QF-3 rings

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In this paper, by a left QF-3 ring we shall always understand a ring with a (unique) minimal faithful left module, and by a QF-3 ring we mean a ring which is both left and right QF-3. While many generalizations of the notion of a QF-3 algebra to general rings were proposed, cf. [29], [18], [14], this definition is just Thrall's original one of QF-3 without any other assumption on the ring [30].

The duality between the endomorphism rings of the minimal faithful left and right modules over a QF-3 algebra and the reconstruction of QF-3 maximal quotient algebras were already known in [16] and [28]. Recently, the relation between Morita duality and QF-3 rings was extensively studied by several authors [12], [21], [25]. However, these investigations seem to be more or less restrictive. Our first purpose of this paper is to point out that these results are valid for all QF-3 rings. Thus, in section 2 we give a structure theory for QF-3 rings. The QF-3 maximal quotient rings are precisely the endomorphism rings of modules which are linearly compact generators and cogenerators, and it is shown that this module is determined by the endomorphism ring as far as possible. Further, a ring R is a QF-3 ring if and only if R can be embedded into a QF-3 maximal quotient ring Q in such a way that R contains a minimal faithful left ideal and a minimal faithful right ideal of Q. In this case, Q is the maximal quotient ring of R. If Q is a QF-3 ring, then we denote by $C(Q)$ the set of all QF-3 rings $R \leqq Q$ such that Q is a quotient ring of R. We show that $C(Q)$ has minimal elements and that they are isomorphic under an inner automorphism of Q.

Eilenberg and Nakayama [8] proved in 1956 that a left and right noetherian ring R with ${}_RR$ injective is left and right artinian. Later, this result was extended to rings which are either left or right noetherian. The corresponding question for left and right noetherian, left QF-3 rings has an affirmative answer. However, a left QF-3 ring which is only left noetherian (right noetherian) need not to be right noetherian (left noetherian). Also, there exists a left noetherian, left QF-3 ring which is a maximal left quotient ring and which is neither right QF-3 nor semiprimary. On the other hand, it can be shown that for a right noetherian, left QF-3 ring the maximal left quotient ring is a semi-primary QF-3 ring. Since a self-injective ring is its own maximal left quotient ring, this theorem generalizes the classical result. As an immediate consequence we obtain that a left noetherian QF-3 ring is always left artinian. These theorems, together with further remarks on QF-3 rings with chain conditions, are contained in section 3.

Recently, M. Auslander [1] has shown that for artin algebras there is a one-to-one correspondence between Morita equivalence classes of QF-3 maximal quotient rings R with global dimension $\leqq 2$ and of rings A of finite representation type. Here, a ring A

is said to be of finite representation type provided A is left artinian and has only a finite number of finitely generated indecomposable left A-modules. In section 4 we shall give a short proof of this result. From our proof it follows that the condition on R and A to be artin algebras is unnecessary. As a corollary we obtain the following interesting result: If A is a ring of finite representation type, then every indecomposable left A-module is finitely generated, and every left A-module is a direct sum of indecomposable modules.

Following Thrall [30] a ring R is said to be a QF-1 ring if every faithful R-module satisfies the double centralizer condition. When R is at the same time QF-1 and QF-3, we shall call R a QF-13 ring. According to the structure theorem for QF-3 maximal quotient rings, each QF-13 ring is obtained as the endomorphism ring of a linearly compact generator and cogenerator ${}_AM$ over a suitable ring A. We prove in section 5 that for indecomposable direct summands ${}_AX$ and ${}_AY$ of ${}_AM$ with $\mathrm{Hom}_A({}_AX, {}_AY) \neq 0$, either ${}_AX$ is projective or ${}_AY$ is injective with non-zero socle. As a consequence we obtain that every QF-13 is semi-perfect and, up to Morita equivalence, uniquely determined by the endomorphism ring of its minimal faithful left module. For certain QF-3 rings we also show that they are left QF-1 if and only if they are right QF-1, in particular, this is true for serial ("generalized uniserial") rings. We call a module ${}_AM$ minimal fully faithful provided ${}_AM$ is a generator and a cogenerator, but no proper direct summand of ${}_AM$ is a generator and a cogenerator. If A is a serial ring and ${}_AM$ is minimal fully faithful, then $\mathrm{End}({}_AM)$ is serial again. This then gives a characterization of serial QF-1 rings which is different from that in [11]: The ring R is a serial QF-1 ring if and only if R is Morita equivalent to $\mathrm{End}({}_AM)$, where A is serial, and ${}_AM$ is minimal fully faithful such that for given indecomposable direct summands ${}_AX$ and ${}_AY$ of ${}_AM$ with $\mathrm{Hom}({}_AX, {}_AY) \neq 0$, either ${}_AX$ is projective or ${}_AY$ is injective.

1. Preliminaries

In this section we explain the notation used throughout the paper and recall the definitions and elementary properties of left QF-3 and of QF-3 rings.

Rings are usually assumed to have an identity and all modules considered are unital. If R is a ring, then the symbols ${}_RM$ and M_R will be used to underline the fact that the R-module M is a left or a right R-module, respectively. We write homomorphisms of modules always on the side opposite the scalars; in particular, every left R-module ${}_RM$ defines a right C-module M_C, where $C = \mathrm{End}({}_RM)$ is the endomorphism ring of ${}_RM$. The double centralizer $D({}_RM)$ is the endomorphism ring of M_C. There is a canonical ring homomorphism of R into $D({}_RM)$, and if ${}_RM$ is faithful, we may assume $R \leq D({}_RM)$. In the last section we will need that the module ${}_RM$ is said to be balanced or to satisfy the double centralizer condition provided the canonical ring homomorphism $R \to D({}_RM)$ is surjective: The ring R is left QF-1 if every faithful left R-module is balanced, and R is a QF-1 ring provided it is left QF-1 and right QF-1.

If the R-module M has a composition series, denote by $|M|$ its length, and M is called serial provided the set of all submodules of M forms a composition series. A serial ring R is defined by the fact that both ${}_RR$ and R_R are direct sums of serial modules. The radical $\mathrm{Rad}(M)$ of M is the intersection of all maximal submodules of M, and, for the ring R, we write $\mathrm{Rad}(R)$ for $\mathrm{Rad}({}_RR) = \mathrm{Rad}(R_R)$. Similarly, the socle $\mathrm{Soc}(M)$ of M is the sum of all minimal submodules of M. The module M is called local provided it has a maximal submodule which contains all proper submodules of M, and M is called colocal provided it has a minimal submodule which is contained in all non-zero submodules of M. The injective envelop of M will be denoted by $E(M)$. Further, if ${}_RM$ and ${}_RN$ are

R-modules and $\phi : {}_RM \to {}_RN$ is a homomorphism, then $\mathrm{Im}(\phi)$ is the image of ϕ, whereas $\mathrm{Ker}(\phi)$ stands for the kernel of ϕ. The direct sum of M and N is denoted by $M \oplus N$, and similarly, given an arbitrary family $\{M_i \mid i \in I\}$ of left R-modules, their direct sum is denoted by $\oplus_I M_i$ or simply by $\oplus M_i$. In the same way, the product of the family $\{M_i \mid i \in I\}$ is denoted by ΠM_i, and $\Pi_I M$ means a product of I copies of M. It is easy to see that ${}_RM$ is faithful if and only if there is an inclusion ${}_RR \hookrightarrow \Pi_I M$, for some index set I. The R-module ${}_RM$ is a generator provided ${}_RR$ is a direct summand of some $\oplus_n M$, and ${}_RM$ is a cogenerator in case ${}_RM$ contains as submodules the injective hulls of all simple left R-modules.

The module ${}_RM$ is called *minimal faithful*, provided

(a) ${}_RM$ is faithful, and

(b) given a faithful module ${}_RN$, then ${}_RN = {}_RM \oplus *$
(that is, ${}_RM$ is isomorphic to a direct summand of ${}_RN$).

The ring R is a left QF-3 ring if there exists a minimal faithful left R-module. And a QF-3 ring is a ring which is both left QF-3 and right QF-3. The following characterization of left QF-3 rings was proved by Rutter [26].

(1.1) *The following conditions are equivalent for the ring R.*

(i) *R is left QF-3.*

(ii) *There are finitely many, pairwise non-isomorphic simple left ideals L_i such that $\oplus E(L_i)$ is faithful and projective.*

(iii) *R has an injective faithful left ideal L with finitely generated, large socle.*

In case of (ii), *$\oplus E(L_i)$ is minimal faithful.*

Thus, the minimal faithful left module of a left QF-3 ring R is always isomorphic to a left ideal of the form Re, with e an idempotent of R. And, in particular, left QF-3 rings have an injective faithful left ideal. The converse is not true, as the ring of matrices

$$R = \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{pmatrix}$$

with $\mathbb{Z}$ the integers and $\mathbb{Q}$ the rational numbers, shows: R has an injective faithful left ideal, but is not left QF-3. The minimal faithful left module of a left QF-3 ring R will usually be denoted by Re, with $e = e^2 \in R$, and, if R is a QF-3 ring, we will denote the minimal faithful R-modules by Re and fR, where e and f are assumed to be idempotents of R.

If $R \leqq Q$ is an inclusion of rings, then Q is called a left quotient ring of R, provided ${}_RQ$ is a rational extension of ${}_RR$. As is well-known, every ring R has a maximal left quotient ring, namely $D(E({}_RR))$.

(1.2) *Let Q be a left quotient ring of R. If R has an injective faithful left ideal Re, then $Qe = Re$ is an injective faithful left ideal of Q. If Re has large socle, then also Qe has large socle.*

In particular, if R is a left QF-3 ring, then also Q is a left QF-3 ring. For the proof of the first part we refer to Tachikawa [29]. Now assume that Re has large socle. We claim that $\mathrm{Soc}({}_RQe) \leqq \mathrm{Soc}({}_QQe)$. Let Rx be a minimal left ideal of R contained in $Re = Qe$. If $q \in Q$ and $qx \neq 0$, then there is $r \in R$ with $rq \in R$ and $r(qx) \neq 0$, because ${}_RR$ is rational in ${}_RQ$. Now $rqx \neq 0$ belongs to the minimal left ideal Rx of R, so $x = trqx$ for some

$t \in R$. This shows that Qx is a minimal left ideal of Q, and therefore

$$\mathrm{Soc}({}_RQe) \leqq \mathrm{Soc}({}_QQe).$$

Now, since $\mathrm{Soc}({}_RQe)$ is large in ${}_RQe$, we conclude that $\mathrm{Soc}({}_QQe)$ is large in ${}_QQe$.

Left quotient rings of R can be obtained in the following way. Let K_R be a faithful right ideal of R. We claim that the double centralizer $D(K_R)$ of K_R is a left quotient ring of R. For, given two elements $d_1 \neq 0$, d_2 in $D(K_R)$, then there is $x \in K$ with $xd_1 \neq 0$. Now $x \in K \leqq R$ and also $xd_2 \in K \leqq R$, so ${}_RD(K_R)$ is indeed a rational extension of ${}_RR$. Using this remark we get as a corollary from (1. 2):

(1. 3) *Let K_R be a faithful right ideal of the left QF-3 ring R. Then $D(K_R)$ is a left QF-3 ring.*

Another consequence of the remark above is the following lemma (for a generalization we refer to Masaike [15]).

(1. 4) *Let R be a left QF-3 ring. Then the maximal left quotient ring is contained in the maximal right quotient ring.*

Proof. Let Re be a minimal faithful left ideal of R. The maximal left quotient ring of R is given by $D(E_RR)$. Now $E_RR = Re \oplus C_R$ for some module ${}_RC$, since Re is injective. Also, ${}_RC$ is cogenerated by Re. For, ${}_RR \leqq \Pi Re$, since Re is faithful, and since Re is injective, also $E_RR \leqq \Pi Re$. As a consequence

$$D(E_RR) \leqq D(Re),$$

but $D(Re)_R$ is a rational extension of R_R, so a subring of the maximal right quotient ring of R.

Thus, if R is a QF-3 ring, then the maximal left quotient ring and the maximal right quotient ring coincide and we simply speak of the maximal quotient ring of R. If fR is an injective faithful right ideal of the QF-3 ring R, then $D(fR_R)$ is the maximal quotient ring of R. For, $D(fR_R)$ is a quotient ring, and if Q is a quotient ring of R, then $fR = fQ$ by (1. 2), so also $fRf = fQf$, so

$$Q \leqq D(fQ_Q) = \mathrm{End}({}_{fQf}fQ) = \mathrm{End}({}_{fRf}fR) = D(fR_R).$$

(1. 5) *Let R be a left QF-3 ring and Re minimal faithful. If fR is a faithful right ideal (with f an idempotent of R), then*

(1) $\mathrm{Hom}_{fRf}(fR, fRe) \cong Re$, *and*

(2) ${}_{fRf}fRe$ *is injective.*

For the proof, see Tachikawa [29]. This result can be used in order to show that for a QF-3 ring R with minimal faithful modules Re and fR, the rings fRf and eRe are Morita dual. Here, rings A and B are called Morita dual provided there exists a bimodule ${}_AU_B$ such that ${}_AU$ and U_B both are injective cogenerators and $A = \mathrm{End}(U_B)$, $B = \mathrm{End}({}_AU)$. If A and B are Morita dual, then both rings are semi-perfect, as B. Osofsky [23] has shown.

(1. 6) *Let R be a QF-3 ring with minimal faithful modules Re and fR. Then fRf and eRe are Morita dual with respect to ${}_{fRf}fRe_{eRe}$.*

A proof of (1. 6) may be found in Roux [25]. Since we will use Morita duality quite frequently, we have to mention some more details. Let A and B be Morita dual with respect to the bimodule ${}_AU_B$. Given a left A-module ${}_AM$, we denote by M^* the dual of M, namely the right B-module $M^* = \mathrm{Hom}_A({}_AM, {}_AU_B)$. Similarly, for a right B-

module ${}_BX$, we define the left A-module $X^* = \mathrm{Hom}_B(X_B, {}_AU_B)$. The module ${}_AM$ is called reflexive provided the canonical homomorphism $M \to M^{**}$ is an isomorphism, and Morita's theorem [16] asserts that all finitely generated left A-modules are reflexive. Mueller [22] has shown that for a ring A which is Morita dual to some ring B, a left A-module is reflexive if and only if it is linearly compact. Here, the module ${}_RM$ is called linearly compact if any finitely solvable system of congruences $x \equiv x_\alpha (\mathrm{mod}\ M_\alpha)$, where the M_α are submodules of ${}_RM$, is solvable.

(1. 7) *The ring A is Morita dual to some ring B if and only if there exists a linearly compact left A-module which is a generator and a cogenerator.*

Proof. If A and B are Morita dual with respect to the bimodule ${}_AU_B$, then ${}_AA \oplus {}_AU$ is a linearly compact generator and cogenerator. Conversely, assume ${}_AM$ is a linearly compact generator and cogenerator. Note that submodules, factor modules and finite direct sums of linearly compact modules are linearly compact, again. The minimal cogenerator ${}_AV$ is isomorphic to a submodule of ${}_AM$, so linearly compact. Also, ${}_AA$ is a factor module of a finite direct sum of copies of ${}_AM$, and therefore linearly compact. Now it can easily be derived from [22] that A and $\mathrm{End}({}_AV)$ are Morita dual.

Finally, we have to mention a property of the minimal faithful modules over a semi-primary QF-3 ring. Recall that a module M is called Σ-injective provided every direct sum of copies of M is injective, and M is called Π-projective in case every product of copies of M is projective. Then, Colby and Rutter [5] have shown:

(1. 8) *Let R be a semi-primary QF-3 ring with minimal faithful modules Re and fR. Then the R-modules Re and fR both are Σ-injective and Π-projective.*

2. Structure Theorems

If R is a QF-3 ring with minimal faithful modules Re and fR, then fRf and eRe are Morita dual rings and it is rather easy to see that ${}_{fRf}fR$ is a reflexive generator and cogenerator. Moreover, if R is also a maximal quotient ring, then R is just the endomorphism ring of ${}_{fRf}fR$. Thus QF-3 maximal quotient rings are endomorphism rings of linearly compact generator and cogenerator modules. As we will see, also the converse is true and this gives a characterization of QF-3 maximal quotient rings. It should be noted that by (1. 7) the rings which have linearly compact generator and cogenerator modules are just those rings which are Morita dual to some other ring.

(2. 1) Structure Theorem for QF-3 maximal Quotient Rings. *QF-3 maximal quotient rings are precisely the endomorphism rings of modules ${}_AM$ which are linearly compact, generators and cogenerators. Moreover, if ${}_AM$ and ${}_{A'}M'$ are such modules and*

$$\mathrm{End}({}_AM) \cong \mathrm{End}({}_{A'}M'),$$

then there is a categorical equivalence $T: {}_A\mathfrak{M} \to {}_{A'}\mathfrak{M}$ with $T({}_AM) = {}_{A'}M'$.

Proof. First, let R be a QF-3 maximal quotient ring with minimal faithful modules Re and fR. By (1. 6) we know that the rings fRf and eRe are Morita dual with respect to ${}_{fRf}fRe_{eRe}$. Now consider the module ${}_{fRf}fR$. It is a generator, since

$${}_{fRf}fR = {}_{fRf}fRf \oplus {}_{fRf}fR(1-f),$$

and a cogenerator, since ${}_{fRf}fRe$ is a cogenerator and ${}_{fRf}fR = {}_{fRf}fRe \oplus {}_{fRf}fR(1-e)$. Using (1. 5), the dual of ${}_{fRf}fR$ is $fR^* \cong Re_{eRe}$, and similarly, $Re^* \cong {}_{fRf}fR$, so ${}_{fRf}fR$ is reflexive and therefore linearly compact. Also, since fR is a faithful right ideal,

$D(fR_R) = \mathrm{End}({}_{fRf}fR)$ is a left quotient ring of R, so since R is a maximal quotient ring, $R = \mathrm{End}({}_{fRf}fR)$.

Conversely, assume that ${}_AM$ is a linearly compact generator and cogenerator. Then A is Morita dual to some ring B, so there is a bimodule ${}_AU_B$ which defines the duality. Since the property to be a QF-3 ring (and also to be a maximal quotient ring) is Morita equivalent, we may assume

$$M = {}_AX \oplus {}_AY \oplus {}_AZ \oplus {}_AC,$$

where ${}_AX \oplus {}_AY = {}_AA$ and ${}_AY \oplus {}_AZ = {}_AU$, such that neither ${}_AX$ nor ${}_AC$ contains an indecomposable direct summand which is the injective hull of a simple module, whereas neither ${}_AZ$ nor ${}_AC$ contains an indecomposable projective direct summand. Here we use that a linearly compact module is of finite rank.

$$f: {}_AM \to {}_AA = {}_AX \oplus {}_AY \quad \text{and} \quad e: {}_AM \to {}_AU = {}_AY \oplus {}_AZ$$

be the respective projections, considered as elements of R. Now, M can be identified in an obvious way with fR, and then $A = fRf$, $U = fRe$. Also, Re can be identified with $\mathrm{Hom}_A({}_AM, {}_AU_B) = M^*$. Since ${}_AU$ is injective and $M_R = fR$ is projective,

$$_RRe = \mathrm{Hom}_A({}_AM_R, {}_AU)$$

is an injective left ideal. Similarly, since U_B is injective and $M^* = Re$ is a projective left R-module, $fR = M \cong M^{**} = \mathrm{Hom}_B({}_RM^*{}_B, U_B)$ is an injective right ideal, where we use that ${}_AM$ is reflexive. It remains to be shown that the socles of ${}_RRe$ and of fR_R are large.

Let $S = \{s \in fRe \mid \mathrm{Im}(s) \leqq \mathrm{Soc}_A U\}$. We claim that $(1-f)RS = 0$. For, let $r \in R$, $s = fse \in S$ and assume $(1-f)rfse \neq 0$. We consider $(1-f)rf$ and fse as mappings

$$_AZ \oplus {}_AC \xrightarrow{(1-f)rf} {}_AX \oplus {}_AY = {}_AA \xrightarrow{fse} {}_AY \oplus {}_AZ,$$

and by assumption $\mathrm{Im}((1-f)rf) \nsubseteq \mathrm{Ker}(fse)$. Since the A-module ${}_A\mathrm{Im}(fse) \leqq \mathrm{Soc}_A U$ is completely reducible, $\mathrm{Ker}(fse)$ contains the radical $\mathrm{Rad}(A)$ of ${}_AA$. Thus, $\mathrm{Im}((1-f)rf)$ is not contained in $\mathrm{Rad}(A)$. Since A is semi-perfect, either ${}_AZ$ or ${}_AC$ maps onto at least one indecomposable direct summand of ${}_AA$, but this contradicts the assumption that neither ${}_AZ$ nor ${}_AC$ has as direct summand an indecomposable projective A-module.

Now it is easy to see that S is a left ideal. For given $r \in R$ and $s = fse \in S$, we have $rs = frfs$. Since $(1-f)rs = 0$. But $frf \in A$, so

$$\mathrm{Im}(frfs) \leqq A \cdot \mathrm{Im}(s) \leqq A \cdot \mathrm{Soc}_A U \leqq \mathrm{Soc}_A U,$$

and therefore $rs \in S$. Also, under the obvious identification of S and $\mathrm{Soc}_A U$, the left ideals of R contained in S correspond to the A-submodules of $\mathrm{Soc}_A U$. In particular, ${}_RS$ is completely reducible, since the same is true for ${}_A\mathrm{Soc}_A U$.

Next, we show that ${}_RS$ is large in Re. Given $0 \neq r \in Re$ we have $\mathrm{Im}(r) \cap \mathrm{Soc}_A U \neq 0$ since $\mathrm{Soc}_A U$ is large in ${}_AU$. So there is $m \in M$ with $0 \neq mr \in \mathrm{Soc}_A U$. Define an A-homomorphism

$$\phi: {}_AA \oplus {}_AZ \oplus {}_AC \to {}_AM \quad \text{by} \quad (a, z, c)\phi = am$$

for $a \in A$, $z \in Z$, $c \in C$. Then $\phi r = f\phi re \neq 0$, and $\mathrm{Im}(\phi r) \leqq Amr \leqq \mathrm{Soc}_A U$, thus

$$0 \neq \phi r \in S \cap Rr.$$

This proves that ${}_RS$ is large in Re.

Similarly, using the fact that neither ${}_AX$ nor ${}_AC$ contains a direct summand which is the injective hull of a simple module, it can be shown that $SR(1-e)=0$. Thus, R is also a right ideal, and the right ideals of R contained in S correspond to the B-submodules of $\mathrm{Soc}_A U$. But $\mathrm{Soc}_A U = \mathrm{Soc}\, U_B$, in particular $[\mathrm{Soc}_A U]_B$ is completely reducible, so also S_R is completely reducible. In order to show that S_R is large in fR_R, consider a non-zero element $r \in fR$. As an epimorphic image of ${}_AA$, $\mathrm{Im}(r)$ has a maximal submodule ${}_AV \leqq \mathrm{Im}(r)$. Since ${}_AU$ is a cogenerator, there is a mapping $\psi\colon {}_AM \to {}_AU$ with $V\psi = 0$ and $\mathrm{Im}(r\psi) \neq 0$. Then $r\psi = fr\psi e \neq 0$, and $\mathrm{Im}(r\psi) \leqq \mathrm{Soc}_A U$, thus $0 \neq r\psi \in rR \cap S$.

Using (1. 1) we see that R is both left QF-3 and right QF-3. It remains to be shown that R is a maximal quotient ring. As we have seen, fR is an injective faithful right ideal, so we have to show $R = D(fR_R)$. But we have identified ${}_{fRf}fR$ with ${}_AK$, so

$$R = \mathrm{End}({}_AK) = \mathrm{End}({}_{fRf}fR) = D(fR_R).$$

This shows that the endomorphism ring of a linearly compact generator and cogenerator is a QF-3 maximal quotient ring.

Finally, assume A and A' are rings and ${}_AM$ and ${}_{A'}M'$ are linearly compact generators and cogenerators. Observe that A and A' are semi-perfect rings and it is rather obvious that we may assume that A and A' are basic rings. For, taking the basic ring B of A we get a categorical equivalence $T\colon {}_A\mathfrak{M} \to {}_B\mathfrak{M}$, and $T({}_AM)$ is again a linearly compact generator and cogenerator, and $\mathrm{End}({}_AM) \cong \mathrm{End}(T({}_AM))$.

Since A is a basic ring and ${}_AM$ is a generator, we have ${}_AM = {}_AA \oplus {}_AC$ for some complement ${}_AC$. Denote by $f\colon {}_AM \to {}_AA$ the projection considered as an element of $R = \mathrm{End}({}_AM)$. We want to show that fR is a minimal faithful module, so $A = fRf$ can be recovered from R as the endomorphism ring of the minimal faithful right module $fR_R = M_R$, and ${}_AM = {}_{fRf}fR$. Now fR_R is faithful, so fR contains as a direct summand the minimal faithful right module (we know already that R is QF-3). Thus, there are orthogonal idempotents f_1 and f_2 in R with $f = f_1 + f_2$ and f_1R minimal faithful. Assume $f_2 \neq 0$. We have identified fRf and A, so f_1 and f_2 are orthogonal idempotents of A. Since A is a basic ring, every A-homomorphism $Af_1 \to Af_2$ maps Af_1 into $\mathrm{Rad}(Af_2)$. Thus, the image of every element of f_1R is contained in $Af_1 \oplus \mathrm{Rad}(Af_2) \oplus C$. But ${}_AM$ is a cogenerator, so there is a non-zero endomorphism ϕ of ${}_AM$ with $(Af_1 \oplus \mathrm{Rad}(Af_2) \oplus C)\phi = 0$. This shows that $f_1R\phi = 0$, contradicting the fact that f_1R_R is faithful. Consequently, $f_2 = 0$ and $fR = f_1R$ is a minimal faithful module. This implies $A \cong A'$ and proves the second part.

The next theorem shows that the investigation of arbitrary QF-3 rings can be reduced to the case of QF-3 maximal quotient rings. Here, we write $\langle 1, Qe, fQ\rangle$ for the smallest subring of Q containing 1, Qe and fQ.

(2. 2) Structure Theorem for QF-3 Rings. *The ring R is a QF-3 ring if and only if there is a QF-3 maximal quotient ring Q with minimal faithful modules Qe and fQ such that*

$$\langle 1, Qe, fQ\rangle \leqq R \leqq Q.$$

In this case, Q is the maximal quotient ring of R.

Proof. Let R be a QF-3 ring with minimal faithful modules Re and fR. Denote by Q the maximal quotient ring of R. By (1. 2), $Re = Qe$ and $fR = fQ$ are minimal faithful modules for Q.

Conversely, let Q be a QF-3 maximal quotient ring, and let R be a subring of Q with $\langle 1, Qe, fQ\rangle \leqq R$, where Qe and fQ are minimal faithful Q-modules. By (1. 5), ${}_{Qf}fQe$ and fQe_{eQe} both are injective modules and

$$\mathrm{Hom}_{fQf}(fQ, fQe) \cong Qe \quad \text{and} \quad \mathrm{Hom}_{eQe}(Qe, fQe) \cong fQ.$$

Since $Re = Qe$ and $fR = fQ$, we may replace Q by R in the previous sentence and, as a consequence, we see that

$$\mathrm{Hom}_{fRf}(fR, fRe) \cong {}_{R}Re \quad \text{and} \quad \mathrm{Hom}_{eRe}(Re, fRe) \cong fR$$

both are injective R-modules. Obviously, they are also faithful, so it remains to be shown that $\mathrm{Soc}({}_RRe)$ is large in ${}_RRe$ and that $\mathrm{Soc}(fR_R)$ is large in fR_R.

Let $S = Rf \cdot \mathrm{Soc}({}_QQe)$. We claim that ${}_RS$ is completely reducible. For, let N be the radical of R and assume $NS \neq 0$. Then

$$fNf \cdot \mathrm{Soc}({}_QQe) = fR \cdot N \cdot Rf \cdot \mathrm{Soc}({}_QQe) = fR \cdot N \cdot S \neq 0,$$

since fR is faithful. But

$$fNf = \mathrm{Rad}(fRf) = \mathrm{Rad}(fQf) = f \cdot \mathrm{Rad}(Q) \cdot f,$$

and $f \cdot \mathrm{Rad}(Q) \cdot f \cdot S = 0$, since $S \leqq \mathrm{Soc}({}_QQe)$. From this contradiction we know that $S = Rf \cdot \mathrm{Soc}(Qe) = \oplus_{i=1}^{t} Rf_i \cdot \mathrm{Soc}(Qe)$ and $Rf_i \cdot \mathrm{Soc}(Qe)$ is a sum of homomorphic images of Rf_i/Nf_i, where $f = f_1 + \cdots + f_s$ is the decomposition of f into orthogonal primitive idempotents f_i of Q. However Rf_i/Nf_i is simple, because

$$\mathrm{End}(Rf_i/Nf_i) \cong f_iRf_i/f_iNf_i = f_iQf_i/f_i \cdot \mathrm{Rad}(Q) \cdot f_i$$

and $f_iQf_i/f_i\,\mathrm{Rad}(Q)f_i$ is a division ring, and $f_i \in R$.

Hence ${}_RS$ is completely reducible. In order to show that $\mathrm{Soc}({}_RRe)$ is large in ${}_RRe$, let $0 \neq x \in Re$. We know that $\mathrm{Soc}({}_QQe)$ is large in ${}_QRe$, so there is $q \in Q$ with

$$0 \neq qx \in \mathrm{Soc}({}_QQe).$$

Since fR is faithful, there is $r \in R$ with $0 \neq frqx$. But, on the one hand, $frqx \in Rx$ since $frq \in fQ \leqq R$, and, on the other hand, $frqx \in f \cdot \mathrm{Soc}({}_QQe) \leqq S \leqq \mathrm{Soc}({}_RRe)$.

Similarly, $\mathrm{Soc}(fR_R)$ is large in fR_R. As a consequence, R is a QF-3 ring.

Combining the two structure theorems we see that QF-3 rings are obtained in the following way: Take the endomorphism ring Q of a linearly compact generator and cogenerator, let Qe and fQ be minimal faithful Q-modules and take all subrings R with

$$\langle 1, Qe, fQ\rangle \leqq R \leqq Q.$$

It should be noted that this class of rings R was also considered by Kato in [14], where he characterized it using another definition of "QF-3 rings", which seems to be rather complicated.

If Q is a QF-3 ring, then we denote by $C(Q)$ the set of all QF-3 rings $R \leqq Q$ such that Q is a left quotient ring of R. Obviously, $C(Q)$ is partially ordered by inclusion. If $C(Q)$ has minimal elements, then by the last theorem they have to be of the form $\langle 1, Qe, fQ\rangle$ for suitable idempotents e and f. The following theorem shows that minimal elements in $C(Q)$ do exist and are even unique up to an inner automorphism of Q.

(2. 3) Theorem. *Let Q be a QF-3 ring. Then every subring in $C(Q)$ contains as a subring a minimal element of $C(Q)$ and any two minimal elements of $C(Q)$ are isomorphic under an inner automorphism of Q.*

Proof. First, note that every decomposition of 1 as a sum of orthogonal idempotents of Q can be refined to a sum of orthogonal primitive idempotents, since Q is a ring of endomorphisms of a linearly compact module.

The pair e, f of idempotents of Q is called properly chosen, provided there exists a decomposition $1 = \sum_{i=1}^{n} \sum_{j=1}^{m_i} e_{ij}$ with orthogonal primitive idempotents e_{ij} such that ${}_QQe_{ij} \cong {}_QQe_{kl}$ if and only if $i = k$, and

$$e = \sum_{i \in I} e_{il}, \qquad f = \sum_{k \in K} e_{km_k},$$

where $I = \{i \mid {}_QQe_{il}$ is injective with large socle$\}$ and $K = \{k \mid e_{kl}Q_Q$ is injective with large socle$\}$. We want to show that every ring R in $C(Q)$ contains a subring of the form $\langle 1, Qe, fQ\rangle$ with properly chosen idempotents e and f of Q.

Let $R \in C(Q)$, and $1 = \sum_{i=1}^{n} e_i = \sum_{j=1}^{n'} e'_j$ be decompositions into orthogonal primitive idempotents of R. Usually, such decompositions cannot be compared, since R need not to be semi-perfect. But at least, if ${}_RRe_1$ or e_1R_R is injective with large socle, and

$${}_RRe_1 \cong {}_RRe_i \quad \text{for} \quad 1 \leq i \leq s,$$

then there are s idempotents e'_j such that ${}_RRe_1 \cong {}_RRe'_j$. For, in this case, the endomorphism ring e_1Re_1 of ${}_RRe_1$ is a local ring, so ${}_RRe_1$ has the exchange property, according to Warfield [32].

Now, given the ring $R \in C(Q)$, we fix a minimal faithful left ideal Re, and refine $1 = e + (1 - e)$ to a decomposition of 1 as a sum of orthogonal primitive idempotents of R, say $1 = \sum_{i=1}^{t} e_i$ with $e = \sum_{i=1}^{s} e_i$. Also, we fix a decomposition $1 = \sum_{i=1}^{n} \sum_{j=1}^{m_i} e_{ij}$ with orthogonal primitive idempotents e_{ij}, now of Q, and with the property that ${}_QQe_{ij} \cong {}_QQe_{kl}$ if and only if $i = k$. Let $K = \{k \mid e_{kl}Q_Q$ is injective with large socle$\}$. For $k \in K$, we want to define $\pi(k) \in \{1, 2, \ldots, t\}$ such that

(a) $e_{\pi(k)}R_R \cong e_{kl}Q_Q$, and

(b) if $m_k > 1$, then $\pi(k) > s$.

First, we have to notice that a number $\pi(k)$ with (a) exists. Namely, $e_{kl}Q_Q$ is a direct summand of the minimal faithful right Q-module, so $e_{kl}Q_R$ is a direct summand of the minimal faithful right R-module, by (1.2). So there is a primitive idempotent f_k in R with $e_{kl}Q_R \cong f_kR_R$. Then we find $\pi(k)$ with $f_kRe_{\pi(k)} \nsubseteq \mathrm{Rad}(f_kR_R)$, and in this case every element $q \in f_kRe_{\pi(k)} \setminus \mathrm{Rad}(f_kR_R)$ defines an epimorphism $e_{\pi(k)}R \to f_kR$ by left multiplication. Since $e_{\pi(k)}$ is primitive, this epimorphism has to be an isomorphism, so

$$e_{\pi(k)}R_R \cong f_kR_R \cong e_{kl}Q_R,$$

and (a) is satisfied. Now assume $m_k > 1$, but we have chosen $\pi(k) \leq s$. Then both $Qe_{\pi(k)} = Re_{\pi(k)}$ and $e_{\pi(k)}Q = e_{\pi(k)}R$ are contained in R, so also the product $Qe_{\pi(k)}Q \subseteq R$. But this implies that for all $1 \leq j \leq m_k$, we have $Qe_{kj} \subseteq R$. Namely, given e_{kj}, there are elements u_j, v_j in Q with $e_{kj} = u_je_{\pi(k)}v_j$, since $e_{\pi(k)}Q_R \cong e_{kl}Q_R \cong e_{kj}Q_R$. Then

$$Qe_{kj} = Qu_je_{\pi(k)}v_j \subseteq Qe_{\pi(k)}Q \subseteq R.$$

Consequently, all the idempotents e_{kj}, $1 \leq j \leq m_k$ belong to R, and

$${}_RRe_{\pi(k)} = {}_RQe_{\pi(k)} \cong {}_RQe_{kj} = {}_RRe_{kj}.$$

Moreover, $e_{\pi(k)}R$ is injective with large socle, so by the argument above, every decomposition of 1 into orthogonal primitive idempotents of R contains m_k idempotents which generate left ideals isomorphic to ${}_RRe_{\pi(k)}$. In particular, we have $m_k(>1)$ idempotents e_i with ${}_RRe_{\pi(k)} \cong {}_RRe_i$. Since the first s idempotents generate non-isomorphic left ideals, we can replace $\pi(k)$ by another index which satisfies (a) and (b). Now it is easy to see that $e = \sum_{i=1}^{s} e_i$ and $f = \sum_{k\in K} e_{\pi(k)}$ are properly chosen in Q, and obviously $\langle 1, Qe, fQ\rangle \leqq R$.

Next, we want to show that two subrings $\langle 1, Qe, fQ\rangle$ and $\langle 1, Qe', f'Q\rangle$ with properly chosen idempotents e, f and e', f', are isomorphic under an inner automorphism of Q. As before, we fix a decomposition $1 = \sum_{i=1}^{n}\sum_{j=1}^{m_i} e_{ij}$ with orthogonal primitive idempotents of Q such that ${}_QQe_{ij} \cong {}_QQe_{kl}$ if and only if $i = k$, and we introduce

$$\bar{I} = \{(i, 1) \mid {}_QQe_{i1} \text{ is injective with large socle}\} \cup \{(i, m_i) \mid e_{im_i}Q_Q \text{ is injective with large socle}\}.$$

For the pair e, f of properly chosen idempotents we have $ef = fe$, so $d = 1 - e - f + ef$ is an idempotent orthogonal to $e + f - ef$. We can refine the decomposition $1 = d + (e + f - ef)$ to $1 = d + \sum_{t\in\bar{I}} f_t$ with orthogonal idempotents f_t of $\langle 1, Qe, fQ\rangle$ such that ${}_QQf_t \cong {}_QQe_t$. Similarly, for e', f' we define d' and get orthogonal idempotents f'_t of $\langle 1, Qe', f'Q\rangle$ with ${}_QQf'_t \cong {}_QQe_t$. We therefore have direct decompositions of ${}_QQ$,

$$(\oplus_{t\in\bar{I}}\, {}_QQf_t) \oplus {}_QQd = {}_QQ = (\oplus_{t\in\bar{I}}\, {}_QQf'_t) \oplus {}_QQd',$$

with ${}_QQf_t \cong {}_QQf'_t$, for all $t \in \bar{I}$. Since the endomorphism ring f_tQf_t of ${}_QQf_t$ is local, the module ${}_QQf_t$ has the cancellation property, using again [32], so it follows ${}_QQd \cong {}_QQd'$. But in such a situation it is well-known that there exists a regular element $q \in Q$ with

$$q^{-1}f_tq = f'_t \quad \text{for} \quad t \in I, \quad \text{and} \quad q^{-1}dq = d',$$

and therefore $q^{-1}\langle 1, Qe, fQ\rangle q = \langle 1, Qe', f'Q\rangle$.

It follows now that the rings of the form $T = \langle 1, Qe, fQ\rangle$ with properly chosen idempotents are minimal elements of $C(Q)$. For, let $R \in C(Q)$, and $R \leqq T$. As we have seen, there are properly chosen idempotents e', f' such that $T' = \langle 1, Qe', f'Q\rangle \leqq R$. Now T and T' are isomorphic under an inner automorphism of Q, so it is easy to see that $T = T'$, and therefore also $R = T$. This concludes the proof.

For Q a semi-primary QF-3 ring the result (2. 3) was proved by Mueller [20], [21] and our proof follows quite closely his presentation. In the general case considered here, the difficulty comes from the fact that the Krull-Schmidt theorem in Q (and in R) is no longer valid, but fortunately, Warfield's local version of this theorem can be used.

3. QF-3 Rings with chain conditions

We want to consider QF-3 rings with different chain conditions; in particular, we ask under what conditions a QF-3 ring has to be artinian. First, we note that a left perfect QF-3 ring R is at least semi-primary. For, let e and f be idempotents of R such that Re and fR are minimal faithful R-modules. Then, also fRf is left perfect, and since fRf is Morita dual to eRe, it follows from a theorem of B. Osofsky [23], that fRf is left artinian. Now ${}_{fRf}fR$ is a linearly compact module over a left artinian ring, so of finite length. Then the maximal quotient ring $Q = \mathrm{End}({}_{fRf}fR)$ of R is semi-primary. Let $N = \mathrm{Rad}(R)$, $N' = \mathrm{Rad}(Q)$. We know that N' is nilpotent and that N is at least a nil ideal. Now $N + N'/N' \cong N/N \cap N'$ is a nil subring of the semi-simple ring Q/N', so it is nilpotent. Also $N \cap N' \leqq N'$ is nilpotent. Thus N is an extension of the nilpotent ideal

$N \cap N'$ by the nilpotent ring $N/N \cap N'$, so itself nilpotent. This shows that R is semi-primary. Also, a semi-primary QF-3 ring R has finitely generated left socle and finitely generated right socle. For, let fR be a minimal faithful right ideal, then ${}_{fRf}fR$ is finitely generated, so R_R can be embedded into a finite direct sum of copies of fR_R. But the socle of fR_R is finitely generated so the same is true for $\mathrm{Soc}(R_R)$. However, a semi-primary QF-3 ring need not to be left or right artinian. We give an example which is even a maximal quotient ring. Namely, let A and B be Morita dual rings such that A is left artinian but not right artinian, and B is right artinian but not left artinian [23]. Let ${}_AU$ be an injective cogenerator with $B = \mathrm{End}({}_AU)$. Then $\mathrm{End}({}_AA \oplus_A U)$ is a semi-primary QF-3 maximal quotient ring which is obviously neither left nor right artinian.

Next, we consider the case where the ring R is left or right noetherian. Our results are valid not only for left QF-3 rings but for all rings which have an injective faithful left ideal. First, we sharpen a result of Jans [13].

(3. 1) Lemma. *Let R be right noetherian. Then R has an injective faithful left ideal if and only if $E({}_RR)$ is projective.*

Proof. The "only if" part is due to Jans. For the converse, assume that $E({}_RR)$ is projective. Since R is right noetherian, there does not exist an infinite family of orthogonal idempotents. In particular, R has a maximal injective left ideal, say Re, with e an idempotent of R. We want to show that Re is faithful. If the injective hull of a finitely generated module is projective, then it is well-known that it is finitely generated. Since we assume that $E({}_RR)$ is projective, there is a module ${}_RC$ and some natural n with $E({}_RR) \oplus {}_RC = \bigoplus_{i=1}^{n} {}_RR$. The endomorphism ring R_n of $\bigoplus_{i=1}^{n} {}_RR$ is again right noetherian, so does not have infinite families of orthogonal idempotents. Since the idempotents of R_n correspond to the direct summands of $E({}_RR) \oplus {}_RC$, we conclude that $E({}_RR)$ is a finite direct sum of indecomposable modules. Let ${}_RX$ be an indecomposable direct summand of $E({}_RR)$. Since ${}_RX$ is embeddable in

$$\bigoplus_{i=1}^{n} {}_RR = \left(\bigoplus_{i=1}^{n} Re\right) \oplus \left(\bigoplus_{i=1}^{n} R(1-e)\right),$$

there are n homomorphisms $p_i\colon {}_RX \to Re$ and n homomorphisms $q_j\colon {}_RX \to R(1-e)$ such that the intersection of the kernels of all p_i and all q_j is zero. But an indecomposable injective module is uniform, so one of the p_i or q_j is a monomorphism. Now q_j cannot be monomorphism, since otherwise $Re \oplus Xq_j$ is an injective left ideal properly containing Re. Thus ${}_RX$ is embeddable in Re. This shows that $E({}_RR)$ is cogenerated by Re, and consequently, Re is faithful.

(3. 2) Theorem. *Let R be left and right noetherian. If R has an injective faithful left ideal, then R is left and right artinian.*

Proof. By the previous lemma, $E({}_RR)$ is projective, and therefore finitely generated. Now we use a result of Vinsonhaller [31] to derive that R is left artinian. But a semi-primary right noetherian ring is also right artinian.

If we weaken the assumptions and only assume that R is left noetherian or right noetherian, the conclusion does not remain valid, in spite of assertions in [31]. This is even the case for left QF-3 rings. A QF-3 ring which is right artinian but not left artinian, can be constructed as follows. Let D be a division ring with a division subring P such that $\dim_P D = \infty$ and $\dim D_P = 2$; for the existence, see Cohn [3]. Then the ring of matrices

$$\begin{pmatrix} D & 0 & 0 \\ D & P & 0 \\ D & D & D \end{pmatrix}$$

is a right artinian QF-3 ring which is not left artinian. Also, it is possible to construct a left QF-3 ring which is left noetherian but not right noetherian. Let A be a (non-trivial) simple left noetherian domain with an injective simple left module ${}_AS$; for the existence, see Cozzens [6]. Let C be the endomorphism ring of ${}_AS$. Then the ring R of matrices

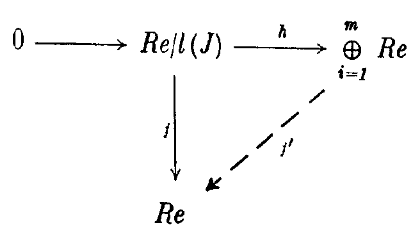
$$\begin{pmatrix} C & 0 \\ S & A \end{pmatrix}$$

is a left noetherian, left QF-3 ring which is not right noetherian, since $\dim S_C = \infty$. If, in addition, we assume that ${}_AS$ is the only simple left A-module, then it is easy to see that R is also its own maximal left quotient ring. Thus, for a left noetherian, left QF-3 ring the maximal left quotient ring does not become nice. For right noetherian, left QF-3 rings the situation is quite different, namely we have the following theorem.

(3. 3) Theorem. *Let R be right noetherian. If R has an injective faithful left ideal, then the maximal left quotient ring Q_l of R is a semi-primary QF-3 ring.*

Proof. Let Re be an injective faithful left ideal of R, where e is an idempotent.

First, we want to show that the ring eRe is right artinian. If J is a right ideal of eRe, denote by $l(J) = \{x \in Re \mid xJ = 0\}$ the left annihilator in Re, and similarly, for an R-submodule K of Re, denote by $r(K) = \{y \in eRe;\ Ky = 0\}$ the right annihilator in eRe. We claim that for every right ideal J of eRe, we have $J = rl(J)$. For, eRe is right noetherian, so $J = \sum_{i=1}^{m} y_i eRe$ for some natural m and elements $y_i \in J$. The mapping $Re \to \bigoplus_{i=1}^{m} Re$, given by $x \to (xy_1, \ldots, xy_m)$, induces a monomorphism h from $Re/l(J)$ into $\bigoplus_{i=1}^{m} Re$. Given $y \in rl(J)$, define $f\colon Re/l(J) \to R$ by $(x + l(J))f = xy$. Then the diagram



can be completed, since ${}_RRe$ is injective. But f' is determined by certain elements $b_i \in eRe$, $1 \leqq i \leqq m$ namely

$$(x_1, \ldots, x_m)f' = \sum_{i=1}^{m} x_i b_i \quad \text{for all } (x_1, \ldots, x_m) \in \bigoplus_{i=1}^{m} Re.$$

Thus, for all $x \in Re$,

$$xy = (x + l(J))f = (x + l(J))hf' = (xy_1, \ldots, xy_m)f' = \bigoplus_{i=1}^{m} xy_i b_i,$$

and therefore $y = \sum_{i=1}^{m} y_i b_i$ belongs to the right ideal J of eRe. This proves $rl(J) \leqq J$, so obviously $rl(J) = J$.

Now let $N = \mathrm{Rad}(eRe)$, and consider the descending chain

$$N \geqq N^2 \geqq N^3 \geqq \cdots.$$

For the left annihilators in Re we get an ascending chain

$$l(N) \leqq l(N^2) \leqq l(N^3) \leqq \cdots \leqq Re.$$

Since N^k is a two-sided ideal of eRe, it follows that $l(N^k)$ is also an eRe-submodule of Re_{eRe}. But R_R is noetherian so also Re_{eRe} is noetherian, and consequently $l(N^n) = l(N^{n+1})$ for some natural n. This implies that

$$N^n = rl(N^n) = rl(N^{n+1}) = N^{n+1},$$

and using Nakayama's lemma for the finitely generated module $(N^n)_{eRe}$, we conclude $N^n = 0$. On the other hand, eRe is the endomorphism ring of ${}_RRe$, and ${}_RRe$ is a finite direct sum of indecomposable injective modules, so eRe is semi-perfect. It follows that eRe is semi-primary, and since eRe is also right noetherian, it is right artinian.

Using the fact that Re_{eRe} is noetherian, so finitely generated, so of finite length, we see that the endomorphism ring $Q = \mathrm{End}(Re_{eRe})$ is semi-primary. We claim that Q is the maximal left quotient ring of R. As we have seen in (3. 1), $E({}_RR)$ is embeddable into $\bigoplus_{i=1}^{n} Re$, for some n, so there is ${}_RC$ with

$$E({}_RR) \oplus {}_RC = \bigoplus_{i=1}^{n} Re.$$

Now ${}_RC$ is generated by $E({}_RR)$, since it is an injective left R-module; also, ${}_RC$ is cogenerated by $E({}_RR)$, since it is cogenerated by Re. Therefore, the double centralizer Q of ${}_RR$ coincides with the double centralizer of $E({}_RR)$, but this is just the maximal left quotient ring Q_l.

Since Q is a left quotient ring of R, we know by (1. 2) that ${}_QQe = {}_QRe$ is an injective faithful left ideal of Q. But Q is semi-primary, so $\mathrm{Soc}({}_QQe)$ is essential in ${}_QQe$, and therefore Q is even a left QF-3 ring. It remains to be shown that Q has also an injective faithful right ideal.

We may assume that ${}_QQe$ is a minimal faithful module. In this case, we find an idempotent f in Q such that $\mathrm{Soc}({}_QQe) \cong Qf/N'f$, where $N' = \mathrm{Rad}(Q)$, since $\mathrm{Soc}({}_QQe)$ is the direct sum of a finite number of mutually non-isomorphic simple modules. Obviously, fQ_Q is faithful, since for $0 \neq q \in Q$ there is $t \in Qe$ with $qt \neq 0$, and then $Qqt \cap \mathrm{Soc}({}_QQe) \neq 0$; but no non-zero submodule of $Qf/N'f$ is annihilated by f, therefore $fQqt \neq 0$, so $fQq \neq 0$. Thus, we only have to prove that fQ_Q is injective.

We may use (1. 5), replacing R by Q, and get that ${}_{fQf}fQe$ is injective and that $\mathrm{Hom}_{fQf}(fQ, fQe) = Qe$. Now ${}_{fQf}fQe$ is also a cogenerator, since obviously every simple left fQf-module occurs in the socle of ${}_{fQf}fQe$. So ${}_{fQf}fQe$ is an injective cogenerator. The equality above implies the following equalities

$$\mathrm{Hom}_{fQf}({}_{fQf}fQe,\ {}_{fQf}fQe_{eQe}) = eQe_{eQe},$$

and

$$\mathrm{Hom}_{fQf}({}_{fQf}fQf,\ {}_{fQf}fQe_{eQe}) = fQe_{eQe},$$

and since both modules eQe_{eQe} and fQe_{eQe} are artinian, it follows that the modules ${}_{fQf}fQe$ and ${}_{fQf}fQf$ are noetherian. Here we use that for a bimodule ${}_AU_B$, where ${}_AU$ is an injective cogenerator, the fact that $\mathrm{Hom}_A({}_AX, {}_AU_B)$ is an artinian B-module implies that ${}_AX$ is noetherian. As a left noetherian semi-primary ring, fQf is left artinian. Consequently, ${}_{fQf}fQe$ is a finitely generated injective cogenerator over a left artinian ring, so by Tachikawa [27], ${}_{fQf}fQe_{eQe}$ defines a duality. In particular, also fQe_{eQe} is injective.

Now we use the equality $\mathrm{End}(Qe_{eQe}) = Q$ in order to derive that

$$\mathrm{Hom}_{eQe}({}_QQe_{eQe}, fQe_{eQe}) = fQ_Q.$$

But ${}_QQe$ is projective, and fQe_{eQe} is injective, so also fQ_Q is injective. This concludes the proof of the theorem.

(3.4) Corollary. *Let R be right noetherian. If R has an injective faithful left ideal and an injective faithful right ideal, then R is right artinian.*

Proof. Denote by Q the maximal quotient ring of R. By the last theorem, Q is a semi-primary QF-3 ring. If fR is an injective faithful right ideal of R, then $fQ = fR$ is an injective faithful right ideal of Q. Since $\mathrm{Soc}(Q_Q)$ is a finitely generated right ideal, we have an embedding

$$Q_Q \hookrightarrow \bigoplus_{i=1}^{n} fQ$$

for some natural n. This yields an embedding

$$R_R \subseteq Q_R \subseteq \bigoplus_{i=1}^{n} fQ_R = \bigoplus_{i=1}^{n} fR_R,$$

which can be extended to an embedding of $E(R_R)$ into $\bigoplus_{i=1}^{n} fR_R$ since fR_R is injective. Consequently, $E(R_R)$ is finitely generated. Again using Vinsonhaller's theorem [31], we conclude that R is right artinian.

Similar to the characterization of semi-primary QF-3 rings by Colby and Rutter [5], we characterize right noetherian rings which have an injective faithful left ideal.

(3.5) Corollary. *Let R be right noetherian. Then the following conditions are equivalent.*

(i) *R contains an injective faithful left ideal.*

(ii) *R contains a Σ-injective faithful left ideal.*

(iii) *R contains an injective, Π-projective faithful left ideal.*

(iv) *The injective hull of every projective left R-module is projective.*

Proof. (i) ⇒ (ii). If Re is an injective faithful left ideal of R, then by theorem (3.3), eRe is right artinian and Re_{eRe} is finitely generated. Thus, by a result of Faith [10], ${}_RRe$ is Σ-injective.

(ii) ⇒ (iii). Let Q be the maximal quotient ring of R, and consider $\Pi_I Re$, where I is an arbitrary index set, and Re is Σ-injective and faithful. Now $Re = Qe$, and also ${}_QQe$ is Σ-injective. We consider $\Pi Qe = \Pi Re$ as a Q-module. Since Q is semi-primary, ${}_Q\Pi Qe$ has a large socle T,

$$T = \mathrm{Soc}({}_Q\Pi Qe) \cong \oplus_J S_\alpha,$$

where J is another index set, and S_α are simple Q-modules. Obviously, every such S_α is embeddable in ${}_QQe$, so the injective hull $E(S_\alpha)$ is a direct summand of Qe, and therefore again Σ-injective. Now there are only finitely many different types of simple Q-modules, consequently $\oplus_J E(S_\alpha)$ is injective and therefore an injective hull of T. Thus we conclude

$${}_Q\Pi Qe \cong \oplus E_J(S_\alpha),$$

since both are injective hulls of T. Now $E(S_\alpha) = Qe_\alpha$ for some primitive idempotent e_α in $Qe = Re$, so

$$_R\Pi Re = {_R\Pi Qe} = \oplus_J E(S_\alpha) = \oplus_J {_RQe_\alpha} = \oplus_J {_RRe_\alpha},$$

and consequently ΠRe is a projective R-module.

(iii) $\to$ (iv). Let $_RM$ be projective, let $_RP$ be an injective Π-projective faithful left ideal. Since $_RP$ is faithful, there is an embedding $_RR \leqq \Pi_I\,{_RP}$, for some set I. Since $_RM$ is projective, there is an embedding

$$_RM \leqq \oplus_J\,{_RR} \leqq \Pi_J \Pi_I\,{_RP},$$

for some set J. Since $_RP$, and so every product of copies of $_RP$ is injective, $E(_RM)$ is embeddable into a product of copies of $_RP$, and is of course a direct summand. But $_RP$ is also Π-projective, so $E(_RM)$ is also projective.

(iv) $\Rightarrow$ (i). The assumption implies, in particular, that $E(_RR)$ is projective. By (3. 1), R has an injective faithful left ideal.

4. Rings of finite representation type

We are interested in the endomorphism ring of the minimal faithful right module of a semi-primary QF-3 ring with left global dimension $\leqq 2$. Recall that the left global dimension l. gl. dim R of a ring R is defined to be the smallest number n such that every left R-module has a projective resolution of length n. Thus, l. gl. dim $R \leqq 2$ if and only if the kernel of every homomorphism between projective left R-modules is projective.

(4. 1) Lemma. *Let R be a semi-primary QF-3 ring with l. gl.* dim $R \leqq 2$. *Let fR be a minimal faithful right ideal and let $A = fRf$. Then every left A-module is a direct sum of indecomposable A-modules, and every indecomposable left A-module is of the form $_AfRe_i$, with e_i a primitive idempotent of R.*

Proof. Let $_AM$ be a left A-module. Let Re be a minimal faithful left ideal. We know that $_AU = {_AfRe}$ is an injective cogenerator, thus there is an exact sequence

$$0 \longrightarrow {_AM} \longrightarrow \Pi_I\,{_AU} \xrightarrow{g} \Pi_J\,{_AU},$$

where I and J are index sets. There are R-isomorphisms

$$\mathrm{Hom}_A({_AfR_R}, \Pi_A U) \cong \Pi\,\mathrm{Hom}_A({_AfR_R}, {_AU}) \cong \Pi_R Re,$$

where the last isomorphism is given by (1. 5). By (1. 8), $_RRe$ is Π-projective, so $\mathrm{Hom}_A({_AfR_R}, \Pi_A U)$ is a projective left R-module. Since l. gl. dim $R \leqq 2$, the kernel of

$$\mathrm{Hom}(fR, g): \mathrm{Hom}_A({_AfR_R}, \Pi_I\,{_AU}) \to \mathrm{Hom}_A({_AfR_R}, \Pi_J\,{_AU})$$

has to be projective. But a projective module over a semi-primary ring is a direct sum of primitive indecomposable modules, so we have an exact sequence

$$0 \longrightarrow \oplus\, Re_k \longrightarrow \mathrm{Hom}_A({_AfR}, \Pi_I\,{_AU}) \xrightarrow{\mathrm{Hom}(fR,g)} \mathrm{Hom}_A({_AfR}, \Pi_J\,{_AU})$$

with primitive idempotants e_k of R. Tensoring with fR_R is an exact functor, and

$$_AfR \otimes_R (\oplus\, Re_k) \cong \oplus\,({_AfR} \otimes_R Re_k) \cong \oplus\,{_AfRe_k}$$

shows that $\oplus_A\, fRe_k$ is the kernel of $fR \otimes \mathrm{Hom}_A(fR, g)$.

Since the functor $fR \otimes_R \operatorname{Hom}_A(fR, -)$ is natural equivalent to the identity, there is a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \oplus fRe_k & \longrightarrow & fR \otimes_R \operatorname{Hom}_A(fR, \Pi U) & \xrightarrow{fR \otimes \operatorname{Hom}(fR, g)} & fR \otimes_R \operatorname{Hom}_A(fR, \Pi U) \\
& \downarrow m & & \downarrow \cong & & \downarrow \cong \\
0 \longrightarrow & {}_AM & \longrightarrow & \Pi_A U & \xrightarrow{\quad g \quad} & \Pi_A U,
\end{array}$$

where m exists and is an isomorphism, since kernels of isomorphic mappings are isomorphic. This shows that

$${}_AM \cong \oplus\, {}_AfRe_k.$$

Since A is left artinian and ${}_AfR$ is of finite length, ${}_AfRe_k$ is a direct sum of finitely many indecomposable left A-modules. On the other hand, if ${}_AM$ is indecomposable then the isomorphism ${}_AM \cong \oplus\, {}_AfRe_k$ shows that ${}_AM$ is of the form ${}_AfRe_i$, for a primitive idempotent e_i of R. This concludes the proof.

The ring A is said to be of *finite representation type* provided A is left artinian and has only a finite number of finitely generated indecomposable left A-modules. The lemma above asserts that the endomorphism ring of the minimal faithful right ideal of a semi-primary QF-3 ring of left global dimension $\leqq 2$ is of finite representation type. The converse is also true, as the following result shows.

(4. 2) Lemma. *Let A be a ring of finite representation type and let $M_1, \ldots, M_n$ be left A-modules representing all isomorphism classes of finitely generated indecomposable left A-modules. Let ${}_AM = \bigoplus_{i=1}^{n} M_i$, and $R = \operatorname{End}({}_AM)$. Then R is a semi-primary QF-3 maximal quotient ring, and l. gl.* dim $R \leqq 2$.

Proof. Since A is of finite representation type, the indecomposable injective left A-modules are finitely generated. Otherwise, A would have indecomposable left modules of arbitrary finite length, since every submodule of an indecomposable injective module is indecomposable. Consequently, all indecomposable injective left A-modules occur as direct summands of M, so M is a cogenerator. Obviously, M is also a generator, and since M is of finite length, M is linearly compact. Thus, by the structure theorem (2. 1), R is a QF-3 maximal quotient ring. We may assume that A is a basic ring, thus ${}_AM$ contains as a direct summand ${}_AA$, and we denote by f the corresponding idempotent of R. We can identify ${}_AfR_R$ with ${}_AM_R$, and we know that fR is a minimal faithful right ideal of R.

It remains to be shown that l. gl. dim $R \leqq 2$, and for this it is enough to prove that the kernel of any homomorphism

$$h : \oplus_I\, {}_AR \to \oplus_J\, {}_RR$$

with I and J arbitrary index sets, is projective. Let h be given. First, assume that I is finite. Note that

$${}_RR = \operatorname{Hom}_A({}_AM_R, {}_AM) = \operatorname{Hom}_A({}_AfR_R, {}_AfR).$$

Also, there is a homomorphism

$$h' : \oplus_I\, {}_AfR \to \oplus_J\, {}_AfR$$

such that h and $\operatorname{Hom}_A({}_AfR_R, h')$ are isomorphic. Namely, take h' isomorphic to $fR \otimes_R h$ and use that the functor $fR \otimes_R \operatorname{Hom}_A({}_AfR_R, -)$ is natural equivalent to the identity.

Now ${}_A fR$, and therefore also $\oplus_I {}_A fR$, is of finite length, so the kernel of h' is a left A-module of finite length, and therefore a direct sum of certain of the M_i's. Thus, we have an exact sequence

$$0 \longrightarrow \oplus_K M_{i_k} \longrightarrow \oplus_I {}_A fR \xrightarrow{h'} \oplus_J {}_A fR,$$

and application of the functor $\mathrm{Hom}_A({}_A fR_R, —)$ yields an exact sequence

$$0 \to \mathrm{Hom}_A({}_A fR_R, \oplus_K M_{i_k}) \to \mathrm{Hom}_A({}_A fR_R, \oplus_A fR) \to \mathrm{Hom}_A({}_A fR_R, \oplus_A fR).$$

If we denote by $e_i \in R$ the projection of M onto M_i, then we see that

$$\mathrm{Hom}_A({}_A fR_R, \oplus_K M_{i_k}) \cong \oplus_K \mathrm{Hom}_A({}_A M_R, {}_A M_{i_k}) = \oplus_K Re_{i_k}$$

is a projective left R-module. Since h is isomorphic to $\mathrm{Hom}_A({}_A fR_R, h')$, it follows that the kernel of h is projective.

Now assume that I is arbitrary. For any finite subset $F \subseteq I$ denote by h_F the restriction of h to the finitely generated free submodule $\oplus_F {}_R R$ of $\oplus_I {}_R R$, so

$$h_F : \oplus_F {}_R R \to \oplus_J {}_R R,$$

and by the considerations above, we know that the kernel of h_F is projective. But the kernel of h is obviously the union of the kernels of the homomorphisms h_F, where F runs through all finite subsets of I. This shows that the kernel of h is the union of a directed set of projective modules, and therefore itself projective, since R is semi-primary.

(4. 3) Theorem. *There is a one-to-one correspondence between*

(I) *Morita equivalence classes of semi-primary QF-3 maximal quotient rings R with l. gl.* dim $R \leq 2$, *and*

(II) *Morita equivalence classes of rings of finite representation type.*

Here, a ring R of (I) *is associated with the endomorphism ring of a minimal faithful right R-module, and conversely, a ring A satisfying* (II) *is associated with the endomorphism ring of a module which is the direct sum of all different finitely generated indecomposable left A-modules.*

Proof. Let R be a ring with the properties given in (I). If fR is a minimal faithful right ideal, then by (4. 1), $A = fRf$ is of finite representation type, and every indecomposable left A-module is a direct summand of ${}_A fR$. Also, ${}_A fR$ is finitely generated. Thus, if ${}_A M$ is the direct sum of all different finitely generated indecomposable left A-modules, then ${}_A fR$ and ${}_A M$ have the same indecomposable direct summands and therefore $\mathrm{End}({}_A M)$ and $R = \mathrm{End}({}_A fR)$ are Morita equivalent.

Conversely, let ${}_A M$ be the direct sum of all different finitely generated indecomposable left A-modules, where A is a ring of finite representation type, then by (4. 2), $R = \mathrm{End}({}_A M)$ satisfies the conditions of (I). Let fR be a minimal faithful right ideal of R. If we assume that A is a basic ring, then $A \simeq fRf$, thus, in general, A is Morita equivalent to fRf.

For "artin algebras", that is for artinian rings which are finitely generated over the center, the theorem above was proved by Auslander [1]. Namely, an artin algebra is a QF-3 maximal quotent ring if and only if its dominant dimension is ≥ 2. Here, a ring R is said to have dominant dimension $\geq n$, provided there exists an exact sequence of left R-modules

$$0 \to {}_R R \to I_1 \to I_2 \to \cdots \to I_n,$$

with I_i injective and projective, for all $1 \leq i \leq n$. By (3.1), a right artinian ring is a left QF-3 ring if and only if the dominant dimension of R is ≥ 1. It is wellknown that R coincides with its maximal left quotient ring if and only if $E(_R R)/_R R$ is cogenerated by $_R R$. But if R is also left artinian (and $E(_R R)$ is projective), then R is a maximal left quotient ring if and only if the dominant dimension of R is ≥ 2. Now we only have to use theorem (3.3) to see that an artin algebra which is a left QF-3 ring and a maximal left quotient ring is in fact QF-3. However, the generalization of Auslander's theorem is of interest, since there are obviously non-trivial examples of rings of finite representation type which are not finitely generated over the center, for example the "exceptional rings" of Dlab and Ringel [7].

There is an important consequence of the previous investigations concerning the module category of a ring of finite representation type. Apparently, this was not yet noticed even in the case of finite dimensional algebras.

(4.4) Corollary. *Let A be of finite representation type. Then every left A-module is a direct sum of finitely generated modules.*

Proof. By (4.3), we may assume $A = fRf$, where R is a semi-primary QF-3 ring with l. gl. dim $R \leq 2$, and fR a minimal faithful right ideal. Now the statement follows from (4.1), since for every idempotent e_i of R, the left A-module $_A fRe_i$ is finitely generated.

A theorem of Eisenbud and Griffith [9] asserts that a ring of finite representation type is also right artinian and has only a finite number of finitely generated indecomposable right modules. Thus, the assertion of (4.4) is also true for right A-modules.

5. QF-13 rings

Thrall used the notion of a QF-13 algebra for algebras which are at the same time QF-1 and QF-3, and we will understand by a QF-13 a ring which is both a QF-1 ring and a QF-3 ring. We have already derived a result for QF-13 ring. Namely, an obvious consequence of theorem (3.3) is that a right noetherian QF-13 ring is right artinian. Since a QF-13 ring R is in particular a maximal quotient ring, we may consider R as the endomorphism ring of a linearly compact generator and cogenerator $_A M$. We are interested in properties of the module $_A M$ which are necessary or sufficient for End$(_A M)$ to be a QF-1 ring. The first theorem gives a necessary condition: every indecomposable direct summand of $_A M$ is either injective with non-zero socle or projective; moreover, there is a torsion theory containing all indecomposable modules which are injective but not projective, as torsion objects, and all indecomposable modules which are projective but not injective or with zero socle, as torsionfree objects.

(5.1) Theorem. *Let $_A M$ be a linearly compact generator and cogenerator, and let $R = $ End$(_A M)$. Then the following conditions are equivalent:*

(i) *Every faithful left R-module which is a direct sum of local and of colocal R-modules is balanced.*

(ii) *Every faithful right R-module which is a direct sum of local and of colocal R-modules is balanced.*

(iii) *If $_A X$ and $_A Y$ are indecomposable direct summands of $_A M$ and if*

$$\mathrm{Hom}_A(_A X, {_A Y}) \neq 0,$$

then either $_A X$ is projective or $_A Y$ is injective with non-zero socle.

Proof. We may assume that ${}_AM = {}_AA \oplus {}_AU \oplus {}_AC$, where ${}_AU$ is a minimal injective cogenerator, and we denote by f the projection onto ${}_AA$, by e the projection onto ${}_AU$. We know from section 2 that fR is an injective faithful right ideal and that Re is an injective faithful left ideal.

Let ${}_AX$ be an indecomposable direct summand of ${}_AM$, and let e' be a projection onto ${}_AX$. Now, ${}_RRe'$ is isomorphic to a direct summand of Re if and only if ${}_AX$ is injective with non-zero socle. For, let ${}_RRe' \simeq {}_RRe_i$ for some i, where $e = \Sigma e_i$ is a decomposition with orthogonal primitive idempotents e_i. Then there is an invertible element in $e'Re_i$, that is, an isomorphism between the A-modules ${}_AX$ and $\mathrm{Im}(e_i)$. Since $\mathrm{Im}(e_i)$ is an indecomposable direct summand of $\mathrm{Im}(e) = {}_AU$, we know that $\mathrm{Im}(e_i)$ is injective with non-zero socle. Conversely, if ${}_AX$ is injective with non-zero socle, then ${}_AX$ is isomorphic to a direct summand $\mathrm{Im}(e_i)$ of ${}_AU$, so ${}_RRe_i \simeq {}_RRe'$, for some i.

Also, let ${}_AY$ be an indecomposable direct summand of ${}_AM$, and f' a projection onto ${}_AY$. Then $f'R_R$ is isomorphic to a direct summand of fR_R if and only if ${}_AY$ is projective. For, let $f'R_R \simeq f_iR_R$ for some i, where $f = \Sigma f_i$ with orthogonal primitive idempotents f_i. Now, $\mathrm{Im}(f_i)$ is a direct summand of $\mathrm{Im}(f) = {}_AA$, so $\mathrm{Im}(f_i)$ is projective. On the other hand, if ${}_AY$ is projective, then isomorphic to a direct summand of ${}_AA$, so to some $\mathrm{Im}(f_i)$. The statement now follows from the fact that the R-modules f_iR_R and $f'R_R$ are isomorphic if and only if the A-modules $\mathrm{Im}(f_i)$ and $\mathrm{Im}(f') = {}_AY$ are isomorphic.

Using the above considerations and the fact that $\mathrm{Hom}_A({}_AX, {}_AY) \neq 0$ if and only if $e'Rf' \neq 0$, we may reformulate condition (iii) in terms of R as follows:

(iv) If e' and f' are primitive idempotents of R and neither Re' is isomorphic to a direct summand of the minimal faithful left R-module, nor $f'R$ is isomorphic to a direct summand of the minimal faithful right R-module, then $f'Re' = 0$.

Since (iv) is left-right symmetric, it is enough to show the equivalence of (i) and (iv). Assume that (i) is satisfied. Let e', f' be primitive idempotents of R with $f'Re' \neq 0$ and take $0 \neq x = f'xe' \in f'Re'$. Let L be a maximal left ideal contained in Re' and not containing x. Then Re'/L has simple top, namely Re'/Ne', where $N = \mathrm{Rad}(R)$, and simple socle, namely $(Rx + L)/L \simeq Rf'/Nf'$. The R-module $Re \oplus Re'/L$ is faithful and a direct sum of local modules, so by assumption has to be balanced. But according to Morita [17], this is only possible, if Re either generates or cogenerates Re'/L. So we see that either the top Re'/Ne' has to be isomorphic to some Re_i/Ne_i, where Re_i is a direct summand of Re, and then also ${}_RRe' \simeq {}_RRe_i$, or else the socle Rf'/Nf' of Re'/L can be imbedded into Re, and then $Rf'/Nf' \simeq Rf_i/Nf_i$ for some idempotent f_i such that f_iR is a direct summand of fR. Since $Rf'/Nf' \simeq Rf_i/Nf_i$ implies $f'R/f'N \simeq f_iR/f_iN$, we see that in this case $f'R \simeq f_iR$ is isomorphic to a direct summand of fR.

Conversely, let (iv) be satisfied. Again using Morita's criterion [17], we see that we have to show that every local and every colocal left R-module ${}_RK$ is generated or cogenerated by Re. Assume first that ${}_RK$ is local, so ${}_RK \simeq Re'/L$ for some primitive idempotent e' and some left ideal $L \leq Re'$. Since ${}_RK$ is not generated by Re, we know that Re' is not isomorphic to a direct summand of Re. By (iv) we see that for every primitive idempotent f' of R with $f'Re' \neq 0$, the right R-module $f'R$ is isomorphic to a direct summand of fR. Thus, given an element $0 \neq x = f'xe' + L \in Re'/L$, the module Rf'/Nf' appears as a submodule of Re, so there is an R-homomorphism $\phi: (Rx + L)/L \to Re$ with $x\phi \neq 0$, and since Re is injective, we can extend ϕ to an R-homomorphism $Re'/L \to Re$. Consequently, ${}_RK = Re'/L$ is cogenerated by Re. Second, assume that ${}_RK$ is colocal and not cogenerated by Re. Let $0 \neq s = f''s \in \mathrm{Soc}({}_RK)$, with f'' a primitive idempotent of R. Since ${}_RK$ is not cogenerated by Re, it follows that $f''R$ is not isomorphic to a direct summand of fR. Given an element $0 \neq m = e''m$ in K, with e'' a primitive idempotent, we have $s \in Re''m$, since s is contained in every non-zero submodule. As a consequence, $f''Re'' \neq 0$, and by (iv), Re'' is isomorphic to a direct summand of Re. Therefore, there is an R-homomorphism $Re \to {}_RK$, mapping e onto m. This shows that Re generates ${}_RK$.

We derive from this theorem several corollaries. Note that condition (iii) in particular implies that every indecomposable direct summand of ${}_RM$ is either injective with non-zero socle or projective.

(5. 2) Corollary. *A QF-13 ring is semi-perfect.*

Proof. The endomorphism ring of an indecomposable module which is either injective or projective is always a local ring. Thus, if R is QF-13, then there is a decomposition $1 = \sum_{i=1}^{n} e_i$ with orthogonal primitive idempotents e_i such that e_iRe_i is local, for all i.

(5. 3) Corollary. *Let R and R' be QF-13 rings with minimal faithful left modules Re and $R'e'$. If the rings eRe and $e'R'e'$ are isomorphic, then R and R' are Morita equivalent.*

Proof. Let fR and $f'R'$ be minimal faithful right ideals, respectively. If

$$eRe \cong e'R'e',$$

then also $fRf \cong f'R'f'$, since Morita dual basic rings determine each other up to isomorphism. By (2. 1), $R \cong \mathrm{End}({}_{fRf}fR)$ and $R' \cong \mathrm{End}({}_{f'R'f'}f'R')$. But ${}_{fRf}fR$ is a direct sum of indecomposable modules which are either injective with non-zero socle or projective, and similarly ${}_{f'R'f'}f'R'$. So, if we identify fRf and $f'R'f'$, then fR and $f'R'$ considered as fRf-modules differ only in the multiplicity of the occurrence of the indecomposable direct summands. Thus R and R' are Morita equivalent.

(5. 4) Corollary. *Let R be an artinian QF-1 ring with l. gl.* dim $R \leq 2$. *Then* $R = \mathrm{End}({}_AM)$, *where A is an artinian ring such that every indecomposable left A-module is either projective or injective, and ${}_AM$ is a finitely generated generator and cogenerator.*

Proof. It was shown by Ringel [24] that an artinian QF-1 ring R with l. gl. dim $R \leq 2$ is a QF-3 ring. By (4. 3), $R = \mathrm{End}({}_AM)$ where ${}_AM = \bigoplus_{i=1}^{n} M_i$ such that every indecomposable left A-module occurs as one of the M_i. By (5. 1), every indecomposable direct summand of ${}_AM$ is either projective or injective.

Artinian rings for which every indecomposable left module is either projective or injective, can be classified, and it can be shown that the converse of (5. 4) is also true.

Recall that a ring R is said to be of local-colocal representation type (or "cyclic-cocyclic representation type"), provided R is of finite representation type and every indecomposable R-module is either local or colocal. By (4. 4), every R-module is a direct sum of local and of colocal R-modules. Now the equivalence of (i) and (ii) in theorem (5. 1) implies:

(5. 5) Corollary. *Let R be a QF-3 ring of local-colocal representation type. Then R is left QF-1 if and only if R is right QF-1.*

In particular, the conclusion holds for serial rings. In the remaining part of this section we will concentrate on this class of rings. As we have mentioned, a module ${}_AM$ is called minimal fully faithful provided ${}_AM$ is a generator and a cogenerator, but no direct summand of ${}_AM$ is a generator and a cogenerator.

(5.6) Lemma. *Let A be a serial ring and ${}_AM$ minimal fully faithful. Then $R = \operatorname{End}({}_AM)$ is serial, again.*

Proof. We may assume that A is a basic ring and two-sided indecomposable. Let $1 = \sum_{i=1}^{n} e_i$ be a decomposition of $1 \in A$ into orthogonal primitive idempotents. There are elements $a_i \in e_i A e_{i+1}$ $(1 \leqq i \leqq n$, with $e_{n+1} = e_1)$ such that

$$A a_i = N e_{i+1} \quad \text{and} \quad a_i A = e_i N,$$

where $N = \operatorname{Rad}(A)$. Let $\partial_i = |A e_i|$, then $\partial_i \geqq \partial_{i+1} - 1$, and we set $f(i) = \partial_i - \partial_{i+1}$, so $f(i) \geqq -1$. Note that $f(i) + 1 = |\operatorname{Ker}(a_i)|$, where we consider a_i as right multiplication, $a_i \colon A e_i \to A e_{i+1}$, for

$$|\operatorname{Ker}(a_i)| = |A e_i| - |\operatorname{Im}(a_i)| = |A e_i| - |A e_{i+1}| + 1 = f(i) + 1.$$

Given an A-module ${}_AM$, denote $S^j M = \{m \in M \mid N^j m = 0\}$, with $j \geqq 0$. Then $S^0 M = 0$, $S^1 M = \operatorname{Soc}(M)$, and, for a serial module, $|S^j M| = j$.

If $0 \leqq j \leqq f(i)$, then $A e_i / S^j A e_i$ is injective. For assume $A e_i / S^j A e_i$ is properly contained in its injective hull. Since the injective hull is indecomposable, it is of the form $A e_k / *$ for some k. So we have a commutative square

$$\begin{array}{ccc} A e_i / S^j A e_i & \xhookrightarrow{\ \mu\ } & A e_k / * \\ \uparrow{\scriptstyle \varepsilon_i} & & \uparrow{\scriptstyle \varepsilon_k} \\ A e_i & \xrightarrow{\ \phi\ } & A e_k, \end{array}$$

with the canonic epimorphisms ε_i, ε_k, and a lifting ϕ of the inclusion μ. Since ϕ is not surjective, $\phi \in e_i N = a_i A$, so $\phi = a_i \phi'$ for some ϕ'. But then

$$S^j A e_i = \operatorname{Ker}(\varepsilon_i \mu) = \operatorname{Ker}(\phi \varepsilon_k) \supseteq \operatorname{Ker}(a_i),$$

a contradiction, since $|S^j A e_i| = j$, and $|\operatorname{Ker}(a_i)| = f(i) + 1 \geqq j$.

We define modules M_{ij} with $1 \leqq i \leqq n$ in the following way:

$$M_{i0} = A e_i, \quad \text{and, if} \quad f(i) \geqq 1, \quad \text{then} \quad M_{ij} = A e_i / S^j A e_i$$

$(1 \leqq j \leqq f(i))$ and denote by $\varepsilon_{ij} \colon A e_i \to M_{ij}$ the canonical epimorphism. Then every M_{ij} is either injective or projective (or both), and every indecomposable projective $A e_i$ occurs as M_{i0}. Also, every indecomposable injective left A-module appears as M_{ij}. Namely, given e_k, we want to construct $E(A e_k / N e_k)$. There is e_i and

$$0 \neq x = e_k x e_i \in \operatorname{Soc}(e_k A).$$

Let $x \in S^{j+1} A e_i \setminus S^j A e_i$, for some $j \geqq 0$. Note that $j \leqq f(i)$, since $x \in \operatorname{Soc}(e_k A) \subseteq \operatorname{Ker}(a_i)$, and therefore

$$j + 1 = |S^{j+1} A e_i| = |A x| \leqq |\operatorname{Ker}(a_i)| = f(i) + 1.$$

Thus M_{ij} exists, is injective, and

$$\operatorname{Soc}(M_{ij}) = A x / N x \cong A e_k / N e_k.$$

This shows that $M_{ij} = E(A e_k / N e_k)$.

We consider now the A-module

$${}_AM = \oplus_{i,j} M_{ij},$$

where the index set is given by

$$I = \{(i,j) \mid 1 \leqq i \leqq n, \text{ and } j = 0 \text{ or } 1 \leqq j \leqq f(i)\}.$$

We order I lexically, and denote by $(ij) + 1$ the successor of (ij). Thus $(ij) + 1 = (i, j+1)$ for $j < f(i)$, and $(ij) + 1 = (i+1, 0)$ or $j = f(i)$. We define A-homomorphisms $r_{ij}: M_{ij} \to M_{(i,j)+1}$ in the following way. If $j < f(i)$, then

$$r_{ij}: M_{ij} = A e_i / S^j A e_i \to A e_i / S^{j+1} A e_i = M_{(i,j+1)}$$

the canonical epimorphism, whereas for $j = f(i)$,

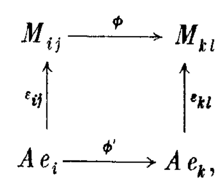
$$r_{if(i)}: M_{if(i)} = A e_i / S^{f(i)} A e_i \to A e_{i+1} = M_{(i+1,0)}$$

is defined by the equation

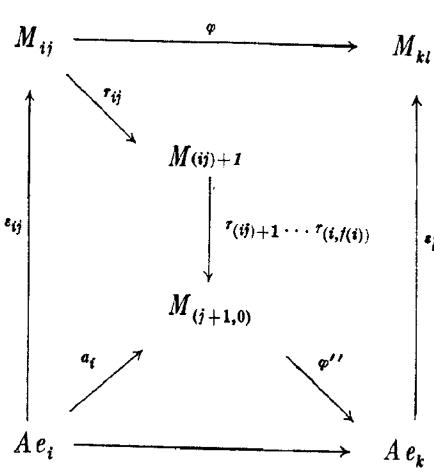
$$\varepsilon_{if(i)} r_{if(i)} = a_i.$$

Here we use that $S^{f(i)} A e_i \leqq \mathrm{Ker}(a_i)$, so a_i factors uniquely through $\varepsilon_{if(i)}$. Note, that with these definitions $\varepsilon_{ij} r_{ij} \cdots r_{il} = \varepsilon_{i,l+1}$, for $j \leqq l$.

Let W be the radical of $R = \mathrm{End}(_A M)$. Let e_{ij} be the idempotent of R given by the projection onto M_{ij}. So we may consider $r_{ij} \in e_{ij} R e_{(ij)+1}$. First, we show that $e_{ij} W = r_{ij} R$. For, assume there is given $\phi: M_{ij} \to M_{kl}$ which is not an isomorphism. We get a commutative diagram



where ϕ' exists since $A e_i$ is projective. If $\phi' \in e_i A e_k$ belongs to the radical N of A, then $\phi' = a_i \phi''$ for some ϕ'', then



is a commutative diagram and thus ϕ factors through r_{ij}. If, on the other hand, $\phi' \notin N$, then $i = k$ and ϕ is an epimorphism. But since ϕ is not an isomorphism, $\mathrm{Ker}(\phi) \neq 0$, so $\mathrm{Ker}(\phi) \geqq \mathrm{Soc}(M_{ij}) = S^{j+1} A e_i / S^j A e_i$. As a consequence, ϕ factors through the canonical epimorphism

$$r_{ij}: M_{ij} \to M_{ij} / \mathrm{Soc}(M_{ij}) \simeq M_{(i,j+1)}.$$

This shows that every element in $e_{ij} W e_{kl}$, with e_{kl} arbitrary, belongs to $r_{ij} R$, and therefore $e_{ij} W = r_{ij} R$.

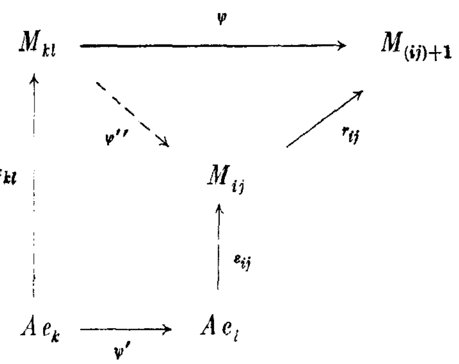
Also, we show that $We_{(ij)+1} = Rr_{ij}$. So assume there is given a non-zero $\psi: M_{kl} \to M_{(ij)+1}$, not an isomorphism. We want to show that there is ψ' with

$$\varepsilon_{kl}\psi = \psi'\varepsilon_{ij}r_{ij}.$$

In the case $j = f(i)$, we have $(ij) + 1 = (i+1, 0)$, and, since $\varepsilon_{kl}\psi: Ae_k \to Ae_{i+1}$ is not an isomorphism, it belongs to $Ne_{i+1} = Ra_i$. So we find ψ' with

$$\varepsilon_{kl}\psi = \psi' a_i = \psi'\varepsilon_{ij(i)}r_i.$$

In the case $j < f(i)$, we have $(ij) + 1 = (i, j+1)$, so $\varepsilon_{ij}r_{ij} = \varepsilon_{(i,j+1)}$ is an epimorphism. Since Ae_k is projective, we find also in this case ψ' with $\varepsilon_{kl}\psi = \psi'\varepsilon_{ij}r_{ij}$. Consider now the corresponding diagram



where we are looking for ψ'' making the diagram commutative. It is enough to show that $\psi'\varepsilon_{ij}$ factors through ε_{kl}, for $\varepsilon_{kl}\psi'' = \psi'\varepsilon_{ij}$ implies that $\psi''r_{ij} = \psi$, since ε_{kl} is an epimorphism. So we have to prove that

$$\operatorname{Ker}(\varepsilon_{kl}) \leqq \operatorname{Ker}(\psi'\varepsilon_{ij}).$$

This is trivial for $l = 0$, for in this case $\varepsilon_{kl} = \text{id}$. If $l > 0$, then, as we have seen above, M_{kl} is injective, so ψ cannot be a monomorphism, since otherwise it would be an isomorphism. As a consequence, $\operatorname{Ker}(\varepsilon_{kl}\psi)$ properly contains $\operatorname{Ker}(\varepsilon_{kl})$, therefore

$$|\operatorname{Ker}(\varepsilon_{kl})| \leqq |\operatorname{Ker}(\varepsilon_{kl}\psi)| - 1 = |\operatorname{Ker}(\psi'\varepsilon_{ij}r_{ij})| - 1 = |\operatorname{Ker}(\psi'\varepsilon_{ij})|,$$

where the last equality stems from the fact that $\psi'\varepsilon_{ij} \neq 0$ and $|\operatorname{Ker}(r_{ij})| = 1$. Thus we have shown that $\operatorname{Ker}(\varepsilon_{kl}) \leqq \operatorname{Ker}(\psi'\varepsilon_{ij})$, so there exists ψ'' with $\varepsilon_{kl}\psi'' = \psi'\varepsilon_{ij}$, and $\psi''r_{ij} = \psi$. Therefore every element in $e_{kl}We_{(ij)+1}$ belongs to Rr_{ij}, and, since e_{kl} was arbitrary, we have $We_{(ij)+1} = Rr_{ij}$.

(5. 7) Theorem. *The ring R is a serial QF-1 ring if and only if R is Morita equivalent to* $\operatorname{End}({}_AM)$, *where A is serial, and ${}_AM$ is minimal fully faithful such that given indecomposable direct summands ${}_AX$ and ${}_AY$ of ${}_AM$ with* $\operatorname{Hom}_A({}_AX, {}_AY) \neq 0$, *either ${}_AX$ is projective or ${}_AY$ is injective.*

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