taking $(0, w_1, 0) + D$, we get $(r_o, 0, 0) + D = (0, w_1, 0) + D$ for a suitable $r_o \in R$, and thus $(-r_o, w_1, 0) \in D$. This is impossible, and therefore $\dim W_Q \leq 2$.

Now, in order to complete the proof, assume that $\dim W_Q = 2$ and take two linearly independent elements u and v of W_Q . Writing $u = w_1$ and $v = w_2$ and denoting by C the centralizer of the indecomposable R-module I_2 of Lemma II.2.2, we have $I_2 = [(1, 0) + D]C$; this follows from Lemma II.2.1, because $\operatorname{Ann}[(1, 0) + D] = 0$. Therefore, taking an arbitrary $r \in R$, there is $\varphi \in C$ such that

$$[(1, 0) + D]\varphi = (r, 0) + D .$$

Lifting φ to

$$R(R \oplus R) \xrightarrow{\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}} R(R \oplus R) ,$$

we get that $(\alpha_{11}-r, \alpha_{12}) \in D$ and thus, in particular, $\alpha_{11}-r \in W$ and $\alpha_{12} \in W$. Also, applying this homomorphism to $(u, v) \in D$, we obtain

$$(u\alpha_{11} + v\alpha_{21}, u\alpha_{12} + v\alpha_{22}) = (ur + v\alpha_{21}, v\alpha_{22}) \in D .$$

Hence,

$$ur + v\alpha_{21} = \lambda u \quad \text{and} \quad v\alpha_{22} = \lambda v \quad \text{for some} \quad \lambda \in R .$$

Therefore $\lambda \in S_v$ and $wr \in vR + S_v u$. Consequently

$$W = uR + vR \subseteq vR + S_v u ,$$

as required.

II.3. EXCEPTIONAL RINGS

Let R be a local ring with the radical W such that $W^2 = 0$. If v is a non-zero element of W, then the subrings

$$T_v = \{\tau | \tau \in R \quad \text{and} \quad v\tau \in Rv\} \quad \text{and} \quad S_v = \{\sigma | \sigma \in R \quad \text{and} \quad \sigma v \in vR\}$$

contained obviously W. Moreover, if τ is a unit belonging to T_v, then

$$v\tau = rv \quad \text{for a suitable} \quad r \in R$$

implies $r^{-1}v = v\tau^{-1}$ and thus $\tau^{-1} \in T_v$, as well. Consequently, T_v/W is a division subring of $Q = R/W$. If ρ is a unit of R, then

$$T_{v\rho} = \{\tau | \tau \in R \quad \text{and} \quad v\rho\tau \in Rv\rho\} = \{\tau | \tau \in R \quad \text{and} \quad v\rho\tau\rho^{-1} \in Rv\} = \rho^{-1}T_v\rho$$

and thus, in particular,

$$\dim_{Q(T_v/W)} = \dim_{Q(T_{v\rho}/W)} .$$

In a similar fashion, S_v/W is a division subring of Q and, for an arbitrary unit λ of R,

$$\dim_{(S_v/W)^Q} = \dim_{(S_{\lambda v}/W)^Q} .$$

DEFINITION II.3.1. *A local ring* R *with the radical* W *is said to be exceptional if*

$$W^2 = 0, \ \dim_Q W \times \dim W_Q = 2$$

and, if $\dim_Q W = 2$, *then*

$$\dim_{Q(T_v/W)} = 2$$

whereas, if $\dim_Q W = 1$, *then*

$$\dim_{(S_v/W)^Q} = 2$$

for a non-zero element $v \in W$.

Let us point out the fact that the notion of an exceptional ring is self-dual : *A ring* R *is exceptional if and only if the opposite ring* R^* *is exceptional.*

PROPOSITION II.3.2. *Let* R *be a local ring, and* W *its radical with*

$$W^2 = 0, \ \dim_Q W = 2 \ \textit{and} \ \dim W_Q = 1 .$$

Then the following statements are equivalent:

(i) R *is exceptional;*

(ii) *there exist two linearly independent elements* u, v *of* ${}_QW$ *such that*

$$W = Rv + uT_v ;$$

(iii) *the indecomposable injective left* R*-module is of length* 2.

Proof. In order to prove the implication (i) $\rightarrow$ (ii), let v be a non-zero element of W, $T = T_v$ and let $\dim Q_{(T/W)} = 2$. Thus, there exists $r \in R\backslash S$ such that $R = T + rT + W$. Taking $u = vr$, one gets

$$W = vR = v(T + rT + W) = vT + uT = Rv + uT ,$$

because $vT = Rv$ in view of $\dim W_Q = 1$.

Conversely, if

$$vR = W = Rv + uT = vT + uT ,$$

then $vR = v(T + rT)$ for a suitable $r \in R$ and hence

$$R = T + rT + W .$$

Now, $r \notin T$; for, otherwise $W = vR = vT = Rv$ in contradiction to $\dim_Q W = 2$. As a consequence, $\dim Q_{(T/W)} = 2$, and we get the equivalence of (i) and (ii).

In order to show that (ii) implies (iii), let u, v be the elements given in (ii). We are going to prove that R/Rv is an injective R-module. To this end, let $\varphi : {}_RW \rightarrow R/Rv$ be a non-zero homomorphism. Since right multiplication by elements of R is transitive on W, we can evidently assume

that $\operatorname{Ker}\varphi = Rv$. Thus, φ is determined by the conditions $v\varphi = 0$ and $u\varphi = w + Rv$ for a suitable $w \in W$. In view of the relation $W = Rv + uT_v$, we have

$$w = rv + u\tau \quad \text{for some} \quad r \in R \quad \text{and} \quad \tau \in T_v \ .$$

Consequently, the homomorphism

$$R^R \xrightarrow{\tau} {}_RR \xrightarrow{\epsilon} R/Rv$$

maps Rv into 0 , u into $w - rv + Rv = w + Rv$, and is thus a required extension of φ to ${}_RR$.

To complete the proof, let us verify the implication (iii) $\rightarrow$ (ii) . Let M be an indecomposable injective left R-module of length 2; hence $M \cong R/Rv$ for some non-zero element $v \in W$. Let $u \in {}_Q W$ so that u and v are linearly independent. Take an arbitrary element $w \in W$ and consider the homomorphism $\varphi : {}_RW \rightarrow M$ such that

$$u\varphi = w + Rv \quad \text{and} \quad v\varphi = 0 \ .$$

Since M is injective, φ can be extended to a homomorphism from ${}_RR$ to M , and therefore lifted to ${}_RR \xrightarrow{\tau} {}_RR$. From here, it follows that $v\tau \in Rv$, and thus $\tau \in T_v$; moreover, $w - u\tau \in Rv$. Consequently,

$$W = Rv + uT_v \ ,$$

completing the proof of Proposition II.3.2.

In a similar way, we can formulate

PROPOSITION II.3.3. *Let* R *be a local ring and* W *its radical with*

$$W^2 = 0,\ \dim_Q W = 1 \text{ and } \dim W_Q = 2 .$$

Then the following statements are equivalent:

(i) R *is exceptional;*

(ii) *there exist two linearly independent elements* u, v *of* W_Q *such that*

$$W = vR + S_v u ;$$

(iii) *the indecomposable injective left* R*-module is of length* 3.

Proof. Both statements (i) and (ii) are dual to those of Proposition II.3.2. and thus they are equivalent. In order to establish that (ii) implies (iii), we are going to show that the indecomposable R-module (cf. Lemma II.2.2)

$$I = {}_R(R \oplus R)/D, \text{ where } D = R(u, v)$$

is injective. Thus, assume that a homomorphism $\varphi : {}_R W \to I$ is given and we are required to extend it to a homomorphism from ${}_R R$ to I. Obviously, φ is determined by the image of v:

$$v\varphi = (v_1, v_2) + D \text{ for some } v_1, v_2 \text{ of } W .$$

But $v_2 = \lambda v$ and thus, for some $w \in W$,

$$(v_1, v_2) + D = (v_1 - \lambda u, 0) + (\lambda u, \lambda v) + D = (w, 0) + D .$$

Now $w \in vR + S_v u$, and therefore there are elements $r_1 \in R$, $\sigma \in S_w$ and $r_2 \in R$ such that

$$w = vr_1 + \sigma u \quad \text{and} \quad \sigma v = -vr_2 \ .$$

We claim that the homomorphism

$$ {}_RR \xrightarrow{(r_1,\ r_2)} {}_R(R \oplus R) \xrightarrow{\epsilon} I \ , $$

where ϵ is the canonical epimorphism, is an extension of φ. Indeed, the element v is mapped into

$$v(r_1,\ r_2) + D = (w - \sigma u,\ -\sigma v) + D = (w,\ 0) + D = v\varphi \ ,$$

as required. Consequently, I is injective and, being of length 3, necessarily indecomposable.

To complete the proof, let us verify the implications (iii) $\rightarrow$ (ii). An indecomposable injective left R-module I of length 3 is necessarily an amalgam of two copies of ${}_RR$ over its socle. Thus,

$$I = {}_R(R \oplus R)/D \quad \text{with} \quad D = R(u,\ v)$$

for suitable u and v of W. Now, take an arbitrary $w \in W$ and consider the homomorphism $\varphi : {}_RW \rightarrow I$ mapping v into $(w,\ 0) + D$. Extend φ to a homomorphism from ${}_RR$ to I and lift the latter to

$$ {}_RR \xrightarrow{(r_1,\ r_2)} {}_R(R \oplus R) \ . $$

Hence,

$$(vr_1,\ vr_2) - (w,\ 0) \in D \ ,$$

and thus

$$(vr_1 - w,\ vr_2) = (\sigma u,\ \sigma v) \quad \text{for some} \quad \sigma \in R \ .$$

Therefore,

$$\sigma \in S_v \quad \text{and} \quad w = vr_1 - \sigma u \in vR + S_v u ,$$

as required.

II.4. EXCEPTIONAL RINGS ARE BALANCED

First, in order to prove that every R-module over an exceptional ring R with $\dim_Q W = 2$ is a direct sum of indecomposable R-modules, we formulate the following two technical lemmas.

LEMMA II.4.1. *Let* R *be a local ring with the radical* W *such that* $W^2 = 0$ *and* $\dim W_Q = 1$. *Let* F *be a free left* R*-module and* $s \neq 0$ *an element of the socle of* F . *Then* s *belongs to a monogenic submodule which is isomorphic to* ${}_RR$.

Proof. The elements of F can be represented by indexed families (r_i) with $r_i \in R$ and the restriction that all but a finite number of the r_i's to be zero. An element (r_i) belongs to the socle $\operatorname{Soc} F$ of F is and only if $r_i \in W$ for all i . Let

$$s = (w_i) \in \operatorname{Soc} F .$$

Let $u \neq 0$ be a fixed element of W . Since $uR = W$, there exists $\rho_i \in R$ such that $w_i = u\rho_i$; here, we take $\rho_i = 0$ if $w_i = 0$. Now, right multiplication by ρ_i yields a homomorphism $\rho_i : {}_RR \to {}_RR$, and thus the family (ρ_i)

defines a homomorphism

$$\varphi : {}_RR \to F .$$

Clearly, $u\varphi = s$, and hence $s \in \mathrm{Im}\varphi$. Furthermore, since $s \neq 0$, there is a unit ρ_{i_o} such that $w_{i_o} = u\rho_{i_o}$; as a consequence, $\mathrm{Im}\varphi \cong {}_RR$.

LEMMA II.4.2. *Let* R *be a local ring with the radical* W *such that* $W^2 = 0$, $\dim_Q W = 2$ *and* $\dim W_Q = 1$. *Let* M *be an* R*-module with submodules* X *and* Y *isomorphic to* ${}_RR$ *such that* $X + Y = M$ *and* $X \cap Y$ *is a minimal submodule. Then* M *contains an indecomposable submodule of length* 2.

Proof. M is obviously isomorphic to the pushout P of the following diagram

$$\begin{array}{ccc} {}_RL & \xrightarrow{\mu} & {}_RR \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota'} \\ {}_RR & \longrightarrow & P \end{array} ,$$

where L is a minimal left ideal of R, ι the inclusion mapping and μ a monomorphism. If $x \neq 0$ is an element of L, then

$$x\mu = (x\iota)\rho \quad \text{for some} \quad \rho \in R ,$$

because $xR = W$. Thus right multiplication by ρ is a mapping from ${}_RR$ into ${}_RR$ satisfying $\iota\rho = \mu$. But this implies, in view of the properties of a pushout, that ι' splits and that the complement is just the cokernel R/L of ι which is obviously of length 2.

Now, we are ready to prove the following

PROPOSITION II.4.3. *Let* R *be a local ring with* $W^2 = 0$, $\dim_Q W = 2$ *and* $\dim W_Q = 1$. *If the indecomposable injective* R*-module is of length* 2, *then every indecomposable* R*-module is either simple, injective or isomorphic to* ${}_RR$, *and any* R*-module is a direct sum of these indecomposable modules.*

Proof. To prove our proposition, we shall show that every R-module can be expressed as a direct sum of modules of the isomorphism types A_1, A_2, A_3 represented by the R-modules ${}_R(R/W)$, ${}_R(R/Ru)$ with a non-zero $u \in W$ and R^R, respectively. Here, A_2 is the injective indecomposable type.

Let M be a left R-module. Take a submodule X of M which is maximal with respect to the property of being a direct sum of modules of type A_2. Since X is injective, $M = X \oplus M'$, where M' is a submodule of M which contains no submodules of type A_2.

Now, let Y be a submodule of M' which is maximal with respect to the property of being a direct sum of modules of type A_3. Let Z be a complement of Soc Y in Soc M'. Then, Z is a direct sum of modules of type A_1 and, evidently, $Y \cap Z = 0$. We want to show that

$$Y \oplus Z = M' .$$

To this end, assume that there is an element $m \in M' \setminus (Y \oplus Z)$. Then Rm must be of type A_3, because $m \notin \operatorname{Soc} M'$ and M' contains no submodule of type A_2. The submodule $Y \cap Rm$ is non-zero; for, otherwise $Y + Rm$ would be a direct sum of modules of type A_3, contradicting the maximality of Y. Take $s \neq 0$ of $Y \cap Rm$. Since $s \in \operatorname{Soc} Y$, Lemma II.4.1 implies that there is a submodule $N \subseteq Y$ of type A_3 containing s. In view of Lemma II.4.2,

$N \cap Rm$ cannot be simple and therefore the length of $N \cap Rm$ is 2 .

If we now assume that $\operatorname{Soc}(N + Rm)$ is of length 2, then $N + Rm$ is isomorphic to the injective hull of $\operatorname{Soc}(N + Rm)$ (because both modules are of length 4). However, since M' has no submodules of type A_2 , this is impossible. Thus, $\operatorname{Soc}(N + Rm)$ has to be of length 3, and therefore

$$N + Rm = N + \operatorname{Soc}(N + Rm) \ .$$

But this means that

$$Rm \subseteq Y + \operatorname{Soc} M' \subseteq Y \oplus Z \ ,$$

and we get a contradiction to our hypothesis. The proof is completed.

In analogy to the preceeding result, we shall prove also that every R-module over an exceptional ring R with $\dim_Q W = 1$ is a direct sum of indecomposable R-modules of the types B_1, B_2 and B_3 represented by the R-modules ${}_R(R/W)$, ${}_RR$ and the injective module I_2 of Lemma II.2.2. Here again, the index of B_i refers to the length of the respective module. Note however that, contrary to the previous situation, B_3 is not a monogenic module.

First, let us prove by induction the following

LEMMA II.4.4. *Let* R *be a local ring with the radical* W *such that* $W^2 = 0$, $\dim_Q W = 1$ *and* $\dim W_Q = 2$.

(a) *Let* M *be an* R*-module of length* $2n + 1$ *generated by* $n + 1$ *monogenic submodules. Let* N *be a submodule of* M *which is a direct sum of* n *copies of modules of type* B_2 . *If, furthermore,* M *does not contain a submodule of type* B_3 , *then*

$$M = N + \operatorname{Soc} M \ .$$

(b) *The only indecomposable* R*-modules of length* $\leq 2n + 1$ *are modules of type* B_1, B_2 *and* B_3 .

Proof. If the length of M is 3 , and if M contains a monogenic submodule N of length 2 , then either $\operatorname{Soc} M$ is simple in which case the injectivity of B_3 yields that M is of type B_3 , or $\operatorname{Soc} M$ is of length ≥ 2 ; in the latter case, evidently

$$M = N + \operatorname{Soc} M .$$

This establishes the validity of both (a) and (b) for $n = 1$.

Now, assume that both assertions hold for all $m \leq n - 1$.

(a) Without loss of generality, we may assume that the $n + 1$ monogenic submodules which generate M are all of length 2. We can consider M as the amalgamation of N with a monogenic module of length 2 with simple submodules identified. Thus, M is isomorphic to the pushout P of the following diagram

$$\begin{array}{ccc} R^W & \xrightarrow{\eta} & {}_RR \oplus {}_RR \oplus \ldots \oplus {}_RR \\ \downarrow \iota & & \downarrow \iota' \\ R^R & \xrightarrow{\eta'} & P \end{array} ,$$

where ι is the inclusion of W in R , η is a monomorphism and ι' corresponds to the inclusion $N \subseteq M$. Let us take a non-zero element $w \in W$; hence, $w\eta$ is of the form $(x_1, x_2, \ldots, x_n)$ with at least one non-zero x_i . Assume that $x_1 \neq 0$ and distinguish three cases:

(i) Let $x_i \in wR$ for all $1 \leq i \leq n$. Then we can find elements σ_i such that $x_i = w\sigma_i$, and thus the morphism

$$(\sigma_1, \sigma_2, \ldots, \sigma_n) : {}_RR \to {}_RR \oplus {}_RR \oplus \ldots \oplus {}_RR ,$$

representing right multiplication, maps w into $(x_1, x_2, \ldots, x_n) = w\eta$. But this means that ${}_RR \oplus {}_RR \oplus \ldots \oplus {}_RR$ is a direct summand of P. Consequently, the complement is simple and therefore $M = N + \operatorname{Soc} M$.

(ii) Let $x_1 \notin wR$ and $x_i \in x_1R$ for all $1 \leq i \leq n$. Then, we can find elements σ_i with $x_i = x_1\sigma_i$; observe that σ_1 is a unit. Now, both $1\eta'$ and $(\sigma_1, \sigma_2, \ldots, \sigma_n)\varepsilon'$ generate submodules of length 2 and the equality

$$w(1\eta') = w\eta' = w\varepsilon\eta' = w\eta\varepsilon' = (x_1, x_2, \ldots, x_n)\varepsilon' =$$

$$= (x_1\sigma_1, x_1\sigma_2, \ldots, x_1\sigma_n)\varepsilon' = x_1(\sigma_1, \sigma_2, \ldots, \sigma_n)\varepsilon'$$

shows that

$$w\eta' \in R(1\eta') \cap R(\sigma_1, \sigma_2, \ldots, \sigma_n)\varepsilon' .$$

Let $X = R(1\eta') + R(\sigma_1, \sigma_2, \ldots, \sigma_n)\varepsilon'$. Assuming that $R(\sigma_1, \sigma_2, \ldots, \sigma_n)\varepsilon'$ is a direct summand of X, we deduce that there is a morphism $R(1\eta) \to R(\sigma_1, \sigma_2, \ldots, \sigma_n)\varepsilon'$ mapping $w\eta'$ into $(x_1\sigma_1, x_1\sigma_2, \ldots, x_1\sigma_n)\varepsilon'$, and thus a morphism ${}_RR \to {}_RR \oplus {}_RR \oplus \ldots \oplus {}_RR$ mapping w into $(x_1\sigma_1, x_1\sigma_2, \ldots, x_1\sigma_n)$. In particular, there is a morphism ${}_RR \to {}_RR$ mapping w into $x_1\sigma_1 = x_1$ and since such a morphism must be induced by right multiplication we get that $x_1 \in wR$, contradicting our hypothesis. Thus, X has to be an indecomposable R-module of length 3 and therefore of type B_3. Since M has no submodule of type B_3, we conclude that the case (ii) cannot happen.

(iii) Let $x_1 \notin wR$ and there is x_i such that $x_i \notin x_1R$. We may assume that $x_2 \notin x_1R$. Thus, $W = x_1R + x_2R$ and therefore there are elements σ_1, σ_2 such that

$$w = x_1\sigma_1 + x_2\sigma_2 .$$

In this case, the pushout P can be considered as the quotient module of $n + 1$ copies of ${}_RR$ by the submodule generated by $(w, -x_1, -x_2, \ldots, -x_n)$. Under the morphism

$$(1, \sigma_1, \sigma_2, 0, \ldots, 0) : {}_RR \oplus {}_RR \oplus \ldots \oplus {}_RR \to {}_RR$$

representing right multiplication, the element $(w, -x_1, -x_2, \ldots, -x_n)$ is mapped into $w - w_1\sigma_1 - x_2\sigma_2 = 0$ and thus the morphism factors through P. As a consequence P has a homomorphic image of type B_2. The latter splits off and we deduce that M is a direct sum of a module of type B_2 and a module M' of length $2n - 1$.

Now, using the induction argument, M' is a direct sum of modules of types B_1, B_2 and B_3. However, since M has no submodules of type B_3, M' is a direct sum of monogenic modules of length 1 and 2. In particular, $\operatorname{Soc} M'$ has to be of length at least n and therefore $\operatorname{Soc} M$ has to be of length at least $n + 1$. Consequently, $M = N + \operatorname{Soc} M$, as required.

The statement (a) is established.

(b) Given an indecomposable R-module M of length $\leq 2n + 1$, we deduce immediately that M has no proper submodule of type B_3; this follows from the fact that B_3 is injective. Now, take a submodule N which is maximal with respect to the property of being a direct sum of copies of modules of type B_2, and let K be a complement of $\operatorname{Soc} N$ in $\operatorname{Soc} M$. In order to

verify (b), it is sufficient to show that $M = N \oplus K$, i.e. to show that every element $x \in M$ generating a submodule of length 2 belongs to $N \oplus K$. Let $M' = N + Rx$. If $x \notin N$, then the length of M' is $2m + 1$, where m is the number of direct summands of type B_2 in N. Since $m \leq n$, we get by induction

$$M' = N + \operatorname{Soc} M' .$$

But this means that $x \in N + K$.

The proof of Lemma II.4.4 is completed.

PROPOSITION II.4.5. *Let* R *be a local ring with* $W^2 = 0$, $\dim_Q W = 1$ *and* $\dim W_Q = 2$. *If the indecomposable injective* R*-module is of length* 3, *then every indecomposable* R*-module is either simple, injective or isomorphic to* ${}_R R$, *and any* R*-module is a direct sum of these indecomposable modules.*

Proof. It is sufficient to show that every R-module M can be expressed as a direct sum of modules of types B_1, B_2 and B_3.

Following the method of proving Proposition II.4.3, we denote by X a submodule of M which is maximal with respect to the property of being a direct sum of modules of type B_3 and observe that $M = X \oplus M'$. In M', take a submodule Y which is a maximal direct sum of modules of type B_2, and denote by Z a complement of $\operatorname{Soc} Y$ in $\operatorname{Soc} M'$. We intend to show that

$$M = X \oplus Y \oplus Z .$$

Assume the contrary, i.e. that there is an element $m \in M' \setminus (Y \oplus Z)$ which generates a submodule of length 2. Clearly, because of maximality of Y, $Y \cap Rm \neq 0$. Thus, there is a direct sum Y' of a finite number of copies

of B_2 contained in Y such that

$$Y' \cap Rm \neq 0 .$$

Now, applying Lemma II.4.4 (a) to the module $Y' + Rm$ and the submodule Y' we get readily that

$$Y' + Rm = Y' + Soc(Y' + Rm) .$$

Consequently, $m \in Y' + Soc(Y' + Rm) \subseteq Y + Soc M' = Y \oplus A$, a contradiction. Proposition II.4.3 follows.

PROPOSITION II.4.6. *Every exceptional ring is both left and right balanced.*

Proof. Because the opposite ring of an exceptional ring is again exceptional, it is sufficient to prove that exceptional rings are left balanced.

Both in the case when $\dim_Q W = 2$, as well as when $\dim_Q W = 1$, it is easy to verify that all indecomposable modules are balanced. This is trivial for modules of the types A_3 and B_2 and also for modules of the simple types A_1 and B_1; and, it follows for modules of the injective types A_2 and B_3 immediately from Lemma I.2.5.

Now, in view of Propositions II.4.3 and II.4.5, we can apply Lemma I.2.1 and complete the proof.

II.5. STRUCTURE OF LOCAL BALANCED RINGS

LEMMA II.5.1. *Let* R *be a local left finitely balanced ring with the*

radical W *such that* $W^3 = 0$. *Let* R *be right uniserial and not left uniserial. Then* R *is exceptional.*

Proof. First, observe that, according to Proposition II.1.1, $\partial_R(W/W^2) = 2$. Thus, in order to prove our Lemma, it is sufficient, in view of Proposition II.2.3 and Definition II.3.1, to show that $W^2 = 0$.

Assume that $W^2 \neq 0$. First, we can see that ${}_RW$ is the direct sum of two monogenic submodules Ru and Rv , where u and v belong to $W\backslash W^2$. This follows from the fact that ${}_RW$ can be considered as a left R/W^2-module and, according to Lemma II.4.3, it is a direct sum of monogenic modules. And, since radical of ${}_RW$ is ${}_R(W^2)$ and since ${}_R(W/W^2)$ is of length 2 , it is a direct sum of two monogenic modules.

Now, since u and v belong to $W\backslash W^2$ and R is right uniserial, there is $\rho \in R$ such that $u\rho = v$. Thus, in particular, $Ru \cong Rv$. Therefore, in view of Proposition II.1.1, $\partial_R(W^2) \leq 2$ and hence, $\partial(Ru) = \partial(Rv) = \partial_R(W^2) = 2$. Write $L = \mathrm{Ann}(u)$. Then ${}_R(R/L) \cong Ru$ and thus $\partial_R L = 3$. But R is right uniserial and therefore

$$LW = L(uR) = 0 ;$$

consequently, L is contained in the right socle W^2 of R . We arrive at a contradiction and conclude that $W^2 = 0$.

LEMMA II.5.2. *Let* R *be a local left finitely balanced ring with the radical* W *such that* $W^3 = 0$. *Let* R *be left uniserial and not right uniserial. Then* R *is exceptional.*

Proof. In view of Proposition II.2.4 and Definition II.3.1, R/W^2 is

an exceptional ring and thus it is sufficient to prove that $W^2 = 0$.

Let us give again an indirect proof. Assume that $W^2 \neq 0$ and consider the *right* R/W^2-module W_{R/W^2} . Applying the dual statements of Proposition II.3.3 and Proposition II.4.3 to this *right* R/W^2-module and taking into account the fact that W_{R/W^2} possesses a completely reducible quotient $(W/W^2)_{R/W^2}$ of length 2 , we can easily conclude that there are elements u and v in W with $uR \cap vR = 0$ such that $u + W^2$ and $v + W^2$ are linearly independent in $(W/W^2)_{R/W} = (W/W^2)_Q$.

Now, let us construct two non-isomorphic R-modules M_1 and M_2 of length 4 such that $M_1 \oplus M_2$ is not balanced.

First, consider

$$M_1 = {}_R(R \oplus R)/D_1, \quad \text{where} \quad D_1 = R(u, v) .$$

The R-module M_1 has no monogenic quotient of length 2 . For, given a homomorphism $\varphi : M_1 \to {}_R(R/W^2)$, we can lift it to a homomorphism

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$
$${}_R(R \oplus R) \longrightarrow {}_R R \quad .$$

Since D_1 is mapped into W^2 , $ur_1 + vr_2$ belongs necessarily to W^2 . But $u + W^2$ and $v + W^2$ are linearly independent in $(W/W^2)_{R/W}$ and therefore both ur_1 and vr_2 belong to W^2 . Consequently, both r_1 and r_2 lie in W and hence φ cannot be surjective. From here it follows easily that $\operatorname{Soc} M_1$ is simple; for, otherwise ${}_R(R \oplus W)/D_1 \cong {}_R R$ would be a direct summand.

Secondly, take a non-zero element $w \in vR \cap W^2$ and define the R-module

$$M_2 = {}_R(R \oplus R)/D_2, \quad \text{where} \quad D_2 = R(u, w) .$$

Again, $\operatorname{Soc} M_2$ is simple. For, if $\operatorname{Soc} M_2$ is not simple, then ${}_R(W \oplus R)/D_2 \cong {}_RR$ is a direct summand of M_2 and M_2 possesses an epimorphic image which is a monogenic R-module of length 3 . But a homomorphism $\varpi : M_2 \to {}_RR$ can be lifted to a homomorphism

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$
$${}_R(R \oplus R) \longrightarrow {}_RR$$

mapping D_2 into 0 . Therefore, $ur_1 + wr_2 = 0$. This relation shows that both r_1 and r_2 belong to W and thus the homomorphism φ cannot be surjective.

Now, M_1 and M_2 are two non-isomorphic R-modules of length 4 . This follows from the fact that M_2 has a monogenic quotient ${}_R(R \oplus R)/(R \oplus W^2)$ of length 2 . Consequently, any homomorphism between M_1 and M_2 must have a non-trivial kernel. Since both $\operatorname{Soc} M_1$ and $\operatorname{Soc} M_2$ are simple, such a homomorphism $\varphi_{ij} : M_i \to M_j$ (with $i \neq j$) satisfies $(\operatorname{Soc} M_i)\varphi = 0$. But then the R-module $M = M_1 \oplus M_2$ is not balanced. For, represent the elements of the centralizer of M by the matrices

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}, \quad \text{where } \varphi_{ij} : M_i \to M_j .$$

Take a non-zero element z of $\operatorname{Soc}_R R$, and define an additive homomorphism $\Psi : M \to M$ by

$$\Psi(m_1, m_2) = (zm_1, 0) \quad \text{for } (m_1, m_2) \text{ in } M_1 \oplus M_2 .$$

Now, zm_i belongs to $\operatorname{Soc} M_i$; so, for $i \neq j$, we have

$$z(m_i\varphi_{ij}) = (zm_i)\varphi_{ij} = 0 .$$

This implies, that Ψ belongs to the double centralizer of M, because of

$$[\Psi(m_1, m_2)]\begin{pmatrix}\varphi_{11} & \varphi_{12}\\ \varphi_{21} & \varphi_{22}\end{pmatrix} = (zm_1\varphi_{11}, zm_1\varphi_{12}) = (zm_1\varphi_{11} + zm_2\varphi_{21}, 0) =$$

$$= \Psi[(m_1, m_2)\begin{pmatrix}\varphi_{11} & \varphi_{12}\\ \varphi_{21} & \varphi_{22}\end{pmatrix}] .$$

Assuming that Ψ is induced by left multiplication by $\rho \in R$, the equation $(zm_1, 0) = (\rho m_1, \rho m_2)$ for all $m_i \in M_i$ implies $\rho = 0$, because M_2 is faithful. But $zM_1 \neq 0$, because M_1 is faithful. Hence M is not balanced.

We conclude that $W^2 = 0$ completing the proof of Lemma II.5.2.

Now, we are ready to formulate the following

PROPOSITION II.5.3. *Let* R *be a local ring with the radical* W *such that* $W^n = 0$ *for some natural* n. *Then* R *is left finitely balanced if and only if it is either uniserial or exceptional.*

Proof. Let R be left finitely balanced. Obviously, without loss of generality, we may assume that $W^3 = 0$. Then the conclusion follows immediately from Proposition II.1.1 and Lemmas II.5.1 and II.5.2.

The opposite implication follows from Proposition I.3.3 and Proposition II.4.6.

III. GENERAL THEOREMS

III.1. MORITA EQUIVALENCE

In this section, we shall give a short proof of the fact that the ring property of being balanced is Morita equivalent. We start with two technical lemmas.

LEMMA III.1.1. *Let* P_R *be a finitely generated projective right* R*-module. Let* $_RM_C$ *be an* R-C*-bimodule. Then the homomorphism*

$$\alpha : P_R \otimes \mathrm{Hom}_C({}_RM_C, {}_RM_C) \to \mathrm{Hom}_C({}_RM_C, P_R \otimes {}_RM_C)$$

given by

$$[\alpha(p \otimes \varphi)]m = p \otimes (\varphi m) \quad \textit{for} \quad p \in P_R, \varphi \in \mathrm{Hom}_C({}_RM_C, {}_RM_C) \quad \textit{and} \quad m \in M,$$

is an isomorphism of right R*-modules.*

Proof. It is easy to see that $\alpha(p \otimes \varphi)$ is a C-homomorphism of M_C into $P_R \otimes {}_RM_C$, and that α is an R-homomorphism. For $P_R = R_R$, the homomorphism α is trivially an isomorphism. It turns out that α is also an isomorphism for a direct summand P_R of a finite direct sum of copies of R_R.

LEMMA III.1.2. *Let* P_R *be a finitely generated projective right* R*-module. Let* $_RM$ *be a balanced* R*-module with the centralizer* C. *Then the morphism*

$$\beta : P_R \to \mathrm{Hom}_C({}_RM_C, P_R \otimes {}_RM_C)$$

defined by

$$(\beta p)m = p \otimes m \quad \textit{for} \quad p \in P \quad \textit{and} \quad m \in M,$$

is an epimorphism of right R*-modules.*

Proof. The morphism β is the composition of

$$P_R \overset{\beta_1}{\to} P_R \otimes {}_RR_R \overset{\beta_2}{\to} P_R \otimes \mathrm{Hom}_C({}_RM_C, {}_RM_C) \overset{\alpha}{\to} \mathrm{Hom}_C({}_RM_C, P_R \otimes {}_RM_C),$$

where $\beta_1 p = p \otimes 1$ for $p \in P$, β_2 is induced by the canonic homomorphism ${}_RR_R \to \mathrm{Hom}_C({}_RM_C, {}_RM_C)$ mapping $r \in R$ onto the left multiplication by r, and α is the morphism defined in Lemma III.1.1. In fact, $\beta_2\beta_1 p = p \otimes \iota$, where ι is the identity automorphism of M_C, and thus $(\alpha\beta_2\beta_1 p)m = [\alpha(p \otimes \iota)]m = p \otimes \iota(m) = p \otimes m$. It is well-known that β_1 is an isomorphism, and the fact that ${}_RM$ is balanced means that β_2 is an epimorphism. Consequently, we deduce that β is an epimorphism, as required.

PROPOSITION III.1.3. *Let* P_R *be a finitely generated projective right* R*-module with the centralizer* A. *If an* R*-module* $_RM$ *is balanced, then also the* B*-module* $_AP_R \otimes {}_RM$ *is balanced.*

Proof. Let C be the centralizer of $_RM$. Then C induces endo-

morphisms of ${}_AP_R \otimes {}_RM$ and, in this way, we get a right C-module $P_R \otimes {}_RM_C$. We shall show that all endomorphisms of $P_R \otimes {}_RM_C$ are induced by left multiplication by the elements of A, as an immediate consequence, we obtain that the canonic mapping of A into the double centralizer of ${}_AP_R \otimes {}_RM$ is surjective.

Since P_R is projective, the epimorphism β of Lemma III.1.2 induces an epimorphism

$$\beta' : \mathrm{Hom}_R(P_R, P_R) \to \mathrm{Hom}_R(P_R, \mathrm{Hom}_C({}_RM_C, P_R \otimes {}_RM_C)),$$

where β' is given by $[(\beta'\lambda)p]m = [\beta(\lambda p)]m = (\lambda p) \otimes m$ for $\lambda \in \mathrm{Hom}_R(P_R, P_R)$, $p \in P$ and $m \in M$. Also, we have the canonic isomorphism

$$\gamma : \mathrm{Hom}_R(P_R, \mathrm{Hom}_C({}_RM_C, P_R \otimes {}_RM_C)) \to \mathrm{Hom}_C(P_R \otimes {}_RM_C, P_R \otimes {}_RM_C),$$

defined by $(\gamma\varphi)(p \otimes m) = (\varphi p)m$ for $\varphi \in \mathrm{Hom}_R(P_R, \mathrm{Hom}_C({}_RM_C, P_R \otimes {}_RM_C))$, $p \in P$ and $m \in M$. Therefore, under β' and γ, an element $\lambda \in \mathrm{Hom}_R(P_R, P_R)$ is mapped onto the C-endomorphism $\gamma\beta'\lambda$ of $P_R \otimes {}_RM_C$, which, by definition, equals

$$(\gamma\beta'\lambda)(p \otimes m) = [(\beta'\lambda)p]m = (\lambda p) \otimes m .$$

But λ is an element of the centralizer $A = \mathrm{Hom}_R(P_R, P_R)$, and hence $\gamma\beta'\lambda$ is just left multiplication by the element $\lambda \in A$ on $P_R \otimes {}_RM_C$. The fact that $\beta'\lambda$ is surjective shows that every C-endomorphism of $P_R \otimes {}_RM_C$ is induced by an element of A. Consequently, ${}_AP_R \otimes {}_RM$ is balanced.

PROPOSITION III.1.4. (K. MORITA & H. TACHIKAWA [21]). *Let* S *be a category isomorphism from* Mod R *onto* Mod A. *If the* R*-module* M *is balanced, then the* A*-module* $S(M)$ *is balanced.*

In particular, let $\underline{R}$ *be a full matrix ring over a ring* R. *Then* $\underline{R}$ *is left balanced, or left finitely balanced, if and only if* R *is left balanced, of left finitely balanced, respectively.*

Proof. Indeed, there is a bimodule ${}_AP_R$ such that $S(M) \cong {}_AP_R \otimes_R M$. But P_R is finitely generated and projective, A is its endomorphism ring, and hence $S(M)$ is balanced by Proposition III.1.3.

As a consequence, the existence of a category isomorphism between Mod R and Mod A implies that the ring R is left balanced, or left finitely balanced, if and only if A is left balanced, or left finitely balanced, respectively.

III.2. LEFT BALANCED RINGS ARE LEFT ARTINIAN

The main purpose of this section is to prove the statement in the title. In order to facilitate the proof we are going to prove first several auxiliary results.

LEMMA III.2.1. *Let* R *be a left balanced ring. Let* M *be a faithful and* S *a simple* R*-module. Then there exists either an injection* $\iota : S \to M$ *or a surjection* $\epsilon : M \to S$. *In particular, every module over a left balanced ring possesses either a minimal or a maximal submodule.*

Proof. If there is no non-zero homomorphism between M and S, then the elements of the centralizer of $M \oplus S$ have the form $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in C(M)$ and $\beta \in C(S)$. Hence, the morphism

$$\psi : M \oplus S \to M \oplus S$$

defined by $\psi(m, s) = (0, s)$ is an element of the double centralizer of $M \oplus S$ which is not induced by the ring multiplication. The lemma follows.

PROPOSITION III.2.2. (V. P. CAMILLO [3]). *Let* R *be a left balanced ring with the radical* W . *Then* R/W *is artinian.*

Proof. Without loss of generality, assume that $W = 0$. Consider the direct sum $N = \bigoplus_{\omega \in \Omega} V_\omega$ of all non-isomorphic simple R-modules V_ω . The module N is obviously faithful and thus R is isomorphic to the double centralizer $\mathcal{D}$ of N . It is easy to see that $\mathcal{D}$ is a cartesian product of full endomorphism rings of vector spaces over the centralizers of the V_ω's . In particular, $\mathrm{Soc}_R R$ is essential in ${}_R R$. To complete the proof it suffices to show that $R = \mathrm{Soc}_R R$. Assume the contrary. Then there is a maximal left ideal L of R containing $\mathrm{Soc}_R R$. Since $\mathrm{Soc}_R R$ is essential in R , the annihilator of the simple R-module $S = R/L$ is essential and therefore S cannot be isomorphic to a submodule of $M = \mathrm{Soc}_R R$. Since R is left balanced and M is faithful we arrive at a contradiction of Lemma III.2.1.

PROPOSITION III.2.3. (K. R. FULLER [16]). *Let* R *be a left finitely balanced ring with the radical* W . *Then, for every natural* n , *the quotient ring* R/W^n *is a finite direct sum of full matrix rings over local left finitely balanced rings. Moreover, if* R *is left balanced, the local rings are also left balanced.*

Proof. Without loss of generality, assume that $W^n = 0$. In view of Lemma III.2.2, we may choose a direct decomposition ${}_R R = \bigoplus_{i=1}^{t} L_i$ of ${}_R R$ into indecomposable left ideals. In order to establish the lemma, we want to show that the simple composition factors of every L_i are isomorphic.

For $n = 2$, this follows from the fact that an extension of a simple R-module S_1 by a non-isomorphic simple R-module S_2 always splits. Indeed,

assuming that such an extension M does not split, we can see easily that the centralizer of M is a skew field. However, then M_C is a vector space and C- homomorphisms act on it transitively. Therefore, since M is balanced and S_1 is a proper R-submodule of M, we arrive at a contradiction. If $n > 2$, we proceed by induction. If we assume that every composition factor of L_i/W^nL_i is isomorphic to L_i/WL_i, we may define for any $x \in W^{n-1}L_i \setminus W^nL_i$ a homomorphism

$$\eta_x : L_i/W^2L_i \to L_i/W^{n+1}L_i$$

with $x \in (L_i/W^2L_i)\eta_x$. The fact that the images of all these homomorphisms cover $W^nL_i/W^{n+1}L_i$ implies that also the composition factors of $W^nL_i/W^{n+1}L_i$ are isomorphic to L_i/WL_i. This proves that the composition factors of L_i are isomorphic.

Therefore, R is a finite direct sum of rings $\underline{R}_i$ which are full matrix rings over local rings R_i. If R is left balanced, or left finitely balanced, also the rings $\underline{R}_i$ are left balanced or left finitely balanced, respectively. For, any $\underline{R}_i$-module M_i can be considered as an R-module and is balanced as an R-module if and only if it is balanced as an $\underline{R}_i$-module. If $\underline{R}_i$ is left balanced or left finitely balanced, then, by Proposition III.1.4, R_i has the respective property, too. This completes the proof.

LEMMA III.2.4. *Let* M *be a module with an endomorphism* φ *such that* $M\varphi \subseteq \operatorname{Rad} M$. *Then the direct limit* X *of the diagram*

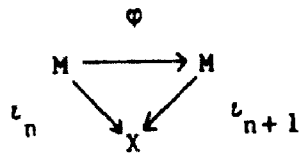
$$M \xrightarrow{\varphi} M \xrightarrow{\varphi} M \xrightarrow{\varphi} \cdots$$

has no maximal submodules.

Proof. Let $K = \bigcup_{n \geq 1} \operatorname{Ker} \varphi^n$. Then $K\varphi \subseteq K$, and φ induces an endomorphism $\varphi' : M/K \to M/K$. It is easy to see that φ' is a monomorphism. Moreover, $(M/K)\varphi' \subseteq \operatorname{Rad} M/K$ and X can be considered as the direct limit of the diagram

$$M/K \xrightarrow{\varphi'} M/K \xrightarrow{\varphi'} M/K \xrightarrow{\varphi'} \cdots .$$

This shows that we may assume that φ is a monomorphism. We denote by $\iota_n : M \to X$ the canonic homomorphisms; for these homomorphisms we have commutative diagrams



Let $M_n = M\iota_n$, and $\varphi_n : M_n \to M_{n+1}$ be induced by φ . Then M_n is a submodule of X and X is the union of the M_n's .

Assume that Y is a maximal submodule of X . Because $Y \neq X$, we find n such that $M_n \not\subseteq Y$. Take $x \in M_n \setminus Y$. Then we have

$$(Y \cap M_{n+1}) + Rx = (Y + Rx) \cap M_{n+1} = X \cap M_{n+1} = M_{n+1} .$$

but $x = (x\varphi_n)$ belongs to $\operatorname{Rad} M_{n+1}$ and therefore $Y \cap M_{n+1} = M_{n+1}$. This implies that $M_n \subseteq M_{n+1} \subseteq Y$, a contradiction.

PROPOSITION III.2.5. (V. P. CAMILLO [3]). *The radical of a left balanced ring is a nil ideal.*

Proof. Let R be a left balanced ring, and W its radical. For $w \in W$, consider the direct limit X of the diagram

$$R^R \xrightarrow{w} R^R \xrightarrow{w} R^R \xrightarrow{w} \cdots ,$$

where w denotes the right multiplication. According to Lemma III.2.4, X has no maximal submodule. If we assume that $w^n \neq 0$ for all natural n, then $X \neq 0$, and every non-zero factor module of X has minimal submodules. This follows from Lemma III.2.1. Thus, if we define by transfinite induction a sequence of submodules X_α, with $X_o = 0$, $X_\alpha/X_{\alpha-1} = \mathrm{Soc}\,(X/X_{\alpha-1})$ for non-limit ordinal α and $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ for a limit ordinal α, then $X = \bigcup X_\alpha$. For each $x \in X_\alpha$, let $h(x)$ be the least ordinal α such that $x \in X_\alpha$. Note, that for $x \neq 0$ and $r \in W$, we have $h(rx) < h(x)$. Let x_n be the image of $1 \in {}_RR$ under the canonic homomorphism $\iota_n : {}_RR \to X$. Then $wx_n = x_n w = x_{n+1}$ and therefore we have, for some n, $x_n = 0$. Indeed, otherwise we would get a strictly decreasing sequence

$$h(x_1) > h(x_2) > \ldots > h(x_n) > \ldots .$$

But $x_n = 0$ implies that $1 \in {}_RR$ is mapped under some morphism $w^m : {}_RR \to {}_RR$ into 0, that is $w^m = 0$. This contradiction shows that W is, in fact, a nil ideal.

LEMMA III.2.6. *Let R be a ring with the radical W. Assume that, for some natural n, $W^n = W^{n+1} \neq 0$. Then there exists a non-zero module which has neither a minimal nor a maximal submodule.*

Proof. Define $X = \{r \in W^n \mid W^n r = 0\}$. Then X is a left R-module and has the property $W^n X = 0$. The module $M = {}_R(W^n/X)$ is non-zero, because otherwise $W^n = X$ would imply $W^n = W^{2n} = W^n X = 0$. Also, M has no maximal submodule. For, assuming the contrary, we get that ${}_R(W^n)$ has a maximal submodule. But $W \cdot W^n$ is contained in every maximal submodule of ${}_R(W^n)$, and therefore $W^{n+1} = W^n$ implies that ${}_R(W^n)$ has no maximal submodule. Let $s + X$

be in the socle of M. Then $W(s + X) \subseteq X$, and thus $Ws \subseteq X$. But $W^n = W^{n+1}$ implies

$$W^n s = W^n \cdot Ws \subseteq W^n X = 0 ;$$

consequently, $s \in X$, and M has no minimal submodule. This shows that M has the required properties.

PROPOSITION III.2.7. *A left balanced ring is left artinian.*

Proof. Let W be the radical of the left balanced ring R. Our first aim is to show the existence of a natural n with $W^n = W^{n+1}$. Let w_i be finitely many elements in W such that the set of the elements $w_i + W^2$ generates ${}_R(W/W^2)$. Such a set exists because R/W^2 is a finite direct sum of full matrix rings over local left balanced rings R_i (Proposition III.2.3), and if W_i is the radical of R_i, then ${}_R(W_i)$ is finitely generated (Proposition II.5.3). Observe also that the same references show that the set $\{w_i + W^n\}$ generates ${}_R(W/W^n)$, for any n. According to Lemma III.2.5, the elements w_i are nilpotent; therefore, there is a natural n with $w_i{}^n = 0$ for all i. It remains to show that $W^n/W^{n+1} = 0$. But this follows from the fact that R/W^{n+1} is a direct sum of full matrix rings over local rings and that its radical W/W^{n+1} is generated by the elements $w_i + W^{n+1}$ which satisfy $(w_i + W^{n+1})^n = 0$.

Now, $W^n = 0$. Otherwise, according to Lemma III.2.6, there exists a non-zero module which has neither minimal nor maximal submodules. But this is impossible, because of Lemma III.2.1.

Finally, the fact that R is artinian follows from another application of Proposition III.2.3 and Proposition II.5.3.

III.3. THE STRUCTURE OF BALANCED RINGS

The preceeding investigations allow to give a complete description of left balanced rings. The following theorem summarizes these results.

THEOREM III.3.1. *The following properties of a ring* R *are equivalent:*

(i) R *is left balanced.*

(ii) R *is left artinian and left finitely balanced.*

(iii) R *is a direct sum of a uniserial ring and finitely many full matrix rings over exceptional rings.*

Proof. The fact, that (i) implies (ii) is shown in Proposition II.2.7. The implication (ii) $\rightarrow$ (iii) follows from the Propositions III.2.3 and II.5.3. Finally, in order to prove (iii) $\rightarrow$ (i) let R be a direct sum of a uniserial ring R_o and finitely many full matrix rings $\underset{=}{R}_i$ over exceptional rings R_i (with $1 \leq i \leq n$). According to Proposition I.3.3, R_o is left balanced. Each of the rings R_i is left balanced because of Proposition II.4.6, and therefore Proposition III.1.4 implies that the rings $\underset{=}{R}_i$ are left balanced. Since R is the direct sum of R_o and the rings $\underset{=}{R}_i$, also R is left balanced. This establishes the theorem.

An immediate consequence of the structure theorem above is the fact that the opposite ring of a left balanced ring is again left balanced.

THEOREM III.3.2. *A ring is left balanced if and only if it is right balanced.*

Proof. The opposite ring of a uniserial or an exceptional ring is again uniserial or exceptional, respectively. Therefore, if a ring R satisfies the condition (iii) of Theorem III.3.1, also the opposite ring of R satisfies this condition.

Thus, we may simply speak of a balanced ring and drop the adjectives "left" and "right" in the notion of a balanced ring.

Another corrollary is the following statement.

THEOREM III.3.3. *A ring* R *with the radical* W *is balanced if and only if* R *is left artinian and* R/W^3 *is balanced.*

Proof. According to Theorem III.3.1, we have to show that a left artinian ring R satisfies the condition (iii), if R/W^3 satisfies the condition (iii). But if R is left artinian and R/W^3, or even R/W^2, is a direct sum of full matrix rings over local rings, then also R itself is a direct sum of full matrix rings $\underset{=i}{R}$ over local rings R_i. If W_i is the radical of R_i, then the condition (iii) for R/W^3 implies, that for all i, R_i/W_i^3 is either uniserial or exceptional. In the first case, R_i itself has to be uniserial. But if R_i/W_i^3 is exceptional, then $W_i^2/W_i^3 = 0$, so $W_i^2 = W_i^3$. Because R_i is left artinian, we may conclude $W_i^2 = 0$ and R_i itself is exceptional. This proves the theorem.

It should be noted that contrary to the case where R is finitely generated over its centre (see below), it is not enough in general to assume here that R/W^2 is balanced. In fact, it is easy to construct local left artinian

rings R with radical W such that R/W^2 is exceptional, whereas $W^2 \neq 0$; this will be done in Section III.7. Such a ring is, of course, not balanced.

REMARK III.3.4. The assumption in Theorem III.3.1 (ii) on the ring R to be left artinian is essential. It is well-known (see [2]) that every principal ideal domain is finitely balanced. In fact, it is not difficult to show that if R is a noetherian integral domain, then R is finitely balanced if and only if R is a Dedekind domain. Or, more generally, *a noetherian commutative ring is finitely balanced if and only if it is a direct sum of a uniserial ring and a finite number of Dedekind domains.* The sufficiency of the condition follows, in view of Lemma I.2.3, from the fact that every faithful finitely generated R-module over a Dedekind domain R possesses a direct summand which is isomorphic to a fractional ideal (see [7]) and a fractional ideal is always a generator. And, an R-module over a Dedekind domain R which is not faithful can be considered as a module over a uniserial ring and is therefore balanced, as well. Conversely, every finitely balanced ring is arithmetical, i.e. has a distributive ideal lattice. *) For, if I is a maximal ideal of such a ring R, then R/I^2 is, by Proposition II.1.1, uniserial. Thus, the ideals of the localization R_I are linearly ordered by inclusion and therefore R is arithmetical. But, a noetherian arithmetical ring is a direct sum of a uniserial ring and a finite number of Dedekind domains.

---

*) The authors are indebted to Professor CH. U. JENSEN for bringing to their attention his paper on Arithmetical rings, *Acta Math. Acad. Sci. Hung.* 17 (1966), 115-123 and suggesting the proof of the necessity of the result.

III.4. RINGS FINITELY GENERATED OVER THEIR CENTRES

In the case, where the ring R or at least the factor ring R/W, where W is the radical of R, is finitely generated over its centre, the description of balanced ring becomes simpler.

THEOREM III.4.1. *Let* R *be a ring finitely generated over its centre. Then* R *is balanced if and only if* R *is uniserial.*

Proof. We first show, that an exceptional ring cannot be finitely generated over its centre. For, assume R is exceptional with the radical W and the centre Z. Clearly, $(Z + W)/W$ is contained in the centre of $Q = R/W$. Let F be the quotient field of $(Z + W)/W$, considered as a subring of Q.

Again, we consider W as a Q-Q-bimodule, so we get the equation

$$fw = wf \quad \text{for} \quad w \in W ,$$

first for the elements $f \in (Z + W)/W$, and therefore also for all $f \in F$. If we assume that R is finitely generated as a Z-module, then R/W is a finite dimensional vector space over F. Let n be the dimension $\dim_F Q$. If $\dim_Q W = m$, then $\dim_F W = mn$. Since the dimension of W over F does not depend on whether we consider the left action or the right action of F on W, we conclude

$$\dim_Q W = \dim W_Q ,$$

contrary to the definition of an exceptional ring.

Let us now assume that R is an arbitrary balanced ring. According to Theorem III.3.1, R is a direct sum of a uniserial ring and a finite number of full matrix rings over exceptional rings R_i. If R is finitely generated over its centre, then any one of the rings R_i is finitely generated over its centre, but as we have seen above, this is impossible for an exceptional ring. So we conclude that R is uniserial.

Let us mention that Theorem III.3.4 applies immediately to the case of a finite dimensional algebra and to the case of a finite ring.

REMARK III.4.2. *Let* R *be a ring finitely generated over its centre; let* W *be its radical. Then* R *is balanced if and only if* R *is left artinian and* R/W^2 *is balanced.*

Proof. A left artinian ring R is uniserial if and only if R/W^2 is uniserial. Therefore, the theorem follows from Theorem III.4.1.

Our next aim is to consider the case, where R/W is finitely generated over its centre. We will need the following lemma which establishes some properties of the subring T_v of a local ring R (for the definition, see II.3).

LEMMA III.4.3. *Let* R *be a local ring with the radical* W *such that* $W^2 = 0$ *and* $\dim W_Q = 1$. *Let* v *be a non-zero element of* W. *Then*

$$T_v/W \cong R/W = Q \quad \textit{and} \quad \dim_{(T_v/W)} Q = \dim_Q W .$$

Proof. Write $T = T_v$ and define a mapping $\alpha : Q \to W$ by

$$\alpha(\tau + W) = v\tau \quad \text{for} \quad \tau \in R .$$

Furthermore, define a mapping $\beta : T/W \to Q$ by

$$\beta(\tau + W) = s + W \quad \text{for} \quad \tau \in T, \quad \text{where} \quad s \in R \quad \text{satisfies} \quad v\tau = sv .$$

Obviously, both α and β are well-defined bijections because of $\mathrm{Ann}(v) = W$ and $vR = W$. And, they are additive. Moreover, β is multiplicative; for, if $\beta(\tau_i + W) = s_i + W$ for τ_1 and τ_2 from T, then

$$v\tau_1\tau_2 = s_1 v\tau_2 = s_1 s_2 v ,$$

and therefore $\beta[(\tau_1 + W) \cdot (\tau_2 + W)] = \beta(\tau_1 + W) \cdot \beta(\tau_2 + W)$. This shows that β defines a ring isomorphism of T/W and Q. Now, the pair

$$(\alpha, \beta) : {}_{(T/W)}Q \to {}_{Q}W$$

satisfies, for any $r \in R$, $\tau \in T$ and $s \in R$ with $v\tau = sv$,

$$\alpha[(\tau + W)(r + W)] = \alpha(\tau r + W) = v\tau r =$$

$$= svr = (s + W) \cdot vr = \beta(r + W)\alpha(r + W) ,$$

which implies the required equality of dimensions.

If R is a local ring with

$$W^2 = 0 , \quad \dim W_Q = 2 \quad \text{and} \quad \dim {}_QW = 1 ,$$

then the Lemma III.4.3 shows that $\dim {}_{(T_v/W)}Q = 2$. But R is exceptional if and only if $\dim Q_{(T_v/W)} = 2$. Therefore, if Q has the property that any

division subring of left index 2, which is isomorphic to Q, has also right index 2, then the above conditions imply that R is exceptional. In particular, this leads to the following theorem.

THEOREM III.4.4. *Let* R *be a ring with radical* W. *Assume that* R/W *is finitely generated over its centre. Then* R *is the direct sum of a uniserial ring and finitely many full matrix rings over local rings* R_i *with*

$$W_i^2 = 0 \text{ and } \dim_{Q_i} W_i \times \dim_{W_i} Q_i = 2 ,$$

where W_i *is the radical of* R_i *and* $Q_i = R_i/W_i$.

Proof. It is sufficient to prove the statement for local rings because, if R is a direct sum of finitely many full matrix rings over local rings R_i and if R/W is finitely generated over its centre, then R_i/W_i is finitely generated over its centre for all i.

But if R is a local ring with radical W and R/W is finite dimensional over its centre, then for any division subring of $Q = R/W$, its right index is 2 if and only if its left index is 2 ([17], p. 158). So R is exceptional if and only if $W^2 = 0$ and $\dim_Q W \times \dim W_Q = 2$.

The last section will deal with the question whether there are rings with $W^2 = 0$ and $\dim_Q W \times \dim W_Q = 2$, which are not exceptional.

III.5. THE MODULE CATEGORY OF A BALANCED RING

It is shown in Proposition III.1.4 that the property of being left

balanced is Morita equivalent. This means that, if the category Mod R of all left R-modules is equivalent to the category Mod R' of all left R'-modules, then R is balanced if and only if R' is balanced. Here, we characterize explicitely the balanced rings R in terms of the module categories Mod R .

LEMMA III.5.1. *Let* R *be a local left artinian ring. Then* R *is balanced if and only if any two indecomposable left* R*-modules of length* 3 *are isomorphic and all other indecomposable left* R*-modules are uniserial. In this case, any two indecomposables of the same length are isomorphic.*

Proof . If R is balanced, then according to Proposition II.5.3, R is either uniserial or exceptional. For a local uniserial ring R , every indecomposable module is uniserial and any two indecomposables of the same length are isomorphic. For an exceptional ring, we may apply Propositions II.3.2 and II.4.3, or Propositions II.3.3 and II.4.5 to show that every indecomposable module of length $\neq 3$ is uniserial and that all indecomposables of the same length are isomorphic. This proves the necessity of the conditions, as well as the last statement.

In order to prove the sufficiency, let us first assume that all indecomposable left R-modules are uniserial. Then R is trivially left uniserial. And, R is right uniserial, too. For, otherwise $({}_RR \oplus {}_RR)/D$, where $D = R(u, v) + (W^2 \oplus W^2)$ with linearly independent elements u, v in $(W/W^2)_{R/W^2}$, is a non-uniserial indecomposable left R-module, according to Lemma II.2.2.

If there is a non-uniserial indecomposable R-module X of length 3 with a simple socle, then X is necessarily injective and R (being a monogenic indecomposable R-module) is left uniserial. Consequently, ${}_RR$ can be embedded in X and therefore $W^2 = 0$ and $\dim_Q W = 1$, where $Q = R/W$. If $\dim W_Q \geq 3$,

then Lemma II.2.2 would give us an indecomposable module I_3 of length 5 which is not uniserial. Hence $\dim W_Q = 2$. Proposition II.3.3 shows that R has to be exceptional and therefore balanced.

If there is a non-uniserial indecomposable R-module Y of length 3 with a non-simple socle, then Y/Rad Y is simple and thus, necessarily, $Y \cong {}_RR$. Consequently, $W^2 = 0$, $\dim {}_QW = 2$ and the indecomposable injective is uniserial (and of length 2). If $\dim W_Q \geq 2$, then Lemma II.2.2 would give us an indecomposable module I_2 of length 5 which is not uniserial. Hence we also have $\dim W_Q = 2$. Proposition II.3.2 shows that R has to be exceptional and therefore balanced.

As a consequence, we can describe the module category Mod R of a balanced ring. Here, we restrict to left artinian rings, because the property of a ring R to be left artinian can easily be described in terms of the module category Mod R .

THEOREM III.5.2. *Let* R *be a left artinian ring. Then* R *is balanced if and only if the category* Mod R *of all left* R*-modules is equivalent to a category* K *with the following properties*

(i) *the composition factors of any indecomposable object of* K *are isomorphic,*

(ii) *every indecomposable object of* K *with length* > 3 *is uniserial, and*

(iii) *any two indecomposable objects of* K *with length* 3 *and isomorphic composition factors are isomorphic.*

Proof. First, note that R is the direct sum of full matrix rings over local rings R_i if and only if condition (i) is satisfied in Mod R . And

then, Mod R satisfies (ii) and (iii) if and only if, for all i, the categories Mod R_i satisfy these conditions. Therefore, Theorem III.5.2 is an immediate consequence of Lemma III.5.1.

THEOREM III.5.3. *A balanced ring has only finitely many isomorphism types of indecomposable modules.*

Proof. A balanced ring is left artinian, therefore the length of the uniserial left R-modules is bounded. For any simple module S, all indecomposable left R-modules with composition factors isomorphic to S are uniserial or of length 3, and any two of them are isomorphic, if there length is equal. This follows from Lemma III.5.1. Since there is only a finite number of non-isomorphic simple R-modules, the number of isomorphism types of indecomposable left modules is finite. By considering the opposite ring, also the number of isomorphism types of indecomposable right modules is finite.

III.6. CENTRALIZERS OF INDECOMPOSABLE MODULES

The main result of this section asserts that, if R is a balanced ring, then the centralizer of every indecomposable R-module is balanced, as well. This is obvious when R is a local uniserial ring; for, in this case an indecomposable R-module is isomorphic to $_R(R/I)$ for a certain ideal I of R and the centralizer $C(R/I)$ is isomorphic to the uniserial (and hence balanced) ring R/I.

PROPOSITION III.6.1. *Let* R *be a local balanced ring. Then the centralizer of every indecomposable* R*-module* M *is a balanced ring.*

Proof. In view of the remark preceeding the proposition, we can assume that R is exceptional.

First, let W be the radical of R, $Q = R/W$, $W^2 = 0$, $\dim_Q W = 2$, $\dim W_Q = 1$ and $W = Rv + uT_v$ for two linearly independent elements u, v of ${}_Q W$. According to Proposition II.4.3, there are three types of indecomposable R-modules, viz. A_1, A_2 and A_3 and, obviously, only the type A_2 needs a consideration. Thus, let $M = R/Rv$ be the injective R-module. Clearly, its centralizer C equals to T_v/Rv; the latter is a local ring with the radical $\mathcal{W} = W/Rv$ and thus $\mathcal{W}^2 = 0$. Denote the elements $x + Rv$ of C simply by $\bar{x}$.

Now, in view of Lemma III.5.1, $\dim_{(T_v/W)} Q = 2$; thus, let $1 + W$ and $r + W$ be a basis of ${}_{(T_v/W)}Q$. Obviously, it is also a basis of $Q_{(T_v/W)}$. Therefore,

$$R = T_v + T_v r + W = T_v + rT_v + W .$$

From here, $Ru = T_v u + T_v ru$ and thus,

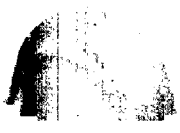
$$\mathcal{W} = (Ru \oplus Rv)/Rv = C\,\bar{u} \oplus C\,\overline{ru} .$$

Also, $W = Rv + uT_v$ implies readily that

$$\mathcal{W} = \bar{u}\,C .$$

Consequently, writing $\mathcal{Q} = C/\mathcal{W}$,

$$\dim_{\mathcal{Q}} \mathcal{W} = 2 \quad \text{and} \quad \dim \mathcal{W}_{\mathcal{Q}} = 1 .$$



Now, for $\bar{u} \in C$, define

$$T_{\bar{u}} = \{\bar{\tau} \mid \bar{\tau} \in C \text{ and } \overline{u\tau} \in C\bar{u}\}$$

and deduce from $Ru = T_v u + rT_v u$ that

$$W = W/Rv = C\bar{u} + \bar{r}C\bar{u} = C\bar{u} + \overline{ru}\, T_{\bar{u}} \quad ,$$

as required. We conclude that C is a balanced (exceptional) ring.

Secondly, in a similar manner, let the exceptional ring R satisfy $\dim_Q W = 1$, $\dim W_Q = 2$ and $W = vR + S_v u$ for two linearly independent elements u, v of W_Q. Again, this time in view of Proposition II.4.5, only R-modules of type B_3 require attention. Thus, let $M = {}_R(R \oplus R)/R(u, v)$ be the injective R-module. Lifting the endomorphisms of M to endomorphisms of ${}_R(R \oplus R)$, we deduce immediately that the elements of the centralizer C of M are induced by

$$(\kappa_{ij}) : {}_R(R \oplus R) \to {}_R(R \oplus R)$$

with $\kappa_{ij} \in R$, $1 \leq i, j \leq 2$ such that

$$u\kappa_{11} + v\kappa_{21} = \lambda u \quad \text{and} \quad u\kappa_{12} + v\kappa_{22} = \lambda v \quad \text{for some} \quad \lambda \in R \, .$$

It is easy to see that the radical W of the local ring C consists of all endomorphisms induced by such matrices with $\kappa_{ij} \in W$. In fact, since in this case there are $\rho_i \in R$, $i = 1, 2$, such that $\kappa_{i2} = \rho_i v$, one can see easily that

$$\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \kappa_{11} - \rho_1\kappa_{12} & 0 \\ \kappa_{21} - \rho_2\kappa_{22} & 0 \end{pmatrix}$$

induce the same endomorphisms. Consequently, there is a one-to-one correspondence between the elements of W and the matrices $\begin{pmatrix} w_1 & 0 \\ w_2 & 0 \end{pmatrix}$ with $w_i \in W$, and thus, we

can identify them.

Now, put

$$\varphi_1 = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} \in W \quad \text{and} \quad \varphi_2 = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \in W .$$

Obviously, given $w_i \in W$, there are $\kappa_{i1} \in R$ satisfying $\kappa_{i1} v = w_1$, $i = 1, 2$, and there are $\kappa_{i2} \in R$ such that

$$u\kappa_{12} + v\kappa_{22} = \lambda v ,$$

where λ is determined by $u\kappa_{11} + v\kappa_{21} = \lambda u$. Hence,

$$\begin{matrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{matrix} \varphi_1 = \begin{matrix} w_1 & 0 \\ w_2 & 0 \end{matrix} ,$$

and writing $Q = C/W$, we have

$$\dim_Q W = 1 .$$

Also, given $w_1 \in W$, we can find $\mu_{11} \in R$ and $\sigma \in S_v$ such that $w_1 = v\mu_{11} + \sigma u$, and $\mu_{12} \in R$ such that $v\mu_{12} = -\sigma v$. Moreover, it is easy to determine μ_{12} and μ_{22} to satisfy the respective equation and thus

$$\varphi_2 \begin{matrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{matrix} = \begin{matrix} w_1 & 0 \\ 0 & 0 \end{matrix} .$$

In a similar manner, given $w_2 \in W$, one can show that

$$\varphi_2 \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ w_2 & 0 \end{pmatrix} \quad \text{for suitable } \nu_{ij}\text{'s} .$$

Hence,

$$\dim W_Q = 2 .$$

Finally, define

$$S_{\varphi_1} = \{\varphi | \varphi \in C \text{ and } \varphi \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} C\} .$$

Thus, if $\sigma_{21} \in W$, $\sigma_{22} \in R$ arbitrary, $\sigma_{12} \in R$ and $\lambda_o \in S_n$ satisfying $u\sigma_{12} + v\sigma_{22} = \lambda_o v$, and σ_{11} such that $\lambda_o u = u\sigma_{11}$, then obviously (σ_{ij}) induces an element of S_{φ_1}.

Now, given $w_1, w_2 \in W$, determine σ_{22} by the relation $\sigma_{22}u = w_2$. Then

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \varphi_2 = \begin{pmatrix} \sigma_{12}v & 0 \\ w_2 & 0 \end{pmatrix} .$$

Therefore, since we can choose $\varphi \in C$ such that

$$\varphi_1 C = \begin{pmatrix} w_1 - \sigma_{12}v & 0 \\ 0 & 0 \end{pmatrix} ,$$

we conclude that

$$\varphi_1 C + S_{\varphi_1} \varphi_2 = W ,$$

as required. The proof of Proposition III.6.1 is completed.

THEOREM III.6.2. *Let* R *be a balanced ring and* M *an indecomposable* R*-module. Then the centralizer of* M *is a local balanced ring.*

Proof. In view of Theorem III.3.1 we may assume that R is the full matrix ring over a local ring R', where R' is either uniserial or exceptional. The rings R and R' are Morita equivalent, so M and the image M' of M

under a categorical isomorphism of Mod R onto Mod R' have isomorphic endomorphism rings. Applying Proposition III.6.1 for M' we see that the centralizer of M' is a local balanced ring.

THEOREM III.6.3. *Let* R *be a balanced ring and* M *a finitely generated injective* R*-module. Then the centralizer of* M *is a balanced ring.*

Proof. A finitely generated injective R-module M is a finite direct sum of indecomposable injective R-modules. It follows from Theorem III.3.1 that any two indecomposable direct summands are either isomorphic or have no non-trivial homomorphism from one to the other. Consequently, the endomorphism ring of M is a direct sum of full matrix rings over endomorphism rings of indecomposable injective R-modules. The latter are local balanced rings, and thus, again by Theorem III.3.1, the centralizer of M is balanced.

III.7. EXISTENCE OF EXCEPTIONAL RINGS

In this last section, exceptional rings are constructed and it is shown the relation to a problem in the theory of division rings.

LEMMA III.7.1. *Let* D *be a division ring with an isomorphic subring* D' *such that* $\dim_{D'}D = 2$. *Let* $\gamma : D \to D'$ *be an isomorphism. Denote by* R *the ring of all pairs* (a, b) *of elements of* D *with component-wise addition and the following multiplication*

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, \gamma(a_1)b_2 + b_1a_2) .$$

Then R *is a local ring, and denoting its radical by* W *and* $Q = R/W$, *we have*

$$W^2 = 0, \dim_Q W = 2 \quad \textit{and} \quad \dim W_Q = 1 .$$

Moreover, R *is exceptional if and only if* $\dim D_{D'} = 2$.

Proof. It is easy to see that (a, b) is a unit if and only if $a \neq 0$. Therefore, the radical W is given by $W = \{(0, b) \mid b \in D\}$ and R is a local ring. Obviously, $W^2 = 0$. If $(0, b) \in W$, then

$$R(0, b) = \{(0, d'b) \mid d' \in D'\} \quad \text{and} \quad (0, b)R = W .$$

The first equation shows that $\dim_Q W = \dim_{D'} D = 2$, the second that $\dim W_Q = 1$.

Now let $v = (0, 1) \in W$; then $T_v = \{\tau \in R \mid v\tau \in Rv\}$ is given by

$$T_v = \{(a, b) \mid a \in D' \text{ and } b \in B\} ,$$

and therefore, $\dim Q_{(T_v/W)} = \dim D_{D'}$. This implies that R is exceptional if and only if $\dim D_{D'} = 2$.

A division ring D with an isomorphic subring D' such that $\dim_{D'} D = \dim D_{D'} = 2$ can easily be constructed: Let D'' be an arbitrary division ring and denote by D the ring of quotients of the polynomial ring $D''[x]$ in one (commuting) indeterminate. The ring endomorphism $D''[x] \to D''[x]$ which fixes D'' and maps x onto x^2 can be extended to an endomorphism $D \to D$, and we denote its image by D'. Then, obviously, D and D' are isomorphic and $\dim_{D'} D = \dim D_{D'} = 2$. This yields the existence of exceptional rings.

As a consequence of the remark above we get the following theorem.

THEOREM III.7.2. *The following assertions are equivalent:*

(i) *There exists a local ring* R *with the radical* W *and* $Q = R/W$ *such that* $W^2 = 0$ *and* $\dim_Q W \times \dim W_Q = 2$, *which is not exceptional.*

(ii) *There exists a division ring* D *with an isomorphic subring* D' *such that* $\dim_{D'} D = 2$ *and* $\dim D_{D'} \neq 2$.

Proof. If (ii) is satisfied, then the ring constructed in LEMMA III.7.1 has all the properties mentioned in (i). Conversely, assume there exists a local ring R with the properties described in (i). We may assume $\dim_Q W = 2$ and $\dim W_Q = 1$; for, otherwise we may consider the ring opposite to R. If w is a non-zero element of W and $T_v = \{\tau \in R \mid v\tau \in Rv\}$, then it follows from Lemma III.4.3 that Q and T_v/W are isomorphic and that $\dim_{(T_v/W)} Q = 2$. Since R is not exceptional, we have $\dim Q_{(T_v/W)} \neq 2$. Therefore, $D = Q$ and $D' = T_v/W$ satisfy the conditions (ii).

In [5], P. M. Cohn has constructed an example of a division ring D with a division subring D' such that $\dim_{D'} D = 2$ and $\dim D_{D'} \neq 2$. Thus, the question is whether such a subring D' exists which is, in addition, isomorphic to D.

The last remark shows that the condition in Theorem III.3.3 that R/W^3 is balanced cannot be replaced by the condition that R/W^2 is balanced.

REMARK III.7.3. *There exists a local ring* R *with the radical* W *such that* $W^3 = 0$, *which is not balanced, although* R/W^2 *is balanced.*

Proof. We start with a division ring D with an isomorphic subring D' such that $\dim_{D'}D = 2 = \dim D_{D'}$, , and $\gamma : D \to D'$ is an isomorphism. Denote by R the ring of all triples (a, b, c) of elements of D with component-wise addition and the following multiplication

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1a_2, \gamma(a_1)b_2 + b_1a_2, \gamma[\gamma(a_1)]c_2 + \gamma(b_1)b_2 + c_1a_2) .$$

Then the radical W is given by $W = \{(0, b, c) \mid b, c \in D\}$. It is easy to see that $W^2 = \{(0, 0, c) \mid c \in D\}$ and $W^3 = 0$. Since R/W^2 is isomorphic to the ring constructed in LEMMA III.7.1, R/W^2 is exceptional. But $W^2 \neq 0$, and thus Proposition II.5.3 shows that R itself is not balanced.

IV. REFERENCES

[1] BASS, H. *Algebraic K-theory*. Benjamin, New York, 1968.

[2] BOURBAKI, N. *Algèbre*, Ch. VIII. Hermann, Paris, 1958.

[3] CAMILLO, V. P. Balanced rings and a problem of Thrall. *Trans. Amer. Soc.* 149 (1970), 143-153.

[4] CAMILLO, V. P. AND FULLER, K. R. Balanced and QF-1 algebras. *To appear.*

[5] COHN, P. M. Quadratic extensions of skew fields. *Proc. London Math. Soc.*11 (1961), 531-556.

[6] CUNNINGHAM, R. S. AND RUTTER, E. A., JR. The double centralizer property is categorical. *To appear.*

[7] CURTIS, C. W. AND REINER, I. *Representation theory of finite groups and associative algebras,* Interscience Publ., New York, 1962.

[8] DICKSON, S. E. AND FULLER, K. R. Commutative QF-1 rings are QF. *Proc. Amer. Math. Soc.*24 (1970), 667-670.

[9] DLAB, V. AND RINGEL, C. M. Rings with the double centralizer property. *To appear.*

[10] DLAB, V. AND RINGEL, C. M. On a class of balanced non-uniserial rings. *Math. Ann.* To appear.

[11] DLAB, V. AND RINGEL, C. M. Balanced local rings with commutative residue fields. *To appear.*

[12] DLAB, V. AND RINGEL, C. M. The structure of balanced rings. *To appear.*

[13] DLAB, V. AND RINGEL, C. M. Exceptional rings. *To appear.*

[14] FLOYD, D. R. On QF-1 algebras. *Pacific J.* 27 (1968), 81-94.

[15] FULLER, K. R. Double centralizers of injectives and projectives over artinian rings. *Illinois J.* 14 (1970), 658-664.

[16] FULLER, K. R. Primary rings and double centralizers. *Pacific J.* 34 (1970), 379-383.

[17] JACOBSON, N. *Structure of rings.* Coll. Publ. Amer. Math. Soc., Vol. 37, 1956.

[18] JANS, J. P. On the double centralizer condition. *Math. Ann.* 188 (1970), 85-89.

[19] MORITA, K. Duality for modules and its application to the theory of rings with minimum condition. *Sci. Rep. Tokyo Kyoiku Daigaku,* Sec. A 6 (1958), 83-142.

[20] MORITA, K. On algebras for which every faithful representation is its own second commutator. *Math. Z.* 69 (1958), 429-434.

[21] MORITA, K. AND TACHIKAWA, H. On QF-3 rings. *To appear.*

[22] NAKAYAMA, T. On Frobeniusean algebras. II. *Ann. of Math.* 42 (1941), 1-21.

[23] NESBITT, C. J. AND THRALL, R. M. Some rings theorems with applications to modular representations. *Ann. of Math.* 47 (1946), 551-567.

[24] TACHIKAWA, H. Duality theorem of character modules for rings with minimum condition. *Math. Z.* 68 (1958), 479-487.

[25] TACHIKAWA, H. On rings for which every indecomposable right module has a unique maximal submodule. *Math. Z.* 71 (1959), 200-222.

[26] TACHIKAWA, H. Double centralizers and dominant dimensions. *Math. Z.* 116 (1970), 79-88.

[27] THRALL, R. M. Some generalizations of quasi-Frobenius algebras. *Trans. Amer. Math. Soc.* 64 (1948), 173-183.