Math. Ann. 214, 19–34 (1975)


# The Indecomposable Representations of the Dihedral 2-Groups

Claus Michael Ringel

Let $K$ be a field. We will give a complete list of the normal forms of pairs $a, b$ of endomorphisms of a $K$-vector space such that $a^2 = b^2 = 0$. Thus, we determine the modules over the ring $R = K\langle X, Y\rangle/(X^2, Y^2)$ which are finite dimensional as $K$-vector spaces; here $(X^2, Y^2)$ stands for the ideal generated by $X^2$ and $Y^2$ in the free associative algebra $K\langle X, Y\rangle$ in the variables $X$ and $Y$.

If $G$ is the dihedral group of order $4q$ (where $q$ is a power of 2) generated by the involutions $g_1$ and $g_2$, and if the characteristic of $K$ is 2, then the group algebra $KG$ is a factor ring of $R$, and the $KG$-modules ${}_{KG}M$ which have no non-zero projective submodule correspond to the $K$-vector spaces (take the underlying space of ${}_{KG}M$) together with two endomorphisms $a$ and $b$ (namely multiplication by $g_1 - 1$ and $g_2 - 1$, respectively) such that, in addition to $a^2 = b^2 = 0$, also $(ab)^q = (ba)^q = 0$ is satisfied.

We use the methods of Gelfand and Ponomarev developped in their joint paper on the representations of the Lorentz group, where they classify pairs of endomorphisms $a, b$ such that $ab = ba = 0$. The presentation given here follows closely the functorial interpretation of the Gelfand-Ponomarev result by Gabriel, which he exposed in a seminar at Bonn, and the author would like to thank him for many helpful conversations.

## 1. Description of the Indecomposable Modules

We are interested in the modules over the ring $R = K\langle X, Y\rangle/(X^2, Y^2)$ which are finite dimensional as $K$-vector spaces. Denote this category by $\mathfrak{M}$. If $a$ and $b$ are the canonical images of $X$ and $Y$ in $R$, respectively, then an $R$-module is given by a triple $M = (M, a, b)$ where $M$ is a $K$-vector space, and $a$ and $b$ are endomorphisms of ${}_KM$, also operating from the left on $M$ with $a^2 = b^2 = 0$. In this section, we are going to give a complete list of the indecomposable objects in $\mathfrak{M}$.

Consider now $a$, $a^{-1}$, $b$, and $b^{-1}$ as "letters" of a formal language, and let $(a^{-1})^{-1} = a$ and $(b^{-1})^{-1} = b$. If $l$ is a letter, we write $l^*$ to mean "either $l$ or $l^{-1}$". A *word* $C = l_1 \ldots l_n$ is given by the sequence $l_1, \ldots, l_n$ of letters subject to the condition that $l_i = a^*$ (for $1 \leqq i < n$) implies $l_{i+1} = b^*$ and similarly that $l_i = b^*$ implies $l_{i+1} = a^*$. The number $n = |C|$ is called the length of $C$. Thus, for example, $ab^{-1}aba^{-1}$ is a word and has length 5. Also, we include into the set $\mathscr{W}$ of all words two words $1_a$ and $1_b$ of length 0, with $(1_a)^{-1} = 1_b$ and $(1_b)^{-1} = 1_a$. If $C = l_1 \ldots l_n$ is a word, then its inverse is given by $C^{-1} = l_n^{-1} \ldots l_1^{-1}$. Let $\varrho$ be the equivalence relation on $\mathscr{W}$ which identifies every word with its inverse, and let $\mathscr{W}_1 = \mathscr{W}/\varrho$.

If $C = l_1 \ldots l_n$ and $D = e_1 \ldots e_m$ are two words with non-zero length, then the product is given by $CD = l_1 \ldots l_n e_1 \ldots e_m$, provided this is again a word. In particular, if $C$ has even length $\neq 0$, then the powers $C^m$ do exist ($m \geq 2$). Let $\mathscr{W}'$ be the subset of $\mathscr{W}$ consisting of all words of even length $\neq 0$ which are not powers of words of smaller length. If $C = l_1 \ldots l_n$ is in $\mathscr{W}'$, denote by $C_{(i)}$, $0 \leq i \leq n-1$, the cyclic permuted words, thus $C_{(0)} = C$, $C_{(1)} = l_2 \ldots l_n l_1$, up to $C_{(n-1)} = l_n l_1 \ldots l_{n-1}$. Let $\varrho'$ be the equivalence relation on $\mathscr{W}'$ which identifies with every word $C$ the cyclic permuted words $C_{(i)}$ and their inverses $C_{(i)}^{-1}$. Thus, if $C = ab^{-1}ab$, then the equivalence class of $C$ with respect to $\varrho'$ contains precisely the words $ab^{-1}ab$, $b^{-1}aba, abab^{-1}, bab^{-1}a, b^{-1}a^{-1}ba^{-1}, a^{-1}b^{-1}a^{-1}b, ba^{-1}b^{-1}a^{-1}, a^{-1}ba^{-1}b^{-1}$. Let $\mathscr{W}_2 = \mathscr{W}/\varrho'$.

To every element of $\mathscr{W}_1$ we are going to construct an indecomposable module, called a *module of the first kind*. Namely, let $C = l_1 \ldots l_n$ be a word of length $n$. Let $M(C)$ be given by a $K$-vector space of dimension $n+1$, say with base $z_0, \ldots, z_n$ on which $a$ and $b$ operate according to the following schema

$$Kz_0 \overset{l_1}{\longleftrightarrow} Kz_1 \overset{l_2}{\longleftrightarrow} Kz_2 \ldots Kz_{n-1} \overset{l_n}{\longleftrightarrow} Kz_n .$$

For example, if $C = ab^{-1}aba^{-1}$, we have the following schema

$$Kz_0 \xleftarrow{a} Kz_1 \xrightarrow{b} Kz_2 \xleftarrow{a} Kz_3 \xleftarrow{b} Kz_4 \xrightarrow{a} Kz_5 ,$$

(note that we have to adjust the direction of the arrows according to whether the letter $l_i$ is equal to a or $a^{-1}$, or to $b$ or $b^{-1}$), and this indicates how the base vectors $z_i$ are mapped into each other or into zero. Namely, $z_1$ goes under a onto $z_0$, and under $b$ onto $z_2$, $z_3$ goes under a onto $z_2$, and so on. If no condition is specified, then the base vector goes to zero, thus $z_0$, $z_2$, and $z_5$ go to zero both under $a$ and $b$, whereas $z_3$ goes to zero under $b$, only. In this way, we get an $R$-module, and it is obvious, that $M(C)$ and $M(C^{-1})$ are isomorphic, [or equal, since $M(C)$ is only defined up to isomorphism].

Next, we construct the *modules of the second kind*. Let $\varphi$ be an automorphism of the (finite dimensional) $K$-vector space $V$. Also, let $C$ be a word in $\mathscr{W}'$, say $C = l_1 \ldots l_n$. Let $M(C, \varphi)$ be given as a vector space by $M(C, \varphi) = \bigoplus_{i=0}^{n-1} V_i$ with $V_i = V$ on which $a$ and $b$ operate according to the following schema

$$V_0 \overset{l_1 = \varphi}{\longleftrightarrow} V_1 \overset{l_2 = \mathrm{id}}{\longleftrightarrow} V_2 \ldots V_{n-2} \overset{l_{n-1} = \mathrm{id}}{\longleftrightarrow} V_{n-1} , \qquad V_0 \overset{l_n = \mathrm{id}}{\longleftrightarrow} V_{n-1}$$

For example, if $C = ab^{-1}ab$, we have the schema

$$V_0 \xleftarrow{a=\varphi} V_1 \xrightarrow{b=\mathrm{id}} V_2 \xleftarrow{a=\mathrm{id}} V_3 , \qquad V_0 \xrightarrow{b=\mathrm{id}} V_3$$

which shows that $V_1 = V$ is mapped under $b$ identically onto $V_2 = V$, and similarly, that $V_3 = V$ is mapped under a identically onto $V_2 = V$. On the other hand, $V_1$ is mapped under a onto $V_0 = V$ by the isomorphism $\varphi$. Again, where no condition is specified, the elements go to zero, thus, for example, $V_3$ is mapped under $b$ into zero. In this way, we get an $R$-module. If $(V, \varphi)$ and $(V', \psi)$ are isomorphic as vectorspaces with automorphisms (as $K[T, T^{-1}]$-modules), the $R$-modules $M(C, \varphi)$ and $M(C, \psi)$ are isomorphic, for fixed $C$. Also, it is easy to see that for two words $C$ and $D$ of $\mathscr{W}'$ which are equivalent with respect to $\varrho'$, and fixed $\varphi$, the modules $M(C, \varphi)$ and $M(D, \varphi)$ are isomorphic.

**Theorem.** *The modules $M(C)$ with $C$ in $\mathscr{W}$ and the modules $M(C, \varphi)$ with $C$ in $\mathscr{W}'$ and $\varphi$ an indecomposable automorphism of a vector space, furnish a complete list of the indecomposable objects in $\mathfrak{M}$. No module of the first kind is isomorphic to a module of the second kind. The modules $M(C)$ and $M(D)$ of the first kind are isomorphic if and only if $C$ and $D$ belong to the same equivalence class with respect to $\varrho$, the modules $M(C, \varphi)$ and $M(D, \psi)$ of the second kind are isomorphic if and only if $C$ and $D$ belong to the same equivalence class with respect to $\varrho'$, and $\varphi$ and $\psi$ are isomorphic as automorphisms of vectorspaces.*

As a consequence, the number of isomorphism classes of indecomposable $R$-modules of given dimension $d$ is as follows: if $d$ is odd, there are precisely $2^{d-1}$ classes, whereas, if $d$ is even and $K$ is infinite, the number of classes is equal to the cardinality of $K$.

## 2. Reminder on Relations

A relation on a vectorspace $V$ is a subspace $C$ of $V \times V$. For example, if $\alpha: V \to V$ is an endomorphism, then we may consider $\alpha$ as the relation $\{(x, \alpha x) \mid x \in V\}$. If $C$ is a relation on $V$, then $C^{-1}$ is given by $\{(x, y) \mid (y, x) \in C\}$. Also, if $C$ and $D$ are relations on $V$, then $CD = \{(x, z) \mid (x, y) \in D$ and $(y, z) \in C$ for some $y \in V\}$. If $x$ is an element of $V$, and $C$ a relation on $V$, we write $Cx$ for $\{y \in V \mid (x, y) \in C\}$, and similarly, for a subset $U$ of $V$, let $CU = \{y \in V \mid$ there is $x \in U$ with $(x, y) \in C\}$.

We will use only relations which are of the form $C = l_1 \dots l_n$ where $l_i$ or $l_i^{-1}$ is a mapping $V \to V$. Note that in this case the definitions mentioned above coincide with the usual ones.

If $C$ is a relation on $V$, let $C' = \bigcup_n C^n 0_V$ ("the stable kernel" of $C^{-1}$) and let $C'' = \bigcap_n C^n V$ ("the stable range" of $C$). Then there is the following important result.

**Lemma.** *If $C$ is a relation on $V$, there are subspaces $U$ and $W$ of $V$ such that $V = U \oplus W$, and $C = [C \cap (U \times U)] \oplus [C \cap (W \times W)]$, with $C \cap (U \times U)$ the graph of an automorphism of $U$, and $C' \oplus U = C''$.*

The lemma can easily be derived from the well-known classification of normal forms of relations on vector spaces.

**Corollary.** *If $C$ is a relation on $V$, then $C$ induces on $C''/C'$ an automorphism $\varphi$ with $\varphi(x + C') = (Cx \cap C'') + C'$, for $x \in C''$.*

The relation $\varphi$ on $C''/C'$ is called the regular part of $C$, and the lemma asserts that the regular part of a relation splits off.

**Corollary.** *If $C$ is a relation on $V$, and $C' \oplus U = C''$ with $C \cap (U \times U)$ the graph of an automorphism of $U$, then also $(C^{-1})' \oplus U = (C^{-1})''$, and $[C' + (C^{-1})'] \oplus U = C'' + (C^{-1})''$.*

Let us mention at the end a rather useful (but trivial) result.

**Lemma.** *Let $a: V \to V$ be an endomorphism with $a^2 = 0$. Let $U_1 \subseteqq U_2 \subseteqq V$ be subspaces. Then*

$$\dim U_2/U_1 \geqq \dim aU_2/aU_1 + \dim a^{-1}U_2/a^{-1}U_1 .$$

*Proof.* The multiplication by a defines isomorphisms

$$a^{-1}U_2/a^{-1}U_1 \to [(U_2 \cap aV) + U_1]/U_1 ,$$

and

$$U_2/[U_2 \cap (a^{-1}0 + U_1)] \to aU_2/aU_1 .$$

On the other hand, we have the following inclusions:

$$U_1 \subseteqq (U_2 \cap aV) + U_1 \subseteqq U_2 \cap (a^{-1}0 + U_1) \subseteqq U_2 .$$

## 3. The Functors Involved in the Proof

We want to apply the following lemma in order to prove the main theorem.

**Lemma.** *Let $I$ be an index set. Let $\mathfrak{M}$ and $\mathfrak{A}_i (i \in I)$ be abelian categories, and let $S_i: \mathfrak{A}_i \to \mathfrak{M}$ and $F_i: \mathfrak{M} \to \mathfrak{A}_i$ $(i \in I)$ be additive functors such that*

(i) *$F_i S_i \simeq \mathrm{id}_{\mathfrak{A}_i}$ and $F_j S_i = 0$, for $i \neq j$ in $I$.*

(ii) *The set $\{F_i \mid i \in I\}$ is locally finite and reflects isomorphisms.*

(iii) *For every $M$ in $\mathfrak{M}$ and every $i \in I$, there is a map $\gamma_{i,M}: S_i F_i(M) \to M$ such that $F_i(\gamma_{i,M})$ is an isomorphism.*

*Then the indecomposable objects in $\mathfrak{M}$ are of the form $S_i(A)$ with $A$ indecomposable in $\mathfrak{A}_i$, all those objects are indecomposable, and $S_i(A)$ and $S_j(B)$ are isomorphic if and only if $i = j$ and $A$ is isomorphic to $B$ in $\mathfrak{A}_i$.*

Here, the set $\{F_i\}$ is called locally finite, provided for every $M$ in $\mathfrak{M}$ there is only a finite number of indices $i$ with $F_i(M) \neq 0$. And it is said to reflect isomorphisms provided every map $\alpha$ in $\mathfrak{M}$ for which $F_i(\alpha)$ is an isomorphism for all $i \in I$, is itself an isomorphism.

*Proof.* Since $\{F_i\}$ is locally finite, the sum $\bigoplus_I S_i F_i(M)$ exists, and we get a map $(\gamma_{i,M})_i: \oplus S_i F_i(M) \to M$, which becomes an isomorphism under all the functors $F_i$ by the second part of (i) and (iii). By (ii), $M$ is isomorphic to $\oplus S_i F_i(M)$.

Now it is easy to see that the functor $(F_i)_i: \mathfrak{M} \to \coprod \mathfrak{A}_i$ is a representation equivalence. Namely, by assumption, it reflects isomorphism. By (i), every object of the target category is isomorphic to an image under this functor, and the functor is full using (i) and the first part of the proof.

Let us describe the situation where we want to apply the lemma. For the moment, we will consider an index set which is far too big, so that the second condition of (i) is not satisfied. Then, in the last section, we will select an appropriate subset. Now, take as index set the disjoint union of $\mathscr{W}'$ and the set of all pairs $(C, D)$ of words in $\mathscr{W}$ such that $C^{-1}D$ is again a word. We have defined the product of two words only in cases where both words have non-zero length. In addition, we define products with $1_a$ and $1_b$ as follows: let $1_a a = a$, $1_a a^{-1} = a^{-1}$, $b 1_a = b$,

$b^{-1}1_a = b^{-1}$, and similarly for longer words. In the same way, $1_b$ shall be right unit for words with last letter $a^*$, and left unit for words with first letter $b^*$.

If $C, D, C^{-1}D$ are words, then let $\mathfrak{U}_{C,D} = {}_K\mathfrak{M}$, the category of all finite dimensional $K$-vector spaces. If $C$ is in $\mathscr{W}'$, let $\mathfrak{U}_C = {}_{K[T,T^{-1}]}\mathfrak{M}$, that is, the category of all $K[T, T^{-1}]$-modules which are finite-dimensional as $K$-vector spaces, or, equivalently, the category of automorphisms of finite dimensional $K$-vector spaces.

Again, if $C, D, C^{-1}D$ are words, define the functor $S_{C,D}: \mathfrak{U}_{C,D} \to \mathfrak{M}$ by $S_{C,D}({}_KK) = M(C^{-1}D)$. Thus, for an arbitrary (finite dimensional) vector space $V$, we have $S_{C,D}(V) = \bigoplus_{i=1}^{n} V_i$ with $V_i = V$, on which $a$ and $b$ act according to the schema

$$V_0 \xleftarrow{l_1(=\mathrm{id})} V_1 \dots V_{n-1} \xleftarrow{l_n(=\mathrm{id})} V_n,$$

where $C^{-1}D = l_1 \dots l_n$. And, if $C$ is in $\mathscr{W}'$, let $S_C(V, \varphi) = M(C, \varphi)$, where $\varphi$ is an automorphism of the vector space $V$.

Our next goal is to describe several functors $\mathfrak{M} \to {}_K\mathfrak{M}$. Recall that the set of all such functors is an abelian category, and that, moreover, it is (partially) ordered by $F \leqq G$ iff $F(M) \subseteqq G(M)$ for all $M$ in $\mathfrak{M}$.

The two letters $a$ and $b$ will be called the *direct* letters, whereas the letters $a^{-1}$ and $b^{-1}$ are said to be *inverse*. If $C$ is a word, there is precisely one direct letter $d$ such that $Cd$ is again a word. Let $C^-(M) = CdM$ and $C^+(M) = Cd^{-1}0_M$, for $M$ in $\mathfrak{M}$. In this way, we define two functors from $\mathfrak{M}$ into ${}_K\mathfrak{M}$.

**Lemma.** *Let $d$ be a letter and $C, D, CdD$ be words. Then we have the following relation*

$$CdD^- \leqq CdD^+ \leqq C^- \leqq C^+ \leqq Cd^{-1}D^- \leqq Cd^{-1}D^+ .$$

*Proof.* Let $e$ be the direct letter with $De$ a word. We have for every $M$ in $\mathfrak{M}$, $eM \subseteqq e^{-1}0_M$, and $dM \subseteqq d^{-1}0_M$, since both $a$ and $b$ operate on $M$ with $a^2 = 0 = b^2$. This gives the first, the third and the last inclusion. But obviously, $CdD^+(M) = CdDe^{-1}0_M \subseteqq CdM = C^-(M)$, and $C^+(M) = Cd^{-1}0_M \subseteqq Cd^{-1}DeM = Cd^{-1}D^-(M)$, which gives the two remaining inclusions.

Also, consider an element $C$ of $\mathscr{W}'$. Note that for every $M$ in $\mathfrak{M}$, we have the following inclusions

$$C0_M \subseteqq C^2 0_M \subseteqq \cdots \subseteqq C^n 0_M \subseteqq \cdots \subseteqq C^n M \subseteqq \cdots \subseteqq C^2 M \subseteqq CM ,$$

thus we also have

$$C'(M) = \bigcup_n C^n 0_M \subseteqq C''(M) = \bigcap_n C^n M ,$$

and obviously, $C'$ and $C''$ are again functors from $\mathfrak{M}$ into ${}_K\mathfrak{M}$.

Denote by $\mathscr{W}_a$ the subset of $\mathscr{W}$ which contains $1_a$ and all words with first letter of the form $a^*$. Similarly, we may define $\mathscr{W}_b$, thus $\mathscr{W}$ is the disjoint union of $\mathscr{W}_a$ and $\mathscr{W}_b$. Also, let $\mathscr{W}_a' = \mathscr{W}_a \cap \mathscr{W}'$ and $\mathscr{W}_b' = \mathscr{W}_b \cap \mathscr{W}'$.

Let $\tilde{\mathscr{W}}_a$ be the set of all functors $\mathfrak{M} \to {}_K\mathfrak{M}$, which are either of the form $(D^+ + C^-) \cap C^+$ or $(D^- + C^-) \cap C^+$ with $C$ in $\mathscr{W}_a$ and $D$ in $\mathscr{W}_b$, or which are of the form $C'$ or $C''$ with $C$ in $\mathscr{W}_a'$.

Now $\tilde{\mathscr{W}}_a$ is an ordered set, and we will show that it even is a chain. Given an arbitrary ordered set $T$, an *intervall* $[t_1, t_2]$ is given bei two elements $t_1, t_2$ in $T$ such that $t_1 \leqq t_2$. Also, two intervalls $[t_1, t_2]$ and $[t_3, t_4]$ are said to *avoid each other*, provided either $t_2 \leqq t_3$ or else $t_4 \leqq t_1$.

**Proposition.** *The ordered set $\tilde{\mathscr{W}}_a$ is a chain, and all the intervalls $[(D^- + C^-) \cap C^+, (D^+ + C^-) \cap C^+]$ with $C \in \mathscr{W}_a$ and $D \in \mathscr{W}_b$, and $[C', C'']$ with $C \in \mathscr{W}'_a$ avoid each other.*

*Proof.* Define an order relation on $\mathscr{W}_a$ by $C < D$ provided either $C = DdE$, or $D = Cd^{-1}E$, or $C = C_1 dE_1$ and $D = C_1 d^{-1} E_2$ for suitable words $C_1, E, E_1, E_2$ and a direct letter $d$. Obviously, $\mathscr{W}_a$ becomes a chain, and the previous lemma shows that $C < D$ in $\mathscr{W}_a$ implies $C^+ \leqq D^-$ in the set $\mathscr{F}$ of all functors. Thus, for $C \neq D$ in $\mathscr{W}_a$, the intervalls $[C^-, C^+]$ and $[D^-, D^+]$ avoid each other in $\mathscr{F}$.

Also, if $C \in \mathscr{W}'_a$ and $D \in \mathscr{W}_a$, then one of the following two possibilities occurs. Either $D < C^n$ for large $n$, then $D^+ \leqq C'$. Namely, if $C^n = Dd^{-1}E$ for some direct letter $d$ and some word $E$, then

$$D^+(M) = Dd^{-1}0_M \subseteqq Dd^{-1}E0_M = C^n 0_M \subseteqq C'(M).$$

Otherwise $D = D_1 dE_1$ and $C^n = D_1 d^{-1} E_2$ for suitable words $D_1, E_1, E_2$ and a direct letter $d$, and using the previous case, we have $D^+ \leqq D_1^+ \leqq C'$. The second possibility is that $C^n < D$ for large $n$. Then we have $C'' \leqq D^-$ in $\mathscr{F}$. Namely, if $C^n = DdE$ for some word $E$ and a direct letter $d$, then

$$C''(M) \subseteqq C^n M = DdEM \subseteqq DdM = D^-(M),$$

and otherwise $C^n = D_1 dE_1$ and $D = D_1 d^{-1} E_2$ implies $C'' \leqq D_1^- \leqq D^-$. As a consequence, the intervalls $[D^-, D^+]$ with $D \in \mathscr{W}_a$ and the intervalls $[C', C'']$ with $C \in \mathscr{W}'_a$ avoid each other. Also, it follows that the intervalls $[C', C'']$ and $[D', D'']$ with $C$ and $D$ in $\mathscr{W}'_a$ avoid each other.

In order to prove the proposition, it is enough to show the second part. Now, the intervall $[(D^- + C^-) \cap C^+, (D^+ + C^-) \cap C^+]$ with $C \in \mathscr{W}_a$ and $D \in \mathscr{W}_b$ lies inside the intervall $[C^-, C^+]$, thus it avoids all the intervalls of the form $[E', E'']$ with $E \in \mathscr{W}'_a$, and also all intervalls of the form $[(D_1^- + C_1^-) \cap C_1^+, (D_1^+ + C_1^-) \cap C_1^+]$ with $C \neq C_1$. In the case where $C = C_1$, but $D \neq D_1$, we use again the previous lemma, this time for the proof that the intervalls $[D^-, D^+]$ and $[D_1^-, D_1^+]$ avoid each other.

As a consequence, $\tilde{\mathscr{W}}_a$ induces a filtration $\tilde{\mathscr{W}}_a(M)$ on every module $M$, given by the set of subspaces $F(M)$, with $F \in \tilde{\mathscr{W}}_a$. We will see later that the intervalls $[(D^- + C^-) \cap C^+, (D^+ + C^-) \cap C^+]$ and $[C', C'']$ (which will be called the elementary intervalls) also *cover* $\tilde{\mathscr{W}}_a$. That means that for every $M$ in $\mathfrak{M}$, and every $0 \neq x$ in $M$, there is one elementary intervall $[F, G]$ such that $x \in G(M)$, but $x \notin F(M)$.

Now we come back to our previous considerations. We have defined the categories $\mathfrak{A}_{C,D}$ (for $C, D, C^{-1}D$ words) and $\mathfrak{A}_C$ (for $C \in \mathscr{W}'$), and also functors $S_{C,D}$ and $S_C$, respectively. It remains to define functors $F_{C,D}$ and $F_C$.

If $C, D, C^{-1}D$ are words, define $F_{C,D} = (C^+ \cap D^+)/[(C^+ \cap D^-) + (C^- \cap D^+)]$. Thus, $F_{C,D}$ does not depend on the order of the pair $C, D$. On the other hand,

$F_{C,D}$ is isomorphic to the functor $[(D^+ + C^-)\cap C^+]/[(D^- + C^-)\cap C^+]$. If we consider $F_{C,D}$ as a factor in the filtration $\tilde{\mathscr{W}}_a$, we will tacitly assume that we have replaced $F_{C,D}$ by the functor $[(D^+ + C^-)\cap C^+]/[D^- + C^-)\cap C^+]$, where $C$ belongs to $\mathscr{W}_a$ (note that always just one of the words $C$ and $D$ belongs to $\mathscr{W}_a$).

**Lemma.** *Let $E$ be a fixed word. The functors $F_{C,D}$ with $C^{-1}D = E$ are all isomorphic.*

*Proof.* It is enough to show that for $E = C^{-1}lD$, with $l$ a letter and $C, D$ words, the functors $F_{C,lD}$ and $F_{l^{-1}C,D}$ are isomorphic. Also, since $F_{C,lD} = F_{lD,C}$, and so on, we may assume that $l$ is a direct letter, say $l = a$. Now it is easily checked that the multiplication by a defines a vector space isomorphism

$$\frac{[(a^{-1}D^+ + C^-)\cap C^+](M)}{[(a^{-1}D^- + C^-)\cap C^+](M)} \to \frac{[(D^+ + aC^-)\cap aC^+](M)}{[(D^- + aC^-)\cap aC^+](M)},$$

which is natural in $M$.

Finally, let $C$ be an element of $\mathscr{W}'$. If we consider $C$ as a relation on $M$, we know that $C$ induces on $C''(M)/C'(M)$ an automorphism, denoted by $\varphi_{C,M}$. Let $F_C(M) = ((C''/C')(M), \varphi_{C,M})$, so this is really an object in $\mathfrak{A}_C = {}_{K[T,T^{-1}]}\mathfrak{M}$.

**Lemma.** *Let $C \in \mathscr{W}'$. The functors $F_C$ and $F_{C_{(i)}}$ are isomorphic. Also,* $\dim F_C(M) = \dim F_{C^{-1}}(M)$.

*Proof.* Let $C = l_1 \ldots l_n$ with letters $l_i$, and let $V_i = (C_{(i)})''/(C_{(i)})'(M)$. Now, if $l_i$ is direct, then it is easily seen that the multiplication by $l_i$ induces an epimorphism $V_{i-1} \leftarrow V_i$. And, if $l_i$ is inverse, then the multiplication by $l_i^{-1}$ induces a monomorphism $V_{i-1} \to V_i$. Thus, in both cases we have $\dim V_{i-1} \leqq \dim V_i$, and

$$\dim V_0 \leqq \dim V_1 \ldots \dim V_{n-1} \leqq \dim V_0$$

shows that all dimensions are equal, and that in both cases the multiplication by the direct letter $l_i$ or $l_i^{-1}$, respectively, is an isomorphism. Therefore, the vector-spaces $V_i$ and the corresponding maps satisfy the schema

$$V_0 \xleftarrow{l_1} V_1 \xleftarrow{l_2} V_2 \ldots V_{n-2} \xleftarrow{l_{n-1}} V_{n-1}, \qquad V_0 \xrightarrow{l_n} V_{n-1}$$

and the map $\varphi_{C,M}$ is given by $l_1 \ldots l_n$, that is, by going once around the circle, counter-clockwise. Note, that the maps $\varphi_{C_{(i)},M}$ are conjugate to $\varphi_{C,M}$.

The last assertion of the lemma is again a statement about the regular part of a relation.

## 4. The Modules of the First Kind

The first lemma deals both with the modules of the first and of the second kind.

Let $M$ be a module of the form $M(C)$ or $M(C, \varphi)$ defined in the first section. Then $M$ has a canonical vector-space decomposition $M = \oplus V_i$, with $V_i = Kz_i$ in the case $M = M(C)$. The elements of $M$ which belong to one of the $V_i$'s are called *homogeneous.* Note that both $a$ and $b$ map homogeneous elements in homogeneous elements. A subspace $U$ of $M$ is said to be homogeneous, provided $U$ can be generated by homogeneous elements.

**Lemma.** *If $U$ is a homogeneous subspace of $M$, and $D$ is a word, then $DU$ is again a homogeneous subspace.*

*Proof.* It is enough to show it for a letter $D=l$. If $U$ is homogeneous, then $U=\Sigma U_i$, with $U_i=U\cap V_i$. Therefore, we may assume that $U$ is contained in one of the $V_i$. If $l$ is direct, say $l=a$, then $aU\subseteq aV_i$, and $aV_i$ is either zero or is just one of the $V_j$'s. Thus, $aU$ contains only homogeneous elements in this case. If $l$ is inverse, say $l=a^{-1}$, then either there is some $V_j$, such that a maps $V_j$ onto $V_i$. Then take a subspace $U'$ of $V_j$ with $aU'=U$, and it follows that $a^{-1}U=U'+a^{-1}0$. Or else $aM\cap V_i=0$, and then $a^{-1}U=a^{-1}0$. Since obviously $a^{-1}0$ is a homogeneous subspace, it follows in both cases that $a^{-1}U$ is homogeneous.

**Lemma.** *Let $C, D$ be words, such that also $C^{-1}D$ is a word. Let $C$ be of length $i$. Then $Kz_i$ embeds naturally into $F_{C,D}(M(C^{-1}D))$.*

*Proof.* Let $M=M(C^{-1}D)$, and $K_i=Kz_i$. Let $d$ be the direct letter with $Dd$ a word.

First, we will show that $K_i\cap D^-(M)=0$. This will be done by induction on the length $j$ of $D$. If $j=0$, then $D=1_d$, thus $D^-(M)=dM$, and obviously, $K_i$ is not contained in $dM$. Now let $l$ be a letter and $E$ a word with $D=lE$. The induction hypothesis implies $K_{i+1}\cap E^-(M)=0$. First, consider the case where $l$ is direct, say $D=aE$, *thus* $K_i=aK_{i+1}$. Assume $K_i\subseteq D^-(M)=aEdM$, thus $K_{i+1}\subseteq EdM+a^{-1}0$. Since $EdM$ and $a^{-1}0$ both are homogeneous, either $K_{i+1}\subseteq a^{-1}0$, which is nonsense, or else $K_{i+1}\subseteq EdM$, which contradicts the induction hypothesis. Next, consider the case where $l$ is inverse, say $D=a^{-1}E$, thus $K_{i+1}=aK_i$. If $K_i\subseteq D^-(M)=a^{-1}EdM$, then $K_{i+1}=aK_i\subseteq EdM$, again a contradiction to the induction hypothesis.

Since $D^-(M)$ is homogeneous, it follows that $D^-(M)$ is contained in $\bigoplus_{k\neq i} K_k$. Similarly, also $K_i\cap C^-(M)=0$, and therefore also $C^-(M)$ is contained in $\bigoplus_{k\neq i} K_k$. As a consequence, $K_i\cap[(C^+\cap D^-)+(C^-\cap D^+)](M)=0$. On the other hand, it is obvious that $K_i$ is contained in $(C^+\cap D^+)(M)$, and both assertions together define the embedding of $K_i$ into

$$F_{C,D}(M)=(C^+\cap D^+)(M)/[(C^+\cap D^-)+(C^-\cap D^+)](M).$$

**Corollary.** *Let $M=M(E)$ for some word $E$. If $C, D$, and $C^{-1}D$ are words, then the functor $F_{C,D}$ takes on $M$ the following value*

a) $F_{C,D}(M)=Kz_i$, *if $C^{-1}D=E$ and the length of $C$ is $i$.*

b) $F_{C,D}(M)=0$, *if both $C^{-1}D\neq E$ and $D^{-1}C\neq E$.*

*For $C$ in $\mathscr{W}'$, we have*

c) $F_C(M)=0$.

*Proof.* The vector spaces $F_{C,D}(M)$ and $F_C(M)$ may be considered as factors in the filtration $\mathscr{W}_a(M)$. Namely, for $C$ in $\mathscr{W}'$ we may assume that $C$ belongs to $\mathscr{W}_a$, since $F_C(M)$ and $F_{C^{-1}}(M)$ have the same dimension.

Let $E$ be of length $n$, thus $M$ is of dimension $n+1$. Also, there are $n+1$ pairs $(C, D)$ with $E=C^{-1}D$. By the previous lemma, the factors $F_{C,D}(M)$ with $C^{-1}D=E$ are non-zero, thus they are just one-dimensional, and all the other factors of the filtration $\tilde{\mathscr{W}}_a(M)$ have to be zero.

**Proposition.** *For every module $M$ in $\mathfrak{M}$, there is an $R$-linear mapping $\gamma_{C,D,M}: S_{C,D}F_{C,D}(M)\to M$ such that $F_{C,D}(\gamma_{C,D,M})$ is an isomorphism.*

*Proof.* Given $x\in(C^+\cap D^+)(M)$, there is an $R$-linear mapping $\gamma: M(C^{-1}D)\to M$, such that $\gamma(z_i)=x$, where $i$ is the length of $C$. Let $U$ be a complement of $[(C^+\cap D^-)+(C^-\cap D^+)](M)$ in $(C^+\cap D^+)(M)$, and let $m$ be the dimension of $U$. Note that $U=F_{C,D}(M)$, and that $S_{C,D}(U)=\bigoplus_m M(C^{-1}D)$. Using a basis of $U$, we get an $R$-linear mapping $\bigoplus_m M(C^{-1}D)\to M$ such that $\bigoplus_m Kz_i$ is mapped onto $U$.

## 5. The Modules of the Second Kind

Given a word $C=l_1\dots l_n$ in $\mathscr{W}'$, there are defined the cyclic permuted words $C_{(0)}=C$, $C_{(1)}=l_2\dots l_n l_1, \dots, C_{(n-1)}=l_n l_1\dots l_{n-1}$. The elements $C_{(i)}$ and $C_{(i)}^{-1}$ with $0\leqq i\leqq n-1$ form an equivalence class with respect to $\varrho'$.

**Lemma.** *The words $C_{(0)},\dots,C_{(n-1)},C_{(0)}^{-1},\dots,C_{(n-1)}^{-1}$ are pairwise different.*

*Proof.* If $C=C_{(0)}$ coincides with some $C_{(i)}$, $i\neq 0$, then $C$ is a non-trivial power of some shorter word in $\mathscr{W}'$, impossible. So assume $C=C_{(i)}^{-1}$ for some $i$. If $C=l_1\dots l_n$, then $C_{(i)}^{-1}=l_i^{-1}\dots l_1^{-1}l_n^{-1}\dots l_{i+1}^{-1}$. Since $l_i^{-1}=l_1$, either both elements $l_1$ and $l_i$ are of the form $a^*$ or both are of the form $b^*$. Thus $i$ is odd, say $i=2j+1$. But the $(j+1)$-th letter of $C_{(i)}^{-1}$ is $l_{j+1}^{-1}$, thus $l_{j+1}^{-1}=l_{j+1}$, impossible.

Recall that we have introduced an ordering on $\mathscr{W}$. In particular, for two words $C, D$ of the same length, we have $C<D$ iff $C=C_1dE_1$ and $D=C_1d^{-1}E_1$ for some direct letter $d$ and words $C_1, E_1, E_2$. Now, the word $C$ in $\mathscr{W}'$ will be called minimal provided it belongs to $\mathscr{W}_a'$ and we have $C<C_{(i)}$ and $C<C_{(j)}^{-1}$ for all $i\neq 0, j$ such that $C_{(i)}$ and $C_{(j)}^{-1}$ belong to $\mathscr{W}_a'$. By the previous lemma, every equivalence class in $\mathscr{W}'$ with respect to $\varrho'$ contains a (uniquely determined) minimal element.

**Lemma.** *Let $C$ be a minimal element in $\mathscr{W}'$, and let $\varphi$ be an automorphism of some vector space $V$. Let $M=M(C,\varphi)$. Then $C0_M=0_M$.*

*Proof.* By definition, $M=\bigoplus_{i=0}^{n-1} V_i$ with $V_i=V$. Let $V_j=V_i$ for $i\equiv j(n)$. Let $C=l_1\dots l_n$. If $l_i=a^*$, then the sequence $i, i-1, i-2,\dots$ is called the $a$-neighbour sequence of $i$, and $i-1, i, i+1,\dots$ is called the $a$-neighbor sequence of $i-1$. In this way, we define for every $i$ its $a$-neighbour sequence. (It corresponds to a walk around the schema, starting at the point $i$ and with direction $a^*$.)

First we show the following. If $D=e_1\dots e_m$ is in $\mathscr{W}_a$, $i_0=i, i_1, i_2,\dots$ is the $a$-neighbor sequence of $i$, and if $D(x)=0$ for some $x\in V_i$, then there are elements $x_{i_k}\in V_{i_k}$, such that $x_{i_0}=x$, $x_{i_m}=0$, and $x_{i_{k-1}}\in e_k x_{i_k}$. The proof follows by induction on $m$. Namely, if $m=1$, then there has nothing to be shown. If $m>1$, then let

$E=e_2\ldots e_m$, and choose an element $y$ such that $x\in d_1 y$ and $y\in E0_M$. If $e_1=a^{-1}$, then $y=ax$ belongs to $V_{i_1}$, so take $x_{i_1}=y$. If $e_1=a$, decompose $y=\sum_{j=0}^{n-1} y_j$ into its homogeneous components. Thus, $x=ay=\Sigma\, ay_j$ implies $x=ay_{i_1}$. On the other hand, with $y$ also $y_{i_1}$ belongs to $E0_M$, since $E0_M$ is homogeneous. So, take in this case $x_{i_1}=y_{i_1}$. Now we can apply induction on the word $E$ (which belong to $\mathscr{W}_b$) and the $b$-neighbor sequence $i_1, i_2, \ldots$ of $i_1$.

We know that $C0_M$ is homogeneous. Thus, let $x\in C0_M\cap V_i$ for some $i$. First, consider the case $i\neq 0$. Let $i_0=i, i_1, i_2, \ldots$ be the a-neighbor sequence for $i$. Let $E=e_1\ldots e_n$ be equal either to $C_{(i)}$ or to $C_{(i)}^{-1}$ whatever word belongs to $\mathscr{W}_a$. Note that $e_k: V_{i_{k-1}}\leftarrow V_{i_k}$ is an isomorphism, for all $k$. Now compare $E$ with $C$, say let $e_1=l_1, \ldots, e_{j-1}=l_{j-1}$, but $e_j\neq l_j$. Since $C<E$, this implies that $l_j$ is direct, and $e_j=l_j^{-1}$. Note that therefore $V_{i_j}=l_jV_{i_{j-1}}$. Now by the first part of the proof, there is a sequence of elements $x_k\in V_{i_k}$ such that $x_0=x$, and $x_{k-1}\in l_kx_k$ (we apply this part for $D=C$). But then $x_{j-1}=l_jx_j$, and $x_j\in V_{i_j}=l_jV_{i_{j-1}}$ together show that $x_{j-1}\in l_j^2M=0$. But this then implies that all the elements $x_k$, with $0\leqq k\leqq j-1$, satisfy $x_k=0$.

The case $i=0$ is even easier. This time, the $a$-neighbor sequence is just $0, 1, 2, \ldots$, and because of $x\in C0_M$, the first part of the proof gives us again a sequence of elements $x_k\in V_k$, with $x_0=x$, $x_{k-1}\in l_kx_k$, and $x_n=0_M$. But since the maps $l_k: V_{k-1}\leftarrow V_k$ all are isomorphisms, it follows that all the elements $x_k=0$.

**Corollary.** *Let $M=M(C,\varphi)$ for some $C$ in $\mathscr{W}'$ and some automorphism $\varphi$ of a vector space $V$. If $D\in\mathscr{W}'$, then the functor $F_D$ takes on $M$ the following value*

a) *$F_D(M)=(V_i,\varphi)$, if $D=C_{(i)}$ for some $i$,*

b) *$F_D(M)=0$, if $D$ is neither of the form $C_{(i)}$ nor of the form $C_{(i)}^{-1}$ for some $i$.*

*If $D$, $E$, and $D^{-1}E$ are words, then*

c) *$F_{D,E}(M)=0$.*

*Proof.* Assume the length of $C$ is $2n$ and that $C$ is minimal. Obviously, $V_0\subseteq C''(M)$. Since $C0_M=0_M$, it follows that $C'(M)=0$, thus, $V_0$ can be embedded into $(C''/C')(M)$. The functors $F_{C(2i+1)}$ and $F_{C_{(2i)}}(0\leqq i\leqq n-1)$ define pairwise different factors in the filtration $\mathscr{W}_a$, and all are of equal dimension, thus it follows that $V_0=(C''/C')(M)$. Of course, the induced automorphism is just $\varphi$. Also it can easily be seen that $V_i$ is a complement to $(C_{(i)})'(M)$ in $(C_{(i)})''(M)$. The other factors in the filtration $\mathscr{W}_a(M)$ have to be zero, this then proves (b) and (c).

**Proposition.** *For every module $M$ in $\mathfrak{M}$, there is an $R$-linear mapping $\gamma_{C,M}: S_CF_C(M)\to M$ such that $F_C(\gamma_{C,M})$ is an isomorphism. Here, $C$ is a word in $\mathscr{W}'$.*

*Proof.* Consider $C$ as a relation on $M$. We know that there is a subspace $U$ such that $C'(M)\oplus U=C''(M)$, and, moreover, $C$ induces an automorphism $\varphi_{C,M}$ on $U$. Thus, if $x_1, \ldots, x_m$ is a basis of $U$, then $Cx_i\cap U$ contains precisely one element, namely $\varphi_{C,M}(x_i)$. Let $C=l_1\ldots l_n$ and choose elements $x_i^{(k)}\in M$ such that $x_i^{(n)}=x_i$, $x_i^{(k-1)}\in l_k(x_i^{(k)})$ and $x_i^{(0)}=\varphi_{C,M}(x_i)$. This then defines a mapping from $S_CF_C(M)$ into $M$. Namely, $F_C(M)$ can be identified with $(U,\varphi_{C,M})$ and we map $U_0$ identically onto $U$, and we map the base elements $x_i$ of $U_k=U$ onto $x_i^{(k)}$. It follows, that this map is $R$-linear and that it goes, under $F_C$, onto an isomorphism.

## 6. The Elementary Intervalls Cover $\tilde{\mathscr{W}}_a$

In this section, we will also consider infinite words in our letters $a, a^{-1}, b, b^{-1}$. An infinite word $l_1 l_2 \dots$ is given by a sequence $l_1, l_2, \dots$ of letters with the same restriction as for finite words, namely that $l_i = a$ implies $l_{i+1} = b$, and that $l_i = b$ implies $l_{i+1} = a$. If $A = l_1 l_2 \dots$ is an infinite word, denote by $A_{[n]} = l_1 l_2 \dots l_n$ its finite part of length $n$. For example, if $C$ belongs to $\mathscr{W}'$ and has length $m$, we may form the infinite word $C^\infty$ with $(C^\infty)_{[km]} = C^k$. Infinite words of this form will be called *periodic*. As in the case of a periodic word, we consider for arbitrary infinite words $A$ the following inclusions

$$A_{[1]}0_M \subseteq A_{[2]}0_M \subseteq \cdots \subseteq A_{[n]}0_M \subseteq \cdots \subseteq A_{[n]}M \subseteq \cdots \subseteq A_{[1]}M\,,$$

so again we may define functors $A'$ and $A''$ from $\mathfrak{M}$ into ${}_K\mathfrak{M}$

$$A'(M) = \bigcup_n A_{[n]}0_M \subseteq \bigcap_n A_{[n]}M = A''(M)\,.$$

Thus, for $C \in \mathscr{W}'$, we have $C' = (C^\infty)'$ and $C'' = (C^\infty)''$. Denote by $\mathscr{W}_a^\infty$ the set of infinite words with first letter $a^*$.

**Lemma.** *Let $M$ be a module, and $0 \neq x \in M$. Then there is either some $C \in \mathscr{W}_a$ with $x \in C^+(M)$ and $x \notin C^-(M)$, or there is some infinite word $A \in \mathscr{W}_a^\infty$ such that $x \in A''(M)$ and $x \notin A'(M)$.*

*Proof.* Recall again that for two words $C, D$ of $\mathscr{W}_a$ of equal length, we write $C < D$ provided there is a direct letter $d$ and words $C_1, E_1, E_2$ with $C = C_1 d E_1$ and $D = C_1 d E_2$. Let $A_{[n]}$ be the smallest word with respect to this ordering of length $n$ such that $x \in A_{[n]}M$. Let $d$ be direct with $A_{[n]}d$ a word. If $x \in A_{[n]}dM$, then let $A_{[n+1]} = A_{[n]}d$, otherwise, let $A_{[n+1]} = A_{[n]}d^{-1}$. It is easy to see that $A_{[n+1]}$ is the smallest word of length $n+1$ with $x \in A_{[n+1]}M$. In this way, we construct an infinite word $A$ in $\mathscr{W}_a^\infty$, and we have $x \in A''(M)$. Now, assume $x$ lies also in $A'(M) = \bigcup A_{[n]}0_M$. Let $n$ be minimal with $x \in A_{[n+1]}0_M$, and let $A_{[n+1]} = A_{[n]}l_{n+1}$. Obviously, $l_{n+1}$ has to be inverse, since otherwise also $x \in A_{[n]}0_M$. Thus, the direct letter $d$ with $A_{[n]}d$ a word satisfies $x \in A_{[n]}d^{-1}0_M = A_{[n]}^+(M)$. Since $A_{[n+1]} = A_{[n]}d^{-1}$, it follows from the construction of $A$, that $x \notin A_{[n]}dM = A_{[n]}^-(M)$.

It remains to be shown that only the intervalls $[A', A'']$ with $A$ a periodic word, are of importance.

**Lemma.** *Let $A$ be an infinite word, and $A' \neq A''$. Then $A$ is periodic.*

*Proof.* Note that we can order the set $\mathscr{W}^\infty$ by the rule $A < B$ provided there is a direct letter $d$ such that $A_{[n]} = B_{[n]}$, $A_{[n+1]} = A_{[n]}d$ and $B_{[n+1]} = B_{[n]}d^{-1}$. (This is just the extension of the ordering of $\mathscr{W}$ to infinite words.) Then $\mathscr{W}^\infty$ is the disjoint union of two chains. namely of $\mathscr{W}_a^\infty$ and $\mathscr{W}_b^\infty$. Also, it is easy to see that $A < B$ implies that $A' \leqq A'' \leqq B' \leqq B''$ as functors. Thus, the intervalls $[A', A'']$ with $A \in \mathscr{W}_a^\infty$ all avoid each other, and similarly, for $\mathscr{W}_b^\infty$.

Now, let $A$ be an infinite word, and $M$ a module with $A'(M) \neq A''(M)$. Let $\mathscr{T}$ be the set of all infinite words $B$ with $B'(M) \neq B''(M)$ for this particular module $M$. Now $M$ is finite dimensional,and the factors $B''(M)/B'(M)$ belong to two filtrations of $M$, thus the number of factors $B''(M)/B'(M) \neq 0$ is finite. Therefore, also $\mathscr{T}$ is a finite set.

Next, if we write $A = A_{[n]}A^{[n]}$ with an infinite word $A^{[n]}$, then also $A^{[n]}$ belongs to $\mathscr{T}$. Since $\mathscr{T}$ is finite, we conclude that $A^{[n]} = A^{[m]}$ for some $n \neq m$, and therefore $A = A_{[n]}C^{\infty}$ with $C$ a finite word, $C \in \mathscr{W}'$. Let $l$ be the last letter of $C$. Since $C$ is of even length, the last letter of $A_{[n]}$ is either $l$ or $l^{-1}$. Now, if we assume that $n$ is minimal, then either $n=0$ and $A$ is periodic, or otherwise the last letter of $A_{[n]}$ has to be $l^{-1}$. We want to show that the latter leads to a contradiction. With $C^{\infty}$ also $lC^{\infty}$ belongs to $\mathscr{T}$, so assume, both $lC^{\infty}$ and $l^{-1}C^{\infty}$ belong to $\mathscr{T}$. On the one hand, we have the inequality, for all $k$,

$$\dim C^kM/C^k0_M \geqq \dim lC^kM/lC^k0_M + \dim l^{-1}C^kM/l^{-1}C^k0_M,$$

since $l$ is a letter, and both $a$ and $b$ act on $M$ with $a^2 = b^2 = 0$. On the other hand, $lC^{\infty} = C^{\infty}_{(n-1)}$, where $C_{(n-1)}$ is the $(n-1)$-th cyclic permuted word to $C$. Thus, for large $k$,

$$\dim C^kM/C^k0_M = \dim lC^kM/lC^k0_M,$$

and therefore, for large $k$, $l^{-1}C^kM = l^{-1}C^k0_M$. This shows that $l^{-1}C^{\infty}$ does not belong to $\mathscr{T}$.

**Proposition.** *The intervalls* $[(D^- + C^-)\cap C^+, (D^+ + C^-)\cap C^+]$ *with* $C \in \mathscr{W}_a$ *and* $D \in \mathscr{W}_b$, *and the intervalls* $[C', C'']$ *with* $C \in \mathscr{W}'_a$, *together cover* $\tilde{\mathscr{W}}_a$.

*Proof.* We only have to show that the intervall $[C^-, C^+]$ with $C \in \mathscr{W}_a$ is covered by the intervalls of the first kind. Since $\mathscr{W}_b$ is covered by the intervalls $[D^-, D^+]$ with $D \in \mathscr{W}_b$ and $[E', E'']$ with $E \in \mathscr{W}'_b$, we only have to show that

$$(E' + C^-)\cap C^+ = (E'' + C^-)\cap C^+,$$

for $C \in \mathscr{W}_a$ and $E \in \mathscr{W}'_b$. Using again the fact that the regular part of a relation splits off, we know that there is a subspace $U$ with $(E' + (E^{-1})')\oplus U = E'' + (E^{-1})''$. Now $E^{-1}$ belongs to $\mathscr{W}'_a$, so we know that either $C^+ \subsetneqq (E^{-1})'$ or that $(E^{-1})'' \subsetneqq C^-$. In both cases, the equality follows immediately.

## 7. Completion of the Proof

We have used throughout the previous sections as index set the disjoint union of $\mathscr{W}'$ and the set of all pairs $C, D$ such that $C, D, C^{-1}D$ are words. It remains to select an appropriate subset $I$ such that the functors $S_i, F_i (i \in I)$ then satisfy the conditions of the first lemma in Section 3.

Given a word $E$, then the functors $F_{C,D}$ with $C^{-1}D = E$ or with $D^{-1}C = E$ are all equivalent, thus we will use just one of those. That is, for every equivalence class $E, E^{-1}$ with respect to $\varrho$, we select one of the words, say $E = 1_dE$, and some decomposition, say $1_d \cdot E$. To be more precise, call $E$ principal, provided either $E$ has even length and belongs to $\mathscr{W}_a$, or $E$ has odd length, say $E = E_1 l E_2$, with words $E_1$ and $E_2$ of equal length, and a letter $l$, and its middle letter $l$ is direct. Let $I$ be the disjoint union of the principal words in $\mathscr{W}$, and the minimal words in $\mathscr{W}'$. If $E$ is a principal word, then let the corresponding functor $F_i$ be given by $F_{1_d,E}$, where $d$ is the direct letter such that $dE$ is *not* a word. Now, it follows easily, that all conditions of the first lemma in Section 3 are satisfied.

## 8. Appendix. Calculation of $\Omega$ and $\Omega^2$

Let $S$ be a quasi-Frobenius ring. If ${}_SM$ is indecomposable and not projective, choose a projective cover $\varepsilon: {}_SP \to {}_SM$, and let $\Omega M = \ker \varepsilon$. In this way, we get a bijective mapping $\Omega$ from the set of isomorphism classes of non-projective indecomposable $S$-modules into itself. It is well known that the mapping $\Omega^2$ is of particular interest.

Fix a natural number $q \geqq 1$. Let $S = S(q) = K\langle X, Y\rangle/(X^2, Y^2, (XY)^q - (YX)^q)$, and denote the canonical images of $X$ and $Y$ in $S$ by $a$ and $b$, respectively. Then $S$ is a quasi-Frobenius ring, and the category ${}_S\mathfrak{M}$ is a full exact subcategory of $\mathfrak{M}$. Let $\mathscr{W}(q)$ be the set of all words in $\mathscr{W}$ which don't contain $(ab)^q$, $(ba)^q$ or their inverses; similarly, let $\mathscr{W}'(q)$ be the set of all words $C$ in $\mathscr{W}'$ such that no power $C^n$ of $C$ contains any of the words $(ab)^q$, $(ba)^q$ or their inverses. Obviously, the indecomposable $S$-modules are ${}_SS = M((ab)^q(ab)^{-q}, \mathrm{id})$, where id is the identity automorphism on the vector space ${}_KK$, and the modules of the first kind $M(C)$ with $C \in \mathscr{W}_1(q)$, and the modules of the second kind $M(C, \varphi)$ with $C \in \mathscr{W}_2(q)$ and $\varphi$ an indecomposable automorphism of some vector space.

Define on $\mathscr{W}(q)$ a set-theoretical mapping $\phi$ as follows. First, denote $A = (ab)^{q-1}a$, $B = (ba)^{q-1}b$. For the words $A$, $B$, $AB^{-1}$ let

$$\phi(A) = A\,, \qquad \phi(B) = B\,, \qquad \phi(AB^{-1}) = A^{-1}B\,.$$

Since we want to have that $\phi$ commutes with forming inverses, this also defines the value of $\phi$ on $A^{-1}$, $B^{-1}$ and $BA^{-1}$. The other words $C \in \mathscr{W}(q)$ will be changed by $\phi$ in two ways, namely by a change on the left side, and one on the right side: If $C$ starts with $Ab^{-1}$ of $Ba^{-1}$, then we cancel this part, otherwise, we multiply from the left either by $A^{-1}b$ or $B^{-1}a$, whatever gives a word. Similarly, if $C$ ends in $aB^{-1}$ or $bA^{-1}$, this part of the word is cancelled under $\phi$, and otherwise $a^{-1}B$ or $b^{-1}A$ is added on the right. For example, consider the word $C = Ab^{-1} \cdot Ab^{-1}aba^{-1}$. If $q > 1$, then $\phi(C) = Ab^{-1}aba^{-1}b^{-1}A$, whereas for $q = 1$, $\phi(C) = Ab^{-1}a$.

Since $\phi$ commutes with forming inverses, it induces a mapping on $\mathscr{W}_1(q)$. It is easy to determine the orbits of $\{\phi^z; z \in \mathbb{Z}\}$ on $\mathscr{W}_1(q)$. Namely, the fixpoints are just the elements $A$, and $B$. All the other orbits have infinite length, and there are infinitely many such orbits.

**Theorem.** *Let $q \geqq 1$ be a natural number. Then*

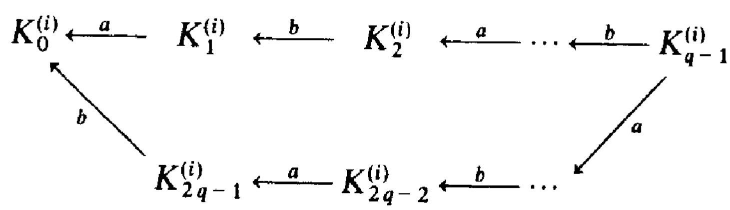
$$\Omega^2 M(C) = M(\phi(C)) \quad \textit{for} \quad C \in \mathscr{W}(q)\,,$$
$$\Omega^2 M(C, \varphi) = M(C, \varphi) \quad \textit{for} \quad C \in \mathscr{W}'(q)\,, \quad \varphi \in {}_{K[T,T^{-1}]}\mathfrak{M}\,.$$

*Proof*. Given a word $C \in \mathscr{W}$, its generation form is given by $C = C_1C_2^{-1} \cdot C_3C_4^{-1} \ldots C_{2g-1}C_{2g}^{-1}$, where all letters in $C_i$ are direct $(1 \leqq i \leqq 2g)$, and such that $|C_i| \geqq 1$ for $1 < i < 2g$ (note that $C_1$ and $C_{2g}$ may be equal to $1_a$ or $1_b$). In this case, $g$ is called the generating number of $C$.

If $C = C_1C_2^{-1} \ldots C_{2g}^{-1}$ is in generation form, then we denote by $K(C)$ the word $K(C) = D_1^{-1}D_2D_3^{-1} \ldots D_{2g}$, where again all letters in the $D_i$'s are direct, and such that, moreover, $D_iC_i$ is a word and of length $2q$. It is easy to see that $K(C)$ exists and is uniquely determined. Namely, given $C$, define the words $D_i$ consisting of direct letters by the property that $D_i C_i$ is a word of length $2q$. Then $D_1^{-1}$ can be multiplied from the right by $D_2$, since $C_1 C_2^{-1}$, $C_1 D_1$ and $C_2 D_2$ are words. Similarly, $D_2 D_3^{-1}$ is a word, and so on, therefore, $K(C)$ exists.

**Lemma.** *Let $C$ be a word in $\mathscr{W}(q)$, and let $c$, $d$ be the direct letters, such that $cCd^{-1}$ is in $\mathscr{W}$. Then $\Omega M(C) = M(K(cCd^{-1}))$.*

*Proof.* Let $C = C_1 C_2^{-1} \dots C_{2g}^{-1}$ be the generation form of $C$. Consider the free module ${}_S P = \bigoplus_{i=1}^{g} S^{(i)}$, where $S^{(i)} = {}_S S$ is given by the following diagramm




with $K_j^{(i)} = K$ for all $i, j$. Also, let $j_i$ be the length of the word $C_1 \dots C_{2i-1}$ $(i = 1, \dots, g)$, thus $M(C)$ has the form

$$V_0 \xleftarrow{C_1} V_{j_1} \xrightarrow{C_2} \xleftarrow{C_3} V_{j_2} \cdots V_{j_g} \xrightarrow{C_{2g}} V_n .$$

Define a map ${}_S P \to M(C)$ by mapping $K_{q-1}^{(i)} = K$ identically onto $V_{j_i} = K$ for $1 \leqq i \leqq g$. It is easy to see that the kernel is just $M(K(cCd^{-1}))$.

**Corollary.** *For $C$ in $\mathscr{W}(q)$, $\Omega^2 M(C) = M(\phi(C))$.*

*Proof.* We may assume that the first letter of $C$ is $a^*$. If $C = A$, then $K(bAb^{-1}) = (1_b)^{-1} A = A$, thus $\Omega M(A) = M(A)$. If $C = AB^{-1}$, then $K(bAB^{-1}a^{-1}) = (1_b)^{-1} 1_a = 1_a$, and $K(b \cdot 1_a \cdot a^{-1}) = A^{-1}B$, thus $\Omega^2 M(AB^{-1}) = M(A^{-1}B)$. So assume $C \neq A, AB^{-1}$.

Let $C_1 C_2^{-1} \dots C_{2g}^{-1}$ be the generation form of $C$, then $(bC_1) C_2^{-1} \dots C_{2g-1} (dC_{2g})^{-1}$ is the generation form of $bCd^{-1}$. Let $K(bCd^{-1}) = D_1^{-1} D_2 \dots D_{2g}$.

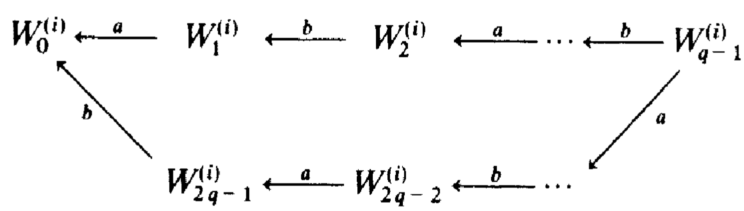
If $|C_1| = 2q - 1$, then $C$ starts with $Ab^{-1}$. In this case, let $C_2 = E_2 b$. Note that $|D_1| = 0$, thus $D = D_2 D_3^{-1} \dots D_{2g}$. Since the last letter of $C_2$ is $b$, the first letter of $D_2 C_2$ is $a$, thus we have to consider $K(bDe^{-1})$, for the suitable direct letter $e$. Now $(bD_2) D_3^{-1} \dots$ is the left hand side of the generation form of $bDe^{-1}$. Since $D_2 C_2 = D_2 E_2 b$ is a word and of length $2q$, also $E_2(bD_2) = (E_2 b) D_2$ is a word and of length $2q$. Thus, the left hand side of $K(bDe^{-1})$ is just $E_2^{-1} C_3 \dots$, and by definition, this is the left hand side of $\phi(C)$.

Next, let $|C_1| < 2q - 1$. Then $|D_1| \neq 0$. Since $D_1 b C_1$ is a word, $D_1^{-1}$ starts with $a^{-1}$, and therefore we have to consider $K(bDe^{-1})$, for some direct letter $e$. Note that now $b \cdot D_1^{-1} D_2 \dots$ is the left hand side of the generation form of $bDe^{-1}$. Therefore, $K(bDe^{-1}) = E_0^{-1} E_1 E_2^{-1} \dots$ where $E_0 b$, $E_1 D_1$, $E_2 D_2, \dots$ are words of length $2q$ and consist only of direct letters. As a consequence, $E_0 = A$, $E_1 = bC_1$, $E_2 = C_2$, and so on. This shows, that $K(bDe^{-1})$ is equal, on the left side, to $A^{-1} b C_1 C_2^{-1} \dots$.

We have shown that $\Omega^2 M(C) = M(E)$, where $E$ coincides, on the left, with $\phi(C)$. By symmetry, the same is true on the right.

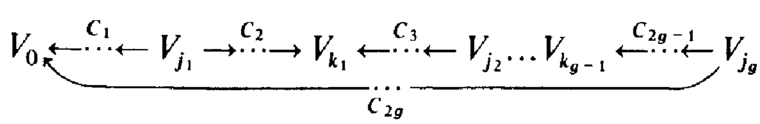
**Lemma.** *Let* $C \in \mathscr{W}''(q)$ *with first letter direct, and last letter inverse. Let* $g$ *be the generating number of* $C$. *Then for any* $\varphi \in {}_{K[T,T^{-1}]}\mathfrak{M}$, $\Omega M(C,\varphi) = M(K(C),(-1)^g \varphi^{-1})$.

*Proof.* Let $\varphi$ be automorphism of the vector space $V$, and $\dim_K V = m$. Again, we have to construct a free module $P = \bigoplus_{i=1}^{g} P^{(i)}$, with $P^{(i)} = S^n$ free of rank $n$, given in a similar way as in the previous lemma, namely



where all $W_j^{(i)} = V$, and all maps are given by the identity map on $V$.

Let $C = C_1 C_2^{-1} \ldots C_{2g}^{-1}$ be the generation form of $C$. Let $j_i$ be the length of the word $C_1 \ldots C_{2i-1}$, and $k_i$ the length of $C_1 \ldots C_{2i}^{-1}$. Then, $M(C,\varphi)$ has the form



where all arrows, except the first one $V_0 \leftarrow V_1$, give the identity map on $V$, whereas $V_0 \leftarrow V_1$ is given by $\varphi$.

Define a mapping ${}_S P \xrightarrow{\varepsilon} M(C,\varphi)$ by the following condition: For odd $i$, map $W_{q-1}^{(i)} = V$ by the identity map id onto $V_{j_i}$, whereas, for even $i$, map $W_{q-1}^{(i)}$ onto $V_{j_i}$ by $-\mathrm{id}$. The kernel of this map can be constructed in the following way. Consider the modules $M(D_1^{-1} D_2)$, $M(D_3^{-1} D_4)$, ..., $M(D_{2g-1}^{-1} D_{2g})$, where $K(C) = D_1^{-1} \ldots D_{2g}$. We want to identify the vectorspaces belonging to consecutive end points, in order to get a module of the form $M(K(C),\psi)$ for some $\psi$. Now, for every $i$ with $1 \leqq i < g$, there are given two indices $u, v$ with $1 \leqq u, v \leqq 2q-1$ and maps $W_u^{(i)} \to V_{k_i}$ and $W_v^{(i+1)} \to V_{k_i}$ which are restrictions of $\varepsilon$. One of these maps is id, the other one is $-\mathrm{id}$, and the kernel of both together identifies two of those vector spaces. In order to identify the first vector space of $M(D_1^{-1} D_2)$ and the last of $M(D_{2g-1}^{-1} D_{2g})$, we use the two maps $W_u^{(g)} \to V_0$ and $W_v^{(1)} \to V_0$ which are restrictions of $\varepsilon$, again for some $u, v$. The first of these maps is of the form $(-\mathrm{id})^{g-1}$, the second is of the form $\varphi$. Thus, the kernel of the combined map $W_u^{(g)} \oplus W_v^{(1)} \to V_0$ is just the graph of the map $(-1)^g \varphi$. Thus, we have constructed to the word $K(C)$ a sequence of vectorspaces and maps which give a module $M(K(C),\psi)$. The only map which is not the identity, is $(-1)^g \varphi$ and is induced by the first letter of $D$. Since this is an inverse letter, we see that $\psi = (-1)^g \varphi^{-1}$.

This concludes the proof of the lemma, and this obviously implies $\Omega^2 M(C,\varphi) = M(C,\varphi)$.

## Reference

Gelfand, I. M., Ponomarev, V. A.: Indecomposable representations of the Lorentz group. Usp. Mat. Nauk, **23**, 1968, pp. 3—60. English Translation: Russ. Math. Surv. **23**, 1 58

Claus Michael Ringel
Sonderforschungsbereich "Theoretische Mathematik"
Universität
D-5300 Bonn
Beringstraße 1
Federal Republic of Germany

(Received June 18, 1974)