Math. Z. 129, 207–230 (1972)

Decomposition of Modules over Right Uniserial Rings

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One of the fundamental problems in the theory of rings is to characterize the rings of bounded representations type in terms of their structure. The problem has been solved for some classes of rings (for references, see Gabriel [8]); however, in general, it is still open. In the present paper, we give a complete solution of the problem for the class of (local) right uniserial rings with a commutative residue field.

Theorem. *Let R be a right uniserial ring with a commutative residue field $Q=R/W$. Then R is of bounded representation type if and only if R is also left uniserial or if $W^2=0$ and ${}_RW$ is of length $\leqq 3$. Moreover, if R is not uniserial and ${}_RW$ is of length 2 or of length 3, then there are just 3, or 5, isomorphism classes of finitely generated indecomposable left R-modules, respectively, and every left R-module is a direct sum of these. Similarly, if R is not uniserial and ${}_RW$ is of length 2 or of length 3, then there are just 3, or 5, isomorphism classes of finitely generated indecomposable right R-modules, respectively, and every right R-module is a direct sum of these.*

Here, by a *right uniserial* ring R we understand a right artinian ring whose right ideals are linearly ordered by inclusion. In particular, such a ring is local.

A ring R is said to be of *bounded, or finite, representation type* if the lengths of the finitely generated indecomposable left R-modules are bounded, or if there is only a finite number of finitely generated indecomposable left R-modules, respectively (cf. [4]). The apparent asymmetry in this definition is removed by Proposition 1.1. The fact that, for a semi-primary ring R, the number of the finitely generated indecomposable left R-modules equals the number of the finitely generated indecomposable right R-modules, implies in conjunction with a recent result of Tachikawa and Ringel [11] that, for a right artinian ring R of finite representation type, every right R-module is a direct sum of finitely generated indecomposable right R-modules, that the last statement of Theorem is a consequence of the preceeding one. We remark that Tachikawa-Ringel theorem is not used to derive direct decompositions of left

R-modules, but that these decompositions are constructed directly. Also, if $W^2=0$ and the length $\partial_R W=3$, five finitely generated indecomposable right R-modules can be described explicitely and an alternative proof of the fact that they are the only ones, based on Tachikawa's duality theory [10], can be given (Remark 6.4).

Now, semi-primary rings of finite representation type are necessarily left artinian, and thus of bounded representation type. And, Theorem asserts that, for right uniserial rings with a commutative residue field, also the converse, i.e. the Brauer-Thrall conjecture, holds. The proof of the essential part of this statement is given in § 5 and involves an extension of Roiter's method in [9]. There, Roiter proved the Brauer-Thrall conjecture for finite-dimensional algebras.

Throughout the paper, R denotes a right uniserial ring, W its radical and $Q=R/W$ the residue division ring which is always assumed to be commutative. Considering W/W^2 as a left or right vector space over Q, it is easy to see that if R is not left uniserial, Q must be infinite. In §§ 2, 3 and 4, we consider the case when $W^2=0$. First, the indecomposable injective left R-module is shown to be of length 2 in § 2. Then, in the case when the length $\partial_R W$ of ${}_R W$ is greater than 4, we construct an infinite family of non-isomorphic local left R-modules in § 3. And, in § 4, we give the description of the five indecomposable left R-modules in the case $\partial_R W=3$, and show that every left R-module is a direct sum of these modules. Let us remark that the case $\partial_R W=2$ was treated in [5]. Finally, in § 6, the investigation of rings R with $W^2 \neq 0$ is reduced to the case $W^2=0$, which completes the proof of Theorem.

We conclude the introduction with a brief remark on our notation. If A is a ring (with unity), all A-modules are assumed to be unital. The symbols ${}_A M$ or M_A will be used to underline the fact that M is a left or a right A-module, respectively. The length of M will be denoted by ∂M, the socle of M by $\operatorname{Soc} M$.

§ 1. Correspondence Between Left and Right A-Modules

In [2], Auslander and Bridger have defined a duality functor by using projective resolutions of finitely presented A-modules. Starting with a finitely presented left A-module ${}_A M$ and a finite presentation f of ${}_A M$, that is a morphism which gives rise to an exact sequence

$$ {}_A P \xrightarrow{f} {}_A Q \to {}_A M \to 0 $$

with ${}_A P$, ${}_A Q$ finitely generated projective, they apply the functor

$$ * = \operatorname{Hom}_A({}_A -, {}_A A_A) $$

to f and consider the cokernel M'_A of f^*,

$$Q^*_A \xrightarrow{f^*} P^*_A \to M'_A \to 0.$$

Obviously, this is a finite presentation of the *right* A-module M'_A. Also, starting with a finitely presented right A-module and its finite presentation g, the application of the functor $\mathrm{Hom}_A(-_A, {}_AA_A)$, which we denote again by $*$, leads to a finitely presented left A-module, namely $\mathrm{Cok}\, g^*$. For a finitely generated projective module ${}_AP$, we have ${}_AP^{**} \cong {}_AP$, and also for a finite presentation f: ${}_AP \to {}_AQ$ of ${}_AM$ we get $f^{**} \cong f$ (as morphisms). Thus, ${}_AM = \mathrm{Cok} f \cong \mathrm{Cok} f^{**}$, that is to say, it is possible to get ${}_AM$ back. However, it should be noted that starting with ${}_AM$, the module M'_A is not uniquely determined, since we may use another presentation; it is only determined up to a "stable equivalence" [2].

Auslander and Bridger usually assume that A is noetherian, but the above procedure works obviously for arbitrary rings. Moreover, for a semi-perfect ring A, the existence of projective covers [3] enables us to define a one-to-one correspondence between finitely presented indecomposable left A-modules and finitely presented indecomposable right A-modules. For, if ${}_AM$ is finitely presented, we consider only minimal presentations

$${}_AP \xrightarrow{f} {}_AQ \xrightarrow{p} {}_AM \to 0,$$

that is p and f are projective covers of ${}_AM$ and $\mathrm{Ker}\, p$, respectively. Two such minimal presentations f_1 and f_2 of ${}_AM$ are isomorphic (as morphisms); therefore also $f_1^* \cong f_2^*$ and $\mathrm{Cok} f_1^* \cong \mathrm{Cok} f_2^*$. Moreover, if ${}_AM$ is indecomposable and not projective, then it is easily seen that f^* is a minimal presentation of $\mathrm{Cok} f^*$. Now, ${}_AM$ is indecomposable if and only if f cannot be decomposed as $f = f_1 \oplus f_2, f_i$: $P_i \to Q_i$, with $P = P_1 \oplus P_2$ and a non-trivial decomposition $Q = Q_1 \oplus Q_2$. In this case, in view of the minimality, there is not even a decomposition $f = f_1 \oplus f_2$ with, say, f_2: $P_2 \to 0$ and $P_2 \neq 0$. We have $f = f_1 \oplus f_2$ if and only if $f^* = f_1^* \oplus f_2^*$; thus, if ${}_AM$ is indecomposable, also $\mathrm{Cok} f^*$ is indecomposable. Altogether we get a one-to-one correspondence between the finitely presented indecomposable left A-modules which are not projective and the finitely presented indecomposable right A-modules which are not projective. But the finitely generated indecomposable projective left A-modules and the finitely generated indecomposable projective right A-modules are in a one-to-one correspondence using the functor $*$. By means of this correspondence we can easily sharpen the result of Eisenbud and Griffith [7] to the following

Proposition 1.1. *Let A be semi-primary.*

(a) *The lengths of the finitely generated indecomposable left A-modules are bounded if and only if the lengths of the finitely generated indecomposable right A-modules are bounded.*

(b) *There is only a finite number of finitely generated indecomposable left A-modules if and only if there is only a finite number of finitely generated indecomposable right A-modules and, in this case, the numbers are equal.*

Moreover, if (a) *or* (b) *holds, then A is left and right artinian.*

Proof. (a) If the lengths of the finitely generated indecomposable left A-modules are bounded, then A is obviously left artinian. Assume that A is not right artinian, and write $A_A = P_A \oplus P'_A$, with an indecomposable P_A which is not artinian. For every natural n, we find a submodule K_n of P_A which is generated by n elements and cannot be generated by less than n elements. Let

$$Q_n \xrightarrow{f_n} P_A \to P_A/K_n \to 0$$

be a minimal presentation of P_A/K_n. Calculating a bound for the length of the left A-module $\operatorname{Cok} f_n^*$, we get

$$\partial \operatorname{Cok} f_n^* \geqq \partial\,_A Q_n^* - \partial\,_A P^* \geqq n - \partial\,_A P^*,$$

and thus (since $\partial\,_A P^*$ is finite), $n - \partial\,_A P^*$ can be arbitrarily large. This shows that A is right artinian. It remains to prove that also the lengths of the finitely generated indecomposable right A-modules are bounded. But these modules are finitely presented (because A is right artinian), and therefore they occur as $\operatorname{Cok} f^*$, where f are minimal presentations of finitely generated indecomposable left A-modules $_AM$, say

$$_A\bar{P} \to {_A\bar{Q}} \to {_AM}.$$

In particular, every $\operatorname{Cok} f^*$ is an epimorphic image of $\bar{P}_A^*$. But, by our assumption, there is a bound m such that every $_A\bar{Q}$ is generated by less than m elements, and consequently there is also a bound m' such that every submodule of $_A\bar{Q}$ is generated by less than m' elements (take, for example, $m' = m \cdot \partial\,_A A$). Therefore, also $_A\bar{P}$ and $\bar{P}_A^*$ are generated by less than m' elements. This proves (a).

(b) Now, assume that there is only a finite number of finitely generated indecomposable left A-modules. Then there is only a finite number of finitely presented indecomposable left A-modules, and therefore also only a finite number of finitely presented indecomposable right A-modules. Consequently, the semiprimary ring A is both left and right artinian. For, assume that A is not left (or right) artinian and let $_AP$ (or P_A) be an indecomposable direct summand of $_AA$ (or A_A) which is not artinian. If Q is a submodule of P of finite length, then P/Q is obviously finitely presented and $P \to P/Q$ is its projective cover. Now, every isomorphism $P/Q \to P/Q'$ can be lifted to an automorphism of P which maps Q onto Q' and thus $P/Q \cong P/Q'$ implies $\partial Q = \partial Q'$. Since P is not left (or right) artinian, there are submodules of P of arbitrarily large length and therefore there is an infinite number of non-isomorphic finitely presented indecomposable left (or right) A-modules, in contradiction to our hypothesis. Finally, finitely generated modules over an artinian ring are finitely presented, and hence (b) follows.

§ 2. Rings with $W^2=0$

As stated in the introduction, R stands always for a right uniserial ring, W for its radical and Q for its residue division ring R/W which is assumed to be commutative.

Here, in addition, we assume that $W^2=0$. Consequently, W can be considered as a left or a right vector space over Q. One of the main tools of our paper is the following generalization of the concept of an algebra over the field Q (cf. [5]).

Definition. A bimodule ${}_QV_Q$ over a field Q with a multiplication $\circ$ is said to be an *algebra*, if $(V,\circ)$ is a ring and if, for all $\kappa_1,\kappa_2\in Q$ and $v_1,v_2\in V$,

$$(\kappa_1 v_1)\circ(v_2\kappa_2)=\kappa_1(v_1\circ v_2)\kappa_2.$$

Thus, if $W^2=0$ and w_1 is an arbitrary (fixed) non-zero element of W, we can define a Q-isomorphism $\varphi\colon Q_Q\to W_Q$ by $\rho\varphi=w_1\rho$ and define a multiplication $\circ$ on W as follows

$$(w_1\rho_1)\circ(w_1\rho_2)=w_1\rho_1\rho_2 \qquad \text{for all } \rho_1,\rho_2\in Q.$$

One can see immediately that ${}_QW_Q$ is an algebra with respect to the operation $\circ$ and that $(W,\circ)$ is isomorphic to Q; it is therefore commutative and w_1 is its identity element. The proofs of the statements which follow will illustrate the use of this concept.

Lemma 2.1. *Let* $W^2=0$ *and* $0\neq w'\in W$ *with*

$$\lambda w'=w'\rho \qquad \textit{for some } \lambda,\rho\in R.$$

Then $\lambda w=w\rho$ *for all* $w\in W$.

Proof. Obviously, we can assume that $\lambda\notin W$ and $\rho\notin W$. Then, we consider the algebra $(W,\circ)$ with the identity $w_1=w'$ and, writing

$$\bar\lambda=\lambda+W\in Q, \qquad \bar\rho=\rho+W\in Q,$$

calculate

$$\begin{aligned}\lambda w=\bar\lambda w=(\bar\lambda w)\circ w'=\bar\lambda(w\circ w')=\bar\lambda(w'\circ w)=(\bar\lambda w')\circ w\\ =(w'\bar\rho)\circ w=w\circ(w'\bar\rho)=(w\circ w')\bar\rho=w\rho,\end{aligned}$$

as required.

Lemma 2.2. *Let* $W^2=0$ *and*

$$T=\{\tau\in R \mid w\tau\in R\,w \text{ for all } w\in W\}.$$

Then $\partial_R W=\partial Q_T$.

Proof. First, notice that

$$\partial_R W=\partial_Q W \quad \text{and} \quad \partial Q_T=\partial Q_{T/W}=\partial_{T/W} Q,$$

where T/W is obviously a subfield of Q. Thus, in order to establish the lemma, it is sufficient to show that

$$\partial_{T/W} Q=\partial_Q W.$$

Notice that, in view of Lemma 2.1,

$$T=\{\tau\in R \mid w_1\tau\in R w_1 \text{ for a (fixed) non-zero } w_1\in W\}.$$

Now, writing $\bar{\rho}=\rho+W\in Q$, $\bar{\lambda}=\lambda+W\in Q$ and $\bar{\tau}=\tau+W\in T/W$, define the morphisms $\alpha\colon Q\to W$ and $\beta\colon T/W\to Q$ by

$$\bar{\rho}\,\alpha=w_1\,\rho$$

and

$$\bar{\tau}\beta=\bar{\lambda} \quad \text{with } \lambda \text{ satisfying } w_1\tau=\lambda w_1.$$

Clearly, both α and β are well-defined bijections. In fact, it is easy to verify that α is an isomorphism between the additive groups of Q and W and β is a ring isomorphism of T/W and Q. Moreover, if $\rho\in R$, $\tau\in T$ and $\lambda\in R$ with $w_1\,\tau=\lambda\,w_1$, then $(\bar{\tau}\,\bar{\rho})\alpha=\overline{\tau\,\rho}\,\alpha=w_1\,\tau\,\rho=\lambda\,w_1\rho=\bar{\lambda}\,w_1\,\rho=(\bar{\tau}\,\beta)(\bar{\rho}\,\alpha)$, and this implies the required equality.

Proposition 2.3. *Let* $W^2=0$ *and* $\dim_Q W=s$. *Then the injective indecomposable left R-module* ${}_R E$ *is of length* 2.

Proof. First, let us show that the right action of Q on W is transitive on hyperplanes, i.e. on $(s-1)$-dimensional subspaces of ${}_Q W$. Let us choose a basis

$$w_1, w_2, \ldots, w_s \quad \text{of } {}_Q W,$$

take the hyperplane H generated by the vectors $w_1, w_2, \ldots, w_{s-1}$ and show that, for any given hyperplane H' generated by $s-1$ vectors

$$v_k=\sum_{j=1}^{s}\mu_{kj}\,w_j, \qquad 1\leqq k\leqq s-1,$$

there exists $\alpha\in Q$ such that $H\alpha=H'$; let

$$w_1\,\alpha=\sum_{j=1}^{s}\alpha_j\,w_j.$$

Notice that

$$w_i\,\alpha=(w_i\circ w_1)\,\alpha=(w_1\,\alpha)\circ w_i=\sum_{j=1}^{s}\alpha_j(w_i\circ w_j)=\sum_{j=1}^{s}\beta_{ij}w_j,$$

where β_{ij} are Q-linear combinations of α_j's. Now, for each $1\leqq i\leqq s-1$, α is required to satisfy

$$\sum_{j=1}^{s}\beta_{ij}w_j=w_i\alpha=\sum_{k=1}^{s-1}\kappa_{ik}v_k=\sum_{k=1}^{s-1}\sum_{j=1}^{s}\kappa_{ik}\mu_{kj}w_j,$$

yielding a homogeneous system of $s(s-1)$ linear equations

$$\beta_{ij}-\sum_{k=1}^{s-1}\kappa_{ik}\mu_{kj}=0$$

for unknowns α_j and κ_{ik} over Q. Since the number of the unknowns is $s+(s-1)^2=s(s-1)+1$, the system has a non-trivial solution. Moreover, it is easy to see that all α_j's cannot be zero and thus there exists a (non-zero) α with $H\alpha=H'$, as required.

Finally, in order to prove that ${}_RE=R/H$ is injective, it is sufficient to show that every morphism

$$\varphi\colon\ {}_RW\to{}_RE$$

can be extended to a morphism from ${}_RR$ to ${}_RE$. In view of the first part of our proof, we can assume that

$$w_i\varphi=0\quad\text{for}\ 1\leqq i\leqq s-1\quad\text{and}\quad w_s\varphi=\lambda w_s+H.$$

Then, taking $\rho\in Q$ such that $w_1\rho=\lambda w_1$ and making use of Lemma 2.1, it is easy to check that the right multiplication ${}_RR\xrightarrow{\rho}{}_RE$ is an extension of φ. The proof is completed.

§ 3. Rings with $W^2=0$ and $\dim_Q W\geqq 4$

The objective of this section is to prove the following

Proposition 3.1. *Let* $W^2=0$ *and* $\dim_Q W=s\geqq 4$. *Then there is an infinite number of non-isomorphic local left R-modules of the same length (equal to* $\partial_R R-2$*).*

Proof. In order to prove Proposition 3.1, we are going to investigate the right action of the field Q on the two-dimensional subspaces ${}_QP$ of ${}_QW$. Observe that two local R-modules of length $\partial_R R-2$, say R/P and R/P', are isomorphic if and only if there is $\alpha\in Q$ such that

$$P\alpha=P';$$

this follows immediately from the fact that an isomorphism between the modules can be lifted.

First assume that $s=4$. Consider the algebra $({}_Q W_Q, \circ)$ and choose a basis for ${}_Q W$. Obviously, there are two cases to be considered: We may assume that the basis is either of the form

$$w_1, w_2, w_3 = w_2^2, w_4 = w_2^3 \quad \text{with} \quad w_2^4 = \sum_{i=1}^{4} \pi_i w_i,$$

or of the form

$$w_1, w_2, w_3, w_4 = w_2 \circ w_3 \quad \text{with} \quad w_2^2 = \sum_{i=1}^{2} \rho_i w_i \quad \text{and} \quad w_3^2 = \sum_{i=1}^{4} \pi_i w_i.$$

Let us observe that there is an infinite subset $\{\kappa_1, \kappa_2, \ldots\}$ of non-zero elements κ_i of the field Q such that

$$\kappa_i + \kappa_j + \kappa_i \kappa_j \pi_4 \neq 0 \qquad \text{for every } i \neq j.$$

Correspondingly, there is an infinite number of distinct planes $P_i \subseteq {}_Q W$ generated by the vectors

$$w_1 \quad \text{and} \quad w_2 + \kappa_i w_3, \qquad i = 1, 2, \ldots.$$

We are going to show that all R/P_i are non-isomorphic.

Assuming the contrary, take $i \neq j$ and $\alpha \in Q$ such that $P_i \alpha \subseteq P_j$; let

$$w_1 \alpha = \sum_{i=1}^{4} \alpha_i w_i \quad \text{with} \quad \alpha_i \in Q.$$

We are going to show that necessarily $\alpha = 0$. First, since $w_1 \alpha \in P_j$, we have $\alpha_4 = 0$ and

$$\sum_{i=1}^{3} \alpha_i w_i = \mu_1 w_1 + \nu_1 (w_2 + \kappa_j w_3).$$

From here, $\mu_1 = \alpha_1$, $\nu_1 = \alpha_2$ and, consequently,

$$\alpha_3 = \alpha_2 \kappa_j.$$

Furthermore,

$$(w_2 + \kappa_i w_3)\alpha = \left(\sum_{i=1}^{3} \alpha_i w_i\right) \circ (w_2 + \kappa_i w_3) = \sum_{i=1}^{4} \beta_i w_i \in P_j,$$

and thus

$$\beta_4 = \alpha_2 \kappa_i + \alpha_3 + \alpha_3 \kappa_i \pi_4 = 0.$$

Therefore,

$$\alpha_2 (\kappa_i + \kappa_j + \kappa_i \kappa_j \pi_4) = 0,$$

and hence,

$$\alpha_2 = 0.$$

Thus, also $\alpha_3=0$. Consequently,

$$(w_2+\kappa_i w_3)\alpha=\alpha_1 w_2+\alpha_1 \kappa_i w_3 \in P_j.$$

Therefore,

$$\alpha_i w_2+\alpha_1 \kappa_i w_3=\mu_2 w_1+\nu_2(w_2+\kappa_j w_3),$$

and we get

$$\mu_2=0, \quad \nu_2=\alpha_1 \quad \text{and} \quad \alpha_1(\kappa_i-\kappa_j)=0.$$

Thus, $\alpha_1=0$ as required.

Now, if $s>4$, then we can always choose a basis of ${}_QW$ which contains a subbasis of one of the following three forms:

either (i) $w_1, w_2, w_3=w_2^2, w_4=w_2^3, w_5=w_2^4$,

or (ii) $w_1, w_2, w_3, w_4=w_2\circ w_3$ with $w_2^2=\rho_1 w_1+\rho_2 w_2$,

or (iii) $w_1, w_2, w_3, w_4=w_2\circ w_3, w_5=w_2^2$.

In either of these three cases, it is a matter of routine to check that the above method applies and to complete the proof of Proposition 3.1.

§ 4. Rings with $W^2=0$ and $\dim {}_QW=3$

Throughout this section, we shall always assume that the ring R satisfies the conditions $W^2=0$ and $\dim {}_QW=3$. And, $\{w_1, w_2, w_3\}$ will be a fixed basis of ${}_QW$.

In view of Proposition 2.3, we can formulate

Lemma 4.1. *Let $W^2=0$ and $\dim {}_QW=3$. Then there are just 4 isomorphism classes of local left R-modules L_1, L_2, L_3 and L_4, represented by the simple module ${}_RR/W$, injective module ${}_RR/(R w_1+R w_2)$, by ${}_RR/R w_1$ and by ${}_RR$, respectively.*

Lemma 4.2. *The left R-module*

$$X_5=({}_RR\oplus{}_RR)/(R(w_1,0)+R(0,w_1)+R(w_2,w_3))$$

does not possess epimorphic images of types L_3 or L_4; consequently, X_5 is an indecomposable module of length 5.

Proof. First, let $\varphi\colon X_5\to{}_RR$. Then, φ can be lifted to

$${}_RR\oplus{}_RR \xrightarrow{\binom{\alpha_1}{\alpha_2}} {}_RR$$

with $w_1\alpha_1=w_1\alpha_2=0$. Hence, α_1 and α_2 lie in W and φ cannot be surjective.

Second, let $\psi\colon X_5\to{}_RR/R w_1$. Again, ψ can be lifted to

$${}_RR\oplus{}_RR \xrightarrow{\binom{\beta_1}{\beta_2}} {}_RR$$

with

$$w_1\beta_1 \in R w_1, \quad w_1\beta_2 \in R w_1 \quad \text{and} \quad w_2\beta_1 + w_3\beta_2 \in R w_1.$$

Now, in view of Lemma 2.1,

$$w_2\beta_1 = \lambda_1 w_2 \quad \text{and} \quad w_3\beta_2 = \lambda_2 w_2, \lambda_1, \lambda_2 \in R;$$

therefore, $w_2\beta_1 = w_3\beta_2 = 0$ and we deduce again that β_1 and β_2 belong to W. Thus ψ is not surjective.

Finally, the fact that X_5 is indecomposable follows easily. For, if X_5 were decomposable, then it would be a direct sum of two local left R-modules one of which would be of type L_3 or L_4.

Lemma 4.3. *Let N be a simple submodule of a direct sum M of modules of type L_3. Then either M/N contains a submodule of type L_2 or M/N is a direct sum of a copy of X_5 and several copies of L_3.*

Thus, in particular, if $\{x, y, z\}$ is another basis of ${}_QW$, then

$$X = ({}_RR \oplus {}_RR)/(R(x, 0) + R(0, x) + R(y, z)) \cong X_5.$$

Proof. Obviously, we may assume that M is a finite direct sum. Let

$$M = (\bigoplus_n {}_RR)/D$$

with $D = \bigoplus_n R w_1$, be a representation of M. Let

$$N = R[(x_1, x_2, \ldots, x_n) + D].$$

Observe that the left ideal $L = R w_1 + \sum_{i=1}^{n} R x_i$ satisfies the relations $R w_1 \neq L \subseteq W$.

First, assume that $L \neq W$ and write, without loss of generality,

$$L = R w_1 + R x_1.$$

Then, for $2 \leq i \leq n$,

$$x_i = \kappa_i w_1 + \lambda_i x_1 \quad \text{with suitable } \kappa_i, \lambda_i \in R,$$

and we have

$$M/N = (\bigoplus_n {}_RR)/(D + R(x_1, x_2, \ldots, x_n)) = (\bigoplus_n {}_RR)/(D + R(x_1, \lambda_2 x_1, \ldots, \lambda_n x_1)).$$

Take $\rho_i \in R$ such that

$$x_1 \rho_i = -\lambda_i x_1 \quad \text{for } 2 \leq i \leq n,$$

and consider the isomorphism $\varphi\colon \bigoplus_n {}_RR \to \bigoplus_n {}_RR$ given by the following triangular matrix of right multiplication

$$\begin{pmatrix} 1 & \rho_2 & \rho_3 & \cdots & \rho_n \\ & 1 & & & 0 \\ & & 1 & & \\ 0 & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Clearly, in view of Lemma 2.1,

$$D_1 = [D + R(x_1, \lambda_2 x_1, \ldots, \lambda_n x_1)]\,\varphi = D + R(x_1, 0, \ldots, 0),$$

and thus

$$M/N \cong (\bigoplus_n {}_RR)/D_1$$

contains a submodule of type L_2, namely

$$R[(1, 0, \ldots, 0) + D] \cong {}_RR/L.$$

Thus, let $L = W$ and let

$$L = R\,w_1 + R\,x_1 + R\,x_2.$$

Then, there are elements ε_j, μ_j, $\nu_j \in R$ such that

$$\varepsilon_j w_1 + \mu_j x_1 + \nu_j x_2 = w_{j+1} \qquad \text{for } j = 1, 2.$$

Moreover, take α_j, β_j $(j = 1, 2)$ such that

$$x_1 \alpha_j = \mu_j x_1 \quad \text{and} \quad x_2 \beta_j = \nu_j x_2,$$

and consider the isomorphism $\psi\colon \bigoplus_n {}_RR \to \bigoplus_n {}_RR$ given by the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 & & \\ \beta_1 & \beta_2 & 0 & & \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ & & 0 & & 1 \end{pmatrix}.$$

Again, using Lemma 2.1, observe that

$$D' = [D + R(x_1, x_2, x_3, \ldots, x_n)]\,\psi = D + R(w_2, w_3, x_3, \ldots, x_n).$$

Now,

$$x_i = \varepsilon_i w_1 + \mu_i w_2 + \nu_i w_3 \quad \text{for } 3 \leqq i \leqq n;$$

take, for $3 \leqq i \leqq n$, α_i and β_i such that

$$w_2 \alpha_i = -\mu_i w_2 \quad \text{and} \quad w_3 \beta_i = -\nu_i w_3,$$

and consider the isomorphism $\eta\colon \bigoplus_n {}_R R \to \bigoplus_n {}_R R$ given by the triangular matrix

$$\begin{pmatrix} 1 & 0 & \alpha_3 & \dots & \alpha_n \\ & 1 & \beta_3 & \dots & \beta_n \\ & & 1 & & \\ & & & \ddots & 0 \\ & & 0 & & \\ & & & & 1 \end{pmatrix}.$$

It is easy to verify that

$$D_2 = D' \eta = D + R(w_2, w_3, 0, \dots, 0),$$

and, consequently, that

$$M/N \cong (\bigoplus {}_R R)/D_2 \cong X_5 \oplus K,$$

where K is the direct sum of $n-2$ R-modules of type L_3.

The final statement of Lemma 4.3 follows trivially.

Lemma 4.4. *Let $x \in \operatorname{Soc} X_5$. Then there exists a submodule of type L_3 in X_5 containing x.*

Proof. Without loss of generality, assume that the basis $\{w_1, w_2, w_3\}$ of ${}_Q W$ satisfies with respect to the multiplication of the algebra $(W, \circ)$ the following conditions: w_1 is the identity, $w_3 = w_2 \circ w_2$ and $w_3 \circ w_2 = \sum_{i=1}^{3} \pi_i w_i$; notice that $\pi_1 + \pi_2 \pi_3 \neq 0$.

Refer to Lemma 4.2 for the definition of X_5 and put

$$x = (x_1, x_2) + D$$

with $D = R(w_1, 0) + R(0, w_1) + R(w_2, w_3)$, $x_1 = \kappa w_3$ and $x_2 = \mu w_2 + \nu w_3$. If $\kappa = 0$, then

$$x \in R[(0, 1) + D] \cong {}_R R / R w_1.$$

Thus, assume $\kappa \neq 0$. Consider the homomorphism

$$\varphi\colon {}_R R \xrightarrow{(\alpha, \beta)} {}_R R \oplus {}_R R \xrightarrow{\varepsilon} X_5,$$

where

$$w_1 \alpha = (\kappa \pi_3 - \nu) w_1 + \kappa w_2,$$

$$w_1 \beta = (\mu - \kappa \pi_2) w_1 + \kappa w_3,$$

and ε is the natural epimorphism. Obviously,

$$w_1\varphi=[(\kappa\pi_3-\nu)w_1+\kappa w_2,(\mu-\kappa\pi_2)w_1+\kappa w_3]\varepsilon=0,$$

$$\begin{aligned}w_2\varphi&=[(\kappa\pi_3-\nu)w_2+\kappa w_3,\kappa\pi_1 w_1+\mu w_2+\kappa\pi_3 w_3]\varepsilon\\&=[\kappa w_3+(\kappa\pi_3-\nu)w_2,(\mu w_2+\nu w_3)+\kappa\pi_1 w_1+(\kappa\pi_3-\nu)w_3]\varepsilon\\&=(x_1,x_2)+D,\end{aligned}$$

and

$$w_3\varphi=(\ldots,\lambda_1 w_1+\kappa(\pi_1+\pi_2\pi_3)w_2+\lambda_3 w_3)\varepsilon\neq 0,$$

because $\kappa(\pi_1+\pi_2\pi_3)\neq 0$. Consequently, $R\varphi$ is of type L_3 and $x\in R\varphi$, as required.

Lemma 4.5. *Let $x\in\mathrm{Soc}(P\oplus Q)$, where $P\cong X_5$ and $Q\cong L_3$. Then there exists a submodule of type L_3 in $P\oplus Q$ containing x.*

Proof. Again, assume that the basis $\{w_1,w_2,w_3\}$ of ${}_QW$ satisfies with respect to the multiplication $\circ$ the same conditions as in the proof of Lemma 4.4.

Consider the following representation of $P\oplus Q$:

$$P\oplus Q\cong({}_RR\oplus{}_RR\oplus{}_RR)/D,$$

where $D=R(w_1,0,0)+R(0,w_1,0)+R(w_2,w_3,0)+R(0,0,w_1)$; let

$$x=(x_1,x_2,x_3)+D.$$

In view of Lemma 4.4, we may assume that $x_3\neq 0$. Furthermore, we may obviously assume that $Rw_1+\sum_{i=1}^{3}Rx_i=W$. But then it is easy to follow the method of the proof of Lemma 4.4 and to verify that there is an automorphism φ of ${}_RR\oplus{}_RR\oplus{}_RR$ such that

$$D'=D\varphi=D+R(0,w_2,w_3).$$

Thus,

$$M'=(P\oplus Q)/Rx\cong({}_RR\oplus{}_RR\oplus{}_RR)/D'.$$

Now, consider the following mapping

$${}_RR\xrightarrow{(\alpha,\beta,\gamma)}{}_RR\oplus{}_RR\oplus{}_RR\xrightarrow{\varepsilon}M',$$

where $w_1\alpha=w_1$, $w_1\beta=\pi_3 w_1+w_2$, $w_1\gamma=-\pi_2 w_1+w_3$ and ε is the natural epimorphism. Obviously,

$$w_1\varphi=(w_1,\pi_3 w_1+w_2,-\pi_2 w_1+w_3)\varepsilon=0,$$

$$w_2\varphi=(w_2,\pi_3 w_2+w_3,\pi_1 w_1+\pi_3 w_3)\varepsilon=0$$

and

$$w_3 \varphi = (w_3, \ldots, \ldots)\varepsilon \neq 0.$$

Thus, ${}_R R\varphi = R(m + Rx)$ is an injective submodule of $(P \oplus Q)/Rx$. But since $P \oplus Q$ does not contain any injective submodule, $x \in Rm$ and Rm is of type L_3, as required.

Lemma 4.6. *Let M be a direct sum of copies of X_5. Then every socle element of M is contained in a submodule of type X_5, and for that matter, in a submodule of type L_3.*

Proof. First, consider the case when $M = P \oplus Q$ is the direct sum of two copies P, Q of X_5. Let $x = (p, q)$ be an element of Soc M. By Lemma 4.4, q belongs to a submodule Q' of Q of type L_3 and thus x belongs to $P \oplus Q'$. Consequently, in accordance with Lemma 4.5, x belongs to a submodule $N \subseteq M$ of type L_3. Therefore, M/Rx contains an injective submodule N/Rx (of type L_2). It follows that

$$M/Rx = N/Rx \oplus C$$

for some complement C. We want to show that C also contains a submodule of type L_2. Let $P = Rm_1 + Rm_2$, $Q = Rm_3 + Rm_4$ with $\partial Rm_i = 3$ for all $1 \leqq i \leqq 4$. Then three of the m_i's generate together with N the entire module M, say

$$M = Rm_1 + Rm_2 + Rm_3 + N = P + Rm_3 + N,$$

and thus C is an epimorphic image of $P + Rm_3$. Since $\partial(P + Rm_3) = 8$ and $\partial C = 7$, there is a socle element $y \in \mathrm{Soc}(P + Rm_3)$ such that

$$C \cong (P + Rm_3)/Ry.$$

But, by Lemma 4.5, y belongs to a submodule of $P + Rm_3$ of type L_3, and therefore C contains a submodule of type L_2. Summarizing,

$$M/Rx = I_1 \oplus I_2 \oplus C' \quad \text{with } I_1 \text{ and } I_2 \text{ of type } L_2.$$

If we lift the homomorphism

$${}_R R \oplus {}_R R \xrightarrow{\varepsilon} I_1 \oplus I_2 \xrightarrow{\iota} M/Rx,$$

with the canonic epimorphism ε and imbedding ι, to a homomorphism ${}_R R \oplus {}_R R \to M$, then both copies of ${}_R R$ are mapped onto submodules of type L_3 which intersect in Rx; this follows from the fact that M has no submodules of type L_2. Now, according to Lemma 4.3, the latter two submodules of M generate a submodule of type X_5 which contains x. And, the lemma is proved in the case that M is the direct sum of two copies of X_5.

In the general case, we may assume that M is a finite direct sum; for, every element of M has only a finite number of non-zero components. Thus, let

$$x=(x_1, x_2, \dots, x_n)\in \operatorname{Soc} M=\operatorname{Soc}(\bigoplus_n X_5).$$

Assuming, by induction, that $(x_1, x_2, \dots, x_{n-1})\in \operatorname{Soc}(\bigoplus_{n-1} X_5)$ belongs to a submodule $P'\subseteq \bigoplus_{n-1} X_5$ of type X_5,

$$x=((x_1, x_2, \dots, x_{n-1}), x_n)\in \operatorname{Soc}(P'\oplus X_5).$$

But then, by the first part of the proof, x belongs to a submodule of M of type X_5, as required.

The last statement of Lemma 4.6 is an immediate consequence of Lemma 4.4.

Lemma 4.7. *Let $x\in \operatorname{Soc}(P\oplus Q)$, where P is a direct sum of modules of type X_5 and Q is a direct sum of modules of type L_3. Then there exists either a submodule of type L_3 in $P\oplus Q$ containing x or there is an automorphism φ of $P\oplus Q$ which is the identity on P and which satisfies $x\varphi\in Q$.*

Proof. Since x has only a finite number of non-zero components and since an automorphism of a direct summand extends to an automorphism of the entire module, we may assume that both P and Q are finite direct sums. Write $x=p+q$ with $p\in P$ and $q\in Q$. Then, by Lemma 4.6, p belongs to a submodule P' of P of type X_5. Assuming that the lemma is true for $P'\oplus Q$, then either x belongs to a submodule of type L_3 in $P'\oplus Q\subseteq P\oplus Q$, or else there is an automorphism φ of $P'\oplus Q$ which is the identity on P' and which satisfies $x\varphi\in Q$. Obviously, we can extend φ to an automorphism of $P\oplus Q$ which is the identity on P. It follows that it is sufficient to consider the case when P is of type X_5 and Q is a finite direct sum of modules of type L_3. Let

$$P\oplus Q=(\bigoplus_n {}_R R)/D,$$

with $D=\bigoplus_n R\,w_1+R(w_2, w_3, 0, \dots, 0)$. In view of Lemmas 4.4 and 4.5, we may assume that $n\geqq 4$. Let

$$x=(x_1, x_2, \dots, x_n)+D.$$

Without loss of generality, assume that

$$x_i\in R\,w_2+R\,w_3 \quad \text{for } 1\leqq i\leqq n.$$

First, suppose that there are two linearly independent elements among $x_3, x_4, \dots, x_n$; say, x_3 and x_4. Then

$$x_j=\kappa_j x_3=\lambda_j x_4 \quad \text{for } j=1, 2,$$

and there are elements $\rho_j, \sigma_j \in R$ such that

$$x_3 \rho_j = -\kappa_j x_3 \quad \text{and} \quad x_4 \sigma_j = -\lambda_j x_4 \qquad (j=1,2).$$

Consider the automorphism of $\bigoplus_n {}_R R$ defined by the triangular matrix

$$\begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ \rho_1 & \rho_2 & 1 & & & 0 & \\ \sigma_1 & \sigma_2 & 0 & 1 & & & \\ & & & & 1 & & \\ & & 0 & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

acting from the right. In view of Lemma 2.1, it maps D into D and thus induces an automorphism φ of $P \oplus Q$. It is easy to see that φ acts as the identity on P and that

$$[(x_1, x_2, \ldots, x_n) + D]\,\varphi = (0, 0, x_3, \ldots, x_n) + D \in Q.$$

Second, if all $x_3, x_4, \ldots, x_n$ are multiples of a certain x_k, then

$$x_i = \lambda_i x_k = x_k \rho_i \quad \text{with suitable } \lambda_i, \rho_i \in R \quad \text{for } 3 \leqq i \leqq n.$$

We may assume that not all ρ_i's belong to W. Consider

$$m = (0, 0, \rho_3, \rho_4, \ldots, \rho_n) + D.$$

Obviously, by Lemma 2.1,

$$x_k m = (0, 0, x_k \rho_3, x_k \rho_4, \ldots, x_k \rho_n) + D = (0, 0, x_3, x_4, \ldots, x_n) + D,$$

and thus $x \in P + Rm$. Since, again according to Lemma 2.1, $w_1 m = 0$ and since $P \oplus Q$ contains no injective submodule, necessarily $\partial R m = 3$; hence, Rm is of type L_3. Consequently, in view of Lemma 4.5, x belongs to a submodule of type L_3, as required.

Lemma 4.8. *Let M be a direct sum of submodules of types L_3, L_4 and X_5. Then every socle element of M is contained in a submodule of type L_3 or L_4.*

Proof. Write

$$M = (\bigoplus_I {}_R R)/D, \quad \text{where } D \subseteq \bigoplus_I {}_R W \text{ with an index set } I.$$

Since, Soc $M = $ Rad M, we have

$$\operatorname{Soc} M = (\bigoplus_I {}_R W)/D,$$

and thus $x \in \operatorname{Soc} M$ can be written in the form

$$x = (w_i) + D \qquad \text{with } (w_i) \in \bigoplus_I {}_R W.$$

Take a fixed non-zero $w \in W$ and define the element

$$(\rho_i) \in \bigoplus_I {}_R R$$

as follows: If $w_i = 0$, then $\rho_i = 0$; if $w_i \neq 0$, then ρ_i satisfies the relation $w\rho_i = w_i$. Obviously,

$$w(\rho_i) = (w_i),$$

and thus

$$w[(\rho_i) + D] = (w_i) + D = x.$$

Since M contains no submodules of type L_2, the submodule

$$R[(\rho_i) + D] \subseteq M$$

containing x must be of type L_3 or L_4.

Proposition 4.9. *Let $W^2 = 0$ and $\partial_R W = 3$. Then there are 5 isomorphism classes of indecomposable left R-modules L_1, L_2, L_3, L_4 and X_5, defined in Lemmas* 4.1 *and* 4.2. *Moreover, every left R-module is a direct sum of these modules.*

Proof. To prove our proposition, we shall show that every left R-module M can be expressed as a direct sum of modules of types L_1, L_2, L_3, L_4 and X_5.

First, take a submodule I of M which is maximal with respect to the property of being a direct sum of modules of type L_2. Since I is injective, M is a direct sum of I and a submodule of M which contains no submodules of type L_2.

Thus, assume that M does not contain any submodule of type L_2 and let X be a submodule of M which is maximal with respect to the property of being a direct sum of modules of type X_5. Then, let Y_3 be a submodule of M which is maximal with respect to the property of being a direct sum of modules of type L_3 which intersects the submodule X trivially. Furthermore, let Y_4 be a submodule of M which is maximal with respect to the property of being a direct sum of modules of type L_4 which intersects the submodule $X \oplus Y_3$ trivially. And finally, let Z be a complement of the socle $\operatorname{Soc}(X \oplus Y_3 \oplus Y_4)$ in $\operatorname{Soc} M$. We want to show that

$$X \oplus Y_3 \oplus Y_4 \oplus Z = M.$$

Assume the contrary, i.e. that there is an element

$$m \in M \setminus (X \oplus Y_3 \oplus Y_4 \oplus Z).$$

Then Rm must be of type L_3 or L_4, because $m \notin \operatorname{Soc} M$ and M contains no submodules of type L_2.

First, consider the case that Rm is of type L_3 and that

$$\partial[(X \oplus Y_3) \cap Rm] = 1.$$

Then, the submodule $N = (X \oplus Y_3) + Rm \subseteq M$ is isomorphic to

$$(X \oplus Y_3 \oplus Q)/Rz,$$

where Q is of type L_3 and $z \in \operatorname{Soc}(X \oplus Y_3 \oplus Q)$. Hence, by Lemma 4.7, N either contains a submodule of type L_2, which is impossible, or there is an automorphism φ of $X \oplus Y_3 \oplus Q$ which is the identity on X and which satisfies $z\varphi \in Y_3 \oplus Q$. In the latter case,

$$(X \oplus Y_3 \oplus Q)/Rz \cong X \oplus [(Y_3 \oplus Q)/R(z\varphi)];$$

however, in view of Lemma 4.3, $(Y_3 \oplus Q)/R(z\varphi)$ contains either a submodule of type L_2 which is impossible, or a copy of X_5, in contradiction to maximality of X.

Thus, consider the second case when Rm is still of type L_3, but

$$\partial[(X \oplus Y_3) \cap Rm] = 2.$$

Writing

$$N' = (X \oplus Y_3) + Rm \cong (X \oplus Y_3 \oplus Q)/(Rz_1 \oplus Rz_2),$$

where Q is of type L_3 and $z_1, z_2 \in \operatorname{Soc}(X \oplus Y_3 \oplus Q)$, we deduce from Lemma 4.7 that there is an automorphism φ of $X \oplus Y_3 \oplus Q$ which is the identity on X and which satisfies $z_1\varphi \in Y_3 \oplus Q$. This follows from the obvious fact that z_1 cannot belong to a submodule of $X \oplus Y_3 \oplus Q$ of type L_3. Now,

$$N' \cong [(X \oplus Y_3 \oplus Q)/R(z_1\varphi)]/R(z_2\varphi).$$

By Lemma 4.3, $(X \oplus Y_3 \oplus Q)/R(z_1\varphi)$ contains either a submodule of type L_2 or a submodule of type X_5. By passage to the quotient module N', a submodule of type L_2 cannot avoid the kernel $R(z_2\varphi)$; but then it produces a simple direct summand of N', a contradiction. Thus, the only possibility is that $(X \oplus Y_3 \oplus Q)/R(z_1\varphi)$ contains a submodule of type X_5 which, in view of maximality of X, cannot avoid the kernel $R(z_2\varphi)$. Thus, N' contains a submodule isomorphic to a quotient module of X_5 by a simple submodule. But such a quotient module contains, according to Lemma 4.4, a submodule of type L_2. This contradiction completes the first part of the proof.

Second, consider the case when Rm is of type L_4. Since

$$(X \oplus Y_3 \oplus Y_4) \cap Rm \neq 0,$$

there is, according to Lemma 4.8, a submodule $Ra \subseteq X \oplus Y_3 \oplus Y_4$ of type L_3 or L_4 such that

$$Ra \cap Rm \neq 0.$$

Let $r_1 a = r_2 m \neq 0$ for some $r_1, r_2 \in R$. Obviously, r_1 and r_2 are non-zero elements of W and thus, since R is right uniserial, there exists $\rho \in R$ with $r_2 \rho = r_1$. Put $b = m - \rho a$. Then

$$r_2 b = r_2 m - r_2 \rho a = 0,$$

and thus $\partial Rb \leqq 3$. Consequently, in view of the first part of the proof, $Rb \subseteq X \oplus Y_3 \oplus Y_4 \oplus Z$, and thus

$$Rm \subseteq Ra + Rb \subseteq X \oplus Y_3 \oplus Y_4 \oplus Z,$$

as required.

§ 5. Brauer-Thrall Conjecture

We have seen in § 1 that, under certain assumptions, R is not of finite representation type, and we want to deduce that R is not even of bounded representation type.

Roiter [9] has shown that every finite-dimensional algebra of bounded representation type is of finite representation type. In his paper, he remarked that the same conclusion holds for an arbitrary left artinian ring and that, to prove it, only the proof of a certain lemma requires slight modifications. However, in order to exclude at least the case where one of the indecomposable injectives is not finitely generated, it is clear that already the statement of that lemma has to be modified. And thus it remains open whether there are left artinian rings which are of bounded representation type, but not of finite representation type.

Here, we shall show that, for a right uniserial ring R with $W^2 = 0$ and commutative R/W, Roiter's method can be used. To this end, the following lemma is crucial.

Lemma 5.1. *Let R be left artinian and $W^2 = 0$. Given a finitely generated left R-module ${}_RM$, there is a local ring T and a bimodule structure ${}_RM_T$ such that, for every finitely generated left R-module ${}_RN$, the left T-module $\mathrm{Ext}^1_R({}_RM_T, {}_RN)$ is finitely generated and is annihilated by the radical $\mathrm{Rad}\, T$.*

Proof. Assume that ${}_RM \neq 0$ and put

$$T = \{\tau \in R \mid w\tau \in Rw \text{ for all } w \in W\}.$$

We will show that M can be considered as a right T-module and, moreover, that M_T is finitely generated and ${}_RM_T$ is a bimodule. For, let

$${}_RM = (\bigoplus_n {}_RR)/D, \qquad \text{where } D \subseteq \bigoplus_n {}_RW.$$

Then the scalar $n \times n$ matrices

$$\begin{pmatrix} \tau & & 0 \\ & \tau & \\ 0 & & \ddots \\ & & & \tau \end{pmatrix} \quad \text{with } \tau \in T,$$

form a subring of $\operatorname{End}(\bigoplus_n {}_R R)$ isomorphic to T, and all these endomorphisms map, in view of Lemma 2.1, D into D. Now, according to our assumption, $\partial_R W$ is finite and thus R_T is finitely generated by Lemma 2.2. Consequently, also

$$M_T = (\bigoplus_n R_T)/D$$

is finitely generated.

Now, we want to prove that, for an arbitrary finitely generated R-module ${}_R N$, the left T-module $\operatorname{Ext}^1_R({}_R M_T, {}_R N)$ is finitely generated and annihilated by the radical $\operatorname{Rad} T$ of T. Denote by $\mathrm{E}\,{}_R N$ the injective hull of ${}_R N$ and consider the exact sequence

$$0 \to {}_R N \to \mathrm{E}\,{}_R N \to (\mathrm{E}\,{}_R N)/N \to 0,$$

which gives rise to the exact sequence of left T-modules

$$\operatorname{Hom}_R({}_R M_T, (\mathrm{E}\,{}_R N)/N) \to \operatorname{Ext}^1_R({}_R M_T, {}_R N) \to \operatorname{Ext}^1_R({}_R M_T, \mathrm{E}\,{}_R N).$$

Obviously, since $\mathrm{E}\,{}_R N$ is injective, the last term is zero. Consequently, the left T-module $\operatorname{Ext}^1_R({}_R M_T, {}_R N)$ is an epimorphic image of

$${}_T H = \operatorname{Hom}_R({}_R M_T, (\mathrm{E}\,{}_R N)/N),$$

and therefore it is sufficient to prove the assertion for ${}_T H$. Since the injective indecomposable module $\mathrm{E}\,{}_R Q$ is of finite length, also $(\mathrm{E}\,{}_R N)/{}_R N$ is of finite length; hence, $(\mathrm{E}\,{}_R N)/{}_R N$ is isomorphic to a finite direct sum $\bigoplus_m {}_R Q$. Also, using the above representation $M = (\bigoplus_n {}_R R)/D$, one can see easily that every homomorphism

$${}_R M \to (\mathrm{E}\,{}_R N)/{}_R N$$

factors through $\bigoplus_n ({}_R R/{}_R W) = \bigoplus_n {}_R Q$, which is again an R-T-bimodule decomposition. Thus, we get the T-isomorphisms

$${}_T H \cong \operatorname{Hom}_R(\bigoplus_n {}_R Q_T, \bigoplus_m {}_R Q) \cong \bigoplus_{mn} \operatorname{Hom}_R({}_R Q_T, {}_R Q),$$

where the last T-module $\operatorname{Hom}_R({}_R Q_T, {}_R Q)$ is obviously annihilated by $W = \operatorname{Rad} T$ and, according to Lemma 2.2, finitely generated.

Now, having proved that, for any two finitely generated left R-modules ${}_R M$ and ${}_R N$, always $\operatorname{Ext}^1_R({}_R M_T, {}_R N)$ can be considered as a finite-dimensional left vector space over $T/\text{Rad}\,T$, only a slight modification of the proof in [10] (consisting in replacing the base field by the division ring $T/\text{Rad}\,T$) is required to establish the following assertion.

If, under the conditions of Lemma 5.1, $M_1, M_2, \ldots, M_l, N$ are finitely generated left R-modules, then there exists a natural number n_0 such that, for every exact sequence of left R-modules

$$0 \to N \to X \to \bigoplus_{i=1}^{l} \bigoplus_{n_i} M_i \to 0 \text{ with } n_i \geqq 0 \text{ and } \partial_R X > n_0,$$

there is a direct decomposition of the left R-module X of the form $X = Y \oplus M_i$ for some i.

Now, using this assertion, the method of [9] yields that R is in this case of bounded representation type if and only if it is of finite representation type. Therefore, if R is left artinian, $W^2=0$ and $\partial_R W \geqq 4$, then, by Proposition 3.1, R is not of bounded representation type. But, if R is not left artinian (and $W^2=0$), then R has obviously local left R-modules of arbitrary finite length. Consequently, we get

Proposition 5.2. *If $W^2=0$ and $\partial_R W \geqq 4$, then R is not of bounded representation type.*

§ 6. Rings with $W^2 \neq 0$

In this final section, we are going to reduce the investigation of rings R with $W^2 \neq 0$ to the case when $W^2=0$. First, we need the following

Lemma 6.1. *Let $W^2 \neq 0$, $W^3=0$ and $\partial_R(W/W^2) \geqq 2$. Then $\partial_R W^2 \geqq 4$.*

Proof. Since R is right uniserial and $W^3=0$, R has only two proper right ideals, namely W and W^2, and these are the only two-sided ideals of R.

Now, for $a \in W \setminus W^2$, the left annihilator $l(a)$ is just W^2. For, since $a \in W$, obviously $W^2 \subseteq l(a)$. On the other hand, if $b \in l(a)$, then

$$bW = baR = 0,$$

because $W = aR$. Thus b belongs to the right socle of R which equals W^2.

Let $S = R/W^2$ and consider W as a left S-module. Evidently

$$\text{Rad}_S W = \text{Rad}_R W = W^2.$$

As we have shown, every element $a \in {}_S W \setminus \text{Rad}_S W$ has zero-annihilator in S and thus,

$$Sa \cong {}_S S.$$

Therefore, if $\partial_S(W/W^2)=\partial_R(W/W^2)\geqq 4$, then

$$\partial\,\mathrm{Soc}(S\,a)=\partial\,\mathrm{Soc}_S S\geqq 4$$

and hence

$$\partial_R W^2=\partial_S W^2=\partial\,\mathrm{Soc}_S W\geqq 4.$$

As a consequence, we may assume that

$$\partial_S(W/W^2)=2 \text{ or } 3.$$

But then, we can apply the decomposition theory to ${}_S W$. Since for every $a\in{}_S W\setminus \mathrm{Rad}\,{}_S W$, $S\,a\cong{}_S S$, it follows from Proposition 3 of [5] or from Proposition 4.9 that ${}_S W$ is a free S-module. Moreover, it is obviously a free S-module on 2 or 3 generators, and thus $\partial_R W^2=\partial\,\mathrm{Soc}_S W\geqq 4$, as required.

Now, our intention is to construct, for every local ring R with $W^2\neq 0$, another local ring with radical-square zero and compare the indecomposable modules of both rings. This will be done in the following lemma which modifies arguments of Auslander [1]. Recall that, given a bimodule ${}_A M_A$ over a ring A, the split extension of ${}_A M_A$ by A is a ring whose additive structure is that of the direct sum $A\oplus M$ and whose multiplication is given by

$$(a,m)(a',m')=(a\,a',\,a\,m'+m\,a').$$

We shall identify M with the two-sided ideal $0\oplus M$. Notice that, if A is a local ring, the split extension is a local ring as well.

Lemma 6.2. *Let A be a local ring with radical J and $S=\mathrm{Soc}\,{}_A A\cap \mathrm{Soc}\,A_A$. Let B be the split extension of ${}_{A/J}S_{A/J}$ by A/J. Then, if A is of bounded representation type, also B is of bounded representation type.*

Proof. We shall define a function F from the set of all isomorphism classes of finitely generated left B-modules into the set of all isomorphism classes of finitely generated left A-modules as follows. Given a finitely generated left B-module ${}_B M$, we consider its representation

$${}_B M=(\bigoplus_n {}_B B)/D \quad \text{with} \quad D\subseteq \bigoplus_n {}_B S.$$

If $(\bigoplus_m {}_B B)/D'$ is another such representation of ${}_B M$, then obviously $m=n$ equals the minimal number of generators of ${}_B M$, and there is an automorphism $\varphi\in\mathrm{End}(\bigoplus_n {}_B B)$ with $D\varphi=D'$. We can write φ as an $n\times n$ matrix $\varphi=(b_{ij})$, where $b_{ij}\in B$. If $\varphi'=(b'_{ij})\in\mathrm{End}(\bigoplus_n {}_B B)$ with $b_{ij}-b'_{ij}\in S$ for all $1\leqq i$, $j\leqq n$, then, in view of $DS=0$, also $D\varphi'=D'$. Thus, we may take

$$b_{ij}=(a_{ij}+J,0) \quad \text{for some } a_{ij}\in A.$$

Now, consider $(a_{ij}) \in \operatorname{End}(\bigoplus_n {}_A A)$. Since $D(a_{ij}) = D'$, (a_{ij}) induces an endomorphism of $(\bigoplus_n {}_A A)/D$. It is easy to verify that (a_{ij}) induces, in fact, an isomorphism between the left A-modules $(\bigoplus_n {}_A A)/D$ and $(\bigoplus_n {}_A A)/D'$. Denote the isomorphism class (or its representative) of these modules by $F(M)$. The above argument can be reversed which shows that F is an injective mapping.

Now, assume that ${}_B M = (\bigoplus_n {}_B B)/D$ is indecomposable. Then also $F(M) = (\bigoplus_n {}_A A)/D$ is indecomposable. For, assume that $F(M) = {}_A X \oplus {}_A Y$. We can write

$$ {}_A X = (\bigoplus_{n_1} {}_A A)/D_1 \text{ and } {}_A Y = (\bigoplus_{n_2} {}_A A)/D_2 \text{ with } n_1 + n_2 = n. $$

Thus, there is an automorphism $\varphi = (a_{ij}) \in \operatorname{End}(\bigoplus_n {}_A A)$ mapping D onto $D_1 \oplus D_2$. Since $D \subseteq \bigoplus_n {}_A S$, also

$$ D_1 \oplus D_2 = D\varphi \subseteq (\bigoplus_n {}_A S)\varphi = \bigoplus_n {}_A S, $$

and therefore $F(M)$ is also the image of

$$ (\bigoplus_{n_1} {}_B B)/D_1 \oplus (\bigoplus_{n_2} {}_B B)/D_2 $$

under F. Hence, the latter module is isomorphic to ${}_B M$. It follows that either $n_1 = 0$ or $n_2 = 0$. Consequently, the image $F(M)$ of an indecomposable B-module ${}_B M$ is an indecomposable A-module.

Finally, it is evident that $\partial_B M \leqq \partial_A F(M)$, and thus, if A is of bounded representation type, so is B.

Proposition 6.3. *If $W^2 \neq 0$ and R is of bounded representation type, then R is left uniserial.*

Proof. Obviously, we may assume that $W^3 = 0$. If R is not left uniserial, then $\partial_R W^2 \geqq 4$, by Lemma 6.1. Denote by B the split extension of ${}_Q W_Q^2$ by $Q = R/W$. Note that $W^2 = \operatorname{Soc}_R R = \operatorname{Soc} R_R$. Now, B is a local ring with $(\operatorname{Rad} B)^2 = 0$, $B/\operatorname{Rad} B \cong Q$ and $\partial_B \operatorname{Rad} B \geqq 4$. Thus, in view of Proposition 5.2 and Lemma 6.2, R is not of bounded representation type, in contradiction to our assumption.

Remark 6.4. In this final remark, we would like to supplement the description of the indecomposable left R-modules given in Lemmas 4.1 and 4.2 by a similar description of the indecomposable right R-modules. It is easy to verify that, if $W^2 = 0$ and $\partial_R W = 3$, then the simple module

$C_1 = R_R/W$, the projective module $C_2 = R_R$, the module

$$C_3 = (R_R \oplus R_R)/(w_1, w_2)R,$$

the injective module

$$C_4 = (R_R \oplus R_R \oplus R_R)/((w_1, w_2, 0)R + (w_1, 0, w_3)R)$$

and the module

$$Y_5 = (R_R \oplus R_R \oplus R_R)/(w_1, w_2, w_3)R$$

are non-isomorphic indecomposable right R-modules.

Proposition 1.1 yields then that these are the only indecomposable right R-modules. The latter statement can be also derived by means of Tachikawa's duality theory [10], using the following statement the proof of which is rather technical: If $W^2=0$ and $\partial_R W = 3$, then the centralizer $\mathscr{C} = \mathrm{End}(C_4)$ is a local ring with radical $\mathscr{W}$ such that $\mathscr{W}^2 = 0$, $\mathscr{Q} = \mathscr{C}/\mathscr{W}$ is commutative, $\partial_{\mathscr{C}} \mathscr{W} = 3$ and $\mathscr{W}_{\mathscr{C}}$ is simple.

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(Received August 11, 1972)