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On Algebras of Finite Representation Type*

VLASTIMIL DLAB AND CLAUS MICHAEL RINGEL

Department of Mathematics, Carleton University, Ottawa, Canada K1S 5B6
and
Mathematisches Institut, Universität Bonn, Beringstr. 1, West Germany

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INTRODUCTION

Throughout the paper, K denotes a fixed commutative field. Let F be a field containing K in its center such that F_K is finite dimensional. A finite (partially) ordered set $\mathscr{S}$ together with an order preserving mapping of $\mathscr{S}$ into the lattice of all subfields of F containing K is called a *K-structure* for F; thus, for $i \in \mathscr{S}$, there is given a subfield F_i of F and, moreover, $K \subseteq F_i \subseteq F_j$ for each $i \leqslant j$ of $\mathscr{S}$. For a fixed subfield G of F containing K and a natural number n, the symbol $\mathscr{I}_n(G)$ will be used to denote the K-structure defined by the chain $\{1 < 2 < \cdots < n\}$ of n elements such that $F_i = G$ for all elements i of the chain. Furthermore, $\mathscr{N}(G)$ will denote the K-structure given by the ordered set $\{i < j > k < l\}$ with $F_i = F_j = F_k = F_l = G$. Given a K-structure $\mathscr{S}$ for F, the *weighted width* of $\mathscr{S}$ is defined as the maximum of all possible sums $\sum_{j \in J} \dim F_{F_j}$, where J is a subset of mutually unrelated elements of $\mathscr{S}$.

An *$\mathscr{S}$-space* (W, W_i) is a right vector space W over F together with an F_i-subspace W_i for each $i \in \mathscr{S}$, such that $i \leqslant j$ implies $W_i \subseteq W_j$. The *weighted dimension* of (W, W_i) is the maximum of all dim W_{F_i}. For a given K-structure $\mathscr{S}$, the $\mathscr{S}$-spaces form an additive category in which the morphisms $(W, W_i) \to (W', W_i')$ are F-linear mappings $\varphi: W \to W'$ satisfying $\varphi W_i \subseteq W_i'$, $i \in \mathscr{S}$. Therefore, the concepts of a direct sum and of an indecomposable $\mathscr{S}$-space are defined. A K-structure $\mathscr{S}$ is said to be of *finite type* if there is only a finite number of finite dimensional indecomposable $\mathscr{S}$-spaces. In the case of a classical K-structure, that is $F_i = F$ for all $i \in \mathscr{S}$, L. A. Nazarova and A. V. Roiter [15] and M. M. Kleiner [11] have characterized the structures of finite type. Their results are extended in the following theorem.

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THEOREM A. *Let $\mathscr{S}$ be a K-structure for F. Then $\mathscr{S}$ is of finite type if and only if $\mathscr{S}$ is of weighted width $\leqslant 3$ and does not contain, as a full ordered subset, any of the following structures:*

(i) $\mathscr{I}_2(F) \sqcup \mathscr{I}_2(F) \sqcup \mathscr{I}_2(F)$;
(ii) $\mathscr{I}_1(F) \sqcup \mathscr{I}_3(F) \sqcup \mathscr{I}_3(F)$;
(iii) $\mathscr{I}_1(F) \sqcup \mathscr{I}_2(F) \sqcup \mathscr{I}_5(F)$;
(iv) $\mathscr{I}_4(F) \sqcup \mathscr{N}(F)$;
(v) $\mathscr{I}_2(G) \sqcup \mathscr{I}_2(F)$ *with* $[F\colon G] = 2$;
(vi) $\mathscr{I}_3(G) \sqcup \mathscr{I}_1(F)$ *with* $[F\colon G] = 2$;
(vii) $\mathscr{I}_2(G)$ *with* $[F\colon G] = 3$.

Here, $\sqcup$ denotes the direct sum (disjoint union) of ordered sets. If $\mathscr{S}$ is of finite type, then every indecomposable $\mathscr{S}$-spaces is of weighted dimension $\leqslant 6$.

Following P. Gabriel [8], a *K-species* $(K_i\,,\, {}_iM_j)_{i,j\in I}$ is a finite set of fields K_i which are finite dimensional over a common central subfield K, together with a set of $K_i - K_j$ — bimodules ${}_iM_j$ such that K operates on ${}_iM_j$ centrally (that is $km = mk$ for all $k \in K$ and $m \in {}_iM_j$) and every ${}_iM_j$ is finite dimensional over K. The *diagram* of the K-species $(K_i\,,\, {}_iM_j)_{i,j\in I}$ is defined as follows: The finite index set I is the set of vertices and there are

$$\dim_{K_i}({}_iM_j) \times \dim({}_iM_j)_{K_j} + \dim_{K_j}({}_jM_i) \times \dim({}_jM_i)_{K_i}$$

edges between the vertices i and j. If ${}_jM_i = 0$ and $\dim_{K_i}({}_iM_j) < \dim({}_iM_j)_{K_j}$, we shall mark this fact by an arrow $i \Rightarrow j$. A *representation* $(V_i\,,\, {}_j\varphi_i)$ of the K-species $(K_i\,,\, {}_iM_j)$ is a set of right vector spaces V_i over K_i together with K_j-linear mappings

$${}_j\varphi_i\colon V_i \otimes_{K_i} {}_iM_j \to V_j \qquad \text{for all } i, j \in I.$$

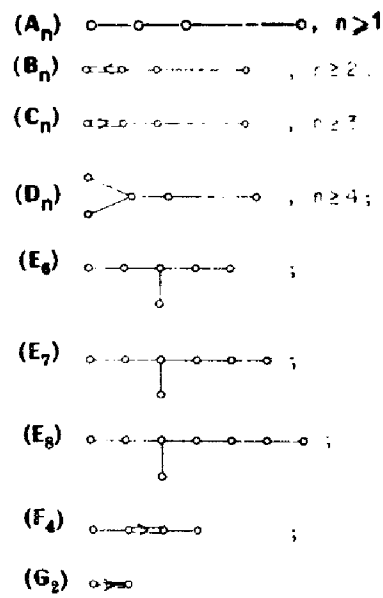
The representations of a given K-species $(K_i\,,\, {}_iM_j)$ form an abelian category in which a morphism $(V_i\,,\, {}_j\varphi_i) \to (V_i'\,,\, {}_j\varphi_i')$ is given by a set of K_i-linear maps $\alpha_i\colon V_i \to V_i'$ satisfying

$${}_j\varphi_i'(\alpha_i \otimes 1) = \alpha_j\, {}_j\varphi_i\,.$$

Again, we have the concepts of a direct sum and of an indecomposable object and we say that the K-species is of *finite type* if the number of its finite dimensional indecomposable representations is finite. In the case when all K_i are equal to a fixed field F and ${}_F({}_iM_j)_F = ({}_FF_F)^{n_{ij}}$ for some natural number n_{ij}, P. Gabriel [7] has characterized K-species of finite type. His result is extended in the following theorem.

THEOREM B. *A K-species is of finite type if and only if its diagram is a finite disjoint union of Dynkin diagrams.*

Recall that the Dynkin diagrams (which occur for example in the theory of simple Lie algebras) are



Given a Dynkin diagram, it is easy to construct a corresponding K-species. Also, P. Gabriel has shown that the numbers of indecomposable representations of the K-species of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$ are, respectively, $\frac{1}{2}n(n+1)$, $n(n-1)$, 36, 63, and 120. We shall prove that there are n^2 indecomposable representations of the K-species of type $\mathbb{B}_n$ or $\mathbb{C}_n$, whereas the numbers of indecomposable representations of the K-species of type $\mathbb{F}_4$ and $\mathbb{G}_2$ are 24 and 6, respectively. Thus, also in each of these remaining cases, the number of indecomposable representations of a K-species of a given type coincides with the number of the positive roots of the corresponding quadratic form (cf. [2]).

To every K-species $\mathscr{Q} = (K_i, {}_iM_j)_{i,j\in I}$, we may associate the tensor algebra $\mathscr{T}(\mathscr{Q}) = \bigoplus_{n=0}^{\infty} M^{(n)}$, where $M^{(0)} = \prod_{i\in I} K_i$, $M^{(1)} = \bigoplus_{i,j\in I} {}_iM_j$, and $M^{(n)}$ is the n-fold tensor product $M^{(1)} \otimes M^{(1)} \otimes \cdots \otimes M^{(1)}$ over $M^{(0)}$; besides, the multiplication is induced by the tensor product. Then the category $\mathfrak{R}(\mathscr{Q})$ of all representations of $\mathscr{Q}$ is equivalent to the category of all right $T(\mathscr{Q})$-modules.

A K-algebra $\mathscr{A}$ (an associative algebra with unity, finite dimensional over K) is said to be of *finite type* if there is only a finite number of indecomposable finite dimensional A-modules. Two classes of K-algebras of finite type, namely hereditary K-algebras and K-algebras with zero square radical, are characterized in the present paper.

THEOREM C. *A finite dimensional K-algebra $\mathscr{A}$ is a hereditary algebra of finite type if and only if $\mathscr{A}$ is Morita equivalent to the tensor algebra $\mathscr{T}(\mathscr{Q})$, where $\mathscr{Q}$ is a K-species of finite type.*

We can attach easily a K-species to an arbitrary finite dimensional K-algebra $\mathscr{A}$. Let $\mathscr{B}$ be the basic algebra of $\mathscr{A}$; thus, $\mathscr{B}/\text{Rad}\,\mathscr{B}$ is a product $K_1 \times K_2 \times \cdots \times K_n$ of fields. We can write

$$\text{Rad}\,\mathscr{B}/(\text{Rad}\,\mathscr{B})^2 = \bigoplus_{1 \leqslant i,j \leqslant n} {}_iM_j$$

with $K_i - K_j$-bimodules ${}_iM_j$. Then $\mathscr{Q}_{\mathscr{A}} = (K_i\,, {}_iM_j)_{1 \leqslant i,j \leqslant n}$ is called the *K-species of* $\mathscr{A}$. The fact that $\mathscr{B}$ is often a quotient ring of $\mathscr{T}(\mathscr{Q}_{\mathscr{A}})$ allows to apply Theorem C in one direction.

Given a K-species $(K_i\,, {}_iM_j)_{i,j \in I}$, define its *separated diagram* as follows. The finite set $I \times \{0, 1\}$ is the set of all vertices, and there are $\dim_{K_i}({}_iM_j) \times \dim({}_iM_j)_{K_j}$ edges between $(i, 0)$ and $(j, 1)$; in addition, there is an arrow $i \rightrightarrows j$ provided $\dim_{K_i}({}_iM_j) < \dim({}_iM_j)_{K_j}$. Note that there are no edges between $(i, 0)$ and $(j, 0)$, nor between $(i, 1)$ and $(j, 1)$.

THEOREM D. *Let $\mathscr{A}$ be a finite dimensional K-algebra with* $(\text{Rad}\,\mathscr{A})^2 = 0$. *Then $\mathscr{A}$ is of finite type if and only if the separated diagram of its K-species is a disjoint union of Dynkin diagrams.*

In the case when the K-species $(K_i\,, {}_iM_j)$ of $\mathscr{A}$ has the property that all K_i are equal to a fixed field F and ${}_F({}_iM_j)_F = ({}_FF_F)^{n_{ij}}$ for some natural n_{ij}, the characterizations given in Theorems C and D are due to P. Gabriel [7, 8] who improved previous results of T. Yoshii [17] (cf. also S. A. Krugljak [12]). Also, P. Gabriel has shown that the structure of a K-algebra $\mathscr{A}$ of finite type with $(\text{Rad}\,\mathscr{A})^2 = 0$ can be recovered from the known results in the case when K is a perfect field. In this way, he has determined for example all $\mathbb{R}$-algebras of finite type, where $\mathbb{R}$ is the real number field. However, his method does not seem to work in the general case.

An additive category $\mathfrak{A}$ will be called a *dimension category* if there exists a mapping dim: $\mathfrak{A} \to \mathbb{N} \cup \{\infty\}$ satisfying the condition

$$\dim(X \oplus Y) = \dim X + \dim Y \qquad \text{for every} \quad X, Y \in \mathfrak{A}.$$

The category $\mathfrak{S}(\mathscr{S})$ of all $\mathscr{S}$-spaces with the weighted dimension $\dim(W, W_i)$, the category $\mathfrak{R}(\mathscr{Q})$ of all representations of a K-species $\mathscr{Q}$ with the dimension defined by $\dim(V_i\,, {}_j\varphi_i) = \sum_i \dim V_{i_K}$ as well as the category $\mathfrak{M}_{\mathscr{A}}$ of all right modules over a K-algebra $\mathscr{A}$ with the dimension defined by $\dim M_{\mathscr{A}} = \dim M_K$ are examples of dimension categories. Generalizing the previously discussed notions of finite type, we define a dimension category $\mathfrak{A}$ to be of *finite type* if there is only a finite number of indecomposable objects of finite dimension in $\mathfrak{A}$. Also, $\mathfrak{A}$ is said to be of ***strongly unbounded type*** if it possesses the following three properties:

(i) $\mathfrak{A}$ has indecomposable objects of arbitrarily large finite dimension;

(ii) If $\mathfrak{A}$ contains a finite dimensional object with an infinite endomorphism ring, then there is an infinite number of (finite) dimensions d such that, for each d, $\mathfrak{A}$ has infinitely many (nonisomorphic) indecomposable objects of dimension d.

(iii) $\mathfrak{A}$ has indecomposable objects of infinite dimension.

R. Brauer and R. M. Thrall have conjectured (see [9]) that a K-algebra is either of finite type or of strongly unbounded type (in the sense that at least the properties (i) and (ii) are satisfied). A. V. Roiter [16] has proved the property (i) for the category $\mathfrak{M}_{\mathscr{A}}$, where $\mathscr{A}$ is a K-algebra which is not of finite type; and, in this case, L. A. Nazarova and A. V. Roiter [14] have announced a proof of (ii) provided K is, in addition, a perfect field. The proof is based on their result of [15] that a classical K-structure $\mathscr{S}$ for an infinite field is either of finite type or that $\mathfrak{S}(\mathscr{S})$ possesses the properties (i) and (ii). Extending this result, we can formulate

THEOREM E. (1) *A K-structure is either of finite or of strongly unbounded type.*

(2) *A K-species is either of finite or of strongly unbounded type.*

(3) *A finite dimensional K-algebra $\mathscr{A}$ which is hereditary or which satisfies* $(\operatorname{Rad}\mathscr{A})^2 = 0$ *is either of finite or of strongly unbounded type.*

Thus, Brauer–Thrall conjecture is proved here for two special classes of K-algebra. Using the ideas indicated in [14], it should be possible to extend Theorem E (3) to arbitrary K-algebras.

The methods used throughout the paper are rather intrinsic. In the case of nonclassical K-structures $\mathscr{S}$, we give explicit constructions of all indecomposable $\mathscr{S}$-spaces and describe a procedure how to decompose all finite dimensional $\mathscr{S}$-spaces. Also, in the critical cases of K-structures and K-species of infinite type, we either construct indecomposable objects of arbitrarily large finite dimensions directly, or we reduce the problem to a known situation by identifying a full subcategory with a module category of infinite type. Besides, some facts from algebraic geometry concerning group action on affine varieties are used, mainly to prove the statements concerning the categories of strongly unbounded type (Theorem E). An algebraic geometry argument is used in the proofs of the structure Theorems A, B, C, and D only once, namely in Lemma 4.1. Let us remark that the statement of

Lemma 4.1 is obvious for a commutative field F and that it would be of interest to provide a direct argument also in the general case.

This work was initiated by a course of lectures on "Indecomposable Representations of Artinian Algebras" given by P. Gabriel in the summer of 1972 at Carleton University. Also, he provided the arguments from algebraic geometry used in this paper. For both, the authors wish to express their gratitude to him.

1. *Preliminaries*

In addition to the notation introduced earlier in the paper, we would like to point out that rings are always assumed to be associative with unity, and modules to be unital. If $\mathscr{A}$ is a ring, the symbols ${}_{\mathscr{A}}M$ or $M_{\mathscr{A}}$ will be used to underline the fact that M is a left or a right module, respectively. It should be noted that homomorphisms always act on the side opposite to that of the operators, which means usually on the left-hand side, because we consider mostly right modules. Homomorphisms are often denoted by Greek letters, in particular, zero homomorphisms, by θ. We denote by $\mathscr{A}^0$ the ring opposite to $\mathscr{A}$, and sometimes we will consider left $\mathscr{A}$-modules just as right $\mathscr{A}^0$-modules. Also, $\mathscr{A}^\times$ denotes the multiplicative group of the invertible elements of the ring $\mathscr{A}$. The letters F, G, and H stand for (noncommutative) fields throughout the paper. If, for $G \subseteq F$, $\dim F_G = \dim {}_GF$, then this common dimension is called the *degree* of F over G and denoted by $[F: G]$; this is for example in the case when G is finite dimensional over a central subfield of F. If $H \subseteq F$ and $G \subseteq F$, we always assume that the bimodule ${}_HF_G$ and module F_G are endowed with their natural structure. The symbol $M_G \otimes {}_GM'$ will frequently stand for the tensor product of M_G and ${}_GM'$ over G. Also, the image of $\varphi: X \otimes Y \to Z$ in the natural isomorphism $\mathrm{Hom}(X \otimes Y, Z) \approx \mathrm{Hom}[X, \mathrm{Hom}(Y, Z)]$ will be always denoted by $\varphi^*: X \to \mathrm{Hom}(Y, Z)$. Finally, the symbol $\times$ is used to denote the cartesian product (of sets or vector spaces) and the symbol $\sqcup$ the disjoint union (of ordered sets). Thus, $M' \times M''$ stands for the external direct sum of two vector spaces, whereas $M' \oplus M'' = M$ stands for the internal direct sum (that is, M' and M'' are two fixed submodules of M such that $M' + M'' = M$ and $M' \cap M'' = 0$).

Let W_F be a vector space over F and U an (additive) subgroup of W. Then $\underline{U}$ denotes the largest F-subspace of W_F contained in U and $\overline{U}$ the least F-subspace of W_F containing U. Thus,

$$\underline{U} = \{x \in U \mid xF \subseteq U\} = \sum_{V_F \subseteq U} V_F$$

and

$$\overline{U} = \sum_{x \in U} xF = \bigcap_{U \subseteq V_F} V_F.$$

The construction of $\underline{U}$ and $\overline{U}$ is used throughout the paper, mainly in the following situation: There are given two fields $G \subseteq F$, and U_G is a G-subspace of the F-vector space W_F. In this case, $\underline{U}$ and $\overline{U}$ always refer to the field F. Also, if U is a subgroup of W_F and $W_F = X_F \oplus Y_F$, then U is said to be compatible with this direct decomposition if

$$U = (U \cap X) \oplus (U \cap Y).$$

Of course, given a K-structure $\mathscr{S}$ for F and an $\mathscr{S}$-space (W, W_i), $W_F = X_F \oplus Y_F$ defines a decomposition of $\mathscr{S}$-spaces if and only if all W_i, $i \in \mathscr{S}$, are compatible with the decomposition.

The following simple observation will be used repeatedly: *If U and V are subgroups of a vector space W_F, then every direct decomposition of W_F which is compatible with U and V is also compatible with $\underline{U}$, $\overline{U}$, $U + V$, and $U \cap V$.* The proof follows easily from the fact that, if $W_F = X_F \oplus Y_F$ and $U = (U \cap X) \oplus (U \cap Y)$, then

$$\underline{U} \subseteq \underline{U \cap X} + \underline{U \cap Y} \subseteq (\underline{U} \cap \underline{X}) + (\underline{U} \cap \underline{Y}) = (\underline{U} \cap X) \oplus (\underline{U} \cap Y).$$

And, a similar argument shows that the decomposition is compatible with $\overline{U}$. If, in addition, $V = (V \cap X) \oplus (V \cap Y)$, then every $u \in U$ and $v \in V$ can be written in the form

$$u = u_1 + u_2, \; u_1 \in U \cap X, \; u_2 \in U \cap Y$$

and

$$v = v_1 + v_2, \; v_1 \in V \cap X, \; v_2 \in V \cap Y,$$

and it follows easily that

$$U + V = [(U + V) \cap X] \oplus [(U + V) \cap Y]$$

and

$$U \cap V = (U \cap V \cap X) \oplus (U \cap V \cap Y).$$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be dimension categories. An additive functor $\mathsf{T}: \mathfrak{A} \to \mathfrak{B}$ is called a *dimension functor* if there exist positive real numbers r, s such that

$$r \dim A \leqslant \dim \mathsf{T}A \leqslant s \dim A$$

for all objects A of $\mathfrak{A}$. In particular, this implies that A is finite dimensional if and only if $\mathsf{T}\mathfrak{A}$ is finite dimensional. If T is an equivalence and T' is inverse to T, then T is a dimension functor if and only if T' is a dimension functor. If $\mathfrak{A}$ is an additive category, then two dimensions d, d' on $\mathfrak{A}$ are called equivalent if the identity functor $id_{\mathfrak{A}}$ is a dimension functor $(\mathfrak{A}, d) \to (\mathfrak{A}, d')$. For example, if $\mathscr{S}$ is a K-structure for F, and $[F: K] = n$, then the weighted dimension $\dim(W, W_i) = \max_{i\in\mathcal{S}} \dim W_{F_i}$ is equivalent to the K-dimension $d(W, W_i) = \dim W_K$ and the F-dimension $d'(W, W_i) = \dim W_F$, because $(1/n)\dim(W, W_i) \leqslant d'(W, W_i) \leqslant d(W, W_i) \leqslant n \dim(W, W_i)$. Similarly, for a K-species $\mathcal{Q} = (K_i, {}_iM_j)_{i,j\in I}$, the dimension $\dim(V_i, {}_j\varphi_i) = \sum_{i\in I}(V_i)_K$ is equivalent to $d(V_i, {}_j\varphi_i) = \sum_{i\in I}(V_i)_{K_i}$, because the number of indices in I, as well as all degrees $[K_i: K]$ are finite. An abelian category $\mathfrak{A}$ becomes a dimension category using the length function l: If $A \in \mathfrak{A}$ has no composition series, then $lA = \infty$; otherwise, lA is the length of a composition series of A. Obviously, if $\mathcal{A}$ is a finite dimensional K-algebra, then the K-dimension and the length dimension in $\mathfrak{M}_{\mathcal{A}}$ are equivalent.

Now, assume that $\mathfrak{A}$ and $\mathfrak{B}$ are two dimension categories such that either none or both of them have finite dimensional objects with infinite endomorphism rings. Then, if $\mathsf{T}: \mathfrak{A} \to \mathfrak{B}$ is a full embedding, and $\mathfrak{A}$ is of strongly unbounded type, then $\mathfrak{B}$ is also of strongly unbounded type. For, T maps indecomposable objects to indecomposable objects and nonisomorphic objects to nonisomorphic objects. Thus it is easy to see that $\mathfrak{B}$ satisfies the conditions (i) and (iii) of the definition of strongly unbounded type whenever $\mathfrak{A}$ does. Moreover, if $\mathfrak{A}$ has infinitely many nonisomorphic indecomposable objects A_i of dimension d, then $rd \leqslant \dim \mathsf{T}A_i \leqslant sd$ for some positive real r, s (determined by T) and all i and therefore, since there is only a finite number of integers between rd and sd, there exists $d' \in \mathbb{N}$ satisfying $rd \leqslant d' \leqslant sd$ and such that there are infinitely many nonisomorphic indecomposable objects of the form $\mathsf{T}A_i$ in $\mathfrak{B}$ with $\dim \mathsf{T}A_i = d'$. Consequently, if $\mathfrak{A}$ is of strongly unbounded type, then the unbounded sequence of natural numbers d of the condition (ii) produces an unbounded sequence of natural numbers d', and thus $\mathfrak{B}$ satisfies (ii), as well. In particular, if d and d' are equivalent dimensions on $\mathfrak{A}$, then $\mathfrak{A}$ is of strongly unbounded type with respect to d if and only if $\mathfrak{A}$ is of strongly unbounded type with respect to d'. Let us point out that nearly all functors constructed in the paper are dimension functors; the proofs are usually rather obvious and are left to the reader.

This paper is divided into three sections. The first section consists of three subsections dealing with particular types of K-structures $\mathcal{S}$ of finite type and the decomposition theories in the corresponding categories $\mathfrak{S}(\mathcal{S})$ of all $\mathcal{S}$-spaces. In this section, we assume that all vector spaces are finite dimensional. For $G \subseteq F$ and $[F: G] = 2$, the structures $\mathcal{I}_m(G)$ and $\mathcal{I}_1(G) \sqcup \mathcal{I}_n(F)$ are investigated in Section 2 and the structure $\mathcal{I}_2(G) \sqcup \mathcal{I}_1(F)$ is dealt with in Section 3. In these two sections, we do not use the existence of the central subfield K; we assume the weaker condition that $\dim {}_GF = \dim F_G = 2$. Let us remark that a shorter proof could be given using duality arguments in the case that a central subfield K exists. For $K \subseteq G \subseteq F$ and $[F: G] = 3$, the K-structure $\mathcal{I}_1(G)$ is studied in Section 4; here, the existence of K is used heavily.

The second section comprizes again three subsections. In Section 5, a criterion for objects in a Grothendieck category to be indecomposable is described and it is used in showing that two elementary types of K-species are of strongly unbounded type. In particular, a very useful dimension condition for a K-species to be of a strongly unbounded type is given. The next two subsections deal with K-structures of strongly unbounded type. In Section 6, it is shown that certain full subcategories of a category of all S-spaces are abelian, and in Section 7 such subcategories are identified with full module categories over a K-algebra of strongly unbounded type. In this way, it is proved that the K-structures $\mathscr{I}_1(G) \sqcup \mathscr{I}_1(F) \sqcup \mathscr{I}_1(F)$, $\mathscr{I}_2(G) \sqcup \mathscr{I}_2(F)$, and $\mathscr{I}_3(G) \sqcup \mathscr{I}_1(F)$ with $[F: G] = 2$ and also $\mathscr{I}_1(G) \sqcup \mathscr{I}_1(F)$ with $[F: G] = 3$ are of strongly unbounded type.

The three subsections of Section 3 are devoted, respectively, to $\mathscr{S}$-spaces, to representations of K-species and to modules over K-algebras. Proofs of Theorems A and E(1) are presented in Section 8. A translation of the results on $\mathscr{S}$-spaces to K-species, including a proof of Theorems B and E(2), is given in Section 9. And, the final Section 10 contains a further translation to the representation theory of K-algebras and offers proofs of Theorems C, D, and E(3).

I. K-Structures of Finite Type

2. *Structures* $\mathscr{I}_m(G)$ *and* $\mathscr{I}_1(G) \sqcup \mathscr{I}_n(F)$ *with* $[F: G] = 2$

Throughout this section, G is a subfield of F such that

$$\dim F_G = \dim {}_GF = 2.$$

Let $\{1, f\}$ be a basis of F_G (and thus of ${}_GF$). First, we shall introduce several lemmas which will be needed in the sequel.

Lemma 2.1. *Let $U_G \subseteq U_G'$ be two G-subspaces of a vector space W_F. Then*

$$\overline{\underline{U} \cap \underline{U}'} = \overline{U} \cap \underline{U}' \quad \textit{and} \quad \underline{\overline{U} + U'} = \overline{U} + \underline{U}'.$$

Proof. In order to prove the first equality, only the inclusion

$$\overline{U} \cap \underline{U}' \subset \overline{U \cap \underline{U}'}$$

requires a verification. Let $x \in \overline{U} \cap \underline{U}'$. Since $F = G + Gf$, $x = u_1 + u_2 f$

with suitable $u_1, u_2 \in U$. Thus, $u_2 f \in U'$; for, $x \in \underline{U}' \subseteq U'$ and $u_1 \in U \subseteq U'$. Consequently, $u_2 \in U \cap \underline{U}'$. Moreover,

$$u_1 f^{-1} = -u_2 + xf^{-1} \in U';$$

hence, $u_1 \in U \cap \underline{U}'$ and the inclusion follows.

Similarly, we can easily prove that $\underline{\bar{U} + U'} \subseteq \bar{U} + \underline{U}'$. Let $x \in \underline{\bar{U} + U'}$. Then

$$x = y + z \qquad \text{and} \qquad xf = yf + zf = y' + z'$$

with $y, y' \in \bar{U}$ and $z, z' \in U'$. Hence,

$$w = zf - z' = y' - yf \in \bar{U},$$

and thus $w = w_1 + w_2 f$ with $w_1, w_2 \in U$. Therefore,

$$(z - w_2)f = w_1 + z' \in U'$$

and consequently, $z - w_2 \in \underline{U}'$. We conclude that

$$x = y + z = (y + w_2) + (z - w_2) \in \bar{U} + \underline{U}'.$$

This completes the proof of Lemma 2.1.

LEMMA 2.2. *Let X_G, Y_G, U_G be three G-subspaces of a vector space W_F such that*

$$X \cap Y = 0, \qquad \overline{X \cap U} = \bar{X}, \qquad \overline{Y \cap U} = \bar{Y} \qquad \textit{and} \qquad \underline{U} \subseteq X.$$

Then

$$\bar{X} \cap \bar{Y} = 0.$$

Proof. Let

$$x_1 + x_2 f = y_1 + y_2 f \quad \text{with } x_i \in X \cap U \quad \text{and} \quad y_i \in Y \cap U, \quad i = 1, 2.$$

Then $x_1 - y_1 = (y_2 - x_2)f$, and thus $y_2 - x_2 \in \underline{U} \subseteq X$. Consequently, $y_2 \in X \cap Y$ and therefore $y_2 = 0$. But then

$$x_2 f = y_1 - x_1 \in U,$$

and thus $x_2 \in \underline{U}$, which implies that also $y_1 - x_1 \in \underline{U} \subseteq X$. Hence,

$$y_1 \in X \cap Y = 0$$

and we conclude that $\bar{X} \cap \bar{Y} = 0$, as required.

As an immediate consequence of Lemma 2.2, we may formulate the following

Remark 2.3. If $U = \underline{U} \oplus \oplus_i C_i$ is a direct decomposition of the G-subspace U_G of W_F, then $\bar{U} = \underline{U} \oplus \oplus_i \bar{C}_i$ is a direct decomposition of the F-subspace $\bar{U}$. In particular, if $U_G \subseteq W_F$ such that $\bar{U} = W$ and $\underline{U} = 0$, then every G-basis of U_G is also an F-basis of W_F. Consequently,

$$\dim W_G = 2 \dim W_F = 2 \dim U_G .$$

LEMMA 2.4. *Let U_G be a G-subspace of W_F such that $\bar{U} = W$ and $\underline{U} = 0$. Let V_F be an F-subspace of W_F such that $U \cap V = 0$. Let $\{v_1, v_2, ..., v_n\}$ be a basis of V_F. Then there exists a basis*

$$B = \{x_1, x_2, ..., x_n, y_1, y_2, ..., y_n, z_1, z_2, ..., z_t\}$$

of W_F such that $B \subseteq U$, $v_i = x_i + y_i f$ for $1 \leqslant i \leqslant n$. Moreover, the G-subspace U' generated by $\{x_1, x_2, ..., x_n, y_1, y_2, ..., y_n\}$ has the property $\bar{U}' = U' + V$. Also, if U'' is a G-subspace of U and $v_i \in \bar{U}''$ for some i, then $x_i, y_i \in U''$.

Proof. As additive groups, $U \oplus Uf = W$. For, if $u \in U \cap Uf$, then $u = u'f$ for some $u' \in U$, and thus $u'F = u'(G + fG) = u'G + uG \subseteq U$. Since $\underline{U} = 0$, this implies that $u' = 0$, and therefore $U \cap Uf = 0$. On the other hand, every element of F has the form $g_1 + g_2 f$ with $g_i \in G$, and hence $u(g_1 + g_2 f) \in U + Uf$ for every $u \in U$. This shows that $W = \bar{U} = U \oplus Uf$.

From here, it follows that every element v_i can be written as $v_i = x_i + y_i f$ with $x_i, y_i \in U$. Moreover, the elements x_i, y_i are uniquely determined. As a result, we get the following consequence: If $v_i \in \overline{U''}$ and $U''_G \subseteq U_G$, then $v_i = x_i'' + y_i'' f$ with $x_i'', y_i'' \in U''$ because of $\overline{U''} = U'' \oplus U''f$; and, in view of uniqueness, $x_i = x_i''$ and $y_i = y_i''$.

Now, consider the G-subspace U_G' and the F-subspace W_F' generated by $\{x_1, x_2, ..., x_n, y_1, y_2, ..., y_n\}$; thus $\underline{U}' = 0$ and $\bar{U}' = W'$. First, we shall show that

$$W' = U' + V.$$

Trivially, $x_i \in U' + V$ and

$$x_i f^{-1} = -y_i + v_i f^{-1} \in U' + V.$$

Hence,

$$x_i F = x_i(G + f^{-1}G) \subseteq U' + V \qquad \text{for every} \quad 1 \leqslant i \leqslant n.$$

Similarly,

$$y_i F = y_i(G + fG) \subseteq U' + V \qquad \text{for every} \quad 1 \leqslant i \leqslant n,$$

and therefore $\bar{U}' \subseteq U' + V \subseteq W'$.

From here it follows (see Remark 2.3) that

$$2 \dim U'_G = 2 \dim W'_F = \dim W'_G = \dim U'_G + \dim V_G$$
$$= \dim U'_G + 2 \dim V_F,$$

resulting in $\dim U'_G = 2 \dim V_F$. Therefore $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ is a basis of U'_G and we may complement it by $\{z_1, z_2, \ldots, z_t\}$ to a basis of U_G. The lemma follows.

PROPOSITION 2.5. *Let* $\mathscr{S} = \mathscr{I}_m(G)$ *be a structure such that* $\dim F_G = \dim {}_GF = 2$. *Then there are exactly* $\frac{1}{2}(m+1)(m+2)$ *nonisomorphic indecomposable* $\mathscr{S}$*-spaces.*

Proof. We are going to prove that every finitely generated $\mathscr{S}$-space $(W_F, U(1)_G \subseteq U(2)_G \subseteq \cdots \subseteq U(m)_G)$ is a direct sum of indecomposable $\mathscr{S}$-spaces

$$\mathsf{L}_{rs} = (F,\ U(i) = F \text{ for } s+r+1 \leqslant i \leqslant m,\ U(i) = G$$
$$\text{for } s+1 \leqslant i \leqslant s+r \text{ and } U(i) = 0 \text{ otherwise}),$$
$$0 \leqslant s, \quad 0 \leqslant r, \quad s+r \leqslant m.$$

Proceed by induction. Obviously

$$W = \underline{U(1)} \oplus W',$$

where W' is an arbitrary F-complement of $\underline{U(1)}$ in W, is a decomposition of the $\mathscr{S}$-space $(W, U(i))$. Here, $\underline{U(1)}$ is a direct sum of L_{00}'s. Assume that, Assume that, for a certain k, $1 \leqslant k \leqslant m$,

$$\underline{U(k)} = 0.$$

Thus, W_F contains no copy of L_{rs} for $r+s < k$. Consider the G-subspaces

$$U(l) \cap \underline{U(k+1)} \subseteq U(l) \qquad \text{for all } 1 \leqslant l \leqslant k,$$

and assume that for every $l \leqslant t \leqslant k$,

$$U(l) \cap \underline{U(t+1)} = 0.$$

This means that W_F contains no copy of L_{rs} with $s < t$. Write

$$U(k)_G = (U(t+1) \cap \underline{U(k+1)})_G \oplus U(t)_G \oplus C_G,$$

where the direct summand C has the property that $(C \cap U(i))_G$ complements the other two direct summands in $U(i)_G$ for all $t+1 \leqslant i \leqslant k$. Let D_F be an F-complement of $\overline{U(k)}$ in W_F. Then, applying Lemma 2.1 and Remark 2.3, we have

$$W_F = (\overline{U(t+1)} \cap \underline{U(k+1)}) \oplus W'_F \qquad \text{with} \quad W'_F = [\overline{U(t)} \oplus C \oplus D].$$

This is a decomposition compatible with all $U(i)$ for $1 \leqslant i \leqslant t$, because $U(i) \subseteq W'_F$, and with all $U(i)$ for $k+1 \leqslant i \leqslant m$, because

$$\overline{U(t+1)} \cap \underline{U(k+1)} \subseteq \underline{U(k+1)} \subseteq U(k+1) \subseteq U(i).$$

Moreover, it is compatible with all the remaining $U(i)$ $(t+1 \leqslant i \leqslant k)$ in view of the construction of the G-complement C in $U(k)$. Consequently, $\overline{U(t+1)} \cap \underline{U(k+1)}$ is a direct sum of $\mathsf{L}_{k-t\,t}$'s and we may assume that the $\mathcal{S}$-space satisfies $\overline{U(t+1)} \cap \underline{U(k+1)} = 0$.

In this way, we split off successively all $\mathsf{L}_{k-t\,t}$ for $t < k$ and may assume that $\underline{U(k+1)} = 0$. For $k = m-1$ this yields a complete decomposition of the $\mathcal{S}$-space $(W, U(i))$, because $U(m) = W$ and thus the condition $U(i) \cap \underline{U(m)} = 0$ simply means that $U(i) = 0$.

A case similar to that of Proposition 2.5 for $n = 1$, was investigated in [4]; it can be treated by the method introduced here.

PROPOSITION 2.6. *Let* $\mathcal{S} = \mathcal{I}_1(G) \sqcup \mathcal{I}_n(F)$ *be a structure such that* $\dim F_G = \dim {}_GF = 2$. *Then there are exactly* $\frac{1}{2}(n+1)(n+6)$ *non-isomorphic indecomposable* $\mathcal{S}$*-spaces.*

Proof. We shall prove that every finitely generated $\mathcal{S}$-space

$$(W_F, U_G, V(1)_G \subseteq V(2)_G \subseteq \cdots \subseteq V(n)_G)$$

is a direct sum of indecomposable $\mathcal{S}$-spaces

$$\mathsf{M}_s = (F, 0, V(i) = F \text{ for } s+1 \leqslant i \leqslant n \text{ and } V(i) = 0 \text{ otherwise}), \quad 0 \leqslant s \leqslant n,$$

$$\mathsf{N}_s = (F, G, V(i) = F \text{ for } s+1 \leqslant i \leqslant n \text{ and } V(i) = 0 \text{ otherwise}), \quad 0 \leqslant s \leqslant n,$$

$$\mathsf{P}_s = (F, F, V(i) = F \text{ for } s+1 \leqslant i \leqslant n \text{ and } V(i) = 0 \text{ otherwise}), \quad 0 \leqslant s \leqslant n$$

and

$$\mathsf{Q}_{rs} = (F \times F, G \times G, V(i) = F \times F \text{ for } s+r+1 \leqslant i \leqslant n, V(i) = (1, f)F \text{ for } s+1 \leqslant i \leqslant s+r \text{ and } V(i) = 0 \text{ otherwise}), \quad 0 \leqslant s, \ 1 \leqslant r, \ s+r \leqslant n.$$

First, we reduce the problem of decomposition to the case when $\underline{U} = 0$. Write $W = V(n+1)$. Obviously,

$$W = [\underline{U} \cap V(1)] \oplus W'$$

with an arbitrary F-complement W' in W, yields a decomposition of the $\mathscr{S}$-space $(W, U, V(i))$; here, $\underline{U} \cap V(1)$ is a direct sum of copies of P_0. Thus we assume inductively that

$$\underline{U} \cap V(k) = 0 \qquad \text{for some} \quad 1 \leqslant k \leqslant n.$$

Consider the decomposition

$$W = [\underline{U} \cap V(k+1)] \oplus W',$$

where $W' = V(k) \oplus C$ with an arbitrary F-complement C. This decomposition is compatible with U because of $\underline{U} \cap V(k+1) \subseteq U$, with $V(i)$ for $1 \leqslant i \leqslant k$ because of $V(i) \subseteq W'$, and with $V(i)$ for $k+1 \leqslant i \leqslant n$ because of $\underline{U} \cap V(k+1) \subseteq V(i)$. Of course, $\underline{U} \cap V(k+1)$ is a direct sum of copies of P_k. Consequently, we may assume that

$$\underline{U} = \underline{U} \cap W = \underline{U} \cap V(n+1) = 0.$$

Proceeding dually, we can decompose

$$W = [\overline{U} + V(n)] \oplus C_n$$

with an arbitrary F-complement C_n. Here, C_n is a direct sum of copies of M_n. Thus, write $V(0) = 0$, and assume that

$$\overline{U} + V(k) = W \qquad \text{for some} \quad 1 \leqslant k \leqslant n.$$

Consider the decomposition

$$W = [\overline{U} + V(k-1)] \oplus C_{k-1}$$

with a complement C_{k-1} of

$$[\overline{U} \cap V(k)] + V(k-1) = [\overline{U} + V(k-1)] \cap V(k)$$

in $V(k)$. This decomposition is compatible with U because of

$$U \subseteq \overline{U} + V(k-1),$$

with $V(i)$ for $0 \leqslant i \leqslant k-1$ because of $V(i) \subseteq \overline{U} + V(k-1)$, and finally with $V(i)$ for $k \leqslant i \leqslant n$ because of $C_{k-1} \subseteq V(i)$. Here, C_{k-1} is a direct sum of copies of M_{k-1}. Thus, in addition to $\underline{U} = 0$, we may assume also

$$\overline{U} = W.$$

Let

$$U_G = [U \cap V(1)] \oplus C$$

be a G-decomposition of U. Then, in view of Remark 2.3,

$$W = \bar{U} = \overline{U \cap V(1)} \oplus \bar{C},$$

which yields obviously a decomposition of the $\mathscr{S}$-space $(W, U, V(i))$. And, $\overline{U \cap V(1)}$ is a direct sum of copies of N_0. Hence, assume that

$$U \cap V(k) = 0 \qquad \text{for some} \quad 1 \leqslant k \leqslant n+1.$$

Consider the F-subspace $\overline{U \cap V(k+1)}$ of $V(k+1)$. Let $B_1 = \{v_1, v_2, ..., v_p\}$ be an F-basis of $\overline{U \cap V(k+1)} \cap V(k)$ such that $B_1 \cap V(l)$ is an F-basis of $\overline{U \cap V(k+1)} \cap V(l)$ for each $1 \leqslant l \leqslant k$. Furthermore, choose $B_2 = \{v_{p+1}, v_{p+2}, ..., v_{p+q}\} \subseteq V(k)$ such that, for each $1 \leqslant l \leqslant k$, $(B_1 \cup B_2) \cap V(l)$ is an F-basis of $V(l)$. Now, applying Lemma 2.4 (with $V = V(k)$), we get the existence of an F-basis

$$B = \{x_1, x_2, ..., x_{p+q}, y_1, y_2, ..., y_{p+q}, z_1, z_2, ..., z_t\}$$

of W such that $B \subseteq U$,

$$v_j = x_j + y_j f \qquad \text{for} \quad 1 \leqslant j \leqslant p+q,$$

and x_j and y_j belong to $U \cap V(k+1)$ for $1 \leqslant j \leqslant p$. From here, it follows that

$$W = \bigoplus_{j=1}^{p} (x_j F \oplus y_j F) \oplus W',$$

where W' is generated by $\{x_{p+1}, ..., x_{p+q}, y_{p+1}, ..., y_{p+q}, z_1, ..., z_t\}$ is a direct decomposition of the $\mathscr{S}$-space $(W, U, V(i))$. Moreover,

$$\bigoplus_{j=1}^{p} (x_j F \oplus y_j F) \supseteq \overline{U \cap V(k+1)} \cap V(k)$$

is a direct sum of copies of Q_{rs} with $r + s = k$. Thus, we may assume that

$$\overline{U \cap V(k+1)} \cap V(k) = 0.$$

Then, using the above notation, B_2 is a basis of $V(k)$ and denoting by U' the G-space generated by $\{x_{p+1}, ..., x_{p+q}, y_{p+1}, ..., y_{p+q}\}$ we have in view of Lemma 2.4, $U' \subseteq U$ and $\bar{U}' = U' + V(k)$. Hence,

$$U' \cap [U \cap V(k+1)] = U' \cap V(k+1) = 0.$$

For, if $U' \cap V(k+1) \neq 0$, then, again by Lemma 2.4 (applied to $U = U'$, $V = V(k)$), we would have that $\overline{U' \cap V(k+1)} \cap V(k) \neq 0$, a contradiction. Therefore, there is a decomposition of the G-space U:

$$U = U' \oplus [U \cap V(k+1)] + C'$$

with an arbitrary G-complement C'. And, according to Remark 2.3, this decomposition induces the following decomposition of the $\mathcal{S}$-space $(W, U, V(i))$:

$$W = \overline{U \cap V(k+1)} \oplus W' \quad \text{with} \quad W' = \bar{U}' \oplus \bar{C}'.$$

Here, $\overline{U \cap V(k+1)}$ is a direct sum of copies of N_k. Thus, by an induction argument, after splitting off the copies of N_n, the remaining complement satisfies the condition

$$0 = U \cap V(n) = U \cap W = U,$$

which implies that $W = \bar{U} = 0$.

The proof of Proposition 2.6 is completed.

3. *Structure* $\mathcal{I}_2(G) \sqcup \mathcal{I}_1(F)$ *with* $[F: G] = 2$

Again, throughout this section, G is a subfield of F satisfying $\dim F_G = \dim {}_GF = 2$ and $\{1, f\}$ is a basis of F_G (as well as of ${}_GF$). Also, all vector spaces will be assumed to be finite dimensional. Two results will be proved in this section: Proposition 3.1 asserting that $\mathcal{I}_2(G) \sqcup \mathcal{I}_1(F)$ is of finite type and Proposition 3.2 which is a consequence of Propositions 2.6, 2.7, and 3.1 and will be applied to prove the main theorem on $\mathcal{S}$-spaces in Section 8.

PROPOSITION 3.1. *Let* $\mathcal{S} = \mathcal{I}_2(G) \sqcup \mathcal{I}_1(F)$ *be a structure such that* $\dim F_G = \dim {}_GF = 2$. *Then there are exactly* 20 *nonisomorphic indecomposable* $\mathcal{S}$*-spaces.*

Proof. We shall show that every finitely generated $\mathcal{S}$-space $(W_F, (U_1)_G \subseteq (U_2)_G, V_F)$ is a direct sum of the following indecomposable $\mathcal{S}$-spaces:

$\mathsf{A}_1 = (F, 0, 0, 0)$, $\quad \mathsf{B}_1 = (F, 0, 0, F)$,
$\mathsf{A}_2 = (F, 0, G, 0)$, $\quad \mathsf{B}_2 = (F, 0, G, F)$,
$\mathsf{A}_3 = (F, G, G, 0)$, $\quad \mathsf{B}_3 = (F, G, G, F)$,
$\mathsf{A}_4 = (F, 0, F, 0)$, $\quad \mathsf{B}_4 = (F, 0, F, F)$,
$\mathsf{A}_5 = (F, G, F, 0)$, $\quad \mathsf{B}_5 = (F, G, F, F)$,
$\mathsf{A}_6 = (F, F, F, 0)$, $\quad \mathsf{B}_6 = (F, F, F, F)$;

$$\mathsf{C}_1 = (F \times F, 0 \times 0, G \times G, (1,f)F),$$
$$\mathsf{C}_2 = (F \times F, G \times 0, G \times G, (1,f)F),$$
$$\mathsf{C}_3 = (F \times F, G \times G, G \times G, (1,f)F),$$
$$\mathsf{C}_4 = (F \times F, G \times G, G \times F, (1,f)F),$$
$$\mathsf{C}_5 = (F \times F, G \times 0, G \times F, (1,f)F),$$
$$\mathsf{C}_6 = (F \times F, G \times G, F \times F, (1,f)F),$$
$$\mathsf{D}_1 = (F \times F \times F, G \times G \times 0, G \times G \times F, (1, f, 1)F),$$
$$\mathsf{D}_2 = (F \times F \times F, G \times G \times 0, G \times G \times F, (f, 1, 0)F + (0, f, 1)F).$$

The proof will be given in several "reduction" steps.

(i) First, decompose the $\mathscr{S}'$-space (W, U_1, V); thus, by Proposition 2.7,

$$W = \oplus\ \mathsf{P}_0 \oplus \oplus\ \mathsf{P}_1 \oplus W',$$

where W' is a direct sum of copies of M_0, M_1, N_0, N_1, and Q_{10}. This decomposition is also compatible with U_2, because $(\oplus\ P_0 \oplus \oplus\ P_1) \subseteq U_2$. Thus, the $\mathscr{S}$-space (W, U_1, U_2, V) is a direct sum of copies of A_6 and B_6 and an $\mathscr{S}$-space in which

$$\underline{U}_1 = 0. \tag{1}$$

(ii) Similarly, using again Proposition 2.7, decompose (W, U_2, V):

$$W = \oplus\ \mathsf{M}_0 \oplus \oplus\ \mathsf{M}_1 \oplus W',$$

where W' is a direct sum of copies of N_0, N_1, P_0, P_1, and Q_{10}. Obviously, this decomposition is also compatible with U_1, because $U_1 \subseteq U_2 \subseteq W'$. Therefore, we may split off the direct sum of indecomposable $\mathscr{S}$-spaces A_1 and B_1 and assume that that (W, U_1, U_2, V) satisfies

$$\overline{U}_2 = W. \tag{2}$$

(iii) Now, use again Proposition 2.7 and decompose the $\mathscr{S}'$-space $(W, U_1, \underline{U}_2 \cap V)$:

$$W = \oplus\ \mathsf{M}_1 \oplus \oplus\ \mathsf{N}_1 \oplus W',$$

where W' is a direct sum of copies of M_0, N_0, and Q_{10}; observe that, in view of (1), W' contains no copies of P_0 and P_1. Since both U_2 and V contain $\oplus\ \mathsf{M}_1 \oplus \oplus\ \mathsf{N}_1$, this is a decomposition of the $\mathscr{S}$-space (W, U_1, U_2, V). Therefore, after splitting off the copies of B_4 and B_5, we may assume that

$$U_1 \cap \underline{U}_2 \cap V = 0 \quad \text{and} \quad \underline{U}_2 \cap V \subseteq \overline{U}_1. \tag{3}$$

(iv) Similarly, consider the decomposition of $(W, U_2, \bar{U}_1 + V)$:

$$W = \oplus \mathsf{N}_0 \oplus \oplus \mathsf{P}_0 \oplus W',$$

where W' is a direct sum of copies of N_1, P_1, and Q_{10}; in view of (2), W' has no summands of type M_0 and M_1. Again, this decomposition is compatible with U_1 and V, because $\bar{U}_1 + V \subseteq W'$. Hence, we split off copies of A_2 and A_4 and may assume

$$\bar{U}_1 + U_2 + V = W \quad \text{and} \quad \underline{U}_2 \subseteq \bar{U}_1 + V. \tag{4}$$

(v) Consider the $\mathscr{S}'$-space $(W, U_1 \cap V, U_1, U_2)$ and apply Proposition 2.6:

$$W = \oplus \mathsf{L}_{30} \oplus W',$$

where W' is a direct sum of copies of L_{11}, L_{02}, L_{21}, and L_{12}. For, W' has no summands of the types L_{00}, L_{10}, and L_{01} because of (1), no summands of the type L_{03} because of (2) and no summands of the type L_{20} because of (3). Now, since $(U_1 \cap V) \cap W' = 0$, $U_1 \cap V \subseteq \oplus \mathsf{L}_{30}$ and thus

$$\oplus \mathsf{L}_{30} = \overline{U_1 \cap V} \subseteq V;$$

this means that the above decomposition is compatible with V. Consequently, (W, U_1, U_2, V) is a direct sum of copies of B_3 and an $\mathscr{S}$-space in which

$$U_1 \cap V = 0. \tag{5}$$

(vi) In a similar way, apply Proposition 2.6 to the $\mathscr{S}'$-space $(W, U_1, U_2, U_2 + V)$ and decompose

$$W = \oplus \mathsf{L}_{30} \oplus W',$$

where, this time, W' is a direct sum of copies of L_{10}, L_{01}, L_{20}, and L_{11}; this follows from (1), (2), and (4). Now, $\underline{U_2 + V} \cap \mathsf{L}_{30} = 0$ and thus, according to Lemma 2.1, $\underline{U_2 + V} \subseteq W'$. Thus, our decomposition is compatible with $V \subseteq \underline{U_2 + V}$. We split off the copies of A_3 and assume that

$$U_2 + V = W. \tag{6}$$

(vii) In this step, use again Proposition 2.7 and decompose $(W, U_1, \underline{U}_2 \cap V, V)$:

$$W = \oplus \mathsf{Q}_{20} \oplus W',$$

where W' is a direct sum of copies of M_0, M_1, N_0, and Q_{11}; W' contains no copies of M_2 because of (3), no copies of N_1, N_2, and Q_{10} because of (5) and no copies of P_0, P_1, and P_2 because of (1). Now, since

$$\oplus\, \mathsf{Q}_{20} \subseteq U_1 + (\underline{U}_2 \cap V) \subseteq U_2,$$

the above decomposition is compatible with U_2, and we may split off the copies of C_6. Thus, we may assume that

$$\underline{U}_2 \cap V = 0. \tag{7}$$

(viii) Similarly, use Proposition 2.7 to decompose $(W, U_2, V, \overline{U}_1 + V)$:

$$W = \oplus\, \mathsf{Q}_{20} \oplus W',$$

where W' is a direct sum of copies of N_2, P_1, P_2, and Q_{10}; this follows from (2), (4), and (6). Now, since

$$[U_2 \cap (\overline{U}_1 + V)] \cap \oplus\, \mathsf{Q}_{20} = 0,$$

we get in view of Lemma 2.1,

$$U_1 \subseteq U_2 \cap (\overline{U}_1 + V) \subseteq W'.$$

Therefore the above decomposition is compatible with U_1. Hence, (W, U_1, U_2, V) is a direct sum of copies of C_1 and an $\mathscr{S}$-space satisfying

$$\overline{U}_1 + V = W. \tag{8}$$

(ix) Now, using repeatedly the fact that $\overline{U}_2 = W$, we can find easily G-complements C_1 of $\underline{U}_2$ in $(\overline{U}_1 + \underline{U}_2) \cap U_2$, $C \subseteq U_2 \cap V$ such that $\underline{U}_2 \oplus C_1 \oplus C \supseteq U_2 \cap V$ and finally C_2 of $\underline{U}_2 \oplus C_1 \oplus C$ in U_2. In view of Remark 2.4,

$$W = \overline{C} \oplus W', \qquad \text{where} \quad W' = \underline{U}_2 \oplus \overline{C_1} \oplus \overline{C_2}.$$

The construction of this decomposition ensures that it is compatible with U_2; moreover, it is also compatible with U_1 because of $U_1 \subseteq W'$ and with V because of $\overline{C} \subseteq V$. Hence, (W, U_1, U_2, V) is a direct sum of copies of B_2 and an $\mathscr{S}$-space satisfying

$$U_2 \cap V \subseteq \overline{U}_1 + \underline{U}_2. \tag{9}$$

(x) Now, write $T = U_1 + \underline{U}_2 + V$ and put

$$A = (\overline{U}_1 + \underline{U}_2) \cap \underline{T}.$$

Thus, using the modular law,

$$A = (A \cap \overline{U}_1) + \underline{U}_2 .$$

Observe that $A \cap \overline{U}_1$ is U_1-generated: $\overline{A \cap \overline{U}_1 \cap U_1} = A \cap \overline{U}_1$. This follows immediately from Lemma 2.2 applied to $U_1 \subseteq T$:

$$A \cap \overline{U}_1 = \underline{T} \cap \overline{U}_1 = \overline{\underline{T} \cap U_1} = \overline{A \cap U_1} .$$

Consequently,

$$\begin{aligned} \overline{A \cap U_2} &= \overline{[(A \cap \overline{U}_1) + \underline{U}_2] \cap U_2} \\ &= \overline{(A \cap \overline{U}_1 \cap U_2) + \underline{U}_2} \supseteq (A \cap \overline{U}_1) + \underline{U}_2 = A \end{aligned}$$

and thus A is U_2-generated: $\overline{A \cap U_2} = A$.

Now, define a G-complement $C \subseteq U_1$ by

$$A \oplus C = A + U_1 ,$$

and fix a G-basis $\{c_1, c_2, \ldots, c_n\}$ of C. We have therefore

$$\overline{U}_1 \subseteq \overline{A + C} = A + \overline{C} = A \oplus \overline{C}.$$

Since by (6), $U_2 + V = W$, we have also

$$U_2 f + V = W.$$

Hence,

$$c_i = -d_i f + v_i \qquad \text{with} \quad d_i \in U_2 ,\ v_i \in V \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

Let D be the G-subspace generated by $\{d_1, d_2, \ldots, d_n\}$. Thus $D \subseteq U_2$.

First, we shall show that

$$A + C + D = A + U_2 .$$

Obviously, $A + C + D \subseteq A + U_2$. On the other hand,

$$(A + U_2) \cap (A + V) = A.$$

For, by (9),

$$\begin{aligned} (A + U_2) \cap (A + V) &= A + [U_2 \cap (A + V)] \\ &\subseteq A + (U_2 \cap T) = A + [U_2 \cap (U_1 + \underline{U}_2 + V)] \\ &= A + [(U_1 + \underline{U}_2) + U_2 \cap V] \subseteq A + [(\overline{U}_1 + \underline{U}_2) \cap \underline{T}] = A. \end{aligned}$$

Thus, using the modular law, the fact that $\bar{C} \subseteq C + D + V$ and (8), we have

$$\begin{aligned} A + C + D &= (A + C + D) + [(A + U_2) \cap (A + V)] \\ &= (A + U_2) \cap [(A + C + D) + (A + V)] \\ &\supseteq (A + U_2) \cap (A + \bar{C} + V) \supseteq (A + U_2) \cap (\bar{U}_1 + V) \\ &= A + U_2 . \end{aligned}$$

As a consequence of the inclusion $U_2 \subseteq A + C + D$, we have

$$A + \bar{C} + \bar{D} = W.$$

In fact, we claim that this sum is direct. For, since $A + \bar{C} \subseteq \bar{U}_1 + \underline{U}_2$ and $A + V \subseteq \underline{T}$,

$$(A + \bar{C}) \cap (A + V) = A;$$

furthermore, $(A + \bar{C}) + (A + V) \supseteq \bar{U}_1 + V = W$. Also,

$$(A + C + D) \cap (A + V) = A$$

and

$$(A + C + D) + (A + V) \supseteq U_2 + V = W.$$

Hence,

$$\begin{aligned} 2n = \dim \bar{C}_G &= \dim[(A + \bar{C})/A]_G = \dim[W/(A + V)]_G \\ &= \dim[(A + C + D)/A]_G \leqslant \dim(C + D)_G \leqslant 2n, \end{aligned}$$

because $C + D$ is generated by $\{c_1 ,..., c_n , d_1 ,..., d_n\}$. Consequently,

$$A + C + D = A \oplus C \oplus D.$$

Now, applying Remark 2.3 to

$$(A \cap U_2) \oplus C \oplus D = U'$$

we get

$$\bar{U}' = \overline{A \cap U_2} \oplus \bar{C} + \bar{D} = A \oplus \bar{C} \oplus \bar{D};$$

for, $\underline{U}' \subseteq \underline{U}_2 \subseteq A \cap U_2$. Thus, since $U_2 \subseteq A + C + D$ and $\bar{U}_2 = W$, we have

$$W = A \oplus W' \qquad \text{with} \quad W' = \bar{C} \oplus \bar{D} = \bigoplus_{i=1}^{n} (c_i F \oplus d_i F).$$

This decomposition is compatible with U_1, because

$$\begin{aligned} (U_1 \cap A) \oplus [U_1 \cap (C \oplus D)] &= (U_1 \cap A) \oplus C = U_1 \cap (A \oplus C) \\ &= U_1 \cap (A + U_1) = U_1 . \end{aligned}$$

It is compatible with U_2, because of $U_2 \supseteq C \oplus D$. Finally, in order to show compatibility with V, observe first that $D \subseteq \bar{C} + (V \cap \overline{C \oplus D})$ and therefore

$$A + \bar{C} + (V \cap \overline{C \oplus D}) = W.$$

Using this relation, we get

$$\begin{aligned}(V \cap A) \oplus (V \cap \overline{C \oplus D}) &= V \cap [A + (V \cap \overline{C \oplus D})] \\ &= V \cap \{[(A + \bar{C}) \cap (A + V)] + (V \cap \overline{C \oplus D})\} \\ &= V \cap (A + V) \cap [A + \bar{C} + (V \cap \overline{C \oplus D})] = V.\end{aligned}$$

Thus, the $\mathscr{S}$-space (W, U_1, U_2, V) is a direct sum of copies of C_2 and of an $\mathscr{S}$-space satisfying

$$\bar{U}_1 + \underline{U}_2 = W \quad \text{and} \quad U_1 + \underline{U}_2 + V = W. \tag{10}$$

Note that an immediate consequence of (10) is

$$U_1 + \underline{U}_2 = U_2. \tag{10'}$$

Indeed, if $x \in U_2$, then $x = u + u'f + y$ with u, $u' \in U_1$ and $y \in \underline{U}_2$. But $u'f = x - u - y$ belongs to U_2 and thus $u'f \in \underline{U}_2$. Consequently, $x = u + (u'f + y) \in U_1 + \underline{U}_2$. The other inclusion is trivial.

(xi) Now, consider a basis $\{v_1, v_2, \dots, v_p\} \subseteq \bar{U}_1 \cap U_2 \cap V$ of $V_1 = \overline{\bar{U}_1 \cap U_2 \cap V}$ and extend it by $\{v_{p+1}, v_{p+2}, \dots, v_n\} \subseteq \bar{U}_1 \cap V$ to a basis of $\bar{U}_1 \cap V$. Write $V_2 = \oplus_{i=p+1}^{n} v_i F$. For each v_i, $1 \leqslant i \leqslant n$, there exist, by Lemma 2.4, elements $x_i, y_i \in U_1$ such that $v_i = x_i + y_i f$ and such that $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$ is F-independent. Also, for $1 \leqslant i \leqslant p$, $y_i \in \underline{U}_2$; for,

$$y_i f = v_i - x_i \in U_2.$$

Let X be a G-complement of $\oplus_{i=1}^{p} y_i G$ in $U_1 \cap \underline{U}_2$. Then, obviously

$$B = \overline{U_1 \cap \underline{U}_2} \oplus V_1 = \bar{X} \oplus \bigoplus_{i=1}^{p} (x_i F \oplus y_i F).$$

This follows easily from $\bar{X} \cap \oplus_{i=1}^{p} (x_i F \oplus y_i F) = 0$. The latter relation is a consequence of the fact that there is no nontrivial relation

$$\sum_{i=1}^{p} x_i \kappa_i \in \underline{U}_2 \quad \text{with} \quad \kappa_i \in F.$$

For, assuming that such a relation exists, we get

$$\sum_{i=1}^{p} (v_i - y_i f)\kappa_i \in \underline{U}_2$$

by substitution $x_i = v_i - y_i f$. As a result, $\sum_{i=1}^{p} v_i\kappa_i \in \underline{U}_2$ because of $y_i \in \underline{U}_2$, and thus, in view of (7), $\sum_{i=1}^{p} v_i\kappa_i = 0$; hence, all $\kappa_i = 0$. Now, denote by Y the G-subspace of W generated by $\{x_{p+1}, x_{p+2}, \ldots, x_n, y_{p+1}, y_{p+2}, \ldots, y_n\}$.

We are going to prove that

$$B \cap \overline{Y} = 0.$$

Since $\overline{B \cap U_1} = B$ and $Y \subseteq U_1$, it is sufficient to show that $B \cap Y = 0$. First, if $0 \neq y \in Y$, then $yF \not\subseteq V_2$, so $V_2 \cap yF = 0$, and consequently

$$\dim_G(V_2 \oplus yF) = \dim_G V_2 + 2$$

and

$$\dim_G \overline{Y} = 2 \dim_G Y = \dim_G Y + \dim_G V_2 .$$

Thus,

$$\dim_G[(V_2 \oplus yF) \cap Y] \geqslant 2,$$

and there is $x \in (V_2 \oplus yF) \cap Y$ independent of y. Moreover,

$$v' = x + yf' \in V_2 \text{ with a suitable } f' \in F.$$

Consequently, assuming that

$$0 \neq y \in B \cap Y,$$

we have

$$y = u + v_1 \qquad \text{with} \quad u \in \overline{U_1 \cap \underline{U}_2} \quad \text{and} \quad v_1 \in V_1,$$

and thus there exists $0 \neq v' \in V_2$ such that

$$v' = x + uf' + v_1 f' \text{ with suitable } x \in Y \subseteq U_1 \qquad \text{and} \qquad f' \in F.$$

From here,

$$v' - v_1 f' = x + uf' \in U_1 + \underline{U}_2 \subseteq U_2 ,$$

and therefore

$$v' - v_1 f' \in \overline{U_1 \cap U_2 \cap V} = V_1 .$$

But this yields $v' \in V_1 \cap V_2$, a contradiction.

As the next step, take a G-complement C containing Y of $B \cap U_1$ in U_1. Furthermore, complement the basis $\{v_1, v_2, \ldots, v_n\}$ of $V_1 \oplus V_2$ by

$\{w_1, w_2, \ldots, w_m\}$ to a basis of V. We claim that, for each w_j, $1 \leqslant j \leqslant m$, there is $z_j \in V_1$ such that

$$\bar{w}_j = w_j - z_j \in \bar{C} + \underline{U}_2 .$$

This follows from the fact that, by (10), $\bar{U}_1 + \underline{U}_2 = W$:

$$w_j = c_j + u_j + v_j \qquad \text{with} \quad c_j \in \bar{C},\ u_j \in \underline{U}_2 \qquad \text{and} \qquad v_j \in V_1 .$$

For, since

$$v_j = \sum_i x_i \kappa_{ij} + \sum_i y_i \lambda_{ij} = \sum_i (x_i + y_i f)\kappa_{ij} + \sum_i y_i(\lambda_{ij} - f\kappa_{ij}) = z_j + u_j'$$

$$\text{with} \quad z_j \in V_1 \quad \text{and} \quad u_j' \in \underline{U}_2 ,$$

we get immediately

$$w_j - z_j = c_j + u_j + u_j' \in \bar{C} + \underline{U}_2 ,$$

as required.

Write $V' = \oplus_{j=1}^{m} \bar{w}_j F_j$ and put

$$D = (\bar{C} + V') \cap \underline{U}_2 .$$

It is easy to see that

$$\bar{U}_1 \cap D = \bar{U}_1 \cap \underline{U}_2 \cap (\bar{C} + V') = 0.$$

For, by the modular law,

$$\begin{aligned}(\bar{C} + V') \cap (\bar{U}_1 \cap \underline{U}_2) &\subseteq [(\bar{C} + V') \cap \bar{U}_1] \cap (\bar{U}_1 \cap \underline{U}_2)\\ &= [(\bar{U}_1 \cap V') + \bar{C}] \cap (\bar{U}_1 \cap \underline{U}_2) = \bar{C} \cap \bar{U}_1 \cap \underline{U}_2 = 0.\end{aligned}$$

Also,

$$(B \oplus \bar{C}) + D \supseteq W.$$

Indeed, since by (8), $(B + \bar{C}) + V = W$, we have for every $w \in W$,

$$w = a + \bar{c} + v = a' + \bar{c} + \sum_{j=1}^{m} \bar{w}_j \kappa_j = a' + \bar{c} + c' + u',$$

where $a, a' \in B$, $\bar{c}, c' \in \bar{C}$, and $u' \in \underline{U}_2 \cap (\bar{C} + V') = D$, as required. Thus

$$W = \bar{X} \oplus \bigoplus_{i=1}^{p} (x_i F \oplus y_i F) \oplus W'$$

with

$$W' = \bar{C} \oplus D.$$

Now, in view of the direct decomposition $U_1 = X \oplus \oplus_{i=1}^{p} (x_iG \oplus y_iG) \oplus C$, the above decomposition of W is compatible with U_1. Also, it is obviously compatible with $\underline{U}_2$. Consequently, it is compatible, in view of (10′) with $U_2 = U_1 + \underline{U}_2$. Finally,

$$(\overline{X} \cap V) \oplus \left\{\left[\bigoplus_{i=1}^{p} (x_iF \oplus y_iF)\right] \cap V\right\} \oplus [(C \oplus D) \cap V]$$
$$= 0 \oplus V_1 \oplus V_2 \oplus V' = V,$$

because

$$(\overline{C} \oplus D) \cap V = \{\overline{C} \oplus [(\overline{C} \oplus V') \cap \underline{U}_2]\} \cap V$$
$$= (\overline{C} \oplus V') \cap (\overline{C} \oplus \underline{U}_2) \cap V \supseteq V_2 \oplus V'.$$

Hence, the $\mathscr{S}$-space (W, U_1, U_2, V) is a direct sum of copies of A_5 (comprizing $\overline{X}$), of copies of C_4 and of an $\mathscr{S}$-space satisfying

$$\overline{U}_1 \cap \underline{U}_2 = 0 \qquad \text{and} \qquad \overline{U}_1 \cap U_2 \cap V = 0 \tag{11}$$

(xii) Now, consider the F-subspaces $\overline{U_2 \cap V}$, V_1 and V_2 of V defined by

$$V_1 \subseteq \overline{U}_1 \cap V \qquad \text{and} \qquad \overline{U_2 \cap V} + (\overline{U}_1 \cap V) = \overline{U_2 \cap V} \oplus V_1$$

and

$$\overline{U_2 \cap V} \oplus V_1 \oplus V_2 = V,$$

respectively. Put $V' = V_1 \oplus V_2$.

First, define

$$P = \overline{U}_1 \cap (\underline{U}_2 + \overline{U_2 \cap V}) \qquad \text{and} \qquad Q = \underline{U}_2 \cap (\overline{U}_1 + \overline{U_2 \cap V}).$$

Using the modular law, (10) and (11), it is easy to see that

$$P \oplus Q = (\overline{U}_1 + \overline{U_2 \cap V}) \cap (\underline{U}_2 + \overline{U_2 \cap V}) \supseteq \overline{U_2 \cap V}.$$

Also, according to Lemma 2.2 applied to $U_1 \subseteq U_2 + \overline{U_2 \cap V}$ and $U_2 \cap V \subseteq U_2$, we have

$$\overline{U_1 \cap (\underline{U_2 + \overline{U_2 \cap V}})} = U_1 \cap (\underline{U_2 + \overline{U_2 \cap V}})$$

and

$$\underline{U_2 + \overline{U_2 \cap V}} = \underline{U}_2 + \overline{U_2 \cap V},$$

and thus,

$$\overline{P \cap U_1} = \overline{U_1 \cap (\underline{U}_2 + \overline{U_2 \cap V})} = \overline{U}_1 \cap (\underline{U}_2 + \overline{U_2 \cap V}) = P.$$

Second, put

$$C = \overline{U}_1 \cap \underline{U_2 + V'} \qquad \text{and} \qquad D = \underline{U}_2 \cap \overline{U_1 + V'}.$$

Again, using the modular law, (10) and (11), we get easily that

$$C \oplus D \supseteq (\overline{U}_1 + V') \cap (\underline{U}_2 + V') \supseteq V'.$$

And, moreover, by Lemma 2.2 applied to $U_1 \subseteq U_2 + V'$ we get

$$\overline{C \cap U_1} = \overline{U_1 \cap \underline{U_2 + V'}} = \overline{U}_1 \cap \underline{U_2 + V'} = C.$$

Now, we shall prove that

$$\overline{U}_1 = C \oplus P \qquad \text{and} \qquad \underline{U}_2 = D \oplus Q.$$

This will be achieved if we show that (a) $C \cap P = 0$, (b) $C + P = \overline{U}_1$, (c) $D \cap Q = 0$, and (d) $D + Q = \underline{U}_2$.

(a) Show first that

$$\underline{U_2 + V'} \cap \overline{U_2 \cap V} = 0.$$

Indeed, let

$$z = u + v' \qquad \text{with} \quad z \in \overline{U_2 \cap V}, \quad u \in U_2, \quad v' \in V',$$

and

$$(u + v')f \in U_2 + V'.$$

Then $z - v' = u \in U_2 \cap V$ and thus $v' = z - u \in \overline{U_2 \cap V}$. Therefore, $v' = 0$. Also, $uf \in U_2 + V'$, and thus $uf = x + y$ with $x \in U_2$ and $y \in V'$. Consequently, $x = uf - y = zf - y \in U_2 \cap V$, so

$$y = zf - x \in \overline{U_2 \cap V} \cap V' = 0,$$

and therefore $zf = x \in V \cap U_2 = 0$. Hence

$$C \cap P = \overline{U}_1 \cap \underline{U_2 + V'} \cap (\underline{U}_2 + \overline{U_2 \cap V})$$

$$= \overline{U}_1 \cap \{[\underline{U_2 + V'} \cap \overline{U_2 \cap V}] + \underline{U}_2\} = \overline{U}_1 \cap \underline{U}_2 = 0,$$

as required.

(b) Let X be a G-complement of $(C \cap U_1) \oplus (P \cap U_1)$ in U_1. Then $C \oplus P \oplus \overline{X} = \overline{U}_1$. Now, $C \oplus P \oplus \underline{U}_2$ contains $P \oplus Q$, and therefore

$\overline{U_2 \cap V}$; moreover, it contains $C \oplus D$, and therefore V'. Consequently, it contains all of V, and thus

$$C \oplus P \oplus \underline{U}_2 \oplus X \supseteq U_1 + \underline{U}_2 + V = W.$$

But $C \oplus P \oplus \underline{U}_2 \oplus X$ is a proper subset of W unless $X = 0$. Hence, $\overline{U}_1 = C \oplus P$, as required.

(c) We claim that

$$(\overline{U}_1 + V') \cap (\overline{U}_1 + \overline{U_2 \cap V}) = \overline{U}_1 .$$

For, let x belong to the intersection:

$$x = u' + v' = u + v \qquad \text{with} \quad u, u' \in \overline{U}_1, \quad v' \in V', \quad \text{and} \quad v \in \overline{U_2 \cap V}.$$

Then, $u' - u = v - v' \in \overline{U}_1 \cap V$ and thus

$$v' = v + u - u' \in (\overline{U}_1 \cap V) + \overline{U_2 \cap V} = \overline{U_2 \cap V} \oplus V_1 .$$

Therefore, $v' \in \overline{U}_1$ and $x = u' + v' \in \overline{U}_1$, as required. Now, since $\overline{U}_1 + V' = \overline{U_1 + V'}$,

$$\begin{aligned} D \cap Q &= \underline{U}_2 \cap (\overline{U}_1 + V') \cap (\overline{U}_1 + \overline{U_2 + V}) \\ &= \underline{U}_2 \cap \overline{U}_1 = 0. \end{aligned}$$

(d) Finally, using (8) and (10),

$$\begin{aligned} \underline{U}_2 &= \underline{U}_2 \cap \{[(\overline{U}_1 + \overline{U_2 \cap V}) \cap (\overline{U}_1 + \underline{U}_2)] + V'\} \\ &= \underline{U}_2 \cap (Q + \overline{U}_1 + V') = Q + D. \end{aligned}$$

Summarizing,

$$W = W_1 \oplus W_2 \qquad \text{with} \quad W_1 = P \oplus Q \qquad \text{and} \qquad W_2 = C \oplus D$$

is a decomposition which is compatible with U_1, with $\underline{U}_2$, and thus, by (10′) with $U_2 = U_1 + \underline{U}_2$, as well as with V. Moreover, since

$$\overline{(U_2 \cap W_1) \cap (V \cap W_1)} = V \cap W_1$$

and

$$\overline{(U_2 \cap W_2) \cap (V \cap W_2)} = 0,$$

we have decomposed our $\mathscr{S}$-space (W, U_1, U_2, V) into a direct sum of $\mathscr{S}$-spaces satisfying the conditions

$$\overline{U_2 \cap V} = V \tag{12'}$$

and

$$U_2 \cap V = 0, \tag{12''}$$

respectively.

(xii') Assume that the $\mathscr{S}$-space (W, U_1, U_2, V) satisfies, in addition to (1)–(11), $\overline{U_2 \cap V} = V$.

Let $\{v_1, v_2, \dots, v_p\}$ be a basis of $\overline{U}_1 \cap V$. By Lemma 2.4 applied to $U_2 \cap V$ and $\overline{U}_1 \cap V$, there exists a basis

$$\{r_1, r_2, \dots, r_p, s_1, s_2, \dots, s_p, t_1, t_2, \dots, t_q\} \subseteq U_2 \cap V$$

such that

$$v_i = r_i + s_i f \qquad \text{for all} \quad 1 \leqslant i \leqslant p.$$

Also, by Lemma 2.4, there is an independent subset

$$\{x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_p\} \subseteq U_1$$

such that

$$v_i = x_i + y_i f \qquad \text{for all} \quad 1 \leqslant i \leqslant p.$$

Thus, for each i, $1 \leqslant i \leqslant p$,

$$z_i = r_i - x_i = (y_i - s_i)f;$$

from here it follows that $z_i \in \underline{U}_2$. Moreover, in view of (10) and (11), we have a unique decomposition

$$t_j = a_j + b_j f \qquad \text{with} \quad a_j \in \overline{U}_1 \qquad \text{and} \qquad b_j \in \underline{U}_2$$

for each j, $1 \leqslant j \leqslant q$. Furthermore, $a_j = a_j' + a_j'' f$ with $a_j', a_j'' \in U_1$. Hence, $t_j - a_j' - b_j f = a_j'' f \in U_2$; this implies that $a_j'' \in \underline{U}_2$. Since $U_1 \cap \underline{U}_2 = 0$, $a_j'' = 0$ which means that $a_j \in U_1$ for all $1 \leqslant j \leqslant q$.

Now, we claim that both subsets

$$\{x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_p, a_1, a_2, \dots, a_q\} \subseteq U_1$$

and

$$\{z_1, z_2, \dots, z_p, b_1, b_2, \dots, b_q\} \subseteq \underline{U}_2$$

are independent. This follows easily from the facts that the G-subspaces U_1, $\underline{U}_2$ and $U_2 \cap V$ have pair-wise zero intersections and that the independent elements r_i, s_i, and t_j of $U_2 \cap V$ satisfy the relations

$$r_i = x_i + z_i,$$
$$s_i = y_i - z_i f^{-1},$$

and

$$t_j = a_j + b_j f.$$

Denote by C a G-complement of the G-space generated by $\{x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_p, a_1, a_2, \ldots, a_q\}$ in U_1 and by D an F-complement of the F-space generated by $\{z_1, z_2, \ldots, z_p, b_1, b_2, \ldots, b_q\}$ in $\underline{U}_2$. Then

$$W = \bigoplus_{i=1}^{p} (x_iF \oplus y_iF \oplus z_iF) \oplus \bigoplus_{j=1}^{p} (a_jF \oplus b_jF) \oplus W',$$

where $W' = \bar{C} \oplus D$ is obviously a decomposition of W which is compatible with U_1, $\underline{U}_2$ and thus with U_2, as well as with V. Hence, the $\mathscr{S}$-space (W, U_1, U_2, V) is a direct sum of copies of D_2 and of copies of C_5, and an $\mathscr{S}$-space (W', U_1', U_2', V') satisfying the condition $V' = 0$. However, then $U_2' = W'$ and therefore $\underline{U}_2' = W'$ in view of (6), $\overline{U_1'} = W'$ in view of (8), and thus $W' = 0$ by (11).

(xii″) Finally, assume that the $\mathscr{S}$-space (W, U_1, U_2, V) satisfies $U_2 \cap V = 0$.

Let $\{v_1, v_2, \ldots, v_p, v_{p+1}, \ldots, v_n\}$ be an F-basis of V such that $\{v_1, v_2, \ldots, v_p\}$ is a basis of $\bar{U}_1 \cap V$. For each i, $1 \leqslant i \leqslant n$,

$$v_i = u_i + z_i \qquad \text{with} \quad u_i \in \bar{U}_1 \qquad \text{and} \qquad z_i \in \underline{U}_2 .$$

Obviously, for $1 \leqslant i \leqslant p$, $z_i = 0$. Furthermore, one can see easily (as in previous (xii′) that both $\{u_1, u_2, \ldots, u_n\}$ and $\{z_{p+1}, z_{p+2}, \ldots, z_n\}$ are independent. By Lemma 2.4, we get that

$$v_i = x_i + y_i f + z_i \qquad \text{with} \quad x_i, y_i \in U_1$$

such that $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ is independent. Again, let C be a complement of the G-space generated by $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ in U_1 and D a complement of the F-space generated by $\{z_{p+1}, z_{p+2}, \ldots, z_n\}$ in $\underline{U}_2$. Then

$$W = \bigoplus_{i=1}^{p} (x_iF \oplus y_iF) \oplus \bigoplus_{i=p+1}^{n} (x_iF \oplus y_iF \oplus z_iF) \oplus W',$$

where $W' = \bar{C} \oplus D$, is a decomposition of W which is compatible with U_2, as well as with V. Moreover, using the same argument as in the previous section (xii′), W' must be 0. Hence the $\mathscr{S}$-space (W, U_1, U_2, V) is a direct sum of copies of C_3 and of copies of D_1.

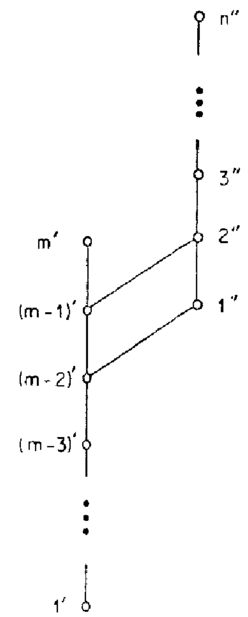
The proof of Proposition 3.1 is completed.

PROPOSITION 3.2. *Let $\mathscr{I}_m(G)$ be given by the chain $\{1' < 2' < \cdots < m'\}$ and $\mathscr{I}_n(F)$ by the chain $\{1'' < 2'' < \cdots < n''\}$. Let $\mathscr{S}$ be a structure given by $\mathscr{I}_m(G) \sqcup \mathscr{I}_n(F)$ together with two additional relations $(m-2)' \leqslant 1''$ and*

$(m-1)' \leqslant 2''$. *Let, moreover,* $\dim F_G = \dim {}_G F = 2$. *Assume that* $m \geqslant 2$ *and* $n \geqslant 1$. *Then there are exactly*

$$\tfrac{1}{2}(m+1)(m+2) + \tfrac{1}{2}(n+1)(n+6) + 7$$

nonisomorphic indecomposable $\mathscr{S}$*-spaces.*



Proof. Let $(W_F, U(i')_G, V(i'')_F)$ be the given $\mathscr{S}$-space. First, decompose the vector space W_F as the $\mathscr{S}'$-space $(W_F, U[(m-1)']_G, U(m')_G, V(1'')_F)$. Write

$$W = W_1 \oplus W_2 \oplus W_3,$$

where W_1 is the direct sum of all copies of A_1, A_2, A_4, and C_1, W_2 is the direct sum of all copies of B_3, B_5, and B_6 and W_3 is the direct sum of all copies of the remaining types. Observe that A_1, A_2, A_4, and C_1 are the only types X for which $\overline{(X \cap U(m-1)')} + (X \cap V(1'')) \neq X$ and that $X \cap U(m-1)' = 0$ in all these cases. Moreover, observe that B_3, B_5, and B_6 are the only types X for which $X \cap U(m-1)' \cap V(m'') \neq 0$ and that $X \cap V(1'') = X$ in all these cases. It follows that

$$U(m-1)' \cap W_3 = 0 \qquad \text{and} \qquad V(1'') \cap W_3 = W_3.$$

Consequently, the above is a decomposition of the $\mathscr{S}$-space $(W, U(i'), V(i''))$ and it is a matter of routine to calculate the number of the indecomposable $\mathscr{S}$-spaces: By Proposition 2.5, there are $\frac{1}{2}(m+1)(m+2)$ of those for which $V(i'')$ is the whole space; by Proposition 2.6, there are $\frac{1}{2}(n+1)(n+6)$ of those for which $U(m-1)' = 0$; in this process we have calculated 3 $\mathscr{S}$-spaces (namely those with $U(m') = 0$, G or F and $V(1'') = F$) twice; to complete our list we have to add the types B_3, B_5, B_6, $\mathsf{C}_2 - \mathsf{C}_6$, D_1, and D_2 of Proposition 3.1. The proof is completed.

Remark 3.3 The proof of Proposition 3.1 can be shortened if we assume that F contains a central subfield K with $[F: K]$ finite and $K \subseteq G \subseteq F$. For, in this case we can make use of the following duality with respect to K.

If W is a finite dimensional K-vector space, let $W^* = \mathrm{Hom}_K(W, K)$ be the K-*dual* of W. Then $\dim_K W = \dim_K W^*$, and we can identify W and W^{**}. If U is a K-subspace of W, let $U^\perp = \{\varphi \in W^* \mid U\varphi = 0\}$. Then $U^\perp \approx (W/U)^*$, and under the identification of W and W^{**}, we get also $U = U^{\perp\perp}$. If V is another K-subspace of W, then $U \subseteq V$ if and only if $V^\perp \subseteq U^\perp$, and $(U + V)^\perp = U^\perp \cap V^\perp$ and $(U \cap V)^\perp = U^\perp + V^\perp$. Now, assume that F is a field which contains K in its center and $[F: K]$ is finite. If W_F is a right finite dimensional F-vector space, then the K-dual W^* becomes a left F-vector space, and thus a right F^0-vector space. If $K \subseteq G \subseteq F$, and U_G is a G-subspace of W_F, then $U^\perp$ is a G^0-subspace of W^*. Also, since $\underline{U} = \sum_{V_F \subseteq U} V_F$ and $\overline{U} = \bigcap_{V_F \supseteq U} V_F$, it follows that

$$\underline{U}^\perp = \overline{(U^\perp)} \qquad \text{and} \qquad \overline{U}^\perp = \underline{(U^\perp)}.$$

If the K-structure $\mathscr{S}$ for F satisfies the condition $F_i = F_j$ for $i \leqslant j$, then we may define a K-structure $\mathscr{S}^*$ for F^0 taking $\mathscr{S}$ with the inverse order and mapping each $i \in \mathscr{S}^*$ into the field $(F_i)^0$. The K-structure $\mathscr{S}^*$ is called the *dual* K-structure of $\mathscr{S}$. Furthermore, every finite dimensional $\mathscr{S}$-space (W, W_i) defines an $\mathscr{S}^*$-space, namely $(W, W_i)^* = (W^*, W_i^\perp)$, and $(W, W_i)^{**} = (W, W_i)$.

Now, in the case of $\mathscr{S} = \mathscr{I}_2(G) \sqcup \mathscr{I}_1(F)$, the dual K-structure $\mathscr{S}^*$ of $\mathscr{S}$ is just $\mathscr{S}^* = \mathscr{I}_2(G^0) \sqcup \mathscr{I}_1(F^0)$; thus, every result on the decomposition of $\mathscr{S}$-spaces can be applied to the K-structure $\mathscr{S}^*$, and yields therefore a dual result for $\mathscr{S}$-spaces. For example, in the first step of the proof of Proposition 3.1, we have shown that every $\mathscr{S}$-space is the direct sum of copies of A_6 and B_6, and an $\mathscr{S}$-space in which $\underline{U}_1 = 0$. Now, if (W, U_1, U_2, V) is an $\mathscr{S}$-space, then we may apply this result to the $\mathscr{S}^*$-space $(W^*, U_2^\perp, U_1^\perp, V^\perp)$; hence, $(W^*, U_2^\perp, U_1^\perp, V^\perp)$ is the direct sum of copies of $(F^0, F^0, F^0, 0)$ and (F^0, F^0, F^0, F^0), and an $\mathscr{S}^*$-space in which $\underline{(U_2^\perp)} = 0$. Applying the dual argument again, we see that $(W, U_1, U_2, V) = (W^*, U_2^\perp, U_1^\perp, V^\perp)^*$ is the direct sum of copies of $\mathsf{B}_1 = (F^0, F^0, F^0, 0)^*$ and $\mathsf{A}_1 = (F^0, F^0, F^0, F^0)^*$, and an $\mathscr{S}$-space in which $\overline{U}_2 = \overline{U}_2^{\perp\perp} = \underline{(U_2^\perp)}^\perp = 0^\perp = W$. In this way, we may replace every second argument in the proof by a reference to the duality.

4. *K-Structure* $\mathscr{I}_1(G)$ *with* $[F: G] = 3$

Throughout this section, we assume that $K \subseteq G \subseteq F$ are three fields such that K is central in F, $[F: G] = 3$ and $[G: K] = n$. Let $\mathscr{S} = \mathscr{I}_1(G)$. First, we want to show the existence of an indecomposable $\mathscr{S}$-space (W_F, U_G)

with dim $W_F = 2$. To this end, we need the following lemma which will also be used to prove Proposition 7.4.

LEMMA 4.1. *Let* $[F: G] = 3$ *and* $e \in F \setminus G$. *Then there exists a G-subspace* U_G *of* $F \times F$ *such that*

$$\dim U_G = 3, \qquad U \cap Ue = 0 \qquad \textit{and} \qquad U \cap (F \times 0) = 0.$$

Proof. First, assume that F is commutative. Obviously, $\{1, e, e^2\}$ is a basis of F_G. If $e^3 \notin G + eG$, let $f = e^2$, otherwise, put $f = e + e^2$. $\{1, e, f\}$ as well as $\{1, e, ef\}$ are bases of F_G. Let

$$U = (0, 1)G + (e, e)G + (1, f)G.$$

Since $\{1, e, f\}$ is a basis of F_G, we have $U + F \times 0 = F \times F$, so

$$\dim U_G = 3 \qquad \text{and} \qquad U \cap F \times 0 = 0.$$

If $x \in U \cap Ue$, say

$$x = (eg_2 + g_3, g_1 + eg_2 + fg_3) = (e^2g_2' + eg_3', eg_1' + e^2g_2' + efg_3')$$

with $g_i, g_i' \in G$ for $1 \leqslant i \leqslant 3$, then

$$g_3 + e(g_2 - g_3') - e^2g_2' = 0$$

implies that $g_3 = 0 = g_2'$ and $g_2 = g_3'$, because $\{1, e, e^2\}$ is a basis of F_G. Comparing the second components of x, we have

$$g_1 + e(g_2 - g_1') - efg_3' = 0,$$

and since $\{1, e, ef\}$ is a basis of F_G, we conclude that $g_1 = 0 = g_3'$ and $g_2 = g_1'$. Consequently, since $g_2 = g_3' = 0$, $x = 0$. Therefore $U \cap Ue = 0$, which proves the lemma in the case when F is commutative.

Thus, we can assume that the field F, and therefore also K, are infinite. Let $W_F = F \times F$ and $U_G = G \times G \times G$. Consider the affine variety

$$\mathrm{Hom}_G(U_G, W_G) \approx K^{3 \cdot 6 \cdot n}$$

over K. Denote by Aut U_G the automorphism group of U_G. Also, consider the ring $T_2(F)$ of all 2×2 upper triangular matrices over F as the subring of $\mathrm{Hom}_F(W_F, W_F)$ of all elements mapping $F \times 0$ into itself, and denote by $T_2(F)^{\times}$ its group of units. Both Aut U_G and $T_2(F)^{\times}$ are open subvarieties of affine varieties, namely

$$\mathrm{Aut}\, U_G \subseteq \mathrm{Hom}_G(U_G, U_G) \approx K^{3 \cdot 3 \cdot n}$$

and

$$T_2(F)^{\times} \subsetneq T_2(F) \approx K^{3\cdot 3\cdot n}.$$

Moreover, Aut $U_G \times T_2(F)^{\times}$ operates on $\mathrm{Hom}_G(U_G, W_G)$ via

$(\alpha, \beta)\varphi = \beta\varphi\alpha^{-1}$ for $\alpha \in \mathrm{Aut}\, U_G$, $\beta \in T_2(F)^{\times}$ and $\varphi \in \mathrm{Hom}_G(U_G, W_G)$.

Here, the diagonal $K^{\times} = \{(k, k) \mid k \in K^{\times}\} \subsetneq \mathrm{Aut}\, U_G \times T_2(F)^{\times}$ operates on $\mathrm{Hom}_G(U_G, W_G)$ trivially. Since

$$\mathrm{Dim}(\mathrm{Aut}\, U_G \times T_2(F)^{\times}/K^{\times}) = 9n + 9n - 1 = 18n - 1,$$

whereas

$$\mathrm{Dim}\, \mathrm{Hom}_G(U_G, W_G) = 18n,$$

we conclude that there are infinitely many orbits.

Now, denoting by $\mathscr{S}'$ the K-structure $\mathscr{I}_1(G) \sqcup \mathscr{I}_1(F)$, it is easy to see that these orbits correspond bijectively to the isomorphism classes of $\mathscr{S}'$-spaces of the form $(F \times F, U_G', F \times 0)$ with $\dim U_G' \leqslant 3$. For, given

$$\varphi \in \mathrm{Hom}_G(U_G, W_G),$$

we can construct the $\mathscr{S}'$-space $(F \times F, \varphi(U), F \times 0)$ and, obviously, we get in this way all such $\mathscr{S}'$-spaces. If $\varphi, \psi \in \mathrm{Hom}_G(U_G, W_G)$ belong to the same orbit, that is, if there are $\alpha \in \mathrm{Aut}\, U_G$ and $\beta \in T_2(F)^{\times}$ such that $\psi = \beta\varphi\alpha^{-1}$, then the automorphism $\beta: W_F \to W_F$ maps $F \times 0$ into $F \times 0$, and

$$\beta(\varphi(U)) = \beta(\varphi\alpha^{-1}(U)) = \beta\varphi\alpha^{-1}(U) = \psi(U);$$

thus, $(F \times F, \varphi(U), F \times 0)$ and $(F \times F, \psi(U), F \times 0)$ are isomorphic $\mathscr{S}'$-spaces. Conversely, if there is an isomorphism of $\mathscr{S}'$-spaces

$$\beta: (F \times F, \varphi(U), F \times 0) \to (F \times F, \psi(U), F \times 0),$$

then $\beta \in T_2(F)^{\times}$ and, since both $\beta\varphi$ and ψ map U onto $\psi(U)$, there is also $\alpha \in \mathrm{Aut}\, U_G$ with $\psi = \beta\varphi\alpha^{-1}$.

We want to show that there is only a finite number of (nonisomorphic) $\mathscr{S}'$-spaces of the form $(F \times F, U_G, F \times 0)$ with $\dim U_G \leqslant 3$ which satisfy either $U \cap Ue \neq 0$ or $U \cap F \times 0 \neq 0$.

First, there is only a finite number of decomposable $\mathscr{S}'$-spaces $(F \times F, U, F \times 0)$. Indeed, an $\mathscr{S}'$-space (W_F, U', V') with $\dim W_F = 1$ is either of the form (F, U, F) or of the form $(F, U, 0)$, and thus there are just 8 such nonisomorphic $\mathscr{S}'$-spaces, corresponding to whether $\dim U_G = 0, 1, 2$, or 3. Perhaps only the case when $\dim U_G = 2$ requires to show that every $\mathscr{S}'$-space (F, U, V') with $V' = 0$ or F, is isomorphic to $(F, G + eG, V')$, where

$e \in F \setminus G$. Indeed, both U and Ue^{-1} are K-subspaces of F_K of dimension $2n$, and therefore, since $\dim F_K = 3n$, there exists $0 \neq x \in U \cap Ue^{-1}$. And, it is easy to see that the left multiplication by x maps $(F, G + eG, V')$ into $(F, xG + xeG, V') = (F, U, V')$, as required.

Also, there is (up to an isomorphism) only one indecomposable $\mathscr{S}'$-space $(F \times F, U, F \times 0)$ with $\dim U_G = 2$. Obviously, we can assume that $(0, 1) \in U$ and that $U = (0, 1)G + (1, f)G$ with a suitable $f \in F \setminus G$. If $U' = (0, 1)G + (1, f')G$ with another $f' \in F \setminus G$, then

$$(G + f'G) \cap (G + f'G)f \neq 0,$$

because both $G + f'G$ and $(G + f'G)f$ are K-subspaces of F of dimension $2n$, and $\dim F_K = 3n$. Thus, there is a nonzero element $\gamma \in F$ such that

$$\gamma = g_1 + f'g_2 \qquad \text{and} \qquad \gamma f = g_1' + f'g_2'$$

for some $g_1, g_2, g_1', g_2' \in G$. Define

$$\alpha = g_2' - g_2 f \qquad \text{and} \qquad \beta = g_2,$$

and observe that $\alpha \neq 0$; for, otherwise, $g_2 = 0 = g_2'$, and, consequently, $g_1 = \gamma \neq 0$ and $g_1 f = g_1' \in G$. Then, $\left(\begin{smallmatrix}\alpha & \beta \\ 0 & \gamma\end{smallmatrix}\right)$ is an automorphism of W_F which maps $F \times 0$ into itself and U into U' according to

$$\begin{pmatrix}\alpha & \beta \\ 0 & \gamma\end{pmatrix}\begin{pmatrix}0 \\ 1\end{pmatrix} = \begin{pmatrix}\beta \\ \gamma\end{pmatrix} = \begin{pmatrix}g_2 \\ g_1 + f'g_2\end{pmatrix} = \begin{pmatrix}0 \\ 1\end{pmatrix} g_1 + \begin{pmatrix}1 \\ f'\end{pmatrix} g_2$$

and

$$\begin{pmatrix}\alpha & \beta \\ 0 & \gamma\end{pmatrix}\begin{pmatrix}1 \\ f\end{pmatrix} = \begin{pmatrix}\alpha + \beta f \\ \gamma f\end{pmatrix} = \begin{pmatrix}g_2' - g_2 f + g_2 f \\ g_1' + f'g_2'\end{pmatrix} = \begin{pmatrix}0 \\ 1\end{pmatrix} g_1' + \begin{pmatrix}1 \\ f'\end{pmatrix} g_2'.$$

Similarly, there is at most one indecomposable $\mathscr{S}'$-space $(F \times F, U', F \times 0)$ with $\dim U_G' = 3$ and $U' \cap F \times 0 \neq 0$. For, if such an $\mathscr{S}'$-space exists, then, in view of indecomposability, necessarily $\dim(U' \cap F \times 0)_G = 1$ and $U' \cap U'e = 0$. We may assume that $G \times G \subseteq U'$, and thus let

$$U' = G \times G + (e', f')G \text{ with suitable } e', f' \in F.$$

Let $\{1, e, f\}$ be a basis of ${}_GF$ and let $U = G \times G + (e, f)G$. We are going to show that $(F \times F, U, F \times 0)$ and $(F \times F, U', F \times 0)$ are isomorphic. Since $(G + f'G) \cap (G + f'G)f^{-1} \neq 0$, there is a nonzero $\gamma \in F$ such that

$$\gamma = g_2 + f'g_3 \qquad \text{and} \qquad \gamma f = g_2' + f'g_3'$$

for some $g_2, g_3, g_2', g_3' \in G$. Since $\{1, e, f\}$ is a basis of ${}_GF$, we can write

$$-e'g_3 f + e'g_3' = -g_1' + \alpha e + g_1 f$$

with the coefficients g_1', α and g_1 from G. Defining, furthermore, $\beta = g_1 + e'g_3$, we can verify easily that $\varphi = \begin{pmatrix}\alpha & \beta \\ 0 & \gamma\end{pmatrix}$ is an endomorphism of W_F which maps $F \times 0$ into $F \times 0$ and U into U':

$$\begin{pmatrix}\alpha & \beta \\ 0 & \gamma\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix} = \begin{pmatrix}\alpha \\ 0\end{pmatrix} = \begin{pmatrix}1 \\ 0\end{pmatrix}\alpha,$$

$$\begin{pmatrix}\alpha & \beta \\ 0 & \gamma\end{pmatrix}\begin{pmatrix}0 \\ 1\end{pmatrix} = \begin{pmatrix}\beta \\ \gamma\end{pmatrix} = \begin{pmatrix}1 \\ 0\end{pmatrix}g_1 + \begin{pmatrix}0 \\ 1\end{pmatrix}g_2 + \begin{pmatrix}e' \\ f'\end{pmatrix}g_3,$$

and

$$\begin{pmatrix}\alpha & \beta \\ 0 & \gamma\end{pmatrix}\begin{pmatrix}e \\ f\end{pmatrix} = \begin{pmatrix}\alpha e + \beta f \\ \gamma f\end{pmatrix} = \begin{pmatrix}1 \\ 0\end{pmatrix}g_1' + \begin{pmatrix}0 \\ 1\end{pmatrix}g_2' + \begin{pmatrix}e' \\ f'\end{pmatrix}g_3'.$$

Now, if φ is not an isomorphism, then $\alpha = 0$ and thus $F \times 0$ is its kernel. But $\varphi(U) = (U + \ker \varphi)/\ker \varphi = (U + F \times 0)/F \times 0$ is of dimension 2 over G, and therefore $F \times F = (F \times 0) \oplus \varphi(U)F$ is a direct decomposition of W_F compatible both with $F \times 0$ and U', a contradiction. Hence φ is an isomorphism, as required.

Finally, there is just one indecomposable $\mathscr{S}'$-space $(F \times F, U, F \times 0)$ with $\dim U_G = 3$ and $U \cap Ue \neq 0$. For, such an $\mathscr{S}'$-space necessarily satisfies $U \cap F \times 0 = 0$, and we may assume that $(0, e) \in U \cap Ue$. Hence

$$U = (0, 1)G + (0, e)G + (h, f)G,$$

where $\{1, e, f\}$ is a basis of F_G and $0 \neq h \in F$. Now, if

$$U' = (0, 1)G + (0, e)G + (h', f)G$$

is another such $\mathscr{S}'$-space, one can see immediately that the automorphism $\begin{pmatrix}h'h^{-1} & 0 \\ 0 & 1\end{pmatrix}$ of W_F maps $F \times 0$ into $F \times 0$ and U into U'.

Thus, we have shown that there is only a finite number of (nonisomorphic) $\mathscr{S}'$-spaces of the form $(F \times F, U_G, F \times 0)$ with $\dim U_G \leqslant 3$ satisfying either $U \cap Ue \neq 0$ or $U \cap F \times 0 \neq 0$. Since the number of nonisomorphic $\mathscr{S}'$-spaces $(F \times F, U_G, F \times 0)$ with $\dim U_G \leqslant 3$ is infinite, we conclude that there are $\mathscr{S}'$-spaces $(F \times F, U_G, F \times 0)$ with

$$\dim U_G = 3, \qquad U \cap Ue = 0 \qquad \text{and} \qquad U \cap F \times 0 = 0.$$

This completes the proof of Lemma 4.1.

Now, if we take a subspace U_G of $F \times F$ with $\dim U_G = 3$ and $U \cap Ue = 0$, then the $\mathscr{S}$-space $(F \times F, U_G)$ is indecomposable. For, otherwise the intersection U_G' of U_G with one of the one-dimensional summands would have dimension at least two and thus $U' \cap U'e \neq 0$, in contradiction to $U \cap Ue = 0$. We are ready to formulate the following:

PROPOSITION 4.2. *Let $\mathscr{S} = \mathscr{I}_1(G)$ be a K-structure such that $[F: G] = 3$. Then there are exactly five nonisomorphic indecomposable $\mathscr{S}$-spaces.*

Proof. First, choose a basis $\{1, e, f\}$ of F_G which is a basis for the vector space ${}_GF$, as well. This is possible in view of the fact that if $\{1, e, f'\}$ and $\{1, e, f''\}$ are bases of F_G and ${}_GF$, respectively, then either one of them or $\{1, e, f' + f''\}$ is a basis for both F_G and ${}_GF$.

We are going to prove that every finitely generated $\mathscr{S}$-space (W_F, U_G) is a direct sum of indecomposable $\mathscr{S}$-spaces of the form

(i) $(F, 0)$,
(ii) (F, G),
(iii) $(F, G + eG)$,
(iv) (F, F), and
(v) $(F \times F, (G \times G) + (e, f)G)$.

First, it is obvious that

$$W = \underline{U} \oplus W_1 \oplus W_2,$$

where W_1 is an F-complement of $\bar{U}$ in W and W_2 is an F-complement of $\underline{U} \oplus W_1$ in W, is a decomposition of the $\mathscr{S}$-space (W, U). Here $\underline{U}$ is a direct sum of copies of (iv) and W_1 is a direct sum of copies of (i). Hence, we may assume that $\underline{U} = 0$ and that $\bar{U} = W$.

Now, consider the F-subspace $U \cap Ue^{-1}$ and let $\{u_1, u_2, \ldots, u_r\} \subseteq U \cap Ue^{-1}$ be an F-basis of it. Thus

$$\{u_1, u_2, \ldots, u_r, u_1e, u_2e, \ldots, u_re\}$$

is a G-independent subset of U. Since $\bar{U} = W$, $\{u_1, u_2, \ldots, u_r\}$ can be extended to an F-basis

$$\{u_1, u_2, \ldots, u_r, u_{r+1}, \ldots, u_s\} \subseteq U$$

of W. Define

$$X = u_1G \oplus u_2G \oplus \cdots \oplus u_sG \subseteq U \subseteq u_1F \oplus u_2F \oplus \cdots \oplus u_sF = W.$$

Obviously, since $X + Xe + Xf = W$, every element $w \in W$ has the form

$$w = x_0 + x_1e + x_2f \qquad \text{with} \quad x_i \in X.$$

Now, the G-independent subset

$$\{u_1, u_2, \ldots, u_r, u_{r+1}, \ldots, u_s, u_1e, u_2e, \ldots, u_re\} \subseteq U$$

can be extended by $\{v_1, v_2, \ldots, v_t\}$ to a G-basis of U. We may assume that $v_j \in Xe + Xf$; let

$$v_j = x_j'e + x_j''f \qquad \text{with} \quad x_j', x_j'' \in X, \quad 1 \leqslant j \leqslant t.$$

Put

$$A_G = \sum_{i=1}^{r} u_i G + \sum_{j=1}^{t} x_j' G + \sum_{j=1}^{t} x_j'' G \subseteq X_G ,$$

and observe that

$$U \subseteq \bar{A} + X.$$

Choose a G-complement B of A in X: $X = A \oplus B$. Then, $W = \bar{A} \oplus \bar{B}$ is a decomposition of the $\mathscr{S}$-space (W, U). For, since $U \subseteq \bar{A} + X \subseteq \bar{A} + B$, every element $u \in U$ can be written

$$u = \bar{a} + b \qquad \text{with} \quad \bar{a} \in A \qquad \text{and} \qquad b \in B;$$

hence, $\bar{a} \in A \cap U$ and $b \in B \cap U \subseteq \bar{B} \cap U$, as required. Here, $\bar{B}$ is obviously a direct sum of copies of (ii). Thus, we may assume that $A = X$ and thus $\bar{A} = W$.

Now, since $[G: K] = n$,

$$\dim U_K = (r + s + t)n \qquad \text{and} \qquad \dim W_K = 3sn.$$

Also, considering the G-subspaces $U \cap V$, $Ue^{-1} \cap V$, and $Uf^{-1} \cap V$ of $V = \overline{U \cap Ue^{-1}}$, we see easily that

$$\dim(U \cap V)_K = \dim(Ue^{-1} \cap V)_K = \dim(Uf^{-1} \cap V)_K \geqslant 2rn$$

and $\dim V_K = 3rn$. Thus,

$$\dim(U \cap Ue^{-1})_K = \dim[(U \cap V) \cap (Ue^{-1} \cap V)]_K \geqslant 4rn - 3rn = rn.$$

Moreover, since $\underline{U} = 0$, $U \cap Ue^{-1} \cap Uf^{-1} = (U \cap Ue^{-1}) \cap (Uf^{-1} \cap V) = 0$, and we deduce immediately that $\dim(U \cap Ue^{-1})_K = rn$. Hence,

$$\begin{aligned} 3sn \geqslant \dim(U + Ue^{-1})_K &= \dim U_K + \dim(Ue^{-1})_K - \dim(U \cap Ue^{-1})_K \\ &= (r + 2s + 2t)n \end{aligned}$$

and thus

$$\dim W_F = s \geqslant r + 2t.$$

Consequently, in view of the fact that the set $\{u_1, u_2, \ldots, u_r, x_1', x_1'', \ldots, x_t', x_t''\}$ F-generates W, we obtain the decomposition

$$W = \bigoplus_{i=1}^{r} u_i F \oplus \bigoplus_{j=1}^{t} (x_j' F \oplus x_j'' F)$$

of the $\mathscr{S}$-space $\{U_G \subseteq W_F\}$ into indecomposable $\mathscr{S}$-spaces of types (iii) and (v). The proof is completed.

A case similar to that of Proposition 4.2 was treated in [5]; it can be dealt with by the same method as here.

II. Categories of Strongly Unbounded Type

5. *Construction of Large Indecomposable Representations*

The aim of this section is to study the representations of certain rather simple K-species and to construct large indecomposable representations. As P. Gabriel [7] has pointed out (in the classical case), there is a strong affinity between K-structures and K-species. In some cases, there is a correspondence between a K-structure $\mathscr{S}$ and a K-species $\mathscr{Q}$ in such a way that the categories $\mathfrak{S}(\mathscr{S})$ of all $\mathscr{S}$-spaces and $\mathfrak{R}(\mathscr{Q})$ of all representations of $\mathscr{Q}$ are nearly equivalent to each other. Since it is usually easier to operate within an $\mathscr{S}$-space than within a representation of a K-species, the final classification of K-species of finite type is derived from the corresponding classification of K-structures of finite type. However, in some instances, there is an advantage in working in the category $\mathfrak{R}(\mathscr{Q})$, because $\mathfrak{R}(\mathscr{Q})$ is an abelian category, whereas this is usually not true for $\mathfrak{S}(\mathscr{S})$. In the sequel, we use the fact that $\mathfrak{R}(\mathscr{Q})$ is abelian, mainly in order to be able to apply the following criterion for indecomposability.

LEMMA 5.1. *Let $\mathfrak{A}$ be a Grothendieck category and let B, B' be indecomposable objects of $\mathfrak{A}$.*

(a) *Let*

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_k = A$$

be a sequence of subobjects $A_i \in \mathfrak{A}$ of A such that $A_i/A_{i-1} \approx B$ for all $1 \leqslant i \leqslant k$. If every morphism $B \to A/A_{i-1}$ with $1 \leqslant i \leqslant k$ maps B into A_i/A_{i-1}, then A is indecomposable.

(b) *In addition, let*

$$0 = A_0' \subseteq A_1' \subseteq \cdots \subseteq A_l' = A'$$

be a sequence of subobjects $A_j' \in \mathfrak{A}$ of A' such that $A_j'/A_{j-1}' \approx B'$ for all $1 \leqslant j \leqslant l$. If $\mathrm{Hom}(B, B') = 0$, *then also* $\mathrm{Hom}(A, A') = 0$.

(c) *Let*

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{i-1} \subseteq A_i \subseteq \cdots \subseteq A = \bigcup_{i \in \mathbb{N}} A_i$$

be a sequence of subobjects of A such that $A_i/A_{i-1} \approx B$ for all $i \in \mathbb{N}$. If every morphism $B \to A/A_{i-1}$, $i \in \mathbb{N}$, maps B into A_i/A_{i-1}, then A is indecomposable.

Proof. We prove first (a) by induction on k. We show that every idempotent ϵ in $\mathrm{Hom}(A, A)$ is either 0 or 1. Since with ϵ also $1 - \epsilon$ is an idempotent and either $\epsilon A_1 \neq 0$, or $(1 - \epsilon) A_1 \neq 0$, we may assume that $\epsilon A_1 \neq 0$. Let $\mu: A_1 \to A$ be the inclusion of A_1 in A. Then $\epsilon\mu: A_1 \to A$ maps $A_1 \approx B$ into A_1, so there is ϵ' in $\mathrm{Hom}(A_1, A_1)$ with $\epsilon\mu = \mu\epsilon'$. Obviously, ϵ' is also an idempotent, and since A_1 is indecomposable, either $\epsilon' = 1$ or $\epsilon' = 0$. But $\epsilon' = 0$ is impossible since $\epsilon A_1 \neq 0$; hence, $\epsilon' = 1$. If we denote by $\pi: A \to A/A_1$ the canonical projection, then there is ϵ'' in $\mathrm{Hom}(A/A_1, A/A_1)$ with $\epsilon''\pi = \pi\epsilon$, and so ϵ'' is an idempotent. All the assumptions are satisfied for A/A_1 with the series

$$0 = A_1/A_1 \subsetneq A_2/A_1 \subsetneq \cdots \subsetneq A_k/A_1 = A/A_1,$$

of subobjects. Thus, by induction, either $\epsilon'' = 1$ or $\epsilon'' = 0$. Now, $\epsilon'' = 1$ together with $\epsilon' = 1$ implies that ϵ is an isomorphism and then $\epsilon = 1$. Hence, we may assume that $\epsilon'' = 0$. But then $\pi\epsilon = 0$, so $\epsilon = \mu\rho$ for some $\rho: A \to A_1$, since μ is the kernel of π. It follows that $\rho\mu = 1$. As a consequence, also the inclusion $A_1 \subseteq A_2$ splits, that is $A_2 = A_1 \oplus C$ with $C \approx A_2/A_1$. But $A_2/A_1 \approx B$ shows that there exists an embedding of B into $A_2 \subseteq A$ which avoids A_1. This contradiction shows that $\epsilon = 1$, as required.

Now, let $\varphi: A \to A'$ be a morphism. In order to prove (b), we will use induction with respect to $k + l$. If $k > 1$, let $\mu: A_1 \to A$ be the inclusion. Then $\varphi\mu: A_1 \to A'$ is zero, so $A_1 \subseteq \ker \varphi$, and φ induces a map $\varphi': A/A_1 \to A'$ which, again by induction, has to be zero. Therefore, $\varphi = 0$. Similarly, if $l > 1$, let $\pi': A' \to A'/A_1'$ be the projection. Then $\pi'\varphi: A \to A'/A_1'$ is, by induction, zero. Consequently, φ maps into A_1', but then also $\varphi = 0$.

Finally, in order to verify (c), we note that every endomorphism $\varphi \in \mathrm{Hom}(A, A)$ maps A_i into A_i for all $i \in \mathbb{N}$. For, assuming that $\varphi A_{i-1} \subseteq A_{i-1}$, denote by φ' the induced map $A/A_{i-1} \to A/A_{i-1}$. Then, by assumption, $\varphi'(A_i/A_{i-1})$ lies in A_i/A_{i-1}, since $A_i/A_{i-1} \approx B$. Thus also $\varphi A_i \subseteq A_i$. In particular, every idempotent endomorphism $\epsilon \in \mathrm{Hom}(A, A)$ maps A_i into itself and so induces a decomposition of A_i. But note that, by (a), A_i is indecomposable. Thus, if ϵ is an idempotent such that $\epsilon A_1 \neq 0$, then $\epsilon A_i \neq 0$ for all i, and therefore the restriction of ϵ to A_i is the identity. Hence, since $A = \bigcup_{i \in \mathbb{N}} A_i$, we conclude that $\epsilon = 1$. This completes the proof.

Now, we consider a K-species $(K_i, {}_iM_j)_{i,j \in I}$ with $I = \{1, 2\}$ and ${}_1M_1 = {}_2M_2 = {}_2M_1 = 0$. That is, there is given just a bimodule ${}_{K_1}({}_1M_2)_{K_2}$ on which K acts centrally and which is finite dimensional over K. A representation is given by a pair $(V_1)_{K_1}$, $(V_2)_{K_2}$ of vector spaces together with a map of the form $\varphi: (V_1)_{K_1} \otimes {}_{K_1}({}_1M_2)_{K_2} \to (V_2)_{K_2}$. To avoid indices, we write $K_1 = F$, $K_2 = G$, ${}_1M_2 = M$, $V_1 = X$, $V_2 = Y$ and simply speak about the *K-species* ${}_FM_G$ and its representations (X, Y, φ). We shall denote by $\varphi^*: X_F \to \mathrm{Hom}_G({}_FM_G, Y_G)$ the adjoint mapping of φ. Maps from (X, Y, φ) to (X', Y', φ') are pairs of mappings $\alpha: X_F \to X_F'$, $\beta: Y_G \to Y_G'$ satisfying $\beta\varphi = \varphi'(\alpha \otimes 1)$, or, equivalently, $\mathrm{Hom}(1, \beta)\, \varphi^* = \varphi'^*\alpha$. In this way, we get the abelian category $\mathfrak{R}({}_FM_G)$ of all representations of ${}_FM_G$.

If (X, Y, φ) is a representation of ${}_FM_G$, and φ is not an epimorphism, let C_G be a complement of $\varphi(X \otimes M)$ in Y_G and $\theta: 0 \otimes M \to C$ the zero map. Then

$$(X, Y, \varphi) = (X, \varphi(X \otimes M), \varphi) \oplus (0, C, \theta)$$

is a decomposition in $\mathfrak{R}({}_FM_G)$. Thus, for an indecomposable object (X, Y, φ) in $\mathfrak{R}({}_FM_G)$, either φ is an epimorphism or $(X, Y, \varphi) \approx (0, G_G, \theta)$ with the zero mapping θ. We denote by $\mathfrak{Re}({}_FM_G)$ the full subcategory of $\mathfrak{R}({}_FM_G)$ of all objects (X, Y, φ) with an epimorphism φ. Similarly, we denote by $\mathfrak{Rm}({}_FM_G)$ the full subcategory of $\mathfrak{R}({}_FM_G)$ of all objects (X, Y, φ) such that φ^* is monomorphism. Then, an indecomposable object of $\mathfrak{R}({}_FM_G)$ either belongs to $\mathfrak{Rm}({}_FM_G)$ or is isomorphic to $(F_F, 0, F_F \otimes {}_FM_G \to 0)$. The K-species ${}_FM_G$ is therefore of finite type if and only if one of the categories $\mathfrak{Re}({}_FM_G)$ and $\mathfrak{Rm}({}_FM_G)$ (and, therefore both) is of finite type. And similarly, if one of the categories $\mathfrak{R}({}_FM_G)$, $\mathfrak{Re}({}_FM_G)$, and $\mathfrak{Rm}({}_FM_G)$ is of strongly unbounded type, then all three categories are of strongly unbounded type. The following proposition is of great importance.

PROPOSITION 5.2. *Let ${}_FM_G$ be a K-species. If* $(\dim {}_FM) \times (\dim M_G) \geqslant 4$, *then ${}_FM_G$ is of strongly unbounded type.*

Proof. Let $\mathfrak{A} = \mathfrak{A}({}_FM_G)$ be the category defined as follows: the objects of $\mathfrak{A}$ are of the form (U_G, X_F, ψ), where $\psi: U_G \to X_F \otimes {}_FM_G$ is G-linear and the morphisms $(U_G, X_F, \psi) \to (U_G', X_F', \psi')$ are given by pairs (γ, α) with $\gamma: U_G \to U_G'$, $\alpha: X_F \to X_F'$ such that $\psi'\gamma = (\alpha \otimes 1)\psi$. Obviously, $\mathfrak{A}$ is an abelian category. We are mainly interested in the full subcategory $\mathfrak{B} = \mathfrak{B}({}_FM_G)$ of all objects (U_G, X_F, ψ) of $\mathfrak{A}$, where ψ is an inclusion (usually denoted by ι), in which case a morphism is fully determined by a single mapping $\alpha: X_F \to X_F'$ and so we write instead of (γ, α) simply α. Also, if $X_F = \bigoplus_I F_F$ with an index set I, then

$$X_F \otimes {}_FM_G \approx \bigoplus_I (F_F \otimes {}_FM_G) \approx \bigoplus_I M_G,$$

and we will frequently identify $X_F \otimes {}_FM_G$ and $\oplus_I M_G$. *The category* $\mathfrak{B}$ *is equivalent to* $\mathfrak{Re}({}_FM_G)$. For, consider the functor $\mathfrak{Re}({}_FM_G) \to \mathfrak{B}$ which sends (X_F, Y_G, φ) to $(\ker \varphi, X_F, \iota)$, where $\iota\colon \ker \varphi \to X_F \otimes {}_FM_G$ is the inclusion of the kernel of $\varphi\colon X_F \otimes {}_FM_G \to Y_G$. Since φ is an epimorphism, φ is (up to a canonical isomorphism) determined by ι, and therefore the functor is an equivalence. If we define a dimension in $\mathfrak{B}$ by

$$\dim(U_G, X_F, \psi) = \dim X_K,$$

then our functor is a dimension functor. For, if $\dim {}_FM = n$, then $\dim Y_K \leqslant \dim(X_F \otimes {}_FM)_K = n \dim X_F$ and therefore

$$\frac{1}{n+1}(\dim X_K + \dim Y_K) \leqslant \dim X_K \leqslant \dim X_K + \dim Y_K.$$

Let $d = \dim M_G$, $d' = \dim {}_FM$, and $d'd \geqslant 4$. First, we assume that $d' \leqslant d$, and show that in this case $\mathfrak{B} = \mathfrak{B}({}_FM_G)$, and therefore $\mathfrak{Re}({}_FM_G)$, is of strongly unbounded type. To prove that $\mathfrak{B}$ has many indecomposable objects, we will work in the abelian category $\mathfrak{A}$ and construct objects which satisfy the conditions of Lemma 5.1 and which belong to $\mathfrak{B}$. Note that objects of $\mathfrak{B}$ of the form (U_G, F_F, ι) are indecomposable in $\mathfrak{A}$. For, in a direct decomposition $(U_G, F_F, \iota) = (X, Y, \varphi) \oplus (X', Y', \varphi')$, also φ and φ' are monomorphisms, and since $Y_F \oplus Y'_F = F_F$, either $(X, Y, \varphi) = (0, 0, \theta)$ or $(X', Y', \varphi') = (0, 0, \theta)$.

We will consider two cases, namely the case where $d' = 1$ and $d \geqslant 4$, and the case $d \geqslant d' \geqslant 2$.

(i) Let $d \geqslant 4$, $d' = 1$. Hence ${}_FM_G = {}_FF_G$, and G can be viewed as a subfield of F. If $x \in F \setminus G$, consider the object $B = B_x = (G + xG, F, \iota)$, where $\iota\colon G + xG \subseteq F_G = F_F \otimes {}_FF_G$ is the inclusion map. We are going to construct in $\mathfrak{A}$ a sequence

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{i-1} \subseteq A_i \subseteq \cdots \subseteq A = \bigcup_{i \in \mathbb{N}} A_i$$

with $A, A_i \in \mathfrak{B}$ and $A_i/A_{i-1} \approx B$ for all $i \in \mathbb{N}$, and such that every morphism $B \to A$ factors through A_1, and that, for every $j \in \mathbb{N}$, there is an isomorphism $A \to A/A_j$ mapping A_i onto A_{i+j}/A_j for all $i \in \mathbb{N}$. Then, in view of Lemma 5.1, A as well as all A_i are indecomposable.

Since $G + xG + Gx + xGx$ is a proper K-subspace of F_K (for, $\dim(G + xG + Gx + xGx)_K < 4 \dim G_K$, because $xG \cap Gx \neq 0$ and $4 \dim G_K \leqslant \dim F_K$), there is an element $y \in F \setminus G + xG + Gx + xGx$. Let

U_i be the G-subspace of $(F^i)_G = F_G \times \cdots \times F_G$ (i copies), generated by $G^i = G \times \cdots \times G$ and the i elements

$$\begin{aligned} u_1 & (x, 0,\dots, 0), \\ u_2 &= (y, x, 0,\dots, 0), \\ &\vdots \\ u_{i-1} &= (0,\dots, 0, y, x, 0), \text{ and} \\ u_i &= (0,\dots, 0, y, x). \end{aligned}$$

Let $A_i = (U_i, F^i, \iota)$ and, for $k < i$, consider A_k embedded into A_i via

$$(U_k, F^k, \iota) \approx (U_k \times 0^{i-k}, F^k \times 0^{i-k}, \iota) \subseteq (U_i, F^i, \iota).$$

Also, let $A = \bigcup_{i \in \mathbb{N}} A_i$.

Every map $B \to A$ maps B into some A_i, so we may assume that there is given a mapping

$$\alpha\colon B = (G + xG, F, \iota) \to A_i = (U_i, F^i, \iota)$$

which does not factor through A_{i-1}; we want to show that $i \leqslant 1$. The mapping α determines elements $f_1, \dots, f_{i-1} \in F$ with

$$(f_1, \dots, f_{i-1}, 1) \in U_i \quad \text{and} \quad (f_1x, \dots, f_{i-1}x, x) \in U_i.$$

For, let $\alpha(1) = (f_1', \dots, f_i') \in F \times \cdots \times F$. By assumption, $f_i' \neq 0$. Now, α maps $G + xG$ into U_i, so $\alpha(1) = (f_1', \dots, f_i')$ and $\alpha(x) = (f_1'x, \dots, f_i'x)$ both belong to U_i. Since f_i' and $f_i'x$ are right G-independent, and belong to $G + xG$, they generate $G + xG$, so there are elements $g_1, g_2 \in G$ with $f_i'g_1 + f_i'xg_2 = 1$ and $g_3, g_4 \in G$ with $f_i'g_3 + f_i'xg_4 = x$. Now, the last component of $\alpha(1)(g_1 + xg_2)$ is 1, so let

$$\alpha(g_1 + xg_2) = \alpha(1)(g_1 + xg_2) = (f_1, \dots, f_{i-1}, 1).$$

The last component of $\alpha(1)(g_3 + xg_4)$ is x, and since this is an F-multiple of $\alpha(1)(g_1 + xg_2)$, we get

$$\alpha(g_3 + xg_4) = \alpha(1)(g_1 + xg_2)x = (f_1x, \dots, f_{i-1}x, x),$$

as required.

Now, assume that $i \geqslant 2$. Then, from $(f_1, \dots, f_{i-1}, 1) \in U_i$, we derive that $f_{i-1} \in G + xG$, and thus $f_{i-1}x \in Gx + xGx$. Similarly, from $(f_1x, \dots, f_{i-1}x, x) \in U_i$, we derive that there are elements $g, g' \in G$ with

$$f_{i-1}x = g + xg' + y,$$

and therefore $y \in G + Gx + xG + xGx$, a contradiction. This shows that α maps B into A_1.

We have shown that $\mathfrak{B}$ has indecomposable objects of arbitrarily large finite dimension and of infinite dimension. Now assume that K is infinite, and let $[G: K] = n$. We want to show that there is an infinite set $E \subseteq F \setminus G$ with $\text{Hom}(B_x, B_{x'}) = 0$ for $x \neq x'$ in E. Consider the affine variety

$$\text{Hom}_G(G_G \times G_G, F_G) \approx K^{2dn}$$

over K. The algebraic group $\text{Aut}(F_F) \times \text{Aut}(G_G \times G_G)$ operates on it via $(\alpha, \beta)\varphi = \alpha\varphi\beta^{-1}$ for $\alpha \in \text{Aut}(F_F)$, $\beta \in \text{Aut}(G_G \times G_G)$ and $\varphi \in \text{Hom}_G(G \times F, F)$. Obviously, the diagonal $K^\times = \{(k, k) \mid k \in K^\times\} \subseteq \text{Aut}(F_F) \times \text{Aut}(G_G \times G_G)$ operates trivially on $\text{Hom}_G(G \times G, F)$, and

$$\text{Dim}(\text{Aut}(F_F) \times \text{Aut}(G_G \times G_G)/K^\times) = dn + 4n - 1.$$

Since

$$dn + 4n - 1 \leqslant 2\,dn - 1 < 2\,dn,$$

it follows that there are infinitely many orbits. But there are even infinitely many orbits containing only monomorphisms. For, if $\varphi \in \text{Hom}_G(G \times G, F)$ is not a monomorphism, then $\dim \text{Im}(\varphi)_G \leqslant 1$. Since there is just one φ with $\text{Im}(\varphi) = 0$, consider φ, φ' with

$$\dim \text{Im}(\varphi)_G = 1 = \dim \text{Im}(\varphi')_G.$$

Let $\text{Im}(\varphi) = fG$ and $\text{Im}(\varphi') = f'G$, where f, f' are nonzero elements of F and let $\alpha \in \text{Aut}(F_F)$ be the left multiplication by $f'f^{-1}$. Then there is obviously an automorphism β of $G_G \times G_G$ such that $\alpha\varphi = \varphi'\beta$, and thus φ and φ' belong to the same orbit. Consequently, we have infinitely many orbits containing only monomorphisms. Since $\text{Aut}(F_F)$ is transitive on the nonzero elements of F, every such orbit contains a map φ with $1 \in \text{Im}(\varphi)$. We may select a set E of elements $x \in F \setminus G$ such that the maps $G_G \times G_G \approx G + xG \to F_G$ belong to different orbits. Now, we want to show that $\text{Hom}(B_x, B_{x'}) = 0$ for $x \neq x'$ of E. Let $\varphi: G_G \times G_G \approx G + xG \hookrightarrow F_G$, and $\varphi': G_G \times G_G \approx G + x'G \hookrightarrow F_G$, and assume there is a nonzero mapping $\alpha: (G + xG, F, \iota) \to (G + x'G, F, \iota)$ in $\mathfrak{B}({}_FM_G)$. Then α is an automorphism of F_F, and therefore also the restriction of α to $G + xG$ is an isomorphism. But then there exists $\beta \in \text{Aut}(G_G \times G_G)$ making the following diagram commutative

$$\begin{array}{ccc} G_G \times G_G \approx G + xG & \hookrightarrow & F \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ G_G \times G_G \approx G + x'G & \hookrightarrow & F \end{array},$$

so $\alpha\varphi = \varphi'\beta$, and φ and φ' belong to the same orbit. This contradiction shows that $\mathrm{Hom}(B_x, B_{x'}) = 0$ for $x \neq x' \in E$. Hence, by Lemma 5.1.(b), it follows that there are infinitely many dimensions for which there are infinitely many nonisomorphic indecomposable objects. This proves the result in the case when $d \geqslant 4$, $d' = 1$.

(ii) Let $d \geqslant d' \geqslant 2$ and let $[G: K] = n, [F: K] = n'$. Hence $dn = d'n'$, and thus $n \leqslant n'$. For $0 \neq x \in M$, consider the object $B = B_x = (xG, F_F, \iota)$, where $\iota: xG \subseteq M_G = F_F \otimes {}_F M_G$ is the inclusion map. Again, we are going to construct in $\mathfrak{A}$ a sequence

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subset A_{i-1} \subseteq A_i \subseteq \cdots \subseteq A = \bigcup_{i \in \mathbb{N}} A_i$$

with A, A_i in $\mathfrak{B}$ and $A_i/A_{i-1} \approx B$ for all $i \in \mathbb{N}$. First, take $y \in M \setminus (xG + Fx)$. This is possible, because $xG + Fx$ is a K-subspace of M_K, and $\dim(xG + Fx)_K \leqslant \dim(xG)_K + \dim(Fx)_K - 1 = n + n' - 1 \leqslant 2n' - 1$, whereas $\dim M_K = (\dim M_G)(\dim G_K) = dn = d'n' \geqslant 2n'$. Let U_i be the G-subspace of $M^i = M_G \times \cdots \times M_G$ (i copies) generated by the elements

$$\begin{aligned} u_1 &= (x, 0, \dots, 0), \\ u_2 &= (y, x, 0, \dots, 0), \\ &\vdots \\ u_{i-1} &= (0, \dots, 0, y, x, 0), \text{ and} \\ u_i &= (0, \dots, 0, y, x). \end{aligned}$$

Let $A_i = (U_i, F^i, \iota)$ with the inclusion $\iota: U_i \subseteq M^i = F_F{}^i \otimes {}_F M_G$. As in the previous case, we embed, for $k < i$, A_k into A_i by $(U_k, F^k, \iota) \approx (U_k \times 0^{i-k}, F^k \times 0^{i-k}, \iota) \subseteq (U_i, F^i, \iota)$, and define $A = \bigcup_{i \in \mathbb{N}} A_i$. Again, we assume that there is a mapping

$$\alpha: B = (xG, F, \iota) \to A_i = (U_i, F^i, \iota),$$

which does not factor through A_{i-1}, and we want to show that $i \leqslant 1$. Now $\alpha: F_F \to F_F{}^i$ is determined by $\alpha(1) = (f_1, \dots, f_i) \in F^i$, and by assumption $f_i \neq 0$. Then $\alpha \otimes 1: M = F_F \otimes {}_F M \to F_F{}^i \otimes {}_F M = M^i$ maps $z \in M$ into $(f_1 z, \dots, f_i z)$. In particular, x is mapped into $(f_1 x, \dots, f_i x)$, and thus $(f_1 x, \dots, f_i x)$ belongs to U_i. Therefore, there are elements $g_k \in G$, $1 \leqslant k \leqslant i$, with

$$(f_1 x, \dots, f_i x) = \sum_{k=1}^{i} u_k g_k .$$

If $i \geqslant 2$, we may compare the last two components

$$f_{i-1}x = xg_{i-1} + yg_i \quad \text{and} \quad f_i x = xg_i .$$

Since $f_i \neq 0$, $g_i \neq 0$, so the last equality can be written as $xg_i^{-1} = f_i^{-1}x$, and multiplying the first equality from the right by g_i^{-1}, we get after substitution

$$y = f_{i-1}f_i^{-1}x - xg_{i-1}g_i^{-1} \in Fx + xG,$$

a contradiction. This shows that $i \leqslant 1$ and hence, in view of Lemma 5.1, all the A_i and also A, are indecomposable. This shows that in $\mathfrak{B}$ are indecomposable objects of arbitrarily large dimension.

Now, assume that K is infinite. Consider the affine variety $M \approx K^{dn}$ over K together with the algebraic group $F^\times \times G^\times$ ($F^\times$ and $G^\times$ are the multiplicative group of F and G, respectively) operating via $(f, g)x = fxg^{-1}$ for $f \in F^\times$, $g \in G^\times$, $x \in M$. Obviously, the diagonal $K^\times = \{(k, k) \mid k \in K^\times\} \subseteq G^\times \times F^\times$ operates trivially. Now,

$$\operatorname{Dim}(G^\times \times F^\times)/K^\times = n + n' - 1 \leqslant 2n - 1 < dn,$$

and consequently, there are infinitely many orbits. Choose an infinite set $E \subseteq M \setminus \{0\}$ of representatives of different orbits. We claim that

$$\operatorname{Hom}(B_x, B_{x'}) = 0 \qquad \text{for} \quad x \neq x' \in E.$$

For, a homomorphism $\alpha\colon B_x \to B_{x'}$ is given by a commutative diagram

$$\begin{array}{ccc} xG \to M_G \approx & F_F \otimes {}_F M_G \\ \Big\downarrow{\scriptstyle\beta} & \Big\downarrow{\scriptstyle\alpha\otimes 1} \\ x'G \to M_G \approx & F_F \otimes {}_F M_G \end{array},$$

where β is the restriction of $\alpha \otimes 1$, and if $\alpha \neq 0$, then $\beta(x) = x'g$ for some $g \in G^\times$, and $(\alpha \otimes 1)x = fx$ for some $f \in F^\times$, so that

$$fx = (\alpha \otimes 1)x = \beta x = x'g,$$

and x, x' belong to the same orbit. As in the previous case, it follows that there are infinitely many dimensions for which there are infinitely many nonisomorphic indecomposable objects in $\mathfrak{B}$.

Cases 1 and 2 together show that for $d' \leqslant d$ (and $d'd \geqslant 4$), $\mathfrak{B}({}_F M_G) \approx \mathfrak{Re}({}_F M_G)$ is of strongly unbounded type. It remains to reduce the case $d' \geqslant d$ to this situation. Denote by M^* the G-F-bimodule

$${}_G M^*{}_F = \operatorname{Hom}_G({}_F M_G, {}_G G_G).$$

Since

$$\mathrm{Hom}_G(M_G, G_G) = \mathrm{Hom}_G\Big(\bigoplus_d G_G, G_G\Big) = \bigoplus_d \mathrm{Hom}_G(G_G, G_G) = \bigoplus_d G,$$

it follows that $\dim M^*_K = \dim M_K = dn$, so $\dim {}_GM^* = d$ and $\dim M^*_F = d'$. If we assume $d' \geqslant d$, then by the previous investigations, $\mathfrak{B}({}_GM^*_F)$ is of strongly unbounded type. We claim that $\mathfrak{B}({}_GM^*_F) \approx \mathfrak{Rm}({}_FM_G)$. Indeed, an object of $\mathfrak{B}({}_GM^*_F)$ has the form (U_F, X_G, ι), where ι is a monomorphism $U_F \to X_G \otimes {}_GM^*_F$. Let $X_G = \bigoplus_I G_G$ for some index set I. Then

$$X_G \otimes {}_GM^*_F \approx \bigoplus_I (G_G \otimes {}_GM^*_F) \approx \bigoplus_I M^*_F = \bigoplus_I \mathrm{Hom}_G({}_FM_G, G_G)$$

$$\approx \mathrm{Hom}_G\Big({}_FM_G, \bigoplus_I G_G\Big) = \mathrm{Hom}_G({}_FM_G, X_G).$$

Of course, we have used here again the fact that M_G is finite dimensional. Thus, denoting by ι' the canonical monomorphism

$$\iota' \colon U_F \hookrightarrow \mathrm{Hom}_G({}_FM_G, X_G),$$

the functor $\mathfrak{B}({}_FM_G) \to \mathfrak{Rm}({}_FM_G)$ mapping (U_F, X_G, ι) into (U_F, X_G, ι') is an equivalence, and also a dimension functor. This completes the proof of Proposition 5.2.

We like to use the same technique also for another type of K-species. Let G be a subfield of F containing K. Let $I = \{1, 2, 3\}$. Define the K-species $\mathcal{Q} = (K_i, {}_iM_j)_{i,j\in I}$ by $K_1 = K_2 = G$, $K_3 = F$, ${}_1M_2 = {}_GG_G$, ${}_2M_3 = {}_GF_F$, and ${}_iM_j = 0$ otherwise. A representation of $\mathcal{Q}$ is given by $(X_G, Y_G, Z_F, \varphi, \psi)$ with $\varphi\colon X_G \to \mathrm{Hom}_G({}_GG_G, Y_G) \approx Y_G$, and $\psi\colon Y_G \to \mathrm{Hom}_F({}_GF_F, Z_F) \approx Z_G$. We denote by $\mathfrak{Rm}(\mathcal{Q})$ the full subcategory of $\mathfrak{R}(\mathcal{Q})$ of all representations (X, Y, Z, φ, ψ) such that φ and ψ are monomorphisms. In case that φ and ψ are inclusions, we denote both φ and ψ by ι. It is easy to see (but we shall not need it) that $\mathfrak{R}(\mathcal{Q})$ is of finite type if and only if $\mathfrak{Rm}(\mathcal{Q})$ is of finite type.

PROPOSITION 5.3. *Let G be a subfield of F containing K with $[F\colon G] = 3$. Let $I = \{1, 2, 3\}$, and $\mathcal{Q} = (K_i, {}_iM_j)_{i,j\in I}$ a K-species with $K_1 = K_2 = G$, $K_3 = F$, ${}_1M_2 = {}_GG_G$, ${}_2M_3 = {}_GF_F$, and ${}_iM_j = 0$ otherwise. Then $\mathfrak{Rm}(\mathcal{Q})$ is of strongly unbounded type.*

Proof. We will work in the abelian category $\mathfrak{R}(\mathcal{Q})$; however, all objects which will be constructed, will belong to $\mathfrak{Rm}(\mathcal{Q})$. Let $x \in F\setminus G$ and consider the object $B = B_x = (G_G, G + xG, F_F, \iota, \iota)$. Obviously, B is indecomposable in $\mathfrak{R}(\mathcal{Q})$. Calculating the K-dimensions of $G + xG + Gx$ and and of F, we conclude from $[F\colon G] = 3$ that there is an element

$y \in F \setminus G + xG + Gx$. Let U_i be again the G-subspace of $F_G{}^i$ generated by $G_G{}^i$ and the i elements

$$\begin{aligned} u_1 &= (x, 0, \ldots, 0), \\ u_2 &= (y, x, 0, \ldots, 0), \\ &\vdots \\ u_{i-1} &= (0, \ldots, 0, y, x, 0), \text{ and} \\ u_i &= (0, \ldots, 0, y, x). \end{aligned}$$

Let $A_i = (G_G{}^i, U_i, F_F{}^i, \iota, \iota)$, and consider for $k < i$, A_k embedded into A_i via the first k components. Also, let $A = \bigcup_{i \in \mathbb{N}} A_i$.

Now, a morphism

$$B = (G_G, G + xG, F_F, \iota, \iota) \to A_i = (G_G{}^i, U_i, F_F{}^i, \iota, \iota)$$

is obviously fully determined by a single mapping $\alpha: F_F \to F_F{}^i$, and we are going to show that every such morphism factors through A_1. Thus, assume that $i > 1$ and that α does not factor through A_{i-1}. We have $\alpha(1) = (g_1, \ldots, g_i)$ for some elements $g_k \in G$, and $g_i \neq 0$. Also, since $\alpha(x) = (g_1x, \ldots, g_1x)$ belongs to U_i, there are elements $g_k', g_k'' \in G$ with

$$(g_1x, \ldots, g_ix) = (g_1', \ldots, g_i') + \sum_{k=1}^{i} u_k g_k''.$$

Comparing the last two components, we derive the equalities

$$g_ix = g_i' + xg_i'' \qquad \text{and} \qquad g_{i-1}x = g_{i-1}' + xg_{i-1}'' + yg_i''.$$

The first equality implies $g_i'' \neq 0$, so

$$x(g_i'')^{-1} = g_i^{-1}g_i'(g_i'')^{-1} + g_i^{-1}x.$$

If we multiply the second equality from the right by $(g_i'')^{-1}$ and substitute the expression for $x(g_i'')^{-1}$, we get

$$y = g_{i-1}g_i^{-1}g_i'(g_i'')^{-1} + g_{i-1}g_i^{-1}x - g_{i-1}'(g_i'')^{-1} - xg_{i-1}''(g_i'')^{-1},$$

in contradiction to the fact that $y \notin G + xG + Gx$. Thus, the morphism α factors through A_1 and we can again apply Lemma 5.1 to B and the subobject sequence

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_i \subseteq A_{i+1} \subseteq \cdots \subseteq A,$$

in the abelian category $\mathfrak{R}(\mathcal{Q})$, and conclude that all the representations A_i and A are indecomposable.

Now, assume that K is infinite and let $[G: K] = n$. We want to construct an infinite set E of elements of $F \setminus G$ with $\mathrm{Hom}(B_x, B_{x'}) = 0$ for $x \neq x'$ in E. Consider the affine variety

$$\mathrm{Hom}_G(G_G \times G_G, F_G) \times \mathrm{Hom}_G(G_G, G_G \times G_G) \approx K^{6n+2n}$$

over K. The group $F^\times \times \mathrm{Aut}(G_G \times G_G) \times G^\times$ operates on the variety via $(f, \alpha, g)(\varphi, \psi) = (f\varphi\alpha^{-1}, \alpha\psi g^{-1})$ with $f \in F^\times$, $\alpha \in \mathrm{Aut}(G_G \times G_G)$, $g \in G^\times$, $\varphi \in \mathrm{Hom}_G(G_G \times G_G, F_G)$, and $\psi \in \mathrm{Hom}(G_G, G_G \times G_G)$. Obviously, the diagonal $K^\times = \{(k, k, k) \mid k \in K^\times\} \subseteq F^\times \times \mathrm{Aut}(G_G \times G_G) \times G^\times$ operates trivially. Since

$$\begin{aligned}\dim(F^\times \times \mathrm{Aut}(G_G \times G_G) \times G^\times/K^\times) &= 3n + 4n + n - 1\\ &= 8n - 1 < 8n,\end{aligned}$$

there are infinitely many orbits.

Now, $F^\times$ operates transitively on the 1-dimensional, subspaces of F_G, as well as on the 2-dimensional subspaces of F_G. For, let $U_G \subseteq F_G$ and $\dim U_G = 2$, then, for $e \in F \setminus G$, we calculate the K-dimension of $U + Ue^{-1}$ and see immediately that $U \cap Ue^{-1} \neq 0$. But if $0 \neq u \in U \cap Ue^{-1}$, then the left multiplication by u on F_G maps $G + eG$ onto U. As a consequence, there are only three orbits of the form $(\varphi, 0)$. Moreover, there are only two orbits of pairs (φ, ψ) such that φ is not a monomorphism and $\psi \neq 0$, namely with $\psi = 0$ and with $\dim(\mathrm{Im}\,\varphi)_G = 1$, respectively. Thus there are infinitely many orbits containing only pairs of monomorphisms, and every such orbit obviously contains a pair of the form (φ, ψ) with $\mathrm{Im}(\varphi\psi) = G \subseteq F$. In this case, $\mathrm{Im}\,\varphi = G + xG$ for some $x \in \mathrm{Im}\,\varphi \setminus G$. Denote by E an infinite set of such x chosen for different orbits. Note that, for every $x \in E$, there is given a pair (φ, ψ) and an isomorphism $\delta_x: G_G \times G_G \to G + xG$ such that the diagram

$$\begin{array}{ccccc} G_G & \xrightarrow{\psi} & G_G \times G_G & \xrightarrow{\varphi} & F_G \\ \| & & \downarrow{\scriptstyle \delta_x} & & \| \\ G_G & \hookrightarrow & G + xG & \hookrightarrow & F_G \end{array}$$

is commutative. Let also $x' \in E$, and (φ', ψ') a corresponding pair. Now assume there is given a map $\alpha: B_x \to B_{x'}$ in $\mathfrak{R}(\mathscr{Q})$. Then we get a commutative diagram

$$\begin{array}{ccccc} G_G & \xrightarrow{\psi} & G_G \times G_G & \xrightarrow{\varphi} & F_G \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta_{x'}^{-1}\beta\delta_x} & & \downarrow{\scriptstyle \alpha} \\ G_G & \xrightarrow{\psi'} & G_G \times G_G & \xrightarrow{\varphi'} & F_G \end{array}$$

where β and γ are the appropriate restrictions of α. If $\alpha \neq 0$, then α is an automorphism of F_F. Hence, $\alpha \in F^\times$, and then $\delta_{x'}^{-1}\beta\delta_x \in \mathrm{Aut}(G_G \times G_G)$ and $\gamma \in G^\times$. The diagram above shows that $(\alpha, \delta_{x'}^{-1}\beta\delta_x, \gamma)(\varphi', \psi') = (\varphi, \psi)$ and then (φ', ψ') and (φ, ψ) belong to the same orbit. This means that for $x \neq x'$ from E, we have $\mathrm{Hom}(B_x, B_{x'}) = 0$. Now, it follows again from Lemma 5.1(b) that there are infinitely many dimensions for which there are infinitely many nonisomorphic indecomposable objects in $\mathfrak{Rm}(\mathscr{Q})$. Thus $\mathfrak{Rm}(\mathscr{Q})$ is of strongly unbounded type, as required.

6. *Abelian Subcategories of* $\mathfrak{S}(\mathscr{S})$

In order to show that certain K-structures $\mathscr{S}$ are not of finite type, we are going to investigate the category $\mathfrak{S}(\mathscr{S})$ of all $\mathscr{S}$-spaces. We will determine full subcategories $\mathfrak{A}$ of $\mathfrak{S}(\mathscr{S})$ which are abelian and which will be shown to be equivalent to categories $\mathfrak{M}_{\mathscr{R}}$ of modules over finite dimensional K-algebras $\mathscr{R}$. This will be done for several particular K-structures in the next section. In the present section, we develop certain auxiliary techniques.

We note that, in $\mathfrak{S}(\mathscr{S})$, a kernel is given by a monomorphism $\alpha: (W, W_i) \to (W', W_i')$ such that $\alpha(W_i) = \alpha(W) \cap W_i'$ for all $i \in \mathscr{S}$, whereas a cokernel is given by an epimorphism $\alpha: (W, W_i) \to (W', W_i')$ with $\alpha(W_i) = W_i'$. Obviously, $\mathfrak{S}(\mathscr{S})$ is usually not an abelian category, and we want to describe a construction for exact subcategories which are abelian. We will assume that $0, \infty \notin \mathscr{S}$, and we form $\mathscr{S}^+ = \mathscr{S} \sqcup \{0\} \sqcup \{\infty\}$. Also, if (W, W_i) is an $\mathscr{S}$-space, let $W_0 = 0$, $W_\infty = W$. By an *equation for* $\mathscr{S}$ we mean either a condition of the form

$$W_i \cap W_j = W_k, \qquad W_i + W_j = W_l, \tag{*}$$

with $i, j, k, l \in \mathscr{S}^+$, or, if $[F: F_i] = 2$ for some $i \in \mathscr{S}$, a condition of the form

$$\underline{W}_i = W_k, \qquad \overline{W}_i = W_l, \tag{**}$$

with $k, l \in \mathscr{S}^+$. Now assume there is given a system $\mathscr{E}$ of equations for $\mathscr{S}$. We define by induction which elements of $\mathscr{S}^+$ will be said to be *determined by* $\mathscr{E}$. First, 0 and ∞ are always determined by $\mathscr{E}$. If k and l are determined by $\mathscr{E}$, and (*) is in $\mathscr{E}$, then also i, j are determined by $\mathscr{E}$. If k, l are determined by $\mathscr{E}$, and (**) is in $\mathscr{E}$, then i is determined by $\mathscr{E}$. The system $\mathscr{E}$ determines $\mathscr{S}$ if it determines all $i \in \mathscr{S}$.

PROPOSITION 6.1. *Let $\mathscr{S}$ be a K-structure, and $\mathscr{E}$ a system of equations for $\mathscr{S}$. Let $\mathfrak{A}$ be the full subcategory of $\mathfrak{S}(\mathscr{S})$ consisting of all objects of $\mathfrak{S}(\mathscr{S})$ which satisfy all equations of $\mathscr{E}$. If $\mathscr{E}$ determines $\mathscr{S}$, then $\mathfrak{A}$ is an abelian, exact subcategory, which is closed under direct sums.*

Proof. Let $\alpha\colon (W, W_i) \to (W', W_i')$ be a morphism in $\mathfrak{A}$. Let $A = \{w \in W \mid \alpha(w) = 0\}$. We want to show that the $\mathscr{S}$-spaces

$$\mathbf{A} = (A, A \cap W_i), \qquad \mathbf{B} = (\alpha(W), \alpha(W_i))$$

and

$$\mathbf{C} = (W'/\alpha(W), W_i' + \alpha(W)/\alpha(W))$$

belong to $\mathfrak{A}$ and that

$$\alpha(W_i) = \alpha(W) \cap W_i' \qquad \text{for all} \quad i \in \mathscr{S}.$$

From here, it will follow that $\mathbf{B}$ is both image and coimage of α, that $\mathbf{A}$ is the kernel of α, and that $\mathbf{C}$ is the cokernel of α. We proceed by induction.

From now on, assume that we have proved for some $k, l \in \mathscr{S}^+$ that

$$\alpha(W_k) = \alpha(W) \cap W_k', \qquad \alpha(W_l) = \alpha(W) \cap W_l';$$

for example, this is always true for $k = 0$, $l = \infty$.

Assume that the equation (*) belongs to $\mathscr{E}$. Then

$$\alpha(W_k) \subseteq \alpha(W_i) \cap \alpha(W_j) \subseteq W_i' \cap W_j' \cap \alpha(W) = W_k' \cap \alpha(W) = \alpha(W_k),$$

and therefore

$$\alpha(W_i) \cap \alpha(W_j) = \alpha(W_k).$$

Since, obviously,

$$\alpha(W_i) + \alpha(W_j) = \alpha(W_i + W_j) = \alpha(W_l),$$

we conclude that $\mathbf{B} = (\alpha(W), \alpha(W_i))$ satisfies (*). Also, we have

$$\alpha(W_k) \subseteq W_i' \cap \alpha(W_j) \subseteq W_i' \cap W_j' \cap \alpha(W) = W_k' \cap \alpha(W) = \alpha(W_k);$$

thus, $\alpha(W_k) = W_i' \cap \alpha(W_j)$, and, as a consequence,

$$\begin{aligned}\alpha(W) \cap W_i' &= \alpha(W) \cap W_l' \cap W_i' = \alpha(W_l) \cap W_i' \\ &= [\alpha(W_i) + \alpha(W_j)] \cap W_i' = \alpha(W_i) + [\alpha(W_j) \cap W_i'] \\ &= \alpha(W_i) + \alpha(W_k) = \alpha(W_i).\end{aligned}$$

Of course, we have similarly

$$\alpha(W) \cap W_j' = \alpha(W_j).$$

Now, we show that $\mathbf{A} = (A, A \cap W_i)$ satisfies (*). Trivially,

$$(A \cap W_i) \cap (A \cap W_j) = A \cap (W_i \cap W_j) = A \cap W_k.$$

Thus, let $x \in A \cap W_l$, so $x = w_i + w_j$ with $w_i \in W_i$, $w_j \in W_j$. Then, $\alpha(w_i) + \alpha(w_j) = \alpha(x) = 0$ implies that $\alpha(w_i) \in \alpha(W_i) \cap \alpha(W_j) = \alpha(W_k)$, and therefore $w_i = a + w_k$ for some $a \in A$, $w_k \in W_k$. Note that $a \in A \cap W_i$. From

$$x = w_i + w_j = a + (w_k + w_j)$$

we get that $w_k + w_j \in A$, and thus

$$(A \cap W_i) + (A \cap W_j) = A \cap W_l .$$

It remains to prove (*) for $\mathbf{C} = (W'/\alpha(W), W_i' + \alpha(W)/\alpha(W))$. If $w_i' \in W_i'$, $w_j' \in W_j'$ and $w \in W$ with $w_i' = w_j' + \alpha(w)$, then $\alpha(w) \in \alpha(W) \cap W_l' = \alpha(W_l)$, and therefore

$$\begin{aligned} W_i' \cap (W_j' + \alpha(W)) &= W_i' \cap (W_j' + \alpha(W_l)) \\ &= W_i' \cap (W_j' + \alpha(W_i) + \alpha(W_j)) \\ &= W_i' \cap (W_j' + \alpha(W_i)) = (W_i' \cap W_j') + \alpha(W_i) \\ &= W_k' + \alpha(W_i); \end{aligned}$$

as an immediate consequence, we get

$$(W_i' + \alpha(W)) \cap (W_j' + \alpha(W)) = W_k' + \alpha(W).$$

On the other hand, trivially

$$(W_i' + \alpha(W)) + (W_j' + \alpha(W)) = W_l' + \alpha(W),$$

and thus $\mathbf{C}$ satisfies (*), as well.

Now, assume that the equation (**) belongs to $\mathscr{E}$, that is

$$\underline{W}_i = W_k, \qquad \overline{W}_i = W_l \qquad \text{and} \qquad \underline{W}_i' = W_k', \qquad \overline{W}_i' = W_l'.$$

Since α is F-linear, we have $\alpha(\underline{W}_i) \subseteq \underline{\alpha(W_i)}$ and $\alpha(\overline{W}_i) = \overline{\alpha(W_i)}$, and thus

$$\alpha(W_k) = \alpha(\underline{W}_i) \subseteq \underline{\alpha(W_i)} \subseteq \underline{W_i'} \cap \alpha(W) = W_k' \cap \alpha(W) = \alpha(W_k),$$

and

$$\overline{\alpha(W_i)} = \alpha(\overline{W}_i) = \alpha(W_l).$$

Hence, $\mathbf{B} = (\alpha(W), \alpha(W_i))$ satisfies (**). Furthermore,

$$\alpha(W_i) = \alpha(W) \cap W_i',$$

because of

$$\alpha(W_k) \subseteq \alpha(W_i) \subseteq \alpha(W) \cap W_i' \subseteq \alpha(W) \cap W_l' = \alpha(W_l),$$

$$\overline{\alpha(W_i)} = \alpha(W_l)$$

and

$$\underline{\alpha(W)\cap W_i'} = \alpha(W)\cap \underline{W_i'} = \alpha(W)\cap W_k' = \alpha(W_k).$$

Next, we show that $\mathbf{A} = (A, A\cap W_i)$ satisfies (**).
Obviously,

$$\underline{A\cap W_i} = A\cap \underline{W_i} = A\cap W_k .$$

Also, $\overline{A\cap W_i} \subseteq A\cap W_l$. Thus, let $x\in A\cap W_l$, so $x = u + u'f$, where $u, u'\in W_i$ and $F = G + Gf$. We have

$$0 = \alpha(x) = \alpha(u) + \alpha(u')f,$$

so both $\alpha(u')$ and $\alpha(u')f = \alpha(-u)$ belong to W_i'. Therefore,

$$\alpha(u')\in\alpha(W)\cap \underline{W_i'} = \alpha(W)\cap W_k' = \alpha(W_k),$$

and also $\alpha(u) = -\alpha(u')f\in\alpha(W_k)$. As a consequence, we find elements a, $a'\in A$ and w, $w'\in W_k$ with $u = a + w$ and $u' = a' + w'$. Note that a, $a'\in W_i$. Then

$$x = u + u'f = (a + w + w'f) + a'f,$$

with $a'\in A\cap W_i$, and $a + w + w'f\in W_i$. Since x and $a'f$ belong to A, the same is true for $a + w + w'f$, so $x\in\overline{A\cap W_i}$. Thus, $\mathbf{A}$ satisfies (**).

Finally, consider $\mathbf{C} = (W'/\alpha(W), W_i' + \alpha(W)/\alpha(W))$. Trivially,

$$\overline{W_i' + \alpha(W)} = \overline{W_i'} + \alpha(W) = W_l' + \alpha(W).$$

Also,

$$W_k' + \alpha(W) = \underline{W_i'} + \alpha(W)\subseteq \underline{W_i' + \alpha(W)}.$$

Thus, let $y\in W_i'$ with $yf\in W_i' + \alpha(W)$. There are elements $z\in W_i'$, $w\in W$ with $yf = z + \alpha(w)$. Now,

$$\alpha(w)\in\alpha(W)\cap\overline{W}_i = \alpha(W)\cap W_l = \alpha(W_l),$$

and therefore we may assume $w\in W_l = \overline{W}_i$. Thus, $w = u + u'f$ for some u, $u'\in W_i$. Both $y - \alpha(u')\in W_i'$ and

$$[y - \alpha(u')]f = z + \alpha(w) + \alpha(u) - \alpha(w) = z + \alpha(u)\in W_i'.$$

Therefore, $y - \alpha(u')\in \underline{W}_k'$, and $y\in W_k' + \alpha(W)$. This shows that

$$\underline{W_i' + \alpha(W)}\subseteq W_k' + \alpha(W),$$

and, consequently, $\mathbf{C}$ satisfies (**).

We have shown that $\mathfrak{A}$ is an exact subcategory of $\mathfrak{S}(\mathscr{S})$ and that for every map α in $\mathfrak{A}$, its image and its coimage coincide. It is obvious that $\mathfrak{A}$ is closed under arbitrary direct sums, thus, in particular, we see also that $\mathfrak{A}$ is abelian. This completes the proof.

Observe that an abelian, full exact subcategory $\mathfrak{A}$ of $\mathfrak{S}(\mathscr{S})$ which is closed under direct sums, is a Grothendieck category. For, by assumption, $\mathfrak{A}$ has arbitrary colimits and these are constructed as in $\mathfrak{S}(\mathscr{S})$. We will use only such systems of equations which force the corresponding subcategory $\mathfrak{A}$ to have a progenerator, that is an object which is small, projective and a generator in $\mathfrak{A}$. Now, finite dimensional objects A of $\mathfrak{A}$ are obviously small (that means, the functor $\mathrm{Hom}(A, -)$ commutes with direct sums), and a projective object A of a Grothendieck category $\mathfrak{A}$ is a generator if and only if every simple object of $\mathfrak{A}$ is of the form A/A' with a subobject A' of A. Thus, we need a criterion that a given object P of $\mathfrak{A}$ is projective.

In the applications, $\mathfrak{A}$ will usually have precisely two simple objects S and T such that every object A in $\mathfrak{A}$ has a subobject $A' \approx \bigoplus T$ with $A/A' \approx \bigoplus S$, and, moreover, $\bigoplus_J S$ will be injective for every index set J. In this situation, we have the following criterion for projectivity.

LEMMA 6.2. *Let $\mathscr{S}$ be a K-structure. Let $\mathfrak{A}$ be a full exact subcategory of $\mathfrak{S}(\mathscr{S})$ which is abelian, and closed under direct sums. Let $\mathbf{T}$ be an object in $\mathfrak{A}$ such that every object $\mathbf{A}$ in $\mathfrak{A}$ has a subobject $\mathbf{A}'$ satisfying the following two properties: $\mathbf{A}'$ is isomorphic to a direct sum of copies of $\mathbf{T}$ and $\mathbf{A}/\mathbf{A}'$ is injective. If $\mathbf{P}$ is an object of $\mathfrak{A}$ which is finite dimensional over K, and if $\mathrm{Ext}_{\mathfrak{A}}(\mathbf{P}, \mathbf{T}) = 0$, then $\mathbf{P}$ is projective.*

Proof. First, we show that $\mathrm{Ext}_{\mathfrak{A}}(\mathbf{P}, \bigoplus_J \mathbf{T}) = 0$ for every index set J. Let $\alpha: \mathbf{E} \to \mathbf{P}$ be an epimorphism in $\mathfrak{A}$ with kernel $\mathbf{A} = \ker(\alpha) = \bigoplus_{j\in J} \mathbf{T}(j)$, where $\mathbf{T}(j) \approx \mathbf{T}$ for all $j \in J$. Write $\mathbf{E} = (E, E_i)$, and similarly $\mathbf{P} = (P, P_i)$, $\mathbf{T} = (T, T_i)$ and $\mathbf{A} = (A, A_i)$, $i \in \mathscr{S}$. Let C be a K-subspace of E with $A \oplus C = E$.

Since $\mathfrak{A}$ is an exact subcategory of $\mathfrak{S}(\mathscr{S})$, we have $A_i = A \cap E_i$ and $P_i = \alpha(E_i)$, so

$$E_i/A_i = E_i/A \cap E_i \approx E_i + A/E_i = P_i$$

is a finite dimensional vector space over K. Let C_i be a K-subspace of E_i with $A_i \oplus C_i = E_i$. Now, C_i is a finite dimensional subspace of

$$E = A \oplus C = \Big(\bigoplus_{j\in J} T(j)\Big) \oplus C,$$

and is therefore contained in a finite direct sum. This is true for all $i \in \mathscr{S}$, and since $\mathscr{S}$ is finite, there is a finite subset $J' \subseteq J$ with $C_i \subseteq (\oplus_{j\in J'} T(j)) \oplus C$ for all $i \in \mathscr{S}$. Let $J'' = J \setminus J'$, and $\mathbf{A}' = \oplus_{j\in J'} \mathbf{T}(j)$, $\mathbf{A}'' = \oplus_{j\in J''} \mathbf{T}(j)$, and note that both $\mathbf{A}'$ and $\mathbf{A}''$ belong to $\mathfrak{A}$. Also, we construct $\mathbf{E}'$ as follows: the total space E' is given by $E' = A' \oplus C$, and $E_i' = E' \cap E_i$.

We have a direct decomposition $\mathbf{E} = \mathbf{A}'' \oplus \mathbf{E}'$ of $\mathscr{S}$-spaces. For,

$$\begin{aligned} A_i'' + E_i' &= A_i'' + (E' \cap E_i) = A_i'' + [E' \cap (A_i + C_i)] \\ &= A_i'' + (E' \cap A_i) + C_i = A_i'' + A_i' + C_i = E_i . \end{aligned}$$

As a consequence, $\mathbf{E}'$ belongs to $\mathfrak{A}$. Also, if we denote the inclusion $\mathbf{E}' \to \mathbf{E}$ by μ, we have an exact sequence

$$0 \longrightarrow \bigoplus_{j\in J'} \mathbf{T}(j) \longrightarrow \mathbf{E}' \xrightarrow{\alpha\mu} \mathbf{P} \longrightarrow 0$$

in $\mathfrak{A}$. Now, J' is finite, so $\mathrm{Ext}_{\mathfrak{A}}(\mathbf{P}, \oplus_{J'} \mathbf{T}) = \oplus_{J'} \mathrm{Ext}_{\mathfrak{A}}(\mathbf{P}, \mathbf{T}) = 0$. Hence, there is $\beta\colon \mathbf{P} \to \mathbf{E}'$ with $(\alpha\mu)\beta = 1_{\mathbf{P}}$. But then also $\alpha(\mu\beta) = 1_{\mathbf{P}}$.

If $\mathbf{B}$ is an arbitrary object of $\mathfrak{A}$, then there is a subobject $\mathbf{B}'$ of $\mathbf{B}$ with $\mathbf{B}' \approx \oplus \mathbf{T}$ and injective quotient $\mathbf{B}/\mathbf{B}'$. This gives rise to an exact sequence

$$\mathrm{Ext}_{\mathfrak{A}}(\mathbf{P}, \mathbf{B}') \to \mathrm{Ext}_{\mathfrak{A}}(\mathbf{P}, \mathbf{B}) \to \mathrm{Ext}_{\mathfrak{A}}(\mathbf{P}, \mathbf{B}/\mathbf{B}').$$

As we have seen above, $\mathrm{Ext}_{\mathfrak{A}}(\mathbf{P}, \mathbf{B}') = 0$, and, if $\mathbf{B}/\mathbf{B}'$ is injective, also $\mathrm{Ext}_{\mathfrak{A}}(\mathbf{P}, \mathbf{B}/\mathbf{B}') = 0$. Thus $\mathrm{Ext}_{\mathfrak{A}}(\mathbf{P}, \mathbf{B}) = 0$ and $\mathbf{P}$ is projective.

If a Grothendieck category $\mathfrak{A}$ possesses a progenerator A, then $\mathfrak{A}$ is equivalent to the category $\mathfrak{M}_{\mathscr{R}}$ of all right $\mathscr{R}$-modules, where $\mathscr{R} = \mathrm{Hom}(A, A)$. In order to determine the category $\mathfrak{M}_{\mathscr{R}}$ in several particular cases, we will use in the next section the following

LEMMA 6.3. *Let $\mathscr{R}$ be a finite dimensional basic K-algebra with radical $\mathscr{N}$. Let e_1 and e_2 be idempotents in $\mathscr{R}$ such that $e_1\mathscr{R}e_1$ and $e_2\mathscr{R}e_2$ are fields satisfying $e_1\mathscr{R}e_2 \subseteq \mathscr{N}$ and $e_1\mathscr{R}e_2\mathscr{N} = 0$. Then there is a full embedding of $\mathfrak{Re}({}_{e_1\mathscr{R}e_1}(e_1\mathscr{R}e_2)_{e_2\mathscr{R}e_2})$ into $\mathfrak{M}_{\mathscr{R}}$, which is a dimension functor.*

Proof. Let $F = e_1\mathscr{R}e_1$, $G = e_2\mathscr{R}e_2$ and $M = e_1\mathscr{R}e_2$. If (X, Y, φ) is an object in $\mathfrak{Re}({}_FM_G)$, then φ is just an epimorphism

$$\varphi\colon \bigoplus_I M_G \approx X_F \otimes {}_FM_G \to Y_G,$$

where I is an index set with $X_F = \oplus_I F_F$, and φ is determined by its kernel

$$\ker \varphi \subseteq \bigoplus_I M_G = \bigoplus_I (e_1\mathscr{R}e_2)_{e_2\mathscr{R}e_2}.$$

We may consider $\ker \varphi$ as a subset of $\oplus_I e_1\mathscr{R}$, and we claim that it is, in fact, a right $\mathscr{R}$-submodule of $\oplus_I e_1\mathscr{R}$. Indeed, since $\mathscr{R}$ is a basic algebra, $e_2\mathscr{R}(1 - e_2) \subseteq \mathscr{N}$, and thus,

$$(\ker \varphi)\mathscr{R} = (\ker \varphi)\, e_2\mathscr{R} = (\ker \varphi)\, e_2\mathscr{R}e_2 + (\ker \varphi)\, e_2\mathscr{R}(1 - e_2),$$

where the first equality comes from the inclusion $\ker \varphi \subseteq e_1\mathscr{R}e_2$. The same inclusion together with the assumption $e_1\mathscr{R}e_2\mathscr{N} = 0$ shows that the above equality reduces to

$$(\ker \varphi)\mathscr{R} = (\ker \varphi)\, e_2\mathscr{R}e_2 = \ker \varphi,$$

because φ is $e_2\mathscr{R}e_2$-linear. Thus, we may form the right R-module $V_{\mathscr{R}} = (\oplus_I e_1\mathscr{R})/\ker \varphi$. Moreover, $\ker \varphi \subseteq \oplus_I e_1\mathscr{R}e_2 \subseteq \oplus_I e_1\mathscr{N}$, and therefore the canonical epimorphism $\oplus_I e_1\mathscr{R} \to V_{\mathscr{R}}$ is a projective cover.

Now, assume there is another object (X', Y', φ') in $\mathfrak{Re}({}_FM_G)$, with $X' = \oplus_{I'} F_F$, and an epimorphism

$$\varphi'\colon \bigoplus_{I'} M_G \approx X_F' \otimes {}_FM_G \to Y_G';$$

let $V_{\mathscr{R}}' = (\oplus_{I'} e_1\mathscr{R})/\ker \varphi'$. Given a mapping

$$(\alpha, \beta)\colon (X, Y, \varphi) \to (X', Y', \varphi')$$

in $\mathfrak{Re}({}_FM_G)$,

$$\alpha\colon \bigoplus_I F_F = X_F \to X_F' = \bigoplus_{I'} F_F$$

is a mapping such that

$$\alpha \otimes 1_M\colon \bigoplus_I M_F \approx \Big(\bigoplus_I F_F\Big) \otimes {}_FM_G \to \Big(\bigoplus_{I'} F_F\Big) \otimes {}_FM_G \approx \bigoplus_{I'} M_F$$

maps $\ker \varphi$ into $\ker \varphi'$. Then

$$\alpha \otimes 1_{e_1\mathscr{R}}\colon \bigoplus_I e_1\mathscr{R} \to \bigoplus_{I'} e_1\mathscr{R}$$

maps again $\ker \varphi$ into $\ker \varphi'$, so we get an induced mapping

$$\bar{\alpha}\colon V_{\mathscr{R}} = \Big(\bigoplus_I e_1R\Big)\Big/\ker \varphi \to V_{\mathscr{R}}' = \Big(\bigoplus_{I'} e_1R\Big)\Big/\ker \varphi'.$$

Conversely, given a mapping $\gamma\colon V_{\mathscr{R}} \to V_{\mathscr{R}}'$, we may lift it to the projective cover, and get a mapping $\tilde{\gamma}\colon \oplus_I e_1\mathscr{R} \to \oplus_{I'} e_1\mathscr{R}$ such that $\tilde{\gamma}(\ker \varphi) \subseteq \ker \varphi'$. Since $\operatorname{Hom}_{\mathscr{R}}(e_1\mathscr{R}, e_1\mathscr{R}) = e_1\mathscr{R}e_1 = F$, every mapping from $\oplus_I e_1\mathscr{R}$ into $\oplus_{I'} e_1\mathscr{R}$ is of the form $\alpha \otimes 1_{e_1\mathscr{R}}$ with $\alpha\colon \oplus_I F_F \to \oplus_{I'} F_F$. Thus, $\tilde{\gamma} = \alpha \otimes 1_{e_1\mathscr{R}}$

and $\bar{\alpha} = \gamma$. But α is even uniquely determined, since $F = e_1\mathscr{R}e_1$ is a field. Also, α defines in a unique way β such that (α, β) is a mapping in $\mathfrak{Re}({}_FM_G)$.

This shows that the factor $\mathfrak{Re}({}_FM_G) \to \mathfrak{M}_{\mathscr{R}}$ which maps $(\oplus_I F_F, Y_G, \varphi)$ onto $(\oplus_I e_1\mathscr{R})/\ker\varphi$ is a full embedding. It is also a dimension functor. For, if $\dim {}_FM = k$, $[F: K] = l$ and $\dim(e_1\mathscr{R})_K = m$, we get $\dim Y_K \leqslant \dim(\oplus_I F_F \otimes {}_FM)_K = k \dim(\oplus_I F)_K$ in view of the fact that Y_G is an epimorphic image of $\oplus_I F_F \otimes {}_FM_G$. Therefore,

$$\frac{1}{k+1}\left(\dim\left(\bigoplus_I F\right)_K + \dim Y_K\right) \leqslant \dim\left(\bigoplus_I F\right)_K = \dim\left[\bigoplus_I (e_1\mathscr{R}/e_1\mathscr{N})\right]_K$$

$$\leqslant \dim\left[\left(\bigoplus_I e_1\mathscr{R}\right)\Big/\ker\varphi\right]_K.$$

On the other hand,

$$\dim\left[\left(\bigoplus_I e_1R\right)\Big/\ker\varphi\right]_K \leqslant \dim\left(\bigoplus_I e_1\mathscr{R}\right)_K = \frac{m}{l}\dim\left(\bigoplus_I F\right)_K$$

$$\leqslant \frac{m}{l}\left(\dim\left(\bigoplus_I F\right)_K + \dim Y_K\right).$$

Thus $1/(k+1)$ and m/l are real numbers with

$$\frac{1}{k+1}\dim\left(\bigoplus_I F_F, Y_G, \varphi\right) \leqslant \dim\left(\bigoplus_I e_1\mathscr{R}\right)\Big/\ker\varphi$$

$$\leqslant \frac{m}{l}\dim\left(\bigoplus_I F_F, Y_G, \varphi\right).$$

Finally, we like to present a simple criterion which will be used in the next section to decide whether certain objects of an abelian category are indecomposable. Recall that in an abelian category two composition series of a given object have the same length, called the *length* of the object. Simple objects are just those of length 1, and indecomposable objects of length 2 are serial, that is they have just one composition series.

LEMMA 6.4. *Let $\mathfrak{A}$ be an abelian category, and $A \in \mathfrak{A}$ an object of length* 3. *If A has two simple subobjects T_1 and T_2 such that A/T_1 and A/T_2 are non-isomorphic indecomposable objects, then A is indecomposable.*

Proof. Let $A = X \oplus Y$, and assume X is of length 2, and Y of length 1. Then, for $i = 1, 2$,

$$Y \hookrightarrow X \oplus Y = A \twoheadrightarrow A/T_i$$

maps Y into the socle, and therefore the canonical morphisms

$$X \hookrightarrow X \oplus Y = A \twoheadrightarrow A/T_i$$

cannot map X into the socle. Consequently, we get $X \approx A/T_1$ and $X \approx A/T_2$, a contradiction.

7. *K-Structures of Infinite Type*

In the preceeding section, we have prepared certain auxiliary results which we now intend to apply, and show that the following K-structures are of strongly unbounded type: (1) $\mathscr{S} = \mathscr{I}_1(G) \sqcup \mathscr{I}_1(F) \sqcup \mathscr{I}_1(F)$ with $[F: G] = 2$, (2) $\mathscr{S} = \mathscr{I}_2(G) \sqcup \mathscr{I}_2(F)$ with $[F: G] = 2$, (3) $\mathscr{S} = \mathscr{I}_3(G) \sqcup \mathscr{I}_1(F)$ with $[F: G] = 2$, and (4) $\mathscr{S} = \mathscr{I}_1(G) \sqcup \mathscr{I}_1(F)$ with $[F: G] = 3$.

PROPOSITION 7.1. *Let* $[F: G] = 2$. *The K-structure* $\mathscr{S} = \mathscr{I}_1(G) \sqcup \mathscr{I}_1(F) \sqcup \mathscr{I}_1(F)$ *is of strongly unbounded type.*

Proof. Let $\mathfrak{A}$ be the full subcategory of $\mathfrak{S}(\mathscr{S})$ of all objects (W, U, V_1, V_2) with

$$U \oplus V_1 = W \qquad \text{and} \qquad V_1 \oplus V_2 = W.$$

This system of equations determines $\mathscr{S}$, so by Proposition 6.1, $\mathfrak{A}$ is an abelian, full exact subcategory of $\mathfrak{S}(\mathscr{S})$. We want to determine the simple objects of $\mathfrak{A}$. Let (W, U, V_1, V_2) be a simple object in $\mathfrak{A}$. If $V_1 \neq 0$, take $0 \neq v \in V_1$, and observe that the subobject $(vF, 0, vF, 0)$ belongs to $\mathfrak{A}$; therefore, $(W, U, V_1, V_2) \approx (F, 0, F, 0)$. We will denote this object by $\mathbf{T} = (F, 0, F, 0)$. Now, assume that $V_1 = 0$; thus, $(W, U, V_1, V_2) = (W, W, 0, W)$ is a direct sum of copies of $\mathbf{S} = (F, F, 0, F)$. This shows that $\mathbf{S}$ and $\mathbf{T}$ are the only simple objects in $\mathfrak{A}$. Also, an arbitrary object (W, U, V_1, V_2) has $(V_1, 0, V_1, 0)$ as a subobject, and $(V_1, 0, V_1, 0)$ is a direct sum of copies of $\mathbf{T}$, and, furthermore,

$$(W, U, V_1, V_2)/(V_1, 0, V_1, 0) \approx (W/V_1, W/V_1, 0, W/V_1)$$

is a direct sum of copies of $\mathbf{S}$. We claim that every direct sum of copies of $\mathbf{S}$ is injective. Indeed, consider an inclusion

$$(W', W', 0, W') \hookrightarrow (W, U, V_1, V_2)$$

in $\mathfrak{A}$. Here, W' is an F-subspace of W with $W' \cap V_1 = 0$. Let C be an F-subspace of W with $V_1 \subseteq C$ and $W' \oplus C = W$. Then

$$(W, U, V_1, V_2) = (W', W', 0, W') \oplus (C, C \cap U, V_1, C \cap V_2),$$

as required. Also, $\mathbf{T}$ is projective. By Lemma 6.2, it is enough to show that

$\mathrm{Ext}_{\mathfrak{A}}(\mathbf{T}, \mathbf{T}) = 0$. Let (W, U, V_1, V_2) be an extension of $\mathbf{T}$ by $\mathbf{T}$. Then $U = V_2 = 0$, and thus $V_1 = W$, which means that the extension splits.

Now, let $F = G + fG, f \in F \setminus G$, and consider the $\mathscr{S}$-space

$$\mathbf{P} = (F \times F \times F; (0, 1, 1)G + (1, f, f)G, F \times F \times 0, 0 \times 0 \times F),$$

which obviously belongs to $\mathfrak{A}$. We claim that $\mathbf{P}$ is projective. Using Lemma 6.2 again, assume there is an exact sequence

$$0 \longrightarrow \mathbf{T} \longrightarrow (W, U, V_1, V_2) \xrightarrow{\alpha} \mathbf{P} \longrightarrow 0.$$

Let $C = \bar{U} + V_2$. As an F-subspace, C is generated by three elements, and since

$$\alpha(C) = \alpha(\bar{U} + V_2) = \overline{\alpha(U)} + \alpha(V_2) = (0, 1, 1)F + (1, f, f)F + (0, 0, 1)F$$
$$= F \times F \times F$$

is three-dimensional over F, we conclude that $\dim C_F = 3$ and that $\ker \alpha \cap C = 0$. Now, also $\dim(V_1)_F = 3$, so since both C and V_1 are subspaces of W_F, and $\dim W_F = 4$, we see that $\dim(C \cap V_1)_F = 2$. Thus, we have a direct decomposition

$$(W, U, V_1, V_2) = \mathbf{T} \oplus (C, U, C \cap V_1, V_2)$$

and α splits.

Also, $\mathbf{P}$ has $\mathbf{T}_1 = (F \times 0 \times 0, 0, F \times 0 \times 0, 0)$ and $\mathbf{T}_2 = (0 \times F \times 0, 0, 0 \times F \times 0, 0)$ as subobjects, and

$$\mathbf{P}/\mathbf{T}_1 \approx (F \times F, (1, 1)G + (f, f)G, F \times 0, 0 \times F)$$

has the property that $\dim(U + V_2/V_2)_G = 2$, whereas for

$$\mathbf{P}/\mathbf{T}_2 \approx (F \times F, (0, 1)G + (1, f)G, F \times 0, 0 \times F),$$

$\dim(U + V_2/V_2)_G = 1$. Since both $\mathbf{P}/\mathbf{T}_1$ and $\mathbf{P}/\mathbf{T}_2$ are obviously indecomposable (a proper decomposition would have to be $F \times 0 \oplus 0 \times F$ in order to be compatible with V_1 and V_2, but this is not compatible with U), we conclude, in view of Lemma 6.4, that $\mathbf{P}$ is indecomposable.

Now, $\mathbf{T}_1 \oplus \mathbf{T}_2$, which is isomorphic to $\mathbf{T} \oplus \mathbf{T}$, is the socle of $\mathbf{P}$. Also, $\mathbf{P}/\mathbf{T}_1 \oplus \mathbf{T}_2 \approx \mathbf{S}$. Hence, since $\mathbf{P}$ is indecomposable and projective and since $\mathbf{S}$ and $\mathbf{T}$ are nonisomorphic and therefore $\mathrm{Hom}(\mathbf{S}, \mathbf{T}) = 0$, we have

$$\mathrm{Hom}(\mathbf{P}, \mathbf{P}) \approx \mathrm{Hom}(\mathbf{S}, \mathbf{S}) = F.$$

Also, $\mathrm{Hom}(\mathbf{T}, \mathbf{T}) = F$, and we have

$$\mathrm{Hom}(\mathbf{T}, \mathbf{P}) \approx \mathrm{Hom}(\mathbf{T}, \mathrm{Soc}\, \mathbf{P}) \approx \mathrm{Hom}(\mathbf{T}, \mathbf{T} \oplus \mathbf{T}) \approx F \times F.$$

Thus, as an Hom(**P**, **P**) – Hom(**T**, **T**)-bimodule, we see that Hom(**T**, **P**) is of the form ${}_F(F \times F)_F$.

We consider $\mathbf{P} \oplus \mathbf{T}$. This is a finite-dimensional, and thus small, projective object; moreover, since **S** and **T** are the only simple objects of $\mathfrak{A}$, and $\mathbf{P}/\operatorname{Soc}\mathbf{P} \approx \mathbf{S}$, $\mathbf{P} \oplus \mathbf{T}$ is also a generator, hence a progenerator. Therefore, $\mathfrak{A} \approx \mathfrak{M}_{\mathscr{R}}$, with $\mathscr{R} = \operatorname{Hom}(\mathbf{P} \oplus \mathbf{T}, \mathbf{P} \oplus \mathbf{T})$. The simple objects of $\mathfrak{A}$ are finite dimensional, and hence the dimension in $\mathfrak{A}$ is equivalent to the length function. Since $\mathscr{R}$ is a finite dimensional K-algebra, the K-dimension of $\mathscr{R}$-modules is equivalent to their length. Thus, every equivalence $\mathfrak{A} \approx \mathfrak{M}_{\mathscr{R}}$ is a dimension functor. Consider the two orthogonal idempotents e_1 and e_2 in $\mathscr{R}$ which are given by the projections onto **P** and onto **T**, respectively. Then

$$e_1\mathscr{R}e_1 = \operatorname{Hom}(\mathbf{P}, \mathbf{P}), \qquad \text{and} \qquad e_2\mathscr{R}e_2 = \operatorname{Hom}(\mathbf{T}, \mathbf{T})$$

are isomorphic to F, and

$$e_1\mathscr{R}e_2 = \operatorname{Hom}(\mathbf{T}, \mathbf{P}) = \operatorname{Rad}(\mathscr{R}) \approx {}_F(F \times F)_F.$$

We can apply Lemma 6.3 and conclude that $\mathfrak{Re}({}_F(F \times F)_F)$ is a full subcategory of $\mathfrak{A}$. But

$$\dim {}_F(F \times F) = 2 = \dim(F \times F)_F,$$

because $[F\colon K]$ is finite and K acts centrally on $F \times F = \operatorname{Hom}(\mathbf{T}, \mathbf{P})$. Thus, it follows from Proposition 5.2 that $\mathfrak{A}$ is of strongly unbounded type.

PROPOSITION 7.2. *Let $[F\colon G] = 2$. The K-structure $\mathscr{S} = \mathscr{I}_2(G) \sqcup \mathscr{I}_2(F)$ is of strongly unbounded type.*

Proof. Let $\mathfrak{A}$ be the full subcategory of $\mathfrak{S}(\mathscr{S})$ of all objects (W, U_1, U_2, V_1, V_2) (where $U_1 \subseteq U_2$ are G-subspaces of W_F, and $V_1 \subseteq V_2$ are F-subspaces of W_F) such that

$$\underline{U}_1 = 0 \qquad \text{and} \qquad \overline{U}_1 \oplus V_1 = U_2 \oplus V_1 = U_1 \oplus V_2 = W.$$

We claim that $\mathfrak{A}$ is abelian and an exact subcategory of $\mathfrak{S}(\mathscr{S})$.

In order to be able to use Proposition 6.1, define another K-structure $\mathscr{S}'$ by introducing a new element: Let $\mathscr{S}' = \mathscr{S} \sqcup \mathscr{I}_1(F)$ and denote an $\mathscr{S}'$-space by $(W, U_1, U_2, V_1, V_2, X)$. Consider the following system $\mathscr{E}$ of equations for $\mathscr{S}'$

$$X \oplus V_1 = W, \qquad U_2 \oplus V_1 = W, \qquad U_1 \oplus V_2 = W,$$

and

$$\underline{U}_1 = 0, \qquad \overline{U}_1 = X.$$

Of course, we have introduced just a "label" for $\bar{U}_1$. Now, $\mathscr{E}$ determines $\mathscr{S}'$, and thus as a consequence, the category of all $\mathscr{S}'$-spaces satisfying $\mathscr{E}$ is an abelian category; but, the latter is just $\mathfrak{A}$ under the obvious embedding

$$\mathfrak{A} \subseteq \mathfrak{S}(\mathscr{S}) \xrightarrow{\mathsf{T}} \mathfrak{S}(\mathscr{S}')$$

with $\mathsf{T}(W, U_1, U_2, V_1, V_2) = (W, U_1, U_2, V_1, V_2, \bar{U}_1)$. Since T preserves kernels, cokernels, and direct sums, and since $\mathfrak{A}$ is an exact subcategory of $\mathfrak{S}(\mathscr{S}')$, it is also an exact subcategory of $\mathfrak{S}(\mathscr{S})$.

Again, let $F = G + fG$ with $f \in F \backslash G$. We claim that

$$\mathbf{S} = (F \times F, G \times G, F \times F, 0, (1, f)F) \qquad \text{and} \qquad \mathbf{T} = (F, 0, 0, F, F)$$

are the only simple objects of $\mathfrak{A}$. Note that $\mathbf{S}$ is simple. For, if $(W, U_1, U_2, V_1, V_2) \subseteq \mathbf{S}$ and $\dim W_F = 1$, then either $W = V_2 = (1, f)F$, so $U_1 = (G \times G) \cap (1, f)F = 0$ and $V_1 = 0$, contrary to the equality $\bar{U}_1 \oplus V_1 = W$, or else $V_2 = 0$, and then $W = U_1 \oplus V_2 = U_1$, contrary to $\underline{U}_1 = 0$. Now, if (W, U_1, U_2, V_1, V_2) is an arbitrary object of $\mathfrak{A}$, then $(V_1, 0, 0, V_1, V_1)$ is a subobject of it and it is a direct sum of copies of $\mathbf{T}$. The corresponding quotient is of the form $(W', U, W', 0, V)$, and this is a direct sum of copies of $\mathbf{S}$. Indeed, in order to decompose $(W', U, W', 0, V)$ with $\underline{U} = 0$, $\bar{U} = W'$ and $U \oplus V = W'$, we just use either Lemma 2.5 or the classification of $\mathscr{I}_1(G) \sqcup \mathscr{I}_1(F)$-spaces in Proposition 2.7.

Next, we show that every direct sum of copies of $\mathbf{S}$ is injective. Assume that we have an inclusion

$$(W', U', W', 0, V') \hookrightarrow (W, U_1, U_2, V_1, V_2)$$

in $\mathfrak{A}$. Again, we use Lemma 2.5, this time for $\bar{U}_1$, the G-subspace U_1 and the F-subspace $V = \bar{U}_1 \cap V_2$. We have $V' \subseteq V$ and so we can take a basis of V extending a basis of V'; moreover, we may assume that the basis elements of V' are of the form $x + yf$ with $x, y \in U'$. In this way, we get a decomposition $U_1 = U' \oplus U''$ such that

$$V = V' \oplus (\overline{U''} \cap V).$$

Let $C = \overline{U''} \oplus V_1$. Then

$$(W, U_1, U_2, V_1, V_2) = (W', U', W', 0, V') \oplus (C, U'', C \cap U_2, V_1, C \cap V_2).$$

For, $W = \bar{U}_1 \oplus V = \overline{U'} \oplus \overline{U''} \oplus V = W' \oplus C$, and this decomposition is trivially compatible with U_1, U_2, and V_1, whereas, for V_2, we have

$$\begin{aligned} V' \oplus (C \cap V_2) &= (V' + C) \cap V_2 = (\overline{U'} + \overline{U''} + V_1) \cap V_2 \\ &= (\bar{U}_1 + V_1) \cap V_2 = V_2. \end{aligned}$$

Every extension (W, U_1, U_2, V_1, V_2) of $\mathbf{T}$ by itself satisfies $U_1 = U_2 = 0$ and $V_1 = V_2 = W$ and so it splits. Thus, by Lemma 6.2 we see that $\mathbf{T}$ is projective. Similarly, we can prove that

$$\mathbf{P} = (F \times F \times F \times F, 0 \times 0 \times G \times G, U, F \times F \times 0 \times 0, V)$$

with

$$U = 0 \times 0 \times G \times G + (0, 1, f, 0)G + (1, f, 0, f)G$$

and

$$V = F \times F \times 0 \times 0 + (0, 0, 1, f)F$$

is projective. Assume that there is an exact sequence

$$0 \longrightarrow \mathbf{T} \longrightarrow (W, U_1, U_2, V_1, V_2) \xrightarrow{\alpha} \mathbf{P} \longrightarrow 0$$

in $\mathfrak{A}$. Now, $\dim(U_2)_G = 4$, so $\dim(\bar{U}_2)_F \leqslant 4$. Since $\alpha(\bar{U}_2) = \overline{\alpha(U_2)} = \bar{U}$ is of dimension 4 over F, we have $\dim(\bar{U}_2)_F = 4$, and $\bar{U}_2 \cap \ker \alpha = 0$. As a consequence, $W = \bar{U}_2 \oplus \ker \alpha$, and

$$(\ker \alpha, 0, 0, \ker \alpha, \ker \alpha) \oplus (\bar{U}_2, U_1, U_2, \bar{U}_2 \cap V_1, \bar{U}_2 \cap V_2)$$
$$= (W, U_1, U_2, V_1, V_2)$$

is a direct decomposition, and thus α splits.

In order to see that $\mathbf{P}$ is indecomposable, let

$$\mathbf{T}_1 = (F \times 0 \times 0 \times 0, 0, 0, F \times 0 \times 0 \times 0, F \times 0 \times 0 \times 0)$$

and

$$\mathbf{T}_2 = (0 \times F \times 0 \times 0, 0, 0, 0 \times F \times 0 \times 0, 0 \times F \times 0 \times 0).$$

Then

$$\mathbf{P}/\mathbf{T}_1 \approx (F \times F \times F, 0 \times G \times G, 0 \times G \times G + (1, f, 0)G + (f, 0, f)G, F \times 0 \times 0, F \times 0 \times 0 + (0, 1, f)F)$$

and here is $\dim(U_2 + \bar{U}_1/\bar{U}_1)_G = 2$, whereas

$$\mathbf{P}/\mathbf{T}_2 \approx (F \times F \times F, 0 \times G \times G, 0 \times G \times G + (0, f, 0)G + (1, 0, f)G, F \times 0 \times 0, F \times 0 \times 0 + (0, 1, f)F)$$

with $\dim(U_2 + \bar{U}_1/\bar{U}_1)_G = 1$. Also, both $\mathbf{P}/\mathbf{T}_1$ and $\mathbf{P}/\mathbf{T}_2$ are indecomposable, since in both cases in decomposition has to be of the form $F \times 0 \times 0 \oplus 0 \times F \times F$, but this is not compatible with U_2. By Lemma 6.4, $\mathbf{P}$ is indecomposable.

The socle of $\mathbf{P}$ is $\mathbf{T}_1 \oplus \mathbf{T}_2 \approx \mathbf{T} \oplus \mathbf{T}$, and $\mathbf{P}/(\mathbf{T}_1 \oplus \mathbf{T}_2) \approx \mathbf{S}$. Since $\mathbf{P}$ is the projective cover of $\mathbf{S}$, we see that $\operatorname{Hom}(\mathbf{P}, \mathbf{P}) \approx \operatorname{Hom}(\mathbf{S}, \mathbf{S})$. Let $H = \operatorname{Hom}(\mathbf{S}, \mathbf{S})$. Then H is a field, and since H does not act transitively on $\mathbf{S}$, $\dim_H S \geqslant 2$. Let $F = \operatorname{Hom}(\mathbf{T}, \mathbf{T})$, and consider the H-F-bimodule

$$\operatorname{Hom}(\mathbf{T}, \mathbf{P}) \approx \operatorname{Hom}(\mathbf{T}, \mathbf{T} \oplus \mathbf{T}) \approx F \oplus F.$$

Calculating the dimensions over K (and noting that K acts centrally on $\operatorname{Hom}(\mathbf{T}, \mathbf{P})$), we see that

$$\dim_H(F \oplus F) \geqslant 2, \qquad \text{and} \qquad \dim(F \oplus F)_F = 2.$$

Thus $\mathfrak{Re}({}_H(F \oplus F)_F)$ is of strongly unbounded type, but we can embed $\mathfrak{Re}({}_H(F \oplus F)_F)$ into $\mathfrak{M}_{\mathscr{R}}$ with $\mathscr{R} = \operatorname{Hom}(\mathbf{P} \oplus \mathbf{T}, \mathbf{P} \oplus \mathbf{T})$, and $\mathfrak{M}_{\mathscr{R}} \approx \mathfrak{A}$, and these functors are dimension functors.

Proposition 7.3. *Let $[F: G] = 2$. The K-structure $\mathscr{S} = \mathscr{I}_3(G) \sqcup \mathscr{I}_1(F)$ is of strongly unbounded type.*

Proof. Let $\mathfrak{A}$ be the full subcategory of $\mathfrak{S}(\mathscr{S})$ of all objects (W, U_1, U_2, U_3, V) (where $U_1 \subseteq U_2 \subseteq U_3$ are G-subspaces of W_F, and V is an F-subspace of W_F) such that

$$\underline{U}_1 = 0, \qquad \underline{U_3 \cap V} = 0, \qquad \overline{U_3 \cap V} = V$$

and

$$\overline{U}_1 \oplus V = W, \qquad U_2 \oplus V = W, \qquad \underline{U}_3 \oplus V = W.$$

Note that, from these conditions, we can derive that $U_2 \oplus (U_3 \cap V) = (U_2 + V) \cap U_3 = U_3$, and thus also $\overline{U}_3 = W$. We claim that $\mathfrak{A}$ is abelian.

In order to use Proposition 6.1, we define another K-structure $\mathscr{S}'$ by introducing three new elements. Let $\mathscr{S}' = \mathscr{S} \sqcup \mathscr{I}_1(F) \sqcup \mathscr{I}_1(F) \sqcup \mathscr{I}_1(G)$, and denote an $\mathscr{S}'$-space by $(W, U_1, U_2, U_3, V, X, Y, Z)$. Consider the following system $\mathscr{E}$ of equations for $\mathscr{S}'$:

$$X \oplus V = W, \qquad U_2 \oplus V = W, \qquad Y \oplus V = W, \qquad U_2 \oplus Z = U_3$$

and

$$\underline{U}_1 = 0, \qquad \overline{U}_1 = X, \qquad \underline{U}_3 = Y, \qquad \overline{U}_3 = W, \qquad \underline{Z} = 0, \qquad \overline{Z} = V.$$

It is obvious that we have introduced X as the "label" for $\overline{U}_1$, and Y as the "label" for $\underline{U}_3$. Also, one can see easily that $Z \subseteq V \cap U_3$, and thus, since $U_3 = U_2 \oplus (U_3 \cap V) = U_2 \oplus Z$, we get $Z = U_3 \cap V$.

Now, $\mathscr{E}$ determines X, V, U_2, Y, using the first line of the equations and therefore also U_1, U_3, and Z using the remaining equations. As a consequence, the category of all $\mathscr{S}'$-spaces satisfying $\mathscr{E}$ is an abelian, exact subcategory of $\mathfrak{S}(\mathscr{S}')$. But this category can be identified with $\mathfrak{A}$ under the obvious functor

$$\mathfrak{A} \subseteq \mathfrak{S}(\mathscr{S}) \xrightarrow{\mathsf{T}} \mathfrak{S}(\mathscr{S}')$$

with $\mathsf{T}(W, U_1, U_2, U_3, V) = (W, U_1, U_2, U_3, V, \overline{U}_1, \underline{U}_3, U_3 \cap V)$. Since T preserves kernels, cokernels, and direct sums, we see that $\mathfrak{A}$ is also an exact subcategory of $\mathfrak{S}(\mathscr{S})$.

We claim that the only simple objects in $\mathfrak{A}$ are

$$\mathbf{S} = (F, G, F, F, 0) \qquad \text{and} \qquad \mathbf{T} = (F, 0, 0, G, F)$$

and that every object $\mathbf{A} = (W, U_1, U_2, U_3, V)$ of $\mathbf{A}$ has a subobject $\mathbf{A}' \approx \oplus \mathbf{T}$ with $\mathbf{A}/\mathbf{A}' \approx \oplus \mathbf{S}$. Now $\mathbf{A}' = (V, 0, 0, U_3 \cap V, V)$ is always a subobject of $\mathbf{A}$, and obviously it is a direct sum of copies of $\mathbf{T}$, since $\underline{U_3 \cap V} = 0$ and $\overline{U_3 \cap V} = V$. And $\mathbf{A}/\mathbf{A}'$ is of the form $(W', U', W', W', 0)$ with $\underline{U'} = 0$ and $\overline{U'} = W'$, so it is a direct sum of copies of $\mathbf{S}$. Also, a direct sum of copies of $\mathbf{S}$ is injective. Indeed, consider an inclusion

$$(W', U', W', W', 0) \hookrightarrow (W, U_1, U_2, U_3, V),$$

and take a complement U'' to U' in U_1, thus $U_1 = U' \oplus U''$, and let $C = \overline{U''} \oplus V$. Then

$$(W, U_1, U_2, U_3, V) = (W', U', W', W', 0) \oplus (C, U'', C \cap U_2, C \cap U_3, V).$$

Thus, we may use Lemma 6.2 to conclude that $\mathbf{T}$ is projective. Every extension (W, U_1, U_2, U_3, V) of $\mathbf{T}$ by $\mathbf{T}$ splits, because $U_1 = U_2 = 0$, and thus $\underline{U}_3 = 0$, $\overline{U}_3 = V = W$; hence, $\operatorname{Ext}(\mathbf{T}, \mathbf{T}) = 0$ and $\mathbf{T}$ is projective.

The same reference to Lemma 6.2 can be used to establish projectivity of

$$\mathbf{P} = (F \times F \times F, 0 \times 0 \times G, 0 \times 0 \times G + (1, f, f)G, G \times G \times G + (1, f, f)G, F \times F \times 0).$$

For, let

$$0 \longrightarrow \mathbf{T} \longrightarrow (W, U_1, U_2, U_3, V) \xrightarrow{\alpha} \mathbf{P} \longrightarrow 0$$

be an extension in $\mathfrak{A}$. Since $(\overline{U}_2)_F$ is generated by two elements and since

$$\alpha(\overline{U}_2) = \overline{\alpha(U_2)} = 0 \times 0 \times F + (1, f, f)F$$

is of dimension 2 over F, we conclude that also $\dim(\overline{U}_2)_F = 2$ and that $\ker \alpha \cap \overline{U}_2 = 0$. But $U_3 \cap V \not\subseteq \ker \alpha \cap \overline{U}_2$, because $\dim(\ker \alpha + \overline{U}_2)_F = 3$, whereas $U_2 + \overline{U_3 \cap V} = W$. Thus, there is an element $u \in U_3 \cap V$ with $u \notin \ker \alpha \cap \bar{U}_2$. Consequently, $W = \ker \alpha \oplus \bar{U}_2 \oplus uF$, and

$$(W, U_1, U_2, U_3, V) = \mathbf{T} \oplus (\bar{U}_2 \oplus uF, U_1, U_2, U_2 \oplus uG, uF)$$

gives a splitting of α.

Also, $\mathbf{P}$ is indecomposable. Let $\mathbf{T}_1 = (F \times 0 \times 0, 0, 0, G \times 0 \times 0, F \times 0 \times 0)$; then

$$\mathbf{P}/\mathbf{T}_1 \approx (F \times F, 0 \times G, 0 \times G + (f, f)G, G \times G + (f, f)G, F \times 0)$$

satisfies $U_1 \not\subseteq \underline{U}_3 = (1, 1)F$. On the other side, for $\mathbf{T}_2 = (0 \times F \times 0, 0, 0, 0 \times G \times 0, 0 \times F \times 0)$, we have

$$\mathbf{P}/\mathbf{T}_2 \approx (F \times F, 0 \times G, 0 \times G + (1, f)G, G \times G + (1, f)G, F \times 0)$$

with $U_1 \subseteq \underline{U}_3 = (0, 1)F$. Since $\mathbf{P}/\mathbf{T}_1$ and $\mathbf{P}/\mathbf{T}_2$ are nonisomorphic and indecomposable, it follows from Lemma 6.4 that $\mathbf{P}$ is indecomposable.

Now $\operatorname{Hom}(\mathbf{T}, \mathbf{T}) = F$, and $\operatorname{Hom}(\mathbf{P}, \mathbf{P}) \approx \operatorname{Hom}(\mathbf{S}, \mathbf{S}) = F$, and

$$\operatorname{Hom}(\mathbf{T}, \mathbf{P}) \approx \operatorname{Hom}(\mathbf{T}, \mathbf{T} \oplus \mathbf{T}) \approx F \oplus F.$$

Again we consider the K-species ${}_F\operatorname{Hom}(\mathbf{T}, \mathbf{P})_F$ which is of strongly unbounded type, and embed $\mathfrak{Re}({}_F\operatorname{Hom}(\mathbf{T}, \mathbf{P})_F)$ into $\mathfrak{M}_{\mathscr{R}}$ with $\mathscr{R} = \operatorname{Hom}(\mathbf{P} \oplus \mathbf{T}, \mathbf{P} \oplus \mathbf{T})$. Since $\mathfrak{M}_{\mathscr{R}} \approx \mathfrak{A}$, also $\mathfrak{A}$ is of strongly unbounded type.

PROPOSITION 7.4. *Let* $[F\colon G] = 3$. *The K-structure* $\mathscr{S} = \mathscr{I}_1(G) \sqcup \mathscr{I}_1(F)$ *is of strongly unbounded type.*

Proof. Let $\mathfrak{A}$ be the full subcategory of $\mathfrak{S}(\mathscr{S})$ of all objects (W_F, U_G, V_F) with

$$U \oplus V = W.$$

This equation determines $\mathscr{S}$, so $\mathfrak{A}$ is abelian.

Now, an arbitrary object $\mathbf{A} = (W, U, V)$ in $\mathfrak{A}$ has $\mathbf{A}' = (V, 0, V)$ as subobject and $\mathbf{A}/\mathbf{A}' \approx (W/V, W/V, 0)$. This shows that there are precisely two simple objects, namely

$$\mathbf{S} = (F, F, 0) \qquad \text{and} \qquad \mathbf{T} = (F, 0, F).$$

Obviously, every direct sum of copies of $\mathbf{S}$ is injective. For, if there is an inclusion

$$(W', W', 0) \hookrightarrow (W, U, V),$$

then $W' \cap V = 0$, and thus there is a complement $W_F' \oplus W_F'' = W$, with $V \subseteq W''$, and

$$(W, U, V) = (W', W', 0) \oplus (W'', W'' \cap U, V).$$

In order to see that $\mathbf{T}$ is projective we only have to observe that $\operatorname{Ext}(\mathbf{T}, \mathbf{T}) = 0$. But every extension (W, U, V) of $(F, 0, F)$ by $(F, 0, F)$ satisfies $U = 0$, so it is of the form $(W, 0, W)$ and therefore splits.

Again we want to use Lemma 6.2 for an object $\mathbf{P}$ of length 3 which we are going to construct. Let $\{1, e, f\}$ be a basis both of F_G and of ${}_GF$. By Lemma 4.1, there is an indecomposable $\mathscr{S}$-space $(F \times F, U, F \times 0)$ with $\dim U_G = 3$, $U \cap Ue = 0$ and $U \cap F \times 0 = 0$. Calculating the dimensions, we see that $U + F \times 0 = F \times F$, and thus $(F \times F, U, F \times 0)$ belongs to $\mathfrak{A}$. Also, since $U + F \times 0 = F \times F$, we may choose a G-basis in U_G such that

$$U = (a, 1)G + (b, e)G + (c, f)G,$$

with some elements $a, b, c \in F$. Now, let $\mathbf{P}$ be given by

$$\mathbf{P} = (F \times F \times F, (a, 0, 1)G + (b, 0, e)G + (c, 1, f)G, F \times F \times 0).$$

First, we claim that $\mathbf{P}$ is indecomposable. Let $\mathbf{T}_1 = (F \times 0 \times 0, 0, F \times 0 \times 0)$. Then

$$\mathbf{P}/\mathbf{T}_1 \approx (F \times F, (0, 1)G + (0, e)G + (1, f)G, F \times 0)$$

is indecomposable; for, a decomposition would have to be of the form $F \times 0 \oplus (f_1, f_2)F$, and since $F \times 0 \cap U = 0$, it would be necessary that $(f_1, f_2)F = U$ which is not the case for any $f_1, f_2 \in F$. Moreover, $U \cap Ue \neq 0$, since $(0, e) \in U \cap Ue$. On the other hand, take $\mathbf{T}_2 = (0 \times F \times 0, 0, 0 \times F \times 0)$. Then

$$\mathbf{P}/\mathbf{T}_2 \approx (F \times F, (a, 1)G + (b, e)G + (c, f)G, F \times 0)$$

is also indecomposable, but satisfies $U \cap Ue = 0$. Thus, it follows from Lemma 6.4 that $\mathbf{P}$ is indecomposable.

Since $\mathbf{P}$ is indecomposable, it follows that

$$U' = (a, 0, 1)G + (b, 0, e)G + (c, 1, f)G$$

has the property that $\overline{U'} = F \times F \times F$. For, otherwise we take an F-subspace $V_F' \subseteq F \times F \times 0$ with $\overline{U'} \oplus V' = F \times F \times F$ and this decomposition is obviously compatible with both U' and V.

In order to prove that $\mathbf{P}$ is projective, take an exact sequence

$$0 \longrightarrow \mathbf{T} \longrightarrow (W, U, V) \xrightarrow{\alpha} \mathbf{P} \longrightarrow 0.$$

Since U_G is generated by three elements, $\dim \overline{U}_F \leqslant 3$. But

$$\alpha(\overline{U}) = \overline{\alpha(U)} = \overline{U'} = F \times F \times F;$$

hence, we conclude that $\dim \overline{U}_F = 3$ and $\overline{U} \cap \ker \alpha = 0$. Also, $\ker \alpha \subseteq V$; thus, the decomposition $W = \overline{U} \oplus \ker \alpha$ is compatible both with U and with V, and α splits.

Again, we see that $\mathrm{Hom}(\mathbf{T}, \mathbf{T}) = F$, $\mathrm{Hom}(\mathbf{P}, \mathbf{P}) \approx \mathrm{Hom}(\mathbf{S}, \mathbf{S}) = F$, and $\mathrm{Hom}(\mathbf{T}, \mathbf{P}) \approx \mathrm{Hom}(\mathbf{T}, \mathbf{T} \oplus \mathbf{T}) \approx F \oplus F$. Since the K-species ${}_F\mathrm{Hom}(\mathbf{T}, \mathbf{P})_F$ is of strongly unbounded type, the same is true for $\mathfrak{A} \approx \mathfrak{M}_{\mathscr{R}}$ with $\mathscr{R} = \mathrm{Hom}(\mathbf{P} \oplus \mathbf{T}, \mathbf{P} \oplus \mathbf{T})$.

Let us conclude this section with a remark which, although not used anywhere in the paper, may be found of an interest.

Remark 7.5. In the case of K-structures investigated in Propositions 7.1, 7.2, and 7.3, we can construct by induction explicitly large indecomposable $\mathscr{S}$-spaces. Namely, the following $\mathscr{S}$-spaces are indecomposable for all $i \in \mathbb{N}$:

(1) the $\mathscr{S}$-space (W, U, V_1, V_2) defined by the $2i$-dimensional vector space

$$W_F = F \times F \times \cdots \times F \times F$$

together with

$$U = G \times G \times \cdots \times G \times G,$$
$$V_1 = (1, f)F \times \cdots \times (1, f)F,$$

and

$$V_2 = 0 \times (1, f)F \times \cdots \times (1, f)F \times 0;$$

(2) the $\mathscr{S}$-space (W, U_1, U_2, V_1, V_2) defined by the $3i$-dimensional vector space

$$W_F = F \times F \times F \times \cdots \times F \times F \times F$$

together with

$$U_1 = G \times G \times 0 \times \cdots \times G \times G \times 0,$$
$$U_2 = G \times G \times F \times \cdots \times G \times G \times F,$$
$$V_1 = (1, f, 1)F \times \cdots \times (1, f, 1)F$$

and the $(2i - 1)$-dimensional F-space V_2 generated by V_1 and

$$X = 0 \times 0 \times (f, 1)F \times \cdots \times (f, 1)F \times 0 \times 0; \quad \text{and}$$

(3) the $\mathscr{S}$-space (W, U_1, U_2, U_3, V) defined by the $2i$-dimensional space

$$W_F = F \times F \times \cdots \times F \times F,$$

and

$$\begin{aligned} V &= (1, f)F \times \cdots \times (1, f)F, \\ U_1 &= 0 \times G \times \cdots \times 0 \times G, \\ U_2 &= G \times G \times \cdots \times G \times G \end{aligned}$$

and U_3 is generated by U_2 and

$$X = 0 \times (1, 1)F \times \cdots \times (1, 1)F \times 0.$$

3. Proof of the Theorems

8. *K-Structures*

In this section, we are going to present proofs of Theorem A and Theorem E(1). First, from the results of Section 5, we derive that certain K-structures are of strongly unbounded type.

Proposition 8.1. *The K-structure $\mathscr{S} = \mathscr{I}_1(G)$ with $[F: G] \geqslant 4$, is of strongly unbounded type.*

Proof. Consider the K-species ${}_FF_G$. According to Proposition 5.2, $\mathfrak{Re}({}_FF_G)$ is of strongly bounded type. Furthermore, $\mathfrak{Re}({}_FF_G)$ is equivalent to the category $\mathfrak{B}({}_FF_G)$ with objects of the form

$$(U_G, X_F, \iota: U_G \to X_F \otimes {}_FF_G) = (U_G, X_F, \iota: U_G \subseteq X_G)$$

and this equivalence is given by a dimension functor. It is obvious that $\mathfrak{B}({}_FF_G)$ is just the category of all $\mathscr{S}$-spaces.

Proposition 8.2. *The K-structure $\mathscr{S} = \mathscr{I}_1(G) \sqcup \mathscr{I}_1(H)$ with $[F: G] \geqslant 2$, $[F: H] \geqslant 2$, is of strongly unbounded type.*

Proof. Consider the K-species ${}_GF_H$. Again, by Proposition 5.2, $\mathfrak{Re}({}_GF_H)$ is of strongly unbounded type. Define a dimension functor, which is a full embedding,

$$\mathsf{T}: \mathfrak{Re}({}_GF_H) \to \mathfrak{S}(\mathscr{S})$$

as follows: For (X_G, Y_H, φ) with $\varphi: X_G \otimes {}_GF_H \to Y_H$, put $\mathsf{T}(X, Y, \varphi) = (X_G \otimes {}_GF_F, X_G \otimes {}_GG_G, \ker \varphi)$, and for $(\alpha, \beta): (X, Y, \varphi) \to (X', Y', \varphi')$, put $\mathsf{T}(\alpha, \beta) = \alpha \otimes 1$. Now, $\alpha \otimes 1$ maps $X_G \otimes {}_GG_G$ into $X'_G \otimes {}_GG_G$, and $\ker \varphi$ into $\ker \varphi'$ because $\varphi'(\alpha \otimes 1) = \beta\varphi$. Since φ and φ' are epimorphisms, β is determined by α, and hence T is faithful. Let

$$\gamma: \mathsf{T}(X, Y, \varphi) \to \mathsf{T}(X', Y', \varphi')$$

be a map in $\mathfrak{S}(\mathscr{S})$. Obviously, $\gamma: X_G \otimes {}_G F_F \to X'_G \otimes {}_G F_F$ maps $X_G \otimes {}_G G_G$ into $X'_G \otimes {}_G G_G$, and thus the restriction of γ to $X_G \otimes {}_G G_G$ is of the form $\alpha \otimes 1$ with $\alpha: X_G \to X'_G$. But, since γ is F-linear,

$$\gamma(x \otimes f) = [\gamma(x \otimes 1)]f = (\alpha(x) \otimes 1)f = \alpha(x) \otimes f$$

for every $f \in F$, and therefore $\gamma = \alpha \otimes 1$. Finally, since γ maps $\ker \varphi$ into $\ker \varphi'$, there is $\beta: Y_H \to Y_H'$ with $\varphi'\gamma = \beta\varphi$. Consequently, $\gamma = \mathsf{T}(\alpha, \beta)$ and T is full, as required.

Proposition 8.3. *The K-structure $\mathscr{S} = \mathscr{I}_2(G)$ with $[F: G] = 3$ is of strongly unbounded type.*

Proof. It is immediate that $\mathfrak{S}(\mathscr{S})$ is equivalent to the category $\mathfrak{Rm}(\mathscr{Q})$ of Proposition 5.3, by a dimension functor, and therefore $\mathfrak{S}(\mathscr{S})$ is of strongly unbounded type.

Next, we need a lemma which reduces the investigation of a K-structure to its K-substructures (comp. Lemma 6 of [15]).

Lemma 8.4. *Let $\mathscr{S}$ be a K-structure for F and let $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{S}_3$ be a decomposition of $\mathscr{S}$ such that $\mathscr{S}_3$ (with respect to the induced order) is a chain, $i \leqslant j$ for all $i \in \mathscr{S}_1$ and $j \in \mathscr{S}_2$ and that $F_i = F$ for all $i \in \mathscr{S}_2 \cup \mathscr{S}_3$. Then $\mathscr{S}$ is of finite type if and only if both $\mathscr{S}_1 \cup \mathscr{S}_3$ and $\mathscr{S}_2 \cup \mathscr{S}_3$ are of finite type. Also, the maximal weighted dimension of the indecomposable $\mathscr{S}$-spaces equals the maximum of the weighted dimensions of the indecomposable $(\mathscr{S}_1 \cup \mathscr{S}_3)$-spaces and $(\mathscr{S}_2 \cup \mathscr{S}_3)$-spaces.*

Proof. Me claim that, if (W, W_i) is an indecomposable $\mathscr{S}$-space, then the F-subspace

$$V = \bigcap_{i \in S_2} W_i$$

of W satisfies either $V = 0$ or $V = W$.

Assume that $0 \neq V \neq W$. Let $\mathscr{S}_3 = \{i_1 < i_2 < \cdots < i_n\}$ and define inductively an F-complement U_F of V_F in W in such a way that $U \cap W_i$ is an F-complement of $V \cap W_i$ in W_i for each $i \in \mathscr{S}_3$. It is easy to show that $W = U \oplus V$ defines a decomposition of the $\mathscr{S}$-space (W, W_i). For, if $i \in S_1$, then $V \cap W_i = W_i$, and therefore $W_i = (U \cap W_i) \oplus (V \cap W_i)$; if $i \in S_2$, then $V \cap W_i = V$, and hence again

$$W_i = (U \oplus V) \cap W_i = (U \cap W_i) \oplus V = (U \cap W_i) \oplus (V \cap W_i).$$

And, since the decomposition is obviously compatible with $\mathscr{S}_3$, we get a contradiction to the indecomposability of W.

Thus, if (W, W_i) is an indecomposable $\mathcal{S}$-space, then either $W_i = 0$ for all $i \in \mathcal{S}_1$, or $W_i = W$ for all $i \in \mathcal{S}_2$. The lemma follows.

Now, we are ready to present the following.

Proof of Theroem A *and Theorem* E(1). Assume that $\mathcal{S}$ is a K-structure of F which is not of strongly unbounded type. Then, the width of $\mathcal{S}$ is $\leqslant 3$ and $\mathcal{S}$ cannot contain a K-structure of the form (i)–(iv) according to Kleiner–Nazarova–Roiter theorem. For, otherwise we can embed the category $\mathfrak{M}_{F[t]}$ of all right F[t]-modules into $\mathfrak{S}(\mathcal{S})$, where F[t] denotes the polynomial ring in one indeterminate t over F (cf. [7]). But, if the width is $\leqslant 3$ and the weighted width is >3, then the structure $\mathcal{S}$ must contain one of the following K-substructures: either $\mathcal{I}_1(G) \sqcup \mathcal{I}_1(F) \sqcup \mathcal{I}_1(F)$ with $[F\colon G] = 2$, or $\mathcal{I}_1(G) \sqcup \mathcal{I}_1(F)$ with $[F\colon G] = 3$, or $\mathcal{I}_1(G)$ with $[F\colon G] \geqslant 4$, or $\mathcal{I}_1(G) \sqcup \mathcal{I}_1(H)$ with $[F\colon G] \geqslant 2$ and $[F\colon H] \geqslant 2$. However, then the structure $\mathcal{S}$ is of strongly unbounded type according to Propositions 7.1, 7.4, 8.1, and 8.2, respectively. Finally, $\mathcal{S}$ does not contain any K-substructure of the form (v), (vi), or (vii) in view of Propositions 7.2, 7.3, and 8.3, respectively.

Now, in order to establish the sufficiency, let $\mathcal{S}$ be a K-structure satisfying the conditions of Theorem A. If $F_i = F$ for all $i \in \mathcal{S}$, then the result follows from Kleiner–Nazarova–Roiter theorem. Thus, assume that there exists $j \in \mathcal{S}$ such that $F_j \neq F$. Observe that there is no $i \in \mathcal{S}$ such that $F_i \neq F_j$ and $F_i \neq F$ (both for i, j related and unrelated). This follows easily from the fact that the weighted width of $\mathcal{S}$ is $\leqslant 3$. Thus, there is a chain $\mathcal{I}_m(G) \subseteq \mathcal{S}$ such that $[F\colon G] \leqslant 3$ and $F_i = F$ for all i from the complement $\mathcal{S} \backslash \mathcal{I}_m(G)$.

Now, if $[F\colon G] = 3$, then in view of (vii), $m = 1$: $\mathcal{I}_1(G) = \{1'\}$. Moreover, since the weighted width of $\mathcal{S} \leqslant 3$, $i > 1'$ and $F_i = F$ for all $i \in \mathcal{S}$, $i \neq 1'$. Therefore, by Lemma 8.4, $\mathcal{S}$ is of finite type if (and only if) both $\mathcal{S}_1 = \mathcal{I}_1(G)$ and $\mathcal{S}_2 = \mathcal{S} \diagdown \mathcal{S}_1$ are of finite type which is the case according to Proposition 4.2 and Kleiner–Nazarova–Roiter theorem.

Hence, assume that $[F\colon G] = 2$. Write

$$\mathcal{S}_1 = \mathcal{I}_m(G) = \{1' < 2' < \cdots < m'\}.$$

Then, by a width argument, all $i \in \mathcal{S}$ which are unrelated to m' form a chain

$$\mathcal{S}_3 = \mathcal{I}_n(F) = \{1'' < 2'' < \cdots < n''\}.$$

Furthermore, write $\mathcal{S}_2 = \{i \mid i \in \mathcal{S} \text{ and } i > m'\}$ and apply Lemma 8.4; as a consequence, we may assume that $\mathcal{S}_2 = \varnothing$. Of course, there is no $i \in \mathcal{S} \diagdown \mathcal{I}_m(G)$ such that $i < m'$. Now, if $n = 0$, then $\mathcal{S}$ is of finite type in view of Proposition 2.5. By Proposition 2.6, $\mathcal{S}$ is of finite type if $m = 1$. And, if $m \geqslant 2$ and $n \geqslant 1$, we claim that $\mathcal{S}$ is a K-substructure of the K-structure described in Proposition 3.2 which is of finite type. This is obvious for $m = 2$ and $n = 1$. For $m = 2$ and $n \geqslant 2$, it follows from the condition (v); for, necessarily $(m-1)' < 2''$. And, to prove the statement for $m \geqslant 3$, we make use of the condition (vi) to deduce $(m-2)' < 1''$, and if, moreover, $n \geqslant 2$, of the condition (v) to get, in addition, $(m-1)' < 2''$.

Finally, let us point out that Propositions 2.5, 2.6, 3.1, 3.2, and 4.2 together with Lemma 8.4 and Kleiner's result of [11] imply that the weighted dimensions of indecomposable $\mathscr{S}$-spaces are bounded by 6.

The proof of Theorem A and Theorem E(1) is completed.

9. *K-Species*

In this section, we are going to apply the preceeding results on $\mathscr{S}$-spaces to the classification of K-species of finite type. Our method consists in showing that this problem can be reduced to the investigation of certain "linear" K-species and their relationship with $\mathscr{S}$-spaces. Recall that the index set I is assumed to be finite. If $\mathscr{Q} = (K_i, {}_iM_j)_{i,j\in I}$ is a K-species and ${}_iM_j \neq 0$, then the pair (i, j) of elements of I will be called *direct*. First, we want to show that, if $\mathscr{Q}$ is of finite type, then there are no indices $i, j \in I$ such that both pairs (i, j) and (j, i) are direct; in particular, in such a case, ${}_iM_i = 0$ for all $i \in I$. In fact, we shall prove the following more general Lemma.

LEMMA 9.1. *Let $\mathscr{Q} = (K_i, {}_iM_j)_{i,j\in I}$ be a K-species. Assume that, for a natural $n \geqslant 1$, a sequence $\{1, 2,..., n\}$ belongs to I such that either ${}_iM_{i+1} \neq 0$ or ${}_{i+1}M_i \neq 0$ for all $1 \leqslant i \leqslant n-1$, and that either ${}_nM_1 \neq 0$ or ${}_1M_n \neq 0$. Then $\mathscr{Q}$ is of strongly unbounded type.*

Proof. Without loss of generality, we may assume that ${}_nM_1 \neq 0$. First, for $1 \leqslant i \leqslant n$, define $K_1 - K_i$-bimodules ${}_1N_i$ inductively as follows: ${}_1N_1 = {}_{K_1}(K_1)_{K_1}$, and if $(i, i+1)$ is direct, then ${}_1N_{i+1} = {}_1N_i \otimes_{K_i} {}_iM_{i+1}$, whereas if $(i, i+1)$ is not direct (and hence, by our assumption, $(i+1, i)$ is direct), then ${}_1N_{i+1} = \operatorname{Hom}_{K_i}({}_{i+1}M_i, {}_1N_i)$. Let $W = {}_{K_1}({}_1N_n \otimes {}_nM_1)_{K_1}$; obviously, $W \neq 0$. Now, we are going to consider the K-species with a one-point index set $\mathscr{Q}' = (K_1, {}_{K_1}W_{K_1})$, and show that there is a full embedding T of $\mathfrak{R}(\mathscr{Q}')$ into $\mathfrak{R}(\mathscr{Q})$. Then, it will suffice to prove that $\mathscr{Q}'$ is of strongly unbounded type.

A representation of $\mathscr{Q}'$ is given by a pair (A, φ), where A is a right K_1-vector space and $\varphi\colon A_{K_1} \otimes {}_{K_1}W_{K_1} \to A_{K_1}$ is a K_1-linear map. We define the representation $\mathsf{T}(A, \varphi) = (V_i, {}_j\varphi_i)$ of $\mathscr{Q}$ in the following way: $V_i = A \otimes {}_1N_i$ for $1 \leqslant i \leqslant n$, and $V_i = 0$ otherwise; if $1 \leqslant i \leqslant n-1$ and $(i, i+1)$ is direct, then ${}_{i+1}\varphi_i$ is the identity, of

$$V_i \otimes {}_iM_{i+1} = A \otimes {}_1N_i \otimes {}_iM_{i+1} = A \otimes {}_1N_{i+1} = V_{i+1},$$

whereas if $(i, i+1)$ is not direct, then we define ${}_i\varphi_{i+1}$ by the inclusion

$$ {}_i\varphi_{i+1}^*: V_{i+1} = A \otimes N_{i+1} = A \otimes \mathrm{Hom}({}_{i+1}M_i, {}_1N_i) \hookrightarrow \mathrm{Hom}({}_{i+1}M_i, A \otimes {}_1N_i) $$
$$ = \mathrm{Hom}({}_{i+1}M_i, V_i); $$

moreover, we put ${}_1\varphi_n = \varphi$, that is

$$ {}_1\varphi_n: V_n \otimes {}_nM_1 = A \otimes {}_1N_n \otimes {}_nM_1 = A \otimes W \to A, $$

and ${}_j\varphi_i = 0$ for all remaining pairs (i, j). A map $\alpha: (A, \varphi) \to (A', \varphi')$ in $\mathfrak{R}(\mathscr{Q}')$ is a linear map $\alpha: A \to A'$ satisfying $\varphi'(\alpha \otimes 1) = \alpha\varphi$. Given α, we define $\mathsf{T}(\alpha) = (\alpha_i)$ by $\alpha_i = \alpha \otimes 1$ for $1 \leqslant i \leqslant n$, and $\alpha_i = 0$ otherwise. It is easy to see that T is a faithful functor. In order to show that T is full, assume there is given a map $(\beta_i): \mathsf{T}(A, \varphi) \to \mathsf{T}(A', \varphi')$ in $\mathfrak{R}(\mathscr{Q})$. Me show by induction that, for $1 \leqslant i \leqslant n$, we have $\beta_i = \beta_1 \otimes 1$. This is obvious in the case when $(i, i+1)$ is direct, whereas where $(i, i+1)$ is not direct, it follows from the fact that besides the equality

$$ \mathrm{Hom}(1, \beta_i)\, {}_i\varphi_{i+1}^* = ({}_i\varphi_{i+1})^* \beta_{i+1}, $$

we have also the equality

$$ \mathrm{Hom}(1, \beta_i)\, {}_i\varphi_{i+1}^* = ({}_i\varphi'_{i+1})^* (\beta_i \otimes 1), $$

where $({}_i\varphi'_{i+1})^*$ is a monomorphism. Furthermore, the relation

$$ \beta_1\varphi = \beta_{11}\varphi_n = {}_1\varphi_n'(\beta_n \otimes 1_{{}_nM_1}) = \varphi'(\beta_1 \otimes 1_W) $$

shows that β_1 is a map in $\mathfrak{R}(\mathscr{Q}')$, and thus $(\beta_i) = T(\beta_1)$. This shows that the functor $\mathfrak{R}(\mathscr{Q}') \to \mathfrak{R}(\mathscr{Q})$ defined by $(A, \varphi) \to (A \otimes {}_1N_i, {}_j\varphi_i)$ is a full embedding. Also, it is a dimension functor. For, if $\dim_{K_1}({}_1N_i) = n_i$, $1 \leqslant i \leqslant n$, then

$$ \dim A_K \leqslant \sum_{i=1}^{n} \dim(A \otimes {}_1N_i)_K = \left(\sum_{i=1}^{n} n_i\right) \dim A_K. $$

Now we have to consider $\mathfrak{R}(\mathscr{Q}')$. First, assume that ${}_{K_1}W_{K_1} = {}_{K_1}(K_1)_{K_1}$. Then, obviously, $\mathfrak{R}(\mathscr{Q}')$ is equivalent to the category $\mathfrak{M}_{K_1[\mathrm{t}]}$ of all right $K_1[\mathrm{t}]$-modules, where $K_1[\mathrm{t}]$ is the polynomial ring over K_1 in one indeterminate t; for, a $K_1[\mathrm{t}]$-module can be considered as a pair (A, φ), where A is a K_1-vector space and φ is an endomorphism of A_{K_1} (given by the multiplication by t). If ${}_{K_1}W_{K_1}$ is not isomorphic to ${}_{K_1}(K_1)_{K_1}$, then $\dim {}_{K_1}W \geqslant 2$ and $\dim W_{K_1} \geqslant 2$, because K operates centrally on W and K_1 is finite dimensional over K. Thus, according to Proposition 5.2, the category

$\mathfrak{R}({}_{K_1}W_{K_1})$ is not of finite type. Of course, we have to distinguish carefully between the K-species ${}_{K_1}W_{K_1}$ as defined in Section 5 having the index set with 2 elements) and the K-species $\mathcal{Q}' = (K_1, {}_{K_1}W_{K_1})$ (with a one-point index set). The objects of $\mathfrak{R}(\mathcal{Q}')$ are pairs (A, φ), where A is a K_1-vector space and $\varphi\colon A \otimes {}_{K_1}W_{K_1} \to A$, whereas the objects of $\mathfrak{R}({}_{K_1}W_{K_1})$ are triples (X, Y, ψ), where X, Y are K_1-vector spaces and $\psi\colon X \otimes {}_{K_1}W_{K_1} \to Y$. There is an embedding $\mathsf{T}\colon \mathfrak{R}(\mathcal{Q}') \to \mathfrak{R}({}_{K_1}W_{K_1})$, which maps the object (A, φ) of $\mathfrak{R}(\mathcal{Q}')$ to the triple (A, A, φ), but it should be noted that this functor is not full. A triple (X, Y, ψ) is obviously isomorphic to some $\mathsf{T}(A, \varphi)$ if and only if $\dim X_{K_1} = \dim Y_{K_1}$. So, given an indecomposable representation (X, Y, ψ) of ${}_{K_1}W_{K_1}$, with finite dimensional X_{K_1} and Y_{K_1}, we define a representation of $\mathcal{Q}'$ in the following way. Let $\dim X_{K_1} = m$ and $\dim Y_{K_1} = n$. If $m \leqslant n$, then we put

$$(X \oplus K_1^{n-m}, Y, \varphi) = (X, Y, \psi) \oplus \bigoplus_{n-m} (K, 0, 0),$$

whereas if $m \geqslant n$, we put

$$(X, Y \oplus K_1^{m-n}, \varphi) = (X, Y, \psi) \oplus \bigoplus_{m-n} (0, K, 0).$$

In both cases, the left side is of the form $\mathsf{T}(A, \varphi)$ for some representation (A, φ) of $\mathcal{Q}'$. We show that (A, φ) is always indecomposable and that, starting with nonisomorphic representations (X, Y, ψ) and (X', Y', ψ'), the resulting representations (A, φ) and (A', φ') of $\mathcal{Q}'$ are almost always nonisomorphic again. First, express (A, φ) as a direct sum of k indecomposable representations $(A^{(i)}, \varphi^{(i)})$ of $\mathcal{Q}'$; this is possible, because A is finite dimensional. Then, applying T we get

$$(A, A, \varphi) = \bigoplus_{i=1}^{k} (A^{(i)}, A^{(i)}, \varphi^{(i)}) \approx (X, Y, \psi) \oplus \bigoplus_{n-m} (K, 0, 0);$$

here we write the relation for the first case $n \geqslant m$. By the Krull–Schmidt theorem, we conclude that $n = m$, because $(K, 0, 0)$ cannot be isomorphic to an image under T. However, then $k = 1$, because (X, Y, ψ) was assumed to be indecomposable. Next, assume that (A, φ) and (A', φ') are isomorphic. If both $\mathsf{T}(A, \varphi)$ and $\mathsf{T}(A', \varphi')$ are of the same form, we conclude that the corresponding representations (X, Y, ψ) and (X', Y', ψ') are isomorphic. If $\mathsf{T}(A, \varphi)$ is of the first form, and $\mathsf{T}(A', \varphi')$ is of the second form, then it follows that

$$(X, Y, \psi) \oplus \bigoplus_{n-m} (K, 0, 0) \approx (X', Y', \psi') \oplus \bigoplus_{m-n} (0, K, 0),$$

and therefore, obviously $(X, Y, \psi) \approx (0, K, 0)$ and $(X', Y', \psi') \approx (K, 0, 0)$. Thus, in all other cases the above construction yields, for nonisomorphic indecomposable representations of ${}_{K_1}W_{K_1}$, nonisomorphic indecomposable representations of $\mathscr{Q}'$. Also, the dimension of (X, Y, ψ) is $(m + n)[K_1: K]$, whereas the dimension of the corresponding object in $\mathfrak{R}(\mathscr{Q}')$ is

$$2 \cdot \max\{m, n\} \cdot [K_1: K];$$

of course,

$$(m + n)[K_1: K] \leqslant 2 \cdot \max\{m, n\} \cdot [K_1: K] \leqslant 2(m + n)[K_1: K].$$

Now, $\mathfrak{R}({}_{K_1}W_{K_1})$ is of strongly unbounded type by Proposition 5.2. In the same way as in the case of a full embedding which is a dimension functor, we see also here that, together with $\mathfrak{R}({}_{K_1}W_{K_1})$, the category $\mathfrak{R}(\mathscr{Q}')$ is of strongly unbounded type, as well.

Also, we may use Proposition 5.2 to show that, for a K-species $\mathscr{Q} = (K_i, {}_iM_j)_{i,j \in I}$, the dimensions of the vector spaces ${}_{K_i}({}_iM_j)$ and $({}_iM_j)_{K_j}$ are bounded. Indeed, using a full embedding of $\mathfrak{R}({}_{K_i}({}_iM_j)_{K_j})$ into $\mathfrak{R}(\mathscr{Q})$, we get

LEMMA 9.2. *Let $\mathscr{Q} = (K_i, {}_iM_j)_{i,j \in I}$ be a K-species which is not of strongly unbounded type. Then, for $i, j \in I$, either ${}_iM_j = 0$ or ${}_jM_i = 0$, and*

$$\dim_{K_i}({}_iM_j) \times \dim({}_iM_j)_{K_j} \leqslant 3.$$

Recall that the diagram of a given K-species $\mathscr{Q} = (K_i, {}_iM_j)_{i,j \in I}$ consists of the elements of I as vertices, and has

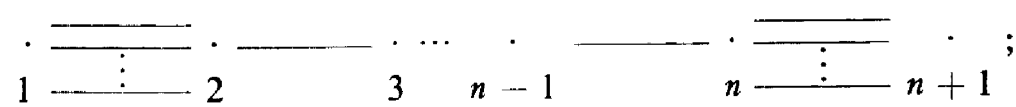
$$\dim_{K_i}({}_iM_j) \times \dim({}_iM_j)_{K_j} + \dim_{K_j}({}_jM_i) \times \dim({}_jM_i)_{K_i}$$

edges between the vertices i and j. In addition, the fact that ${}_jM_i = 0$ and $\dim {}_{K_i}({}_iM_j) < \dim({}_iM_j)_{K_j}$ has been marked by an arrow $\mathrm{i} \Rrightarrow \mathrm{j}$. Thus, according to Lemma 9.1, the diagram of a K-species of finite type does not contain any loops or circles and, between two different vertices, there is either no line, or one line, or a double line with an arrow, or a triple line with an arrow. If there is a single line between i and j, and, say, ${}_iM_j \neq 0$, then we have ${}_{K_i}({}_iM_j) \approx {}_{K_i}(K_i)$, and $({}_iM_j)_{K_j} \approx (K_j)_{K_j}$, and we may identify K_i and K_j in such a way that ${}_{K_i}({}_iM_j)_{K_i} = {}_{K_i}(K_i)_{K_i}$. Now, assume that there is a double or triple line between i and j with an arrow $\mathrm{i} \Longrightarrow \mathrm{j}$ or $\mathrm{i} \Rrightarrow \mathrm{j}$, and that ${}_iM_j \neq 0$. Then, necessarily, $\dim {}_{K_i}({}_iM_j) = 1$, and thus again ${}_{K_i}({}_iM_j) \approx {}_{K_i}(K_i)$. Consequently, we may identify K_j with a subfield of K_i, such that ${}_{K_i}({}_iM_j)_{K_j} = {}_{K_i}(K_i)_{K_j}$ and since $\dim({}_iM_j)_{K_j} = 2$ or 3, $K_j \subset K_i$ is a proper inclusion. Similarly, if ${}_jM_i \neq 0$, we may assume $K_j \subseteq K_i$ and

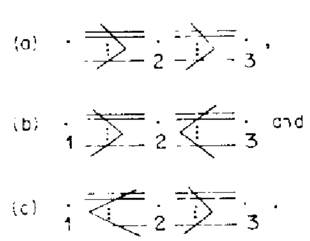
${}_{K_j}({}_jM_i)_{K_i} = {}_{K_j}(K_i)_{K_i}$. It should be noted that for a K-species of finite type these identifications are mutually compatible according to Lemma 9.1. The next lemma deals with the situation, when more than one multiple line occurs in a connected component of the diagram of a species.

LEMMA 9.3. *Let $\mathscr{Q} = (K_i, {}_iM_j)_{i,j \in I}$ be a K-species which is not of strongly unbounded type whose diagram is connected. Then there is at most one pair (i, j) of elements of I such that* $\dim {}_{K_i}({}_iM_j) \times \dim({}_iM_j)_{K_j} > 1$.

Proof. Assuming the contrary, there is a sequence $\{1, 2,..., n, n+1\}$ of elements of I such that 1, 2, and n, $n+1$ are connected by multiple lines, whereas for $2 \leqslant i \leqslant n-1$, each pair $(i, i+1)$ is connected by a single line:



here, we have omitted the arrows. We restrict ourselves to the case $n = 2$ with the additional assumption that $I = \{1, 2, 3\}$, and show that the corresponding K-species are not of finite type. Then, using an appropriate embedding functor, we may translate this result to the general situation. We have to distinguish three different cases, namely



Case (a). There are given three fields $F = K_1$, $G = K_2$, and $H = K_3$ with $H \subset G \subset F$ such that ${}_1M_2 = {}_FF_G$ or ${}_2M_1 = {}_GF_F$, and similarly ${}_2M_3 = {}_GG_H$ or ${}_3M_2 = {}_HG_G$. Since H is a proper subfield of G and G is a proper subfield of F, we have $\dim {}_HF \geqslant 4$, and so we may use Proposition 5.2 for the K-species ${}_HF_F$. Let $\mathfrak{Rm}({}_HF_F)$ be the full subcategory of $\mathfrak{R}({}_HF_F)$ of all objects (X_H, Y_F, φ) with an inclusion φ; thus, X is an H-subspace of Y_F. By Proposition 5.2 this is an additive category of strongly unbounded type. We define a full embedding T, which can be easily seen to be a dimension functor, of $\mathfrak{Rm}({}_HF_F)$ into $\mathfrak{R}(\mathscr{Q})$ as follows. Let $\mathsf{T}(X_H, Y_F, \varphi) = (V_i, {}_j\varphi_i)$ with $V_1 = Y_F$, $V_2 = Y_G$; if (2, 3) is direct, then $V_3 = Y_H/X_H$, whereas $V_3 = X_H$ otherwise. For (1, 2), mappings ${}_j\varphi_i$ are defined by the canonical isomorphisms

$${}_2\varphi_1 \colon Y_F \otimes {}_FF_G \cong Y_G \quad \text{if} \quad (1, 2) \quad \text{is direct,}$$

and by

$${}_1\varphi_2{}^* \colon Y_G \cong \mathrm{Hom}_F({}_GF_F, Y_F) \quad \text{otherwise.}$$

If (2, 3) is direct, then we take for ${}_3\varphi_2$ the canonical projection

$$ {}_3\varphi_2\colon Y_G \otimes {}_G G_H \cong Y_H \to Y_H/X_H; $$

otherwise, we use the inclusion

$$ {}_2\varphi_3{}^*\colon X_H \hookrightarrow Y_H \cong \mathrm{Hom}_G({}_H G_G, Y_G). $$

Case (b). There are given three fields $F = K_1$, $G = K_2$ and $F' = K_3$ with $G \subset F$, $G \subset F'$, such that either ${}_1M_2 = {}_F F_G$ or ${}_2M_1 = {}_G F_F$, and either ${}_2M_3 = {}_G F'{}_{F'}$ or ${}_3M_2 = {}_{F'}F'{}_G$. Moreover, we may assume that at most one of the pairs (1, 2) and (2, 3) is not direct, because otherwise we exchange the indices 1 and 3. We form ${}_F W_{F'} = {}_F F_G \otimes {}_G F'{}_F$. Since $\dim {}_F W = \dim {}_G F' > 1$ and $\dim W_{F'} = \dim F_G > 1$, we know, by Proposition 5.2, that $\mathfrak{R}({}_F W_{F'})$ is of strongly unbounded type. We define now a full embedding $\mathsf{T}\colon \mathfrak{R}({}_F W_{F'}) \to \mathfrak{R}(\mathscr{Q})$, which is a dimension functor. First, assume that (2, 3) is direct.

Given an object $(X_F, Y_{F'}, \varphi)$ of $\mathfrak{R}({}_F W_{F'})$, put $\mathsf{T}(X, Y, \varphi) = (V_i, {}_j\varphi_i)$, where $V_1 = X_F$, $V_2 = X_G$, $V_3 = Y_{F'}$, where ${}_3\varphi_2$ is given by

$$ {}_3\varphi_2\colon X_G \otimes {}_G F'_{F'} \approx X_F \otimes {}_F F_G \otimes {}_G F'_{F'} \xrightarrow{\varphi} Y_{F'}, $$

and where ${}_2\varphi_1$ or ${}_1\varphi_2$ are given by

$$ {}_2\varphi_1\colon X_F \otimes {}_F F_G \cong X_G $$

in the case that (1, 2) is direct, or by the canonical isomorphism

$$ {}_1\varphi_2{}^*\colon X_G \cong \mathrm{Hom}_F({}_G F_F, Y_F) $$

otherwise. On the other hand, if (2, 3) is not direct, then we only have to consider the case where (1, 2) is direct, and we define $\mathsf{T}(X, Y, \varphi) = (V_i, {}_j\varphi_i)$, where $V_1 = X_F$, $V_2 = Y_G$, $V_3 = Y_{F'}$ with ${}_2\varphi_1$ given by

$$ {}_2\varphi_1{}^*\colon X_F \xrightarrow{\varphi^*} \mathrm{Hom}_{F'}({}_F F_G \otimes {}_G F'_{F'}, Y_{F'}) \approx \mathrm{Hom}_G({}_F F_G, \mathrm{Hom}_{F'}({}_G F'_{F'}, Y_{F'})) $$
$$ \approx \mathrm{Hom}_G({}_F F_G, Y_G), $$

and with the canonical isomorphism

$$ {}_2\varphi_3\colon Y_{F'} \otimes {}_{F'} F_G' \cong Y_G. $$

Again, it can be verified immediately that T is a full embedding and a dimension functor.

Case (c). There is given a field $F = K_2$ with two subfields $G = K_1$ and $H = K_3$ such that either ${}_1M_2 = {}_G F_F$ or ${}_2M_1 = {}_F F_G$, and either ${}_2M_3 = {}_F F_H$

or ${}_3M_2 = {}_HF_F$. We consider the K-structure $\mathscr{S} = \mathscr{I}_1(G) \sqcup \mathscr{I}_1(H)$ and, by Proposition 8.2, we know that $\mathscr{S}$ is of strongly unbounded type. Thus, it is enough to define a full embedding T of $\mathfrak{S}(\mathscr{S})$ into $\mathfrak{R}(\mathscr{Q})$. If (W_F, U_G, V_H) is an $\mathscr{S}$-space, we define $\mathsf{T}(W, U, V) = (V_i, {}_i\varphi_j)$ as follows. Always, put $V_2 = W_F$. If (1, 2) is direct, then $V_1 = U_G$ and ${}_2\varphi_1$ is given by the inclusion

$$ {}_2\varphi_1{}^*\colon U_G \hookrightarrow W_G \simeq \mathrm{Hom}_F({}_GF_F, W_F), $$

whereas if (1, 2) is not direct, then $V_1 = W_G/U_G$ and ${}_1\varphi_2$ is the canonical projection

$$ {}_1\varphi_2\colon W_F \otimes {}_FF_G \simeq W_G \to W_G/U_G . $$

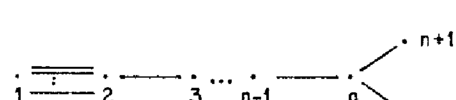
Similarly, if (2, 3) is direct, then $V_3 = W_H/V_H$, and ${}_3\varphi_2$ is the canonical projection, whereas if (2, 3) is not direct, then $V_3 = V_H$ with the inclusion ${}_2\varphi_3{}^*$.

This concludes the proof of Lemma 9.3.

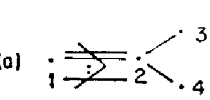
In a similar way we can show that any connected component of a diagram of a K-species of finite type comes either from a "quiver" (see [7]) or is linear. Here, we call a diagram *linear*, if the vertices can be written as $I = \{1, 2,..., n\}$ and, for $i, j \in I$, there are edges between i and j if and only if $j = i + 1$ or $j = i - 1$.

LEMMA 9.4. *Let $\mathscr{Q} = (K_i, {}_iM_j)_{i,j\in I}$ be a K-species which is not of strongly unbounded type whose diagram is connected. If there is a pair (i, j) of elements of I such that* $\dim_{K_i}({}_iM_j) \times \dim({}_iM_j)_{K_j} > 1$, *then the diagram of $\mathscr{Q}$ is linear.*

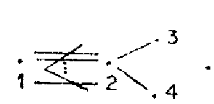
Proof. We may label the elements of I in such a way that $i = 1, j = 2$ (so that 1 and 2 are connected by more than one edge), that $\{1, 2,..., n\} \subseteq I$ with $n \geqslant 2$ is linear and that n and $n + 1$, as well as n and $n + 2$, are connected by an edge:



Again, it is obviously sufficient to prove the lemma for the particular case $n = 2$: $I = \{1, 2, 3, 4\}$. Thus, we have to consider the following two diagrams

(a) 

and

(b) 

Case (a). We are given a field $F = K_1$ and a proper subfield $G = K_2 = K_3 = K_4$ of F. We consider the K-structure $\mathscr{S} = \mathscr{I}_1(G) \sqcup \mathscr{I}_1(G)$, which is of strongly unbounded type according to Proposition 8.2. As in the last case (c) of the proof of the previous Lemma 9.3, we define a full embedding $\mathsf{T}\colon \mathfrak{S}(\mathscr{S}) \to \mathfrak{R}(\mathscr{Q})$ as follows: If (W_F, U_G, V_G) is an $\mathscr{S}$-space, define $\mathsf{T}(W, U, V) = (V_i, {}_j\varphi_i)$ with $V_1 = W_F$, $V_2 = W_G$, and

$$ {}_2\varphi_1\colon W_F \otimes {}_FF_G \simeq W_G \qquad \text{or} \qquad {}_1\varphi_2{}^*\colon W_G \simeq \operatorname{Hom}_F({}_GF_F, W_F), $$

according to whether $(1, 2)$ is direct or not; moreover, define

$$ {}_3\varphi_2\colon W_F \otimes {}_FF_F \simeq W_F \twoheadrightarrow W_F/U_F $$

with the canonical projection if $(2, 3)$ is direct, and

$$ {}_2\varphi_3\colon U_F \otimes {}_FF_F \approx U_F \hookrightarrow W_F \quad \text{otherwise;} $$

${}_4\varphi_2$ or ${}_2\varphi_4$ is defined similarly by using V instead of U.

Case (b). We are given a proper subfield $G = K_1$ of a field $F = K_2 = K_3 = K_4$. The K-structure $\mathscr{S} = \mathscr{I}_1(G) \sqcup \mathscr{I}_1(F) \sqcup \mathscr{I}_1(F)$ is of strongly unbounded type, according to Proposition 7.1 and thus it is sufficient to define a full embedding $\mathsf{T}\colon \mathfrak{S}(\mathscr{S}) \to \mathfrak{R}(\mathscr{Q})$. Denote the image of the S-space (W_F, U_G, V_F, V_F') under T by $(V_i, {}_j\varphi_i)$ and define $V_2 = W_F$. The definition of V_1, V_3 and V_4 and the corresponding ${}_j\varphi_i$'s will, however, depend on the fact, whether $(1, 2)$, $(2, 3)$, and $(2, 4)$, respectively, are direct or not. For example, if $(1, 2)$ is direct, then we define ${}_2\varphi_1$ by the inclusion

$$ {}_2\varphi_1{}^*\colon U_G \hookrightarrow W_F \simeq \operatorname{Hom}_F({}_GF_F, W_F), $$

whereas if $(1, 2)$ is not direct, we take the projection

$$ {}_1\varphi_2\colon W_F \otimes {}_FF_G \simeq W_G \twoheadrightarrow W_G/U_G\,; $$

and, similarly for $(2, 3)$ and $(2, 4)$.

We shall need certain results on K-species of type $\mathbb{A}_{n+1}$, that is on K-species whose diagram is the Dynkin diagram of type $\mathbb{A}_{n+1}$:

$$ \underset{0}{\cdot} \!\!-\!\!-\!\!-\!\! \underset{1}{\cdot} \!\!-\!\!-\!\!-\!\! \underset{2}{\cdot} \cdots \underset{n-1}{\cdot} \!\!-\!\!-\!\!-\!\! \underset{n}{\cdot}\,. $$

Note that a representation of such a K-species is given by $n + 1$ vector spaces $V_0, V_1, \ldots, V_n$ and, for each $0 \leqslant i \leqslant n - 1$, either by a map ${}_{i+1}\varphi_i\colon V_i \to V_{i+1}$ or by a map ${}_i\varphi_{i+1}\colon V_{i+1} \to V_i$, according to whether $(i, i + 1)$ is direct or not. Given a K-species $\mathscr{Q} = (K_i, {}_iM_j)_{i,j \in I}$ of type $\mathbb{A}_{n+1}$ with $n \geqslant 1$ and, say, with $K_i = F$ for all i, we consider the K-structure

$\mathscr{S} = \mathscr{I}_n(F)$ and define a full embedding T of $\mathfrak{S}(\mathscr{S})$ into $\mathfrak{R}(\mathscr{Q})$ as follows. If $(W, U_1, ..., U_n)$ is an $\mathscr{S}$-space, let $\mathsf{T}(W, U_1, ..., U_n) = (V_i, {}_j\varphi_i)$, where $V_0 = W$ and where, if (0, 1) is direct, $V_1 = W/U_1$ and

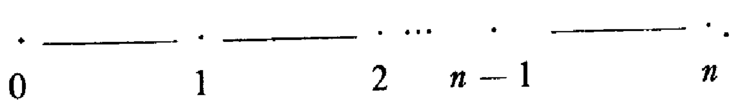
$$_1\varphi_0 = W \twoheadrightarrow W/U_1,$$

whilst, if (0, 1) is not direct, $V_1 = U_n$ and ${}_0\varphi_1$ is the inclusion

$$_0\varphi_1 \colon U_n \hookrightarrow W.$$

Assuming, by induction, that a functor $\mathsf{T}' \colon \mathfrak{S}(\mathscr{S}') \to \mathfrak{R}(\mathscr{Q}')$ has been defined for the K-structure $\mathscr{S}' = \mathscr{I}_{n-1}(F)$ and the K-species $\mathscr{Q}' = (K_i, {}_iM_j)_{i,j \in I \setminus \{0\}}$ of type $\mathbb{A}_n$, we complete the definition of T by applying T′ to the $\mathscr{S}'$-space $(W/U_1, U_2/U_1, ..., U_n/U_1)$ in the case that (0, 1) is direct, and to the $\mathscr{S}'$-space $(U_n, U_1, ..., U_{n-1})$ otherwise (see[7]). In this way, the images under T are just the 0-faithful representations of $\mathscr{Q}$ in the following sense: If $\mathscr{Q} = (K_i, {}_iM_j)_{i,j \in I}$ is a K-species and $k \in I$, then a representation $\mathbf{V} = (V_i, {}_j\varphi_i)$ of $\mathscr{Q}$ is called *k-faithful* if, for every direct decomposition $\mathbf{V} = \mathbf{V}' \oplus \mathbf{V}''$ in $\mathfrak{R}(\mathscr{Q})$ such that $V_k'' = 0$, necessarily $\mathbf{V}'' = 0$.

LEMMA 9.5. (P. GABRIEL [7]). *Let $\mathscr{Q} = (K_i, {}_iM_j)_{i,j \in I}$ be a K-species whose diagram is*



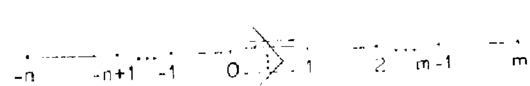
Then,

(a) *Every representation* $\mathbf{V}$ *of* $\mathscr{Q}$ *is a direct sum* $\mathbf{V} = \mathbf{V}' \oplus \mathbf{V}''$ *of a 0-faithful representation* $\mathbf{V}'$ *and a representation* $\mathbf{V}''$ *with* $V_0'' = 0$.

(b) *A representation* $\mathbf{V}$ *of* $\mathscr{Q}$ *is 0-faithful if and only if, for each* $0 \leqslant i \leqslant n-1$, ${}_{i+1}\varphi_i$ *is an epimorphism if the pair* $(i, i+1)$ *is direct, and* ${}_i\varphi_{i+1}$ *is a monomorphism otherwise.*

(c) *The functor* T *provides an equivalence between* $\mathfrak{S}(\mathscr{I}_n(F))$ *and the full subcategory of* $\mathfrak{R}(\mathscr{Q})$ *of all 0-faithful representations and is a dimension functor.*

It follows from Lemmas 9.1 to 9.4 that, investigating a K-species $\mathscr{Q} = (K_i, {}_iM_j)_{i,j \in I}$ of finite type whose diagram contains multiple edges, we may assume that the diagram is of the form

(*) 

with $m \geqslant 1$ and $n \geqslant 0$. In this case, we call 0 the focal point.

LEMMA 9.6. *Let* (*) *be the diagram of a K-species* $\mathcal{Q} = (K_i, {}_iM_j)_{i,j\in I}$. *Let* $K_i = G$ *for* $1 \leqslant i \leqslant m$ *and* $K_i = F$ *for* $-n \leqslant i \leqslant 0$. *Then every representation of* $\mathcal{Q}$ *is the direct sum of a* 0*-faithful representation* $\mathbf{V}'$ *and a representation* $\mathbf{V}''$ *with* $V_0'' = 0$. *Moreover, the full subcategory of* $\mathfrak{R}(\mathcal{Q})$ *of all* 0*-faithful representations is equivalent to* $\mathfrak{S}(\mathcal{S})$, *where* $\mathcal{S} = \mathcal{I}_m(G) \sqcup \mathcal{I}_n(F)$, *with respect to a dimension functor.*

Proof. First, consider the case $n = 0$. Form $\mathcal{Q}$, we derive a K-species $\mathcal{Q}' = (K_i', {}_iM_j')_{i,j\in I}$ of type $\mathbb{A}_{m+1}$ in the following way: $K_i' = G$ for all $i \in I$, and (i, j) is direct for $\mathcal{Q}'$ if and only if (i, j) is direct for $\mathcal{Q}$. If $\mathbf{V} = (V_i, {}_j\varphi_i)$ is a representation of $\mathcal{Q}$, then we may consider $\mathbf{V}$ also as a representation of $\mathcal{Q}'$: We just consider V_0 no longer as an F-space but as a G-space, and if (0, 1) is direct, we deal with

$$ {}_1\varphi_0\colon (V_0)_G \simeq (V_0)_F \otimes {}_FF_G \to V_1, $$

whereas if (0, 1) is not direct, we use

$$ {}_0\varphi_1{}^*\colon V_1 \to \operatorname{Hom}_F({}_GF_F, (V_0)_F) \simeq (V_0)_G. $$

If we decompose the representation $\mathbf{V}$ in accordance with Lemma 9.5(a): $\mathbf{V} = \mathbf{V}' \oplus \mathbf{V}''$, where $\mathbf{V}'$ and $\mathbf{V}''$ are representations of $\mathcal{Q}'$ such that $\mathbf{V}'$ is 0-faithful and $V_0'' = 0$, then, obviously, $\mathbf{V}''$ is also a representation of $\mathcal{Q}$, because the restrictions of $\mathcal{Q}$ and $\mathcal{Q}'$ to $\{1, 2,..., n\}$ coincide. But, also $\mathbf{V}'$ can be considered as a representation of $\mathcal{Q}$, since $V_0' = V_0$, and, if (0, 1) is direct,

$$ {}_1\varphi_0 = {}_1\varphi_0' \oplus {}_1\varphi_0''\colon V_0 \otimes {}_FF_G \simeq V_0' \oplus 0 \to V_1' \oplus V_1'' = V_1, $$

so that ${}_1\varphi_0'$ is a map from $V_0 \otimes {}_FF_G$ to V_1'. Similarly, if (0, 1) is not direct, we have

$$ {}_0\varphi_1{}^* = {}_0\varphi_1' \oplus {}_0\varphi_1''\colon V_1 = V_1' \oplus V_1'' \to V_0' \oplus 0 \simeq \operatorname{Hom}_F({}_GF_F, V_0), $$

and thus ${}_0\varphi_1'$ maps V_1' into $\operatorname{Hom}_F({}_GF_F, V_0)$. This shows that $\mathbf{V} = \mathbf{V}' \oplus \mathbf{V}''$ is a decomposition in $\mathfrak{R}(\mathcal{Q})$ such that $V_0'' = 0$ and such that ${}_{i+1}\varphi_i'$ is an epimorphism for every direct pair $(i, i+1)$, whilst $({}_i\varphi_{i+1}')^*$ is a monomorphism otherwise. Here, the last assertion follows from Lemma 9.5(b). But it is obvious that a representation $\mathbf{V}'$ of $\mathcal{Q}$ satisfying the above conditions on ${}_{i+1}\varphi_i'$ and $({}_i\varphi_{i+1}')^*$ has to be 0-faithful, since every direct summand of $\mathbf{V}'$ satisfies again the same conditions.

Now, in the general case with n not necessarily equal to 0, we consider, for a given representation $\mathbf{V} = (V_i, {}_j\varphi_i)$ of $\mathcal{Q}$, the restrictions to $\{-n,..., -1, 0\}$ and to $\{0, 1,..., m\}$. It is easy to see that in this way we get a decomposition

$\mathbf{V} = \mathbf{V}' \oplus \mathbf{V}'' \oplus \mathbf{V}'''$ with $V_i' = 0$ for $i \leqslant 0$, with $V_i'' = 0$ for $i \geqslant 0$, and with a 0-faithful representation $\mathbf{V}'''$.

Also, starting with the K-structure $\mathscr{S} = \mathscr{I}_m(G) \sqcup \mathscr{I}_n(F)$, we define a full embedding $\mathsf{T}\colon \mathfrak{S}(\mathscr{S}) \to \mathfrak{R}(\mathscr{Q})$ in the following way. Let $I' = \{1, 2, \ldots, m\}$ and $I'' = \{-n, \ldots, -1, 0\}$; furthermore, let $\mathscr{Q}'$ be the restriction of $\mathscr{Q}$ to I', and similarly $\mathscr{Q}''$ the restriction of $\mathscr{Q}$ to I''. Let

$$\mathsf{T}'\colon \mathfrak{S}(\mathscr{I}_{m-1}(G)) \to \mathfrak{R}(\mathscr{Q}') \qquad \text{and} \qquad \mathsf{T}''\colon \mathfrak{S}(\mathscr{I}_n(F)) \to \mathfrak{R}(\mathscr{Q}'')$$

be the corresponding full embeddings defined in the paragraph preceeding Lemma 9.5. Then, if $(W, X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ is an $\mathscr{S}$-space, define its image $(V_i, {}_j\varphi_i)_{i,j\in I}$ under T as follows: If (0, 1) is direct, put

$$(V_i, {}_j\varphi_i)_{i,j\in I'} = \mathsf{T}'(W/X_1, X_2/X_1, \ldots, X_m/X_1)$$

and

$${}_1\varphi_0\colon W_F \otimes {}_FF_G \simeq W_G \twoheadrightarrow W/X_1;$$

if (0, 1) is not direct, put

$$(V_i, {}_j\varphi_i)_{i,j\in I'} = \mathsf{T}'(X_m, X_1, \ldots, X_{m-1})$$

and

$${}_0\varphi_1\colon X_m \hookrightarrow W_G \cong \mathrm{Hom}_F({}_GF_F, W_F)$$

and,

$$(V_i, {}_j\varphi_i)_{i,j\in I''} = \mathsf{T}''(W, Y_1, \ldots, Y_n).$$

Then, as in the proof of Lemma 9.5 (see [7]), it follows that the 0-faithful representations are just those which are isomorphic to the images of $\mathscr{S}$-spaces under T.

This completes the proof of Lemma 9.6.

Now, it is easy to complete the following proof.

Proof of Theorem B *and Theorem* E(2). If $\mathscr{S} = \mathscr{I}_m(G) \sqcup \mathscr{I}_n(F)$ is a K-structure for F, and G is a proper subfield of F, then by Theorem A, $\mathfrak{S}(\mathscr{S})$ is of finite type if and only if $\mathscr{S}$ is of one of the following forms:

(a) $\mathscr{I}_1(G) \sqcup \mathscr{I}_n(F)$, $[F\colon G] = 2$, $n \geqslant 0$;

(b) $\mathscr{I}_m(G)$, $[F\colon G] = 2$, $m \geqslant 1$;

(c) $\mathscr{I}_2(G) \sqcup \mathscr{I}_1(F)$, $[F\colon G] = 2$;

(d) $\mathscr{I}_1(G)$, $[F\colon G] = 3$;

and, if $\mathfrak{S}(\mathscr{S})$ is not of finite type, then it is of strongly unbounded type.

If $\mathscr{Q} = (K_i, {}_iM_j)_{i,j\in I}$ is a K-species as in Lemma 9.6, then there is obviously only a finite number of indecomposable representations $\mathbf{V}$ with

$V_0 = 0$, since these representations can be considered as representations of the restriction of $\mathcal{Q}$ to $\Gamma \setminus \{0\}$, which is the disjoint union of two K-species of type $\mathbb{A}_m$ and $\mathbb{A}_n$, respectively. The restriction on the corresponding K-structure $\mathcal{S}$ shows that $\mathcal{Q}$ is of finite type if and only if its diagram is of one of the following types:

(a) of type $\mathbb{B}_{n+2}$, when $\mathcal{S} = \mathcal{I}_1(G) \sqcup \mathcal{I}_n(F)$ with $[F: G] = 2$, and the number of indecomposable representations is

$$\tfrac{1}{2}(n+1)(n+6) + \tfrac{1}{2}n(n+1) + 1 = (n+2)^2;$$

(b) of type $\mathbb{C}_{m+1}$, when $\mathcal{S} = \mathcal{I}_m(G)$ with $[F: G] = 2$, and the number of indecomposable representations is

$$\tfrac{1}{2}(m+1)(m+2) + \tfrac{1}{2}m(m+1) = (m+1)^2;$$

(c) of type $\mathbb{F}_4$, when $\mathcal{S} = \mathcal{I}_2(G) \sqcup \mathcal{I}_1(F)$ with $[F: G] = 2$, and the number of indecomposable representations is $20 + 1 + 3 = 24$;

(d) of type $\mathbb{G}_2$, when $\mathcal{S} = \mathcal{I}_1(G)$ with $[F: G] = 3$, and the number of indecomposable representations is $5 + 1 = 6$.

10. *K-Algebras*

In this final section, we are going to derive some conditions for a finite dimensional K-algebra to be of finite type.

Let $\mathcal{Q} = (K_i, {}_iM_j)_{i,j \in I}$ be a K-species. Construct the *tensor algebra* $\mathcal{T}(\mathcal{Q})$ of $\mathcal{Q}$ in the following way. First, define the semisimple K-algebra $\Gamma = \prod_{i \in I} K_i$ and consider $M = \bigoplus_{i,j \in I} {}_iM_j$ as a Γ-Γ-bimodule, where Γ acts on ${}_iM_j$ from the left via the projection $\Gamma \to K_i$ and from the right via the projection $\Gamma \to K_j$. We may also write ${}_\Gamma M_\Gamma = \bigoplus_{i,j \in I} {}_iM_j$, because this decomposition is a direct decomposition of Γ-Γ-bimodules. Now, for $n \geqslant 0$, we form the n-fold tensor product $M^{(n)} = M \otimes_\Gamma M \otimes_\Gamma \cdots \otimes_\Gamma M$, where $M^{(0)} = \Gamma$, $M^{(1)} = M$, and put $\mathcal{T}(\mathcal{Q}) = \bigoplus_{n=0}^{\infty} M^{(n)}$.

Obviously, $\mathcal{T}(\mathcal{Q})$ can be made into a ring, or even a K-algebra, defining the multiplication through the canonical isomorphism $M^{(i)} \otimes M^{(j)} \to M^{(i+j)}$ and extending it by the distributivity. The tensor algebra $\mathcal{T}(\mathcal{Q})$ is also called the "maximal" ring of ${}_\Gamma M_\Gamma$ (see [10]).

PROPOSITION 10.1. *Let $\mathcal{Q}$ be a K-species. Then, the category $\mathfrak{R}(\mathcal{Q})$ of all representations of $\mathcal{Q}$ and the category $\mathfrak{M}_{\mathcal{T}(\mathcal{Q})}$ of all right $\mathcal{T}(\mathcal{Q})$-modules are equivalent by a dimension functor.*

Proof. We define two functors $\mathsf{R}: \mathfrak{M}_{\mathcal{T}(\mathcal{Q})} \to \mathfrak{R}(\mathcal{Q})$ and $\mathsf{P}: \mathfrak{R}(\mathcal{Q}) \to \mathfrak{M}_{\mathcal{T}(\mathcal{Q})}$ as follows. If $X_{\mathcal{T}(\mathcal{Q})}$ is a right $\mathcal{T}(\mathcal{Q})$-module, then we may consider X also as a right Γ-module, because Γ is a subring of $\mathcal{T}(\mathcal{Q})$. In this way, X

decomposes uniquely into $X = \oplus_{i\in I} X_i$ with right K_i-modules X_i and $X_i K_j = 0$ for $i \neq j$. Also, since M is a Γ-Γ-submodule of $\mathcal{T}(\mathcal{Q})$, the scalar multiplication of X_Γ gives rise to a right Γ-map $\varphi: X \otimes_\Gamma M_\Gamma \to X_\Gamma$, and, since $X_i \otimes_\Gamma {}_kM_j = 0$ for $i \neq k$, the map φ is given by

$$\varphi: \bigoplus_{i,j} \Big(X_i \bigotimes_\Gamma {}_iM_j\Big) \to \bigoplus_i X_i .$$

But here $X_i \otimes_\Gamma {}_iM_j$ maps into X_j, and thus φ is determined by the right K_j-linear maps ${}_j\varphi_i$: $X_i \otimes_\Gamma {}_iM_j = X_i \otimes_{K_i} {}_iM_j \to X_j$. We define $\mathfrak{R}(X_{\mathcal{T}(\mathcal{Q})}) = (X_i, {}_j\varphi_i)$. Observe that the right $\mathcal{T}(\mathcal{Q})$-module structure on X is uniquely determined by the map $\varphi: X \otimes M_\Gamma \to X_\Gamma$, and since $\varphi = \oplus\, {}_j\varphi_i$, also by the family of the ${}_j\varphi_i$'s. For, the scalar multiplication of $M^{(n)}$ on X can be defined inductively by $\varphi^{(n)}: X \otimes M^{(n)} \to X$ with $\varphi^{(1)} = \varphi$ and

$$\varphi^{(n+1)} = \varphi(\varphi^{(n)} \otimes 1): X \bigotimes_\Gamma M^{(n+1)}$$

$$= \Big(X \bigotimes_\Gamma M^{(n)}\Big) \bigotimes_\Gamma M \xrightarrow{\varphi^{(n)}\otimes 1} X \bigotimes_\Gamma M \xrightarrow{\quad\varphi\quad} X, \qquad (*)$$

because the operation has to be associative.

Also, given $\alpha: X \to Y$ in $\mathfrak{M}_{\mathcal{T}(\mathcal{Q})}$ and $\mathsf{R}(X) = (X_i, {}_j\varphi_i)$, $\mathsf{R}(Y) = (Y_i, {}_j\psi_i)$, we note that α is, in particular, a right Γ-map, and thus $\alpha(X_i) \subseteq Y_i$, and that it is determined by the family of restrictions $\alpha_i: X_i \to Y_i$. We let $\mathsf{R}(\alpha) = (\alpha_i)$. The fact that α is a $\mathcal{T}(\mathcal{Q})$-map implies that ${}_j\psi_i(\alpha_i \otimes 1) = \alpha_j\, {}_j\varphi_i$ for all $i, j \in I$, and therefore (α_i) is, in fact, a map in $\mathfrak{R}(\mathcal{Q})$.

Conversely, given $(X_i, {}_j\varphi_i)$ in $\mathfrak{R}(\mathcal{Q})$, define $\mathsf{P}(X_i, {}_j\varphi_i)$ additively by $X = \oplus_{i\in I} X_i$, where $\Gamma = \prod_{i\in I} K_i$ operates on X_i via the projection $\Gamma \to K_i$. The scalar multiplication by $M^{(n)}$ on X is defined inductively by $\varphi^{(n)}: X \otimes M^{(n)} \to X$ with

$$\varphi^{(1)} = \bigoplus_{i,j\in I} {}_j\varphi_i: X \bigotimes_\Gamma M = \bigoplus_{i,j\in I} \Big(X_i \bigotimes_\Gamma {}_iM_j\Big)$$

$$= \bigoplus_{i,j\in I} \Big(X_i \bigotimes_{K_i} {}_iM_j\Big) \to \bigoplus_{j\in I} X_j = X,$$

(using the fact that, for $i \neq k$, $X_i \otimes_\Gamma {}_kM_j = 0$) and with $\varphi^{(n+1)} = \varphi(\varphi^{(n)} \otimes 1)$ as in (*).

If $(\alpha_i): (X_i, {}_j\varphi_i) \to (Y_i, {}_j\psi_i)$ is a map in $\mathfrak{R}(\mathcal{Q})$, then $\alpha = \oplus\, \alpha_i: X = \oplus_{i\in I} X_i \to Y = \oplus_{i\in I} Y_i$ can easily be seen to be a right $\mathcal{T}(\mathcal{Q})$-map, and hence put $\mathsf{P}(\alpha_i) = \alpha$.

It follows without difficulties that the functors R and P are mutually inverse equivalences of categories, thus proving the proposition.

Let us remark that $\mathscr{T}(\mathscr{Q})$ is semiprimary if and only if $M^{(n)} = 0$ for some n, and that this is the case if and only if every sequence $i_1, i_2, \ldots, i_n$ with ${}_{i_k}M_{i_{k+1}} \neq 0$, for $1 \leqslant k \leqslant n-1$, is of bounded length. If $\mathscr{T}(\mathscr{Q})$ is semiprimary, then its radical is given by Rad $\mathscr{T}(\mathscr{Q}) = \bigoplus_{n=1}^{\infty} M^{(n)}$. Also, in this case, $\mathscr{T}(\mathscr{Q})$ is hereditary, that is gl. dim. $\mathscr{T}(\mathscr{Q}) \leqslant 1$ [10]. We may use this fact to describe $\mathscr{T}(\mathscr{Q})$ in the case when the K-species $\mathscr{Q} = (K_i, {}_iM_j)_{i,j\in I}$ is of finite type. For, then, by Lemma 9.1, every sequence $i_1, i_2, \ldots, i_n$ with ${}_{i_k}M_{i_{k+1}} \neq 0$ for $1 \leqslant k \leqslant n-1$, is of length $\leqslant$ card I, and hence $\mathscr{T}(\mathscr{Q})$ is semiprimary and hereditary. Of course, $\mathscr{T}(\mathscr{Q})$ is even finite dimensional over K. We will show that also the converse is true: a hereditary, finite dimensional K-algebra of finite type is Morita equivalent to a tensor algebra $\mathscr{T}(\mathscr{Q})$, where $\mathscr{Q}$ is a K-species of finite type.

Assume that $\mathscr{A}$ is a basic finite dimensional K-algebra and let $\mathscr{A}/\text{Rad}\,\mathscr{A} = K_1 \times \cdots \times K_n$ with extension fields K_i of K for $1 \leqslant i \leqslant n$; furthermore, let Rad $\mathscr{A}/(\text{Rad}\,\mathscr{A})^2 = \bigoplus_{1\leqslant i,j\leqslant n} {}_iM_j$ be the decomposition with the K_i-K_j-bimodules ${}_iM_j$. Thus, $\mathscr{Q} = (K_i, {}_iM_j)_{1\leqslant i,j\leqslant n}$ is the K-species of $\mathscr{A}$.

PROPOSITION 10.2. *Let $\mathscr{A}$ be a finite dimensional basic K-algebra and $\mathscr{Q}$ its K-species. If $\mathscr{A}$ is hereditary and not of strongly unbounded type, then $\mathscr{A}$ is isomorphic* (*as a ring*) *to $\mathscr{T}(\mathscr{Q})$.*

Proof. Let $\mathscr{N}$ be the radical of $\mathscr{A}$. Let $1 = \sum_{i=1}^n e_i$ be a decomposition of the unity into a sum of orthogonal primitive idempotents. Since $\mathscr{A}$ is hereditary, we may assume that $e_i\mathscr{N}e_j = 0$ for $i \geqslant j$ (see [3]). Since $\mathscr{A}$ is a basic algebra, $e_i\mathscr{A}e_j \subseteq \mathscr{N}$ for $i \neq j$ and, consequently, $\mathscr{N} = \sum_{i\neq j} e_i\mathscr{A}e_j$. Thus, the subring $\Gamma = \sum_{i=1}^n e_i\mathscr{A}e_i$ satisfies $\Gamma + \mathscr{N} = \mathscr{A}$, $\Gamma \cap \mathscr{N} = 0$, and we may identify Γ with $\mathscr{A}/\text{Rad}\,\mathscr{A}$, and K_i with $e_i\mathscr{A}e_i$.

Now, considering $\mathscr{N}$ as a Γ-Γ-bimodule, we are going to show that there is a Γ-Γ-submodule M of $\mathscr{N}$ with $\mathscr{N} = M \oplus \mathscr{N}^2$. If we decompose ${}_\Gamma\mathscr{N}_\Gamma = \bigoplus_{i,j} {}_iN_j$, where ${}_iN_j$ is a K_i-K_j-bimodule, it is sufficient to show that the K_i-K_j-bimodules ${}_iN_j$ are simple; for, in this case, ${}_\Gamma\mathscr{N}_\Gamma$ is semisimple. Hence, assume that ${}_{K_i}({}_iN_j)_{K_j}$ is not simple. But then, by Proposition 5.2, the corresponding K-species (with the two-point index set $\{i, j\}$) is of strongly unbounded type. Let $J = \sum_{j<k} \mathscr{A}e_k$. Then, J is also a right ideal, because

$$J\mathscr{A} = \sum_{j<k} \mathscr{A}e_k\mathscr{A} = \sum_{j<k} \mathscr{A}e_k + \sum_{l\leqslant j<k} \mathscr{A}e_k\mathscr{A}e_l\,;$$

however, since $e_k\mathscr{A}e_l = 0$ for $l < k$, the last summand is zero. Now, for $k \leqslant j$, denote the image of e_k under the ring homomorphism $\mathscr{A} \to \mathscr{A}/J$ again by e_k. Since ${}_iN_j \cap J = 0$, we may identify ${}_iN_j = e_i\mathscr{A}e_j$ and $e_i(\mathscr{A}/J)e_j$. Also, $e_i(\mathscr{A}/J)\,e_j$ belongs to the right socle of $\mathscr{A}/J$, because

$$e_i\mathscr{A}e_j\mathscr{N} = \sum_{j<k} e_i\mathscr{A}e_j\mathscr{N}e_k \subseteq \sum_{j<k} \mathscr{A}e_k = J.$$

This shows that the idempotents e_i, e_j of $\mathscr{A}/J$ satisfy the assumptions of Lemma 6.3, and thus there is a full embedding of $\mathfrak{Re}({}_{K_i}({}_iN_j)_{K_j})$ into $\mathfrak{M}_{\mathscr{A}/J}$ by a dimension functor. Since $\mathfrak{M}_{\mathscr{A}/J}$ is a full subcategory of $\mathfrak{M}_{\mathscr{A}}$, we conclude that $\mathscr{A}$ is of strongly unbounded type, in contradiction with our assumption. Hence, ${}_{\Gamma}\mathscr{N}_{\Gamma}$ is semi-simple and ${}_{\Gamma}\mathscr{N}_{\Gamma} = M \oplus \mathscr{N}^2$ for some Γ-Γ-submodule M.

Now, we define a ring homomorphism $\eta\colon \mathscr{T}(\mathscr{Q}) \to \mathscr{A}$ which is the identity on $\Gamma = \prod_{i \in I} K_i$ and on $M = \bigoplus_{i,j \in I} {}_iM_j$. It is easy to see that η has to be surjective, because M generates $\mathscr{N}$ as a subring. Then, $\mathscr{A} \approx \mathscr{T}(\mathscr{Q})/J'$, where J' is an ideal in $\operatorname{Rad}(\mathscr{T}(\mathscr{Q}))^2$, and $J' = 0$ in view of the fact that gl. dim. $\mathscr{A} \leqslant 1$, by a theorem of Eilenberg and Nakayama [6]. The proof of Proposition 10.2 is completed.

Now, we give a *proof of Theorem* C *and the first part of Theorem* E(3). Thus, let $\mathscr{A}$ be a finite dimensional hereditary K-algebra. The K-dimension of the objects of $\mathfrak{M}_{\mathscr{A}}$ is equivalent to the length dimension, and hence it is a category invariant. As a consequence, we may assume that $\mathscr{A}$ is basic. If $\mathscr{A}$ is not of strongly unbounded type, then, by Proposition 10.2, the ring $\mathscr{A}$ is of the form $\mathscr{T}(\mathscr{Q})$, where $\mathscr{Q}$ is the K-species of $\mathscr{A}$. According to Proposition 10.1, $\mathfrak{M}_{\mathscr{T}(\mathscr{Q})}$ and $\mathfrak{R}(\mathscr{Q})$ are equivalent categories. Since $\mathscr{Q}$ is not of strongly unbounded type, $\mathscr{Q}$ is of finite type, in view of Theorem E(2). Conversely, if $\mathscr{Q}$ is a K-species of finite type and $\mathscr{A} \approx \mathscr{T}(\mathscr{Q})$, then $\mathscr{A}$ is of finite type by Proposition 10.1.

In order to provide a proof of Theorem D, let us consider now the case of an algebra $\mathscr{A}$ with $(\operatorname{Rad} \mathscr{A})^2 = 0$. If $\mathscr{A}$ is a basic finite dimensional K-algebra, the *separated K-species* $\mathscr{Q}'$ of $\mathscr{A}$ is defined in the following way. Let $\mathscr{A}/\operatorname{Rad} \mathscr{A} = K_1 \times \cdots \times K_n$ and $\operatorname{Rad} \mathscr{A}/(\operatorname{Rad} \mathscr{A})^2 = \bigoplus_{1 \leqslant i,j \leqslant n} {}_iM_j$ with K_i-K_j-bimodules ${}_iM_j$. For the index set, we take the set of all pairs (i, t) with $1 \leqslant i \leqslant n$ and $t = 0, 1$, and we let

$$K_{i0} = K_{i1} = K_i \qquad \text{for} \quad 1 \leqslant i \leqslant n,$$

and

$${}_{i0}M_{j1} = {}_iM_j, \quad \text{whereas} \quad {}_{i0}M_{j0} = {}_{i1}M_{j1} = {}_{i1}M_{j0} = 0, \quad \text{for} \quad 1 \leqslant i, j \leqslant n.$$

The separated diagram of $\mathscr{A}$ defined in the introduction is just the diagram of the separated K-species of $\mathscr{A}$. Obviously, in view of Theorem B, Theorem D will be established if we show that $\mathscr{Q}'$ is of finite type if and only if the K-algebra $\mathscr{A}$ is of finite type. This will follow immediately from Lemma 10.3 and Proposition 10.4.

We will consider the following full subcategory of $\mathfrak{R}(\mathscr{Q}')$: If

$$\mathscr{Q} = (K_i, {}_iM_j)_{i,j \in I}$$

is a K-species, then $\mathfrak{R}\mathfrak{m}(\mathscr{Q})$ is the full subcategory of $\mathfrak{R}(\mathscr{Q})$ consisting of all objects $(X_i, {}_j\varphi_i)$ of $\mathfrak{R}(\mathscr{Q})$ such that for all $i \in I$ the map

$$({}_j\varphi_i{}^*)_{j\in I}: X_i \to \bigoplus_{j\in I} \operatorname{Hom}_{K_j}({}_iM_j, X_j)$$

is a monomorphism. Obviously, this is a generalization of $\mathfrak{R}\mathfrak{m}({}_FM_G)$ defined in the Section 5. Under an additional assumption on $\mathscr{Q}$, we will show that nearly all indecomposable objects of $\mathfrak{R}(\mathscr{Q})$ belong to $\mathfrak{R}\mathfrak{m}(\mathscr{Q})$.

LEMMA 10.3. *Let $\mathscr{Q} = (K_i, {}_iM_j)_{i,j\in I}$ be a K-species such that ${}_iM_j \neq 0$ implies ${}_jM_k = 0$ for all $i, j, k \in I$. Then every object in $\mathfrak{R}(\mathscr{Q})$ is a direct sum of an object in $\mathfrak{R}\mathfrak{m}(\mathscr{Q})$ and of simple objects. Moreover, there is only a finite number of simple objects and all of them are finite dimensional.*

Proof. We define I_0 as the set of all $i \in I$ with ${}_iM_j \neq 0$ for some $j \in I$, and set $I_1 = I \setminus I_0$. According to our assumption, I_1 contains all $j \in I$ with ${}_iM_j \neq 0$ for some $i \in I$.

First, we remark that, for $k \in I$, the object $\mathbf{B}(k) = (Y_i, {}_j\psi_i)$ with $Y_i = 0$ except for $Y_k = K_k$, and ${}_j\psi_i = 0$ for all i, j, is simple and that objects of this form are the only simple objects. For, if $\mathbf{X} = (X_i, {}_j\varphi_i)$ is an arbitrary nonzero object, and $X_k \neq 0$ for some $k \in I_1$, then there is a monomorphism $\mathbf{B}(k) \hookrightarrow \mathbf{X}$ (observe that ${}_k\varphi_j = 0$ for all j, because ${}_kM_j = 0$), whereas if $X_k = 0$ for all $k \in I_1$, then all ${}_j\varphi_i = 0$; thus, there are embeddings $\mathbf{B}(k) \hookrightarrow \mathbf{X}$ for all $k \in I_0$ with $X_k \neq 0$.

Now, again, let $\mathbf{X} = (X_i, {}_j\varphi_i)$ be an object in $\mathfrak{R}(\mathscr{Q})$ and let $\mathbf{X}'' = (X_i'', {}_j\varphi_i'')$ be defined in the following way. For $i \in I_0$, let

$$X_i' = \bigcap_{j\in I_1} \ker {}_j\varphi_i{}^*,$$

where ${}_j\varphi_i{}^*: X_i \to \operatorname{Hom}({}_iM_j, X_j)$, and let $X_i'' = X_i/X_i'$. For $j \in I_1$, let $X_j'' = X_j$. Also, for $i \in I_0$, $j \in I_1$, let $({}_j\varphi_i'')^*$ be the map induced by ${}_j\varphi_i{}^*$. Otherwise, of course, we take ${}_j\varphi_i'' = 0$. It is rather obvious that $\mathbf{X}$ is the direct sum of $\mathbf{X}''$ and of copies of $\mathbf{B}(k)$, $k \in I_0$. Here the number of copies of $\mathbf{B}(k)$ is indicated by the dimension of X_k'. The lemma follows.

The assumption of the previous lemma is satisfied, in particular, for the separated K-species $\mathscr{Q}'$ of an algebra $\mathscr{A}$. We claim that the indecomposable objects in $\mathfrak{R}\mathfrak{m}(\mathscr{Q}')$ correspond to the indecomposable right $\mathscr{A}$-modules. Following M. Auslander [1], a functor $\mathsf{P}: \mathfrak{A} \to \mathfrak{B}$ is called a *representation equivalence* if P is full, reflects isomorphisms, and every object in $\mathfrak{B}$ is isomorphic to the image of an object of $\mathfrak{A}$ under P. A representation equivalence $\mathfrak{A} \to \mathfrak{B}$ induces a bijective correspondence between isomorphism classes of indecomposable objects of $\mathfrak{A}$ and of $\mathfrak{B}$. The following proposition is well-known (see [1] and also [8]).

PROPOSITION 10.4. *Let $\mathscr{A}$ be a basic finite-dimensional K-algebra with* $(\text{Rad}\,\mathscr{A})^2 = 0$. *Let $\mathscr{Q}'$ be the separated K-species of $\mathscr{A}$. Then there is a representation equivalence $\mathfrak{M}_{\mathscr{A}} \to \mathfrak{Rm}(\mathscr{Q}')$ which is a dimension functor.*

A proof may be found in M. Auslander [1] (Theorem II. 3.1. and Proposition II. 4.5), where $\mathfrak{Rm}(\mathscr{Q}')$ is called the Grassman category of $\mathscr{A}/\text{Rad}\,\mathscr{A}$ and $\text{Rad}\,\mathscr{A}$. We remark only on the definition of the functor $\mathfrak{M}_{\mathscr{A}} \to \mathfrak{Rm}(\mathscr{Q}')$ on the objects. Let $1 = \sum_{i=1}^{n} e_i$ be a decomposition of the unity as the sum of orthogonal primitive idempotents; let $\mathscr{N} = \text{Rad}\,\mathscr{A}$. Then we may assume that $\mathscr{Q}' = (K_{(i,s)}\,,\,{}_{(i,s)}M_{(j,t)})_{1\leqslant i,j\leqslant n, 0\leqslant s,t\leqslant 1}$ with $K_{(i,0)} = K_{(i,1)} = e_i\mathscr{A}e_i/e_i\mathscr{N}e_i$ and ${}_{(i,0)}M_{(j,1)} = e_i\mathscr{N}e_j$ is the separated K-species of $\mathscr{A}$. Given a right $\mathscr{A}$-module $X_{\mathscr{A}}$, its image under $\mathfrak{M}_{\mathscr{A}} \to \mathfrak{Rm}(\mathscr{Q}')$ is $(X_{(i,s)}\,,\,{}_{(j,t)}\varphi_{(i,s)})$, where

$$X_{(i,0)} = (X/\text{Soc}(X))\,e_i\,, \qquad X_{(i,1)} = \text{Soc}(X)\,e_i\,,$$

and where ${}_{(j,t)}\varphi_{(i,s)}$ is induced by the scalar multiplication on X. In particular, the K-dimensions satisfy the relation

$$\dim X_{\mathscr{A}} = \dim X_K = \sum_{i,s} \dim X_{(i,s)} = \dim(X_{(i,s)}\,,\,{}_{(j,t)}\varphi_{(i,s)}),$$

and thus the functor is a dimension functor. Obviously, this yields a proof of Theorem D and of the remaining part of Theorem E.

Note added in proof. In this note, we want to give a survey on some recent developments. In particular, there are several new techniques for proving the classification theorems of this paper; also some further investigations into categories of unbounded type have been made.

Let $\mathscr{Q} = (K_i\,,\,{}_iM_j)_{1\leqslant i,j\leqslant n}$ be a K-species, and let Γ be its diagram. The K-species $\mathscr{Q}$, or rather its diagram Γ determines a quadratic form q on the n-dimensional rational vector space $\mathbb{R}^n$ by

$$q(x) = \sum_i k_i x_i^2 - \sum_{i\neq j} m_{ij}x_ix_j\,,$$

where $k_i = \dim_K K_i$, $m_{ij} = \dim_K {}_iM_j$, and where $\mathbf{x} = (x_i)$ is an element of $\mathbb{Q}^n$. Let b be the corresponding bilinear form on $\mathbb{Q}^n$, and s_i the reflection of $(\mathbb{Q}^n, b)$ with respect to the ith base vector $\mathbf{e}_i = (0,\ldots, 0, 1, 0,\ldots, 0)$. The roots of Γ are by definition the images of the base vectors $\mathbf{e}_i$ under the action of the Weyl group W (generated by the reflections s_i, $1 \leqslant i \leqslant n$). The base vectors themselves are usually called "simple roots."

If the diagram Γ of $\mathscr{Q}$ is a Dynkin diagram, then the number of indecomposable representations of $\mathscr{Q}$ and the number of positive roots of Γ are equal. In fact, define a map dim from the set of isomorphism classes of finite dimensional representations of $\mathscr{Q}$ into the space $\mathbb{Q}^n$ by $(\dim V)_i = \dim(V_i)_{F_i}$, where $\mathbf{V} = (V_i\,,\,{}_j\varphi_i)$ is a representation of $\mathscr{Q}$. Then dim induces a bijection between the indecomposable representations of $\mathscr{Q}$ and the positive roots of Γ. This follows easily from a case-by-case inspection using the results of this paper.

In the case of the diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$, I. N. Bernstein, I. M. Gelfand, and B. A. Ponomarev have shown that this bijection also can be proved directly, and, in this way, one gets a very elegant proof of Gabriel's classification of the quivers of finite type ("Coxeter functors and Gabriel's theorem," *Uspechi Mat. Nauk* **28** (1973), 19–33, translated in Russian *Math. Surveys* **28** (1973), 17–32). It is obvious, that the simple representations of $\mathcal{Q}$ correspond bijectively to the simple roots, and Bernstein, Gelfand and Ponomarev have defined endo-functors of $\mathfrak{R}(\mathcal{Q})$ (and functors between the categories $\mathfrak{R}(\mathcal{Q})$ and $\mathfrak{R}(\mathcal{Q}')$, where $\mathcal{Q}$ and $\mathcal{Q}'$ are K-species with the same diagram) which construct out of the simple representations all the other indecomposable representations in a similar way as the Weyl group produces all the positive roots out of the simple ones. The most important of these functors are the so-called Coxeter functors, which correspond to the Coxeter transformations in W. It is possible to generalize this method to arbitrary K-species, and, in this way, we get a more conceptual proof of Theorem B (V. Dlab and C. M. Ringel, "Representations of graphs and algebras," to appear. A summary has appeared: "Représentations des graphes valués," *C. R. Acad. Sci. Paris* **278** (1974), 537–540). Theorem B immediately yields Propositions 2.5, 2.6, 3.1, and 4.2; thus in order to obtain Theorem A, we only need the (rather elementary) proof of Proposition 3.2.

A similar technique was developed by W. Müller ("Unzerlegbare Moduln über artinschen Ringen," *Math. Z.* **137** (1974), 197–224) in order to give a new proof of our Theorem D. Given a finite dimensional K-algebra $\mathscr{A}$ with $(\operatorname{rad} \mathscr{A})^2 = 0$, he constructs another finite dimensional K-algebra $\mathscr{B}$ which is weakly symmetric and satisfies $(\operatorname{rad} \mathscr{B})^3 = 0$. For such an algebra $\mathscr{B}$, the images in a minimal projective resolution of a simple module are indecomposable modules, and he shows that in the case of finite representation type, he gets all indecomposable modules in this way. It is rather easy to see that twice the application of his kernel construction just corresponds to the Coxeter functor C^+ on $\mathfrak{R}(\mathcal{Q})$, where $\mathcal{Q}$ is the associated K-species.

Recall that in Section 7 the unboundedness of certain K-structures is proved in the following way: We construct a full subcategory $\mathfrak{A}$ of S-spaces, and prove that $\mathfrak{A}$ is a Grothendieck category with a progenerator $\mathbf{P} \oplus \mathbf{T}$. In all cases, $E_1 = \operatorname{End}(\mathbf{P})$, $E_2 = \operatorname{End}(\mathbf{T})$ are fields, $\operatorname{Hom}(\mathbf{P}, \mathbf{T}) = 0$, and ${}_{E_1}M_{E_2} = \operatorname{Hom}(\mathbf{T}, \mathbf{P})$ satisfies $\dim M_{E_1} = \dim M_{E_2} = 2$. Of course, $\mathfrak{A}$ is then equivalent to the category $\mathfrak{R}({}_{E_1}M_{E_2})$ of representations of the bimodule ${}_{E_1}M_{E_2}$. P. Gabriel has pointed out that it should be easier to determine the bimodule ${}_{E_1}M_{E_2}$ directly, and then to define an appropriate functor $\mathfrak{R}({}_{E_1}M_{E_2}) \to \mathfrak{S}(S)$ such that the image category is precisely $\mathfrak{A}$. For example, in case $\mathscr{S} = \mathscr{I}_1(G) \sqcup \mathscr{I}_1(F) \sqcup \mathscr{I}_1(F)$, let $E_1 = E_2 = E$ and ${}_FM_F = {}_FF_G \otimes {}_GF_F$. Given (X_F, Y_F, φ) in $\mathfrak{R}({}_FM_F)$, with $\varphi\colon X_F \otimes_F F_G \otimes_G F_F \to Y_F$, then we denote by U_G the graph of the map

$$\varphi^*\colon X_G = X_F \otimes_F F_G \to \operatorname{Hom}_F({}_GF_F, Y_F) = Y_G.$$

We define now the functor $\mathfrak{R}({}_FM_F) \to \mathfrak{S}(\mathscr{S})$ by $(X_F, Y_F, \varphi) \mapsto (X_F \times Y_F, U_G, 0 \times Y_F, X_F \times 0)$. It is easy to see that this functor induces an equivalence between the category $\mathfrak{R}({}_FM_F)$ and the full subcategory $\mathfrak{A}$ of $\mathfrak{S}(\mathscr{S})$ consisting of all $\mathscr{S}$-spaces (W, U, V_1, V_2) with $W = U \oplus V_1 = V_1 \oplus V_2$.

Also, the results of Section 7 can be improved considerably. Namely, it turns out that in these cases one may determine *all* indecomposable $\mathscr{S}$-spaces of finite dimension. Of course, we may replace the category $\mathfrak{S}(\mathscr{S})$ by an abelian category, namely the category $\mathfrak{R}(\mathcal{Q})$ of the representations of the corresponding K-species $\mathcal{Q}$. If the quadratic form of $\mathcal{Q}$ is positive semidefinite (and this is true for all cases considered in Section 7, as well as in the situation of Proposition 5.3), then one can write down all indecomposable representations of finite dimension. In addition to using the Coxeter functors, the proof involves several other techniques, in particular a theory of defect of representations (which generalizes the notion of the defect of quadruples introduced by I. M. Gelfand and V. A. Ponomarev in "Problems of linear algebra and classification of quadruples of subspaces in a finite dimensional vector space. *Colloq. Math. Soc. Bolyai* 5, Tihany (Hungary) (1970), 163–237), and nonsymmetric bilinear forms on $\mathbb{Q}^n$ (see the paper of V. Dlab and C. M. Ringel, mentioned above, and C. M. Ringel "Representations of K-species and bimodules," to appear). It turns out that the full subcategories $\mathfrak{A}$ defined in Propositions 7.1–7.4 contain only representations of defect zero, whereas the representations mentioned in Remark 7.5 are of negative defect.

Finally, let us mention that M. Auslander has proved that a finite dimensional K-algebra $\mathscr{A}$ which is not of finite type, always possesses an indecomposable module of infinite dimension (the proof will appear in Auslander's series of papers "Representation theory of artin algebras," in *Comm. Alg.*). Of course, then the same is true for K-species and K-structures. However, his proof is a mere existence proof, and therefore does not reveal a concrete description of such a module.

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