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UNIONS OF CHAINS OF INDECOMPOSABLE MODULES

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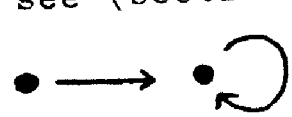
Let K be a commutative field, R a finite dimensional K-algebra (associative, with 1), and $\underline{\underline{M}}_R$ the category of all (right) R-modules. If there are only finitely many indecomposable R-modules, then every R-module is the direct sum of finite dimensional ones [6]. On the other hand, M. Auslander recently has shown [1] that in all other cases, R possesses an indecomposable module X which is infinite dimensional. This module X is constructed as the union of a chain

$$(*) \qquad X_1 \subsetneqq X_2 \subsetneqq \quad \cdots \quad \subsetneqq \bigcup X_i = X$$

of finite dimensional indecomposable R-modules X_i, and he has asked whether for any such chain, the union has to be indecomposable.

The answer to this question is negative. It is rather easy to see (section 1) that the quiver Q



possesses chains (*) of finite dimensional indecomposable representations X_i such that the union X is decomposable, even as the direct sum of infinitely many non-zero representations. We denote by $\underline{\underline{M}}_Q$ the category of all representations of the quiver Q. It is well-known that for many algebras R, there exists an exact embedding $T: \underline{\underline{M}}_Q \longrightarrow \underline{\underline{M}}_R$ which is not necessarily full, but which maps indecomposable objects to indecomposable objects. The "wild" algebras are usually defined by this property (the existence of such an embedding). Since, in most cases, T also preserves unions, we get from a chain of indecomposable objects with decomposable union in $\underline{\underline{M}}_Q$, a similar chain in $\underline{\underline{M}}_R$.

On the other hand, we will show that there are examples of "tame" algebras for which for any chain of finite-dimensional indecomposable modules X_i, the union X is indecomposable. Further investigation of the tame case reveals however a third type of behaviour: it may happen that not all such unions are indecomposable, but that there is a finite bound d on the number of indecomposable summands in any direct decomposition of such a union (section 3).

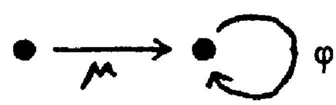
The case of the representations of a K-species S will be solved completely (section 2). If S is tame (equivalently, if the corresponding quadratic form is positive semi-definite), and X is the union of a chain of finite dimensional indecomposable representations X_i , then X is the direct sum of at most six indecomposable representations. The bound d (≤ 6) depends on the quadratic form of S . For example, if S is of type $\tilde{\mathbb{A}}_{12}$, $\tilde{\mathbb{A}}_n$, or $\tilde{\mathbb{C}}_n$, then $d = 1$, and therefore any such union is indecomposable. Also, it turns out that the indecomposable representations X of S which are unions of chains of finite dimensional indecomposable representations, are of two different types: either X admits an endomorphism which is surjective, but not injective. Then any endomorphism $\alpha \neq 0$ is surjective, and its kernel is finite dimensional and indecomposable. These indecomposable representations are quite similar to the Prüfer groups in abelian group theory. Otherwise, any surjective endomorphism is an isomorphism, and then for every endomorphism α of X , the kernels of α^6 and α^7 are equal. It is an open problem whether such an indecomposable representation can have an endomorphism with non-trivial kernel.

The main tool used in this paper are the Coxeter functors and the notion of defect. Both concepts were introduced by Gelfand and Ponomarev [4] in the case of a quiver of type $\tilde{\mathbb{D}}_4$ (the "four subspace problem"), and the

Coxeter functors were defined by Bernstein, Gelfand and Ponomarev [2] for arbitrary quivers. Both concepts have been generalised by Dlab and the author to species [3]. We will recall the main properties which are used in the paper (see the beginning of section 2).

1. Decomposable unions: the wild case

We consider only representations V of Q



with μ an inclusion map. Thus, we can write V in the form $V = (A,B,\varphi)$, where B is a vector space over K, A is a subspace, and $\varphi: B \longrightarrow B$ is a K-linear map.

Let K^{ω} be a countably generated vector space over K with base $e_1, e_2, \ldots$, and let $e_n = 0$ for $n \leq 0$. Let K^n be the n-dimensional subspace of K^{ω} generated by $e_1, \ldots, e_n$, and, for $n \leq 0$, let $K^n = 0$. For $\alpha \in K$, consider the endomorphism $[\alpha]_{\omega}$: $K^{\omega} \longrightarrow K^{\omega}$, defined by

$$[\alpha]_{\omega}(e_i) = \alpha e_i + e_{i-1} \quad , \text{ for all } i \geq 1.$$

The subspaces K^n are invariant with respect to $[\alpha]_{\omega}$ and we denote by $[\alpha]_n$ the restriction of $[\alpha]_{\omega}$ to K^n.

Let $\alpha \neq \beta$ be two fixed elements of K. We define a representation V_n of Q by

$$V_n = (A_n,\ K^n \oplus K^n,\ \lfloor\alpha]_n \oplus [\beta]_n)\ ,$$

where A_n is the subspace of $K^n \oplus K^n$ generated by $K^{n-1} \oplus K^{n-1}$ and the element $e_n \oplus e_n$. Note that we have inclusions

$$V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \cdots ,$$

and that the union V is just

$$V = \bigcup V_i = (K^\omega \oplus K^\omega,\ K^\omega \oplus K^\omega, \lfloor\alpha\rfloor_\omega \oplus [\beta]_\omega).$$

Obviously, V is the direct sum of $(K^\omega, K^\omega, \lfloor\alpha]_\omega)$ and $(K^\omega, K^\omega, \lfloor\beta\rfloor_\omega)$. On the other hand, all the representations V_n are indecomposable. For, any decomposition of V_n induces a decomposition with respect to the sub-quiver $\bullet\!\circlearrowleft \varphi$. But, for $\alpha \neq \beta$, $(K^n \oplus K^n, \lfloor\alpha]_n \oplus [\beta]_n)$ has precisely one non-trivial decomposition, with direct summands $(K^n \oplus 0, [\alpha]_n \oplus 0)$ and $(0 \oplus K^n, 0 \oplus [\beta]_n)$, and this decomposition is not compatible with the subspace A_n.

Similarly, we construct a chain of finite dimensional indecomposable representations with union the direct sum of infinitely many indecomposable representations. Let α_i , $i \in \mathbb{N}$, be pairwise different in K ,

$$W_n = (C_n, \bigoplus_{i \in \mathbb{N}} K^{n-i}, \bigoplus_{i \in \mathbb{N}} [\alpha_i]_{n-i})$$

with C_n the subspace of $\oplus K^{n-i}$ generated by $\oplus K^{n-1-i}$ and the element $\oplus e_{n-i} = (e_n, e_{n-1}, \ldots, e_1, 0, \ldots)$. Then, again, we have a chain of inclusions

$$W_1 \subsetneq W_2 \subsetneq W_3 \subsetneq \cdots \subsetneq W = \bigcup W_n ,$$

and all the representations W_i are indecomposable.

The union W is given by

$$W = (\bigoplus_{i\in\mathbb{N}} K^{\omega}, \bigoplus_{i\in\mathbb{N}} K^{\omega}, \bigoplus_{i\in\mathbb{N}} [\alpha_i]_{\omega}) ,$$

and therefore is the direct sum of the representations $(K^{\omega}, K^{\omega}, [\alpha_i]_{\omega})$.

The last construction was based on the assumption that the field K contains infinitely many elements. However, we easily can modify the construction in order to deal with finite fields: instead of using only one-dimensional representations of  as composition factors of the restriction of W_n to  , we will work with higher dimensional simple representations

Now, let $S = (F_i, {}_iM_j)_{1\leq i,j\leq n}$ be a K-species. [I.e., there are given division rings F_i containing K in the center such that $\dim_K F_i$ is finite, and F_i-F_j-bimodules ${}_iM_j$ with central operation of K and $\dim_K {}_iM_j$ finite.] Let $\underline{\underline{M}}_S$ be the category of all representations of S and $\underline{\underline{m}}_S$ the full subcategory of all finite dimensional ones [a representation $V = (V_i, {}_j\varphi_i)_{1\leq i,j\leq n}$ of S is given by F_i-vector space V_i and F_j-linear maps ${}_j\varphi_i$: $V_i \otimes_{F_i} {}_iM_j \longrightarrow V_j$. In cas all vector spaces V_i are finite dimensional, V itsel is called finite-dimensional. If $V = (V_i, {}_j\varphi_i)$ and $W = (W_i, {}_j\pi_i)$ are representations, a morphism $\alpha = (\alpha_i)$: $V \longrightarrow W$ is given by F_i-linear maps $\alpha_i: V_i \longrightarrow W_i$ such that $\alpha_{jj}\varphi_i = {}_j\pi_i(\alpha_i \otimes 1)$, for all i, j.] The quadratic form q of S on the n-dimensional rational vector space $\mathbb{Q}^n$ is given by

$$q(x) = \sum f_i x_i^2 - \sum m_{ij} x_i x_j \qquad \text{for} \quad x = (x_i) \in \mathbb{Q}^n,$$

where $f_i = \dim_K F_i$ and $m_{ij} = \dim_K {}_iM_j$. In order to use the construction of decomposable unions of indecomposable representations of Q , to get a similar situation in $\underline{\underline{M}}_S$, we need an embedding $\underline{\underline{M}}_Q \longrightarrow \underline{\underline{M}}_S$ with certain properties.

Proposition. Let S be a K-species with indefinite quadratic form. Then there is a finite-extension field K' of K and a proper embedding $T\colon \underline{\underline{M}}_{Q,K'} \longrightarrow \underline{\underline{M}}_S$.

Here, $\underline{\underline{M}}_{Q,K'}$ denotes the category of representations of Q over the field K' , and the embedding T is called proper provided T is exact, union preserving, maps finite dimensional objects to finite dimensional ones, such that, moreover, any direct decomposition of an object of the form $T(X)$ comes from a direct decomposition of the object X .

Proof of the proposition: If S does not contain any oriented cycle, then there is an extension field K' of K and a proper (even full) embedding of $\underline{\underline{M}}_{K'\langle x,y\rangle}$ into $\underline{\underline{M}}_S$, where $\underline{\underline{M}}_{K'\langle x,y\rangle}$ denotes the module category over the free K'-algebra in two (non-commuting) generators x, y [5]. Of course, as is well-known, we further may embed $\underline{\underline{M}}_{Q,K'}$ properly (and full) into $\underline{\underline{M}}_{K'\langle x,y\rangle}$. It remains to consider the case where the diagram of S is of one of the forms

In the first case, we have $S = (F, {}_FM_F)$ with $d = \dim_F M = \dim M_F \geq 2$. Let $S' = (F_1, F_2, {}_1M_2)$ be the K-species with diagram

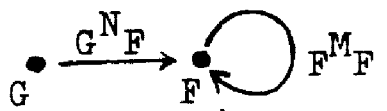
$$F_1 \bullet \xrightarrow{\;{}_1M_2\;} \bullet F_2 \ ,$$

division rings $F_1 = F_2 = F$, and bimodule ${}_1M_2 = {}_FM_F \otimes {}_FM_F$. Now S' has no cycles, and its quadratic form is indefinite, since $2d \cdot 2d > 4$, thus there is a proper embedding of some $\underline{\underline{M}}_{Q,K'}$ into $\underline{\underline{M}}_{S'}$. Define a functor T: $\underline{\underline{M}}_{S'} \longrightarrow \underline{\underline{M}}_S$ by

$$T(X,Y,\varphi) = \left(X \oplus (X \otimes M) \oplus Y, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \varphi & 0 \end{pmatrix}\right),$$

where $\varphi: X \otimes {}_1M_2 \longrightarrow Y$ is an F-linear map. It is easy to see that T is exact, union preserving, and maps finite dimensional objects to finite dimensional ones. Also, its restriction to the full subcategory $\underline{\underline{Me}}_{S'}$ of $\underline{\underline{M}}_{S'}$ consisting of all representations (X,Y,φ) with φ surjectiv, has the property that any decomposition of an object of the form $T(V)$ is induced by a decomposition of V . Since the embedding of $\underline{\underline{M}}_{Q,K'}$ into $\underline{\underline{M}}_{S'}$ constructed in [5], actually embeds $\underline{\underline{M}}_{Q,K'}$ into $\underline{\underline{Me}}_{S'}$, we get a proper embedding of $\underline{\underline{M}}_{Q,K'}$ into $\underline{\underline{M}}_S$.

The remaining two cases are rather similar. Thus, we will only consider the case where S is of the form



Let $N^* = \mathrm{Hom}_F({}_G N_F, {}_F F_F)$, and $\pi\colon N^* \otimes N \longrightarrow F$ the canonical epimorphism. We will consider the K-species $S' = (F, M \oplus M)$ with diagram  . A representation V of S' may be written in the form $V = (X, \varphi_1, \varphi_2)$ with maps $\varphi_i\colon X \otimes M \longrightarrow M$. We define a functor $T\colon \underline{\underline{M}}_{S'} \longrightarrow \underline{\underline{M}}_S$ by

$$T(X, \varphi_1, \varphi_2) = \left((X \otimes N^*) \oplus 0,\ X \oplus X\ ,\ (1 \otimes \pi) \oplus 0, \begin{pmatrix} 0 & \varphi_1 \\ 1 & \varphi_2 \end{pmatrix} \right)$$

and it is rather easy to see that T is a proper embedding. This concludes the proof of the proposition.

<u>Corollary</u>. Let S be a K-species with indefinite quadratic form. Then there exists a chain

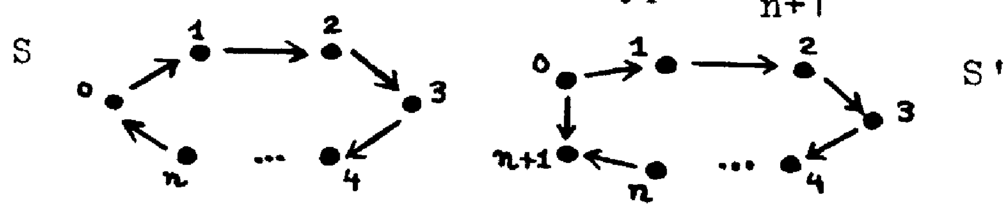
$$V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \cdots$$

of finite dimensional indecomposable representations V_i such that the union $V = \bigcup V_i$ is the direct sum of countably many non-zero representations.

2. Tame K-species

We want to discuss now the case of a tame K-species. Thus, we assume that the quadratic form of S is positive semi-definite, but not definite. Obviously, we may assume that S is connected. In case of a species

$S = (F_i, {}_iM_j)$ of type $\tilde{A}_n$, we will, in addition, assume that S does not have the cyclic orientation. [Otherwise, we add a new edge with opposite direction in order to get a diagram of type $\tilde{A}_{n+1}$



and we let $S' = (F'_i, {}_iM'_j)$ with $F'_i = F_i$ $(o \leq i \leq n)$, and $F'_{n+1} = F_o$, and with ${}_iM'_{i+1} = {}_iM_{i+1}$ $(o \leq i < n)$, ${}_nM'_{n+1} = {}_nM_o$, and ${}_{n+1}M'_o = F_o$ (with the canonical F_o–F_o–bimodule structure). Then, we consider $\underline{\underline{M}}_S$ as the full subcategory of $\underline{\underline{M}}_{S'}$, consisting of all representations $(V_i, {}_j\varphi_i)$ with ${}_o\varphi_{n+1} = \mathrm{id}$.]

Since S is a K–species without oriented cycles, there are defined two endofunctors $C^+, C^-: \underline{\underline{M}}_S \longrightarrow \underline{\underline{M}}_S$, the so–called <u>Coxeter functors</u> and two natural transformations $\pi: C^-C^+ \longrightarrow \mathrm{id}$, and $\gamma: \mathrm{id} \longrightarrow C^+C^-$, which satisfy the following properties:

(C 1) $C^+X = 0$ if and only if X is projective,
$C^-X = 0$ if and only if X is injective.

(C 2) Let X be indecomposable.
If X is not projective, then $C^+(X)$ is indecomposable and $\pi_X: C^-C^+(X) \longrightarrow X$ is isomorphism.
If X is not injective, then $C^-(X)$ is indecomposable and $\gamma_X: X \longrightarrow C^+C^-(X)$ is isomorphism.

(C 3) C^+ is left exact, C^- is right exact.

Also, for any object X in $\underline{\underline{m}}_S$, there is defined an integer δX, called the defect of X, with the following properties:

(D 1) δ is additive on extensions: if

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is an exact sequence in $\underline{\underline{m}}_S$, then we have the following equality: $\delta Y = \delta X + \delta Z$.

(D 2) Let X_i $(o \leq i \leq m)$ be indecomposable in $\underline{\underline{m}}_S$. If X_o can be embedded into $\oplus_{i=1}^m X_i$, and $\delta X_i \geq o$, for all i, then $\delta X_o \leq \sum_{i=1}^m \delta X_i$. If X_o is a homomorphic image of $\oplus_{i=1}^m X_i$, and $\delta X_i \leq o$, for all i, then $\delta X_o \geq \sum_{i=1}^m \delta X_i$.

(D 3) If X is indecomposable, then $|\delta X| \leq 6$.

The Coxeter functors and the defect are related by the following property:

(CD) Let X be indecomposable in $\underline{\underline{m}}_S$. Then:

$\delta X < o$ if and only if $C^{+m}(X) = 0$ for some $m > o$.

$\delta X > o$ if and only if $C^{-m}(X) = 0$ for some $m > o$.

The definitions of C^+, C^-, and δ, and these properties can be found in the joint paper [3] with V.Dlab. An object is called regular, provided it is the direct sum of indecomposable objects of defect zero. These objects form a full, exact, abelian subcategory of $\underline{\underline{m}}_S$, which is denoted by $\underline{\underline{r}}_S$. This follows from (D 1) and (D 2).

The combination of theorem 3.5 of [3] and theorem 1 of [5] yields the following structure theorem for $\underline{\underline{r}}_S$.

(R) The category $\underline{\underline{r}}_S$ is the direct sum of categories $\underline{\underline{r}}_\alpha$, each of which is serial, of global dimension 1 and has only finitely many simple objects.

Recall that an abelian category is called serial provided every indecomposable object has a unique composition series. A serial category which cannot be written as the direct sum of two non-zero categories, has global dimension 1 if and only if there are indecomposable objects of arbitrary length.

After we have collected those known results which we will need in the sequel, we are ready to look at our problem. Thus, assume we have given a chain of indecomposable representations X_i in $\underline{\underline{m}}_S$

$$X_1 \subset X_2 \subset X_3 \subset \ldots$$

and that $X = \bigcup X_i$ is the union in $\underline{\underline{M}}_S$.

<u>Lemma 1</u>. For all i, we have $\delta X_i \leq o$.

Proof: Assume $\delta X_i > o$ for some i, thus there is some $m \geq o$ with $X_i = C^{+m}I$ for an indecomposable injective object I . For $j > i$, we denote by u_j the inclusion $X_i \hookrightarrow X_j$. If we apply C^{-m} to u_j , we get a non-zero map

$$C^{-m}(u_j): C^{-m}X_i = I \longrightarrow C^{-m}X_j \text{ ,}$$

in view of $C^{+m}C^{-m}(u_j) = u_j$. Let J be the image of $C^{-m}(u_j)$. Since J is a homomorphic image of the injective object I , and $\underline{\underline{m}}_S$ has global dimension 1, also J is injective. But J is a subobject of the indecomposable object $C^{-m}X_j$, and therefore $J = C^{-m}X_j$, which implies $X_j = C^{+m}J$. We have shown that for $i < j$, X_j is of the form $X_j = C^{+m}J$, with J indecomposable injective, and fixed m.

However, there is only a finite number of indecomposable injective objects in $\underline{\underline{m}}_S$, and therefore there is only a finite number of objects of the form $C^{+m}J$ (with m fixed!). This contradicts the fact that we have infinitely many X_j with $j > i$.

Lemma 2. Assume there is i with $\delta X_i = 0$. Then X is indecomposable, and its endomorphism ring $\mathrm{End}(X)$ is a complete discrete valuation ring. If $\varphi \neq 0$ is an endomorphism, then φ is surjective, and the kernel $\ker \varphi$ is indecomposable and finite dimensional.

Proof. If $j>i$, then $\delta X_j = 0$. For, assume $\delta X_j \neq 0$, then, by the previous lemma, $\delta X_j < 0$, and therefore $C^{+m}X_j = 0$ for some m. However, C^+ is left exact, thus the inclusion $X_i \hookrightarrow X_j$ gives rise to an inclusion $C^{+m}X_i \hookrightarrow C^{+m}X_j = 0$. Consequently, we would have $\delta X_i < 0$.

It follows that we may assume $\delta X_i = 0$ for all i (deleting, if necessary, the first objects of the chain).

Now, every X_i is regular and indecomposable, thus it belongs to one of the categories $\underline{\underline{r}}_{\alpha_i}$ given in (R). Since the inclusion $X_1 \hookrightarrow X_i$ is a non-zero morphism, we must have $\alpha_1 = \alpha_i$. This shows that all X_i belong to one and the same $\underline{\underline{r}}_\alpha$.

We know that $\underline{\underline{r}}_\alpha$ has only finitely many simple objects, say k. For the investigation of X , we may assume that every X_i (as an object of $\underline{\underline{r}}_\alpha$) has length ki . Namely, since X_i is serial, every subobject of X_i is serial, again, and therefore indecomposable. Thus, we may refine the chain such that X_{i+1}/X_i is simple for all i. If we then consider only every k-th term of the chain, we get a new chain with the same union, such that the length of X_i is precisely ki. Note that X_i/X_{i-1}, for all $i>1$, is isomorphic to X_1 , and that the endomorphism ring $\mathrm{End}(X_1)$ is a division ring.

Let $\varphi \neq 0$ be an endomorphism of X . Let $\varphi(X_i) = 0$, but $\varphi(X_{i+1}) \neq 0$. Consider, for any $s > i$, the object $\varphi(X_s)$ in $\underline{\underline{M}}_S$. Obviously, $\varphi(X_s)$ is finite-dimensional, and therefore a subobject of one of the X_t $(t \geq 1)$. The restriction φ_s of φ to X_s is a morphism $X_s \longrightarrow X_t$ in $\underline{\underline{r}}_\alpha$ with kernel containing X_i , but not X_{i+1}. It follows that the kernel of φ_s is precisely X_i and that φ_s induces an isomorphism of X_s/X_i onto X_{s-i}. Since this is true for all $s > i$, we conclude that $\ker \varphi = \bigcup \ker \varphi_s = X_i$, and that the image $\operatorname{im} \varphi = \bigcup \operatorname{im} \varphi_s = \bigcup_{s>i} X_{s-i} = X$. This shows that φ is surjective, and that its kernel is indecomposable and finite dimensional.

Let $E = \operatorname{End}(X)$. An endomorphism φ of X is a non-unit in E if and only if $\varphi(X_1) = 0$. Thus, E is a local ring, and $M = \{\varphi \in E \mid \varphi(X_1) = 0\}$ is the radical of E. Obviously, X and X/X_1 are isomorphic, and we denote by π an epimorphism $X \longrightarrow X$ with kernel X_1. If $\varphi \in M$, then φ can be factored through X/X_1, and therefore through π, that is $\varphi = \varphi'\pi$ for some $\varphi' \in E$. This shows that M is a principal left ideal of E, and, as a consequence, $M^s = \{\varphi \in E \mid \varphi(X_s) = 0\}$.

In order to show that E is complete, we have to verify that the canonical ring homomorphism $E \longrightarrow \varprojlim E/M^s$ is an isomorphism. Since $\bigcap M^s = 0$, it is obviously a monomorphism. Let $(\overline{\varphi}_s)_s$ be an element of $\varprojlim E/M^s$, with $\overline{\varphi}_s = \varphi_s + M^s$ for certain $\varphi_s \in E$. For $i \leq s$, the restriction of φ_s to X_s does neither depend on the choice of the representative φ_s in $\overline{\varphi}_s$, nor on the index s (as long as $i \leq s$). Thus, we may define $\varphi \in E$ by $\varphi|X_s = \varphi_s|X_s$.

Now, $E/M^s = \operatorname{End}(X_s)$ is a finite dimensional local K-algebra, and M/M^s is a principal left ideal. Then

M/M^s has to be also a principal right ideal, and, since E is complete, M itself is a principal right ideal. This concludes the proof of lemma 2.

<u>Lemma 3</u>. Let d be a natural number with

$$-d \leq \delta X_i < o, \qquad \text{for all } i.$$

Then X is the direct sum of at most d indecomposable representations. If φ is an endomorphism of X, then either $\varphi^d = 0$, or else $\ker \varphi^{d-1} = \ker \varphi^d$. In particular if φ is surjective, then φ is an isomorphism.

Proof: Assume X is the direct sum of $d+1$ non-zero representations Y_j

$$X = \oplus_{j=1}^{d+1} Y_j .$$

Let φ_j be an endomorphism of X with kernel $\oplus_{k=1}^{j} Y_k$. Since X is the union of the X_i, we may choose an index i such that $X_i \cap Y_j \neq 0$ for all j. We define subobjects K_j of X_i by

$$K_j = X_i \cap \oplus_{k=1}^{j} Y_k = X_i \cap \ker \varphi_j ,$$

thus, we have an increasing chain

$$0 = K_o \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_{d+1} = X_i .$$

Here, all inclusions are proper inclusions, since

$$K_j \oplus (X_i \cap Y_{j+1}) \subseteq K_{j+1} .$$

We claim that for every j with $o \leq j \leq d$, the defect of K_{j+1}/K_j is ≤ -1. We have

$$K_{j+1}/K_j \subseteq X_i/X_i \cap \ker\varphi_j \approx \varphi_j(X_i) \subseteq X ,$$

thus K_{j+1}/K_j is (isomorphic to) a subobject of X. Since K_{j+1}/K_j is finite dimensional, it is even a subobject of some X_t, $t\in N$. But X_t is indecomposable with $\delta X < o$, therefore also $\delta(K_{j+1}/K_j) < o$. (Otherwise, an application of the left exact functor C^{+m} to the inclusion $K_{j+1}/K_j \longrightarrow X_t$ would give the contradiction $C^{+m}(K_{j+1}/K_j) \neq 0$, $C^{+m}(X_t) = 0$.)

All factors K_{j+1}/K_j have defect ≤ -1. Therefore, the additivity of the defect yields

$$\delta X_i = \delta K_{d+1} = \sum_{j=1}^{d} \delta(K_{j+1}/K_j) \leq -(d+1),$$

a contradiction.

Next, assume that φ is an endomorphism, and that $\ker\varphi^{d-1} \neq \ker\varphi^d \neq X$. We choose an index i such that, for $o \leq j \leq d-1$, we have

$$X_i \cap \ker\varphi^j \neq X_i \cap \ker\varphi^{j+1} \neq X_i .$$

Let $K_j = X_i \cap \ker\varphi^j$ for $o \leq j \leq d$, and $K_{d+1} = X_i$. Then, again, we have a chain of proper inclusions

$$0 = K_o \subset K_1 \subset K_2 \subset \dots \subset K_{d+1} = X_i ,$$

and the inclusion

$$K_{j+1}/K_j \subseteq X_i/X_i \cap \ker\varphi^j \approx \varphi^j(X_i) \subseteq X,$$

yields, as above, $\delta(K_{j+1}/K_j) \leq -1$. Thus, we get again the contradiction $\delta X_i \leq -(d+1)$. As a consequence, either $\ker\varphi^{d-1} = \ker\varphi^d$, or else $\ker\varphi^d = X$, that is $\varphi^d = 0$. In both cases, $\ker\varphi^d = \ker\varphi^{d+1}$, and this implies $\ker\varphi \cap \operatorname{im}\varphi^d = 0$. Thus, if φ is surjective, then $\operatorname{im}\varphi^d = \operatorname{im}\varphi = X$ yields $\ker\varphi = 0$.

The property (D 3) shows that for any K–species S with semi–definite quadratic form, there is a universal bound d such that $-d \leq \delta X_i$, namely d = 6 . However depending on the type of S , we get a better bound by using the following table

type	$\widetilde{\mathbb{A}}_{11}$	$\widetilde{\mathbb{A}}_{12}$	$\widetilde{\mathbb{A}}_n$	$\widetilde{\mathbb{B}}_n$	$\widetilde{\mathbb{C}}_n$	$\widetilde{\mathbb{BC}}_n$	$\widetilde{\mathbb{BD}}_n$	$\widetilde{\mathbb{CD}}_n$
d	2	1	1	2	1	2	2	2

	$\widetilde{\mathbb{D}}_n$	$\widetilde{\mathbb{E}}_6$	$\widetilde{\mathbb{E}}_7$	$\widetilde{\mathbb{E}}_8$	$\widetilde{\mathbb{F}}_{41}$	$\widetilde{\mathbb{F}}_{42}$	$\widetilde{\mathbb{G}}_{21}$	$\widetilde{\mathbb{G}}_{22}$
	2	3	4	6	4	3	3	2

One obtains this list from [3] by considering the maxima value of $|\delta(P)|$, with P projective (where, of course one assumes that the greatest common divisor of all $\delta(P)$, with P projective, is = 1). We collect the resul of this section in the following theorem.

Theorem. Let S be a K–species with positive semi–definite quadratic form, and let X be the union of a chain of finite dimensional indecomposable represe tations. There is a number $d \leq 6$ such that X is the direct sum of at most d indecomposable representation and X is of one of the following types:

Type I. X admits a surjective, but not injective endomorphism. Then X is indecomposable, every non–zer endomorphism is surjective, with finite–dimensional indecomposable kernel, and its endomorphism ring is a complete discrete valuation ring.

Type II. Every surjective endomorphism is injective. Then, if φ is an endomorphism of X, either $\varphi^d = 0$, or else $\ker \varphi^{d-1} = \ker \varphi^d$.

The cases with $d = 1$ are of particular interest. Namely, the theorem above yields the following corollary:

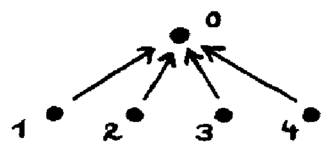
Corollary. Let S be a K-species of one of the types $\widetilde{\mathbb{A}}_{12}$, $\widetilde{\mathbb{A}}_n$, or $\widetilde{\mathbb{C}}_n$. Let X be the union of a chain of finite dimensional indecomposable representations. Then X is indecomposable, and either every non-zero endomorphism is surjective, or every non-zero endomorphism is injective.

In the general case, we do not know any example for a union of type II which actually has a non-zero nilpotent endomorphism, nor an example of an indecomposable union of type II which has a non-zero endomorphism with non-zero kernel. However, in the next section, we will exhibit an example of a union (necessarily of type II) which is decomposable, and which therefore has a non-zero endomorphism with non-zero kernel.

3. Decomposable unions: the tame case

We consider a K-species of type $\widetilde{\mathbb{D}}_4$. Since in this case, $d = 2$, we know that a union of finite dimensional indecomposable representations is the direct sum of at most 2 indecomposable representations. We want to give an example of a union which actually decomposes.

Thus, let Q be the quiver



In fact, we are only interested in representations where all maps are inclusions, and such a representation will be given in the form $X = (X^o|X^1, X^2, X^3, X^4)$, with X^o a K-vector space, and X^i a subspace of X^o, for all $1 \leq i \leq 4$ (thus, we are in the "four subspace" situation). The defect δX of $X = (X^o|X^1, X^2, X^3, X^4)$ is given by

$$\delta X = -2 \cdot \dim X^o + \sum_{i=1}^{4} \dim X^i ,$$

the Coxeter functor C^- maps $X = (X^o|X^1, X^2, X^3, X^4)$ onto $C^- X = Y = (Y^o|Y^1, Y^2, Y^3, Y^4)$, with Y^o given by the exact sequence

$$X^o \longrightarrow \oplus_{i=1}^{4} X^o/X^i \longrightarrow Y^o \longrightarrow 0 ,$$

and Y^j $(1 \leq j \leq 4)$ by

$$X^o \longrightarrow \oplus_{\substack{i=1 \\ i \neq j}}^{4} X^o/X^i \longrightarrow Y^j \longrightarrow 0 .$$

For this case, both the defect and the Coxeter functors were introduced by Gelfand and Ponomarev in [4]. If $X = (X^o|X^1, X^2, X^3, X^4)$ is a representation, its dimension $\dim X$ is defined to be the element of the five-dimensional rational vector space $\mathbb{Q}^5$ with components

$$(\dim X)_i = \dim_K X^i \qquad \text{for } o \leq i \leq 5.$$

If X is indecomposable, and has negative defect, then

$$\dim C^{-2}(X) = \dim X - \delta(X)\cdot(2,1,1,1,1).$$

We will consider the representations

$$X_o = (K|0,0,0,0),\quad Y_o = (K|K,0,0,0),\quad Z_o = (K|0,K,0,0),$$

and their images under C^{-i}, namely

$$X_i = C^{-i}X_o,\quad Y_i = C^{-i}Y_o,\quad Z_i = C^{-i}Z_o\ .$$

Note that

$$X_1 = (KKK|KOO,OKO,OOK,(1,1,1)K),$$

where we denote by KKK the cartesian product of three copies of the vector space K_K , and so on, and by $(1,1,1)K$ the one-dimensional subspace of KKK , generated by the element $(1,1,1)$. Similarly, we have

$$Y_1 = (KK|OO,KO,OK,(1,1)K),\quad Z_1 = (KK|KO,OO,OK,(1,1)K).$$

There are (up to scalar multiples) uniquely determined exact sequences

$$0 \longrightarrow Y_o \xrightarrow{\alpha_o} X_1 \xrightarrow{\gamma_1} Y_1 \longrightarrow 0\ ,$$

$$0 \longrightarrow Z_o \xrightarrow{\beta_o} X_1 \xrightarrow{\delta_1} Z_1 \longrightarrow 0\ .$$

Here, α_o embeds Y_o as $(KOO|KOO,OOO,OOO,OOO)$, and β_o embeds Z_o as $(OKO|OOO,OKO,OOO,OOO)$ into X_1 .

For $i \geq 1$, we define

$$\alpha_i = C^{-i}(\alpha_o),\ \beta_i = C^{-i}(\beta_o),\ \gamma_i = C^{-i+1}(\gamma_1),\ \delta_i = C^{-i+1}(\delta_1),$$

and we claim that all maps

$$\xrightarrow{(\alpha_i,\beta_i)} X \qquad \text{and} \qquad X \xrightarrow{\binom{\gamma_i}{\delta_i}} Y \oplus Z$$

are monomorphisms. Since $\alpha_o(Y_o) \cap \beta_o(Z_o) = 0$, we know that (α_o,β_o) is a monomorphism. Its cokernel is just $U = (K|0,0,K,K)$. Also, $\begin{pmatrix}\gamma_1\\ \delta_1\end{pmatrix}$ is a monomorphism (since its kernel is $\ker\gamma_1 \cap \ker\delta_1 = \alpha_o(Y_o) \cap \beta_o(Z_o) = 0$), and has the same cokernel U. Now, all the representations X_i, Y_i, Z_i and U are indecomposable, and $\delta X_i = -2$, $\delta Y_i = \delta Z_i = -1$, and $\delta U = o$. If we apply the right exact functor C^{-2i} and note that $C^{-2}(U) = U$, we get an exact sequence

$$Y_{2i} \oplus Z_{2i} \xrightarrow{(\alpha_{2i},\beta_{2i})} X_{2i+1} \longrightarrow U \longrightarrow 0 .$$

In order to see that (α_{2i},β_{2i}) is a monomorphism, we show that the dimension $\dim$ of $Y_{2i} \oplus Z_{2i}$ is equal to the dimension of the kernel of the map $X_{2i+1} \twoheadrightarrow U$.

$$\begin{aligned}\dim(Y_{2i} \oplus Z_{2i}) &= \dim Y_o + \dim Z_o + 2\cdot i\cdot(2,1,1,1,1)\\ &= \dim X_1 - \dim U + 2\cdot i\cdot(2,1,1,1,1)\\ &= \dim X_{2i+1} - \dim U .\end{aligned}$$

If we apply to the monomorphism (α_{2i},β_{2i}) with $i \geq 1$ the corresponding left exact functor C^+, we obtain $(\alpha_{2i-1},\beta_{2i-1})$, which therefore also has to be a monomorphism. The same arguement shows that $\begin{pmatrix}\gamma_i\\ \delta_i\end{pmatrix}$ has to be a monomorphism, for all i.

We consider now the chain of monomorphisms

$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} X_3 \longrightarrow \quad \dots ,$$

where

$$\varphi_i = (\alpha_i, \beta_i)\begin{pmatrix}\gamma_i \\ \delta_i\end{pmatrix} : X_i \longrightarrow X_{i+1} .$$

Let $X = \lim X_i$ be the colimit of this chain. Thus, X is the union of a chain of finite dimensional indecomposable representations.

However, since $\varphi_i = (\alpha_i, \beta_i)\begin{pmatrix}\gamma_i \\ \delta_i\end{pmatrix}$, we know that X is also the colimit of the chain of monomorphisms

$$\begin{pmatrix}\gamma_{i+1} \\ \delta_{i+1}\end{pmatrix}(\alpha_i, \beta_i) = \begin{pmatrix} o & \gamma_{i+1}\beta_i \\ \delta_{i+1}\alpha_i & o \end{pmatrix} : Y_i \longrightarrow Y_{i+1} ,$$

where we have used that

$$\gamma_{i+1}\alpha_i = C^{-i}(\gamma_1\alpha_o) = o, \text{ and } \delta_{i+1}\beta_i = C^{-i}(\delta_1\beta_o) = o.$$

We see that this chain decomposes as the direct sum of the two chains

$$Y_o \xrightarrow{\gamma_1\beta_o} Z_1 \xrightarrow{\delta_2\alpha_1} Y_2 \longrightarrow \cdots$$

and

$$Z_o \xrightarrow{\delta_1\alpha_o} Y_1 \xrightarrow{\gamma_2\beta_1} Z_2 \longrightarrow \cdots .$$

As consequence, X is the direct sum of the corresponding two colimits.

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