JOURNAL OF ALGEBRA 41, 269–302 (1976)

Representations of K-Species and Bimodules

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Communicated by P. M. Cohn

Received December 30, 1974

Let K be a commutative field. Let $S = (F_i, {}_iM_j)_{1 \leqslant i,j \leqslant n}$ be a K-species, connected and without oriented cycles. (That is, for all i, j, ${}_iM_j$ is an F_i–F_j-bimodule, where F_i and F_j are division rings containing K in its center, and such that K operates centrally on ${}_iM_j$ and $\dim_K {}_iM_j$ is finite. From S we derive an oriented graph with edges $i \bullet \rightarrow \bullet j$ precisely when ${}_iM_j \neq 0$. Then, S is called connected, provided the corresponding graph is connected, and an oriented cycle of S is given by a sequence $i_1, i_2, ..., i_{k-1}, i_k = i_1$ such that $i_j \bullet \rightarrow \bullet i_{j+1}$ for all $1 \leqslant j \leqslant k-1$. Since we assume that S contains no oriented cycles, we have, in particular, that ${}_iM_i = 0$ for all i, and that ${}_iM_j \neq 0$ implies ${}_jM_i = 0$.) A representation $(V_i, {}_j\varphi_i)$ of S is given by vector spaces $(V_i)_{F_i}$ and F_j-linear mappings $V_i \otimes_i M_j \rightarrow V_j$. Such a representation is called finite dimensional provided all the V_i are finite dimensional vector spaces. A homomorphism $\alpha = (\alpha_i)\colon (V_i, {}_j\varphi_i) \rightarrow (V_i', {}_j\varphi_i')$ is given by F_i-linear mappings $\alpha_i : V_i \rightarrow V_i'$ such that $\alpha_j {}_j\varphi_i = {}_j\varphi_i'(\alpha_i \otimes 1)$. We denote by $\mathfrak{L}(S)$ the category of all representations of S and by $\mathfrak{l}(S)$ the category of finite dimensional representations.

Given a K-species S, there is defined a quadratic form q on the n-dimensional rational vector space $\mathbb{Q}^n$ as follows: for $x \in \mathbb{Q}^n$, let

$$q(x) = \sum f_i x_i^2 - \sum m_{ij} x_i x_j,$$

where $f_i = \dim_K F_i$ and $m_{ij} = \dim_K {}_iM_j$. It is known from our joint work with Dlab [6, 7] that S is of finite representation type if and only if the corresponding quadratic form is positive definite. Also, in [7] the structure of $\mathfrak{l}(S)$ in the positive semidefinite case was reduced to that of a certain abelian full and exact subcategory $\mathfrak{h}$, called the subcategory of all homogeneous objects. Our first aim is to determine the structure of this category. This then shows that in the positive semidefinite case, all indecomposable finite dimensional representations can be classified (S is "tame").

THEOREM 1. *Let S be a connected K-species without cycles and with positive* *semidefinite quadratic form. Then, the category of all homogeneous representations of S is uniserial, and is the direct sum of uniserial categories of global dimension* 1 *with one simple object.*

The theorem follows from a new interpretation of the quadratic form q. Namely, we introduce a (usually nonsymmetric) bilinear form on $\mathbb{Q}^n$ by

$$\tilde{q}(x, y) = \sum f_i x_i y_i - \sum m_{ij} x_i y_j .$$

The corresponding quadratic form obviously is just q, and it turns out that for any two representations X and Y of S,

$$\tilde{q}(\dim X, \dim Y) = \dim_K \operatorname{Hom}(X, Y) - \dim_K \operatorname{Ext}^1(X, Y),$$

where for a representation $(V_i, {}_j\varphi_i)$, we define $[\dim(V_i, {}_j\varphi_i)]_k = \dim (V_k)_{F_k}$.

Thus, we know that in the semidefinite case, there is a complete classification of the indecomposable objects of $\mathfrak{l}(S)$. On the other hand, we show that a similar description is impossible in case the quadratic form is indefinite: a K-species without cycles and with indefinite quadratic form is "wild."

THEOREM 2. *Let $S = (F_i, {}_iM_j)$ be a connected K-species without cycles, and with indefinite quadratic form. Then, for any i, there is a commutative field K' with $K \subseteq K' \subseteq F_i$, and a full exact embedding $\mathfrak{w}(K') \to \mathfrak{l}(S)$.*

Here, $\mathfrak{w}(K)$ denotes the "wild category" over K, defined as follows. Let $R = K\langle x, y\rangle$ be the free associative algebra in two variables over K, and let $\mathfrak{w}(K)$ be the category of all R-modules which are finite dimensional as K-vector spaces. Alternatively, we may consider $\mathfrak{w}(K)$ as the category of all finite dimensional vector spaces over K endowed with two endomorphisms. Also, we denote by $\mathfrak{W}(K)$ the category of all R-modules.

In order to prove Theorem 2, we will have to consider first a special case, namely representations of a bimodule ${}_FM_G$, where F and G are two division rings. Here, we consider ${}_FM_G$ as a species with graph $\bullet \to \bullet$. In dealing with bimodules, we will also consider the "nonalgebraic" case (that is, we do not assume the existence of a central subfield K with $\dim_K M$ finite). In the algebraic case, however, we are able to give a complete classification of the dimension types of the indecomposable representations of ${}_FM_G$. Given a quadratic form q on a vector space $\mathbb{Q}^n$ with a fixed basis, the Weyl roots are by definition the images of the base vectors under the Weyl group. In the given case, where $n = 2$, we will define the imaginary roots as those vectors x in $\mathbb{Z}^2$ which satisfy $q(x) \leqslant 0$ (for a general definition, see [11]).

THEOREM 3. *Let ${}_FM_G$ be an algebraic bimodule. If V is an indecomposable representation of ${}_FM_G$, then* $\dim V$ *is either a positive Weyl root, or a positive* *imaginary root. There is precisely one indecomposable representation for every positive Weyl root, and (assuming that K is infinite) there are infinitely many indecomposable representations for every positive imaginary root.*

Let us mention certain applications. First, consider the bimodule $M = K^n$, where n is a natural number, with the canonic bimodule structure. The objects in $\mathfrak{l}({}_KM_K)$ can be considered as pairs of vector spaces X_K, Y_K together with n linear transformations $X_K \to Y_K$. Thus, we consider the classification problem for n matrices $A_1, ..., A_n$ of equal size (namely, $x \times y$-matrices, with $x = \dim X_K$ and $y = \dim Y_K$). For $n \geqslant 3$, this problem is usually referred to as unsolvable ("an impossible task" [8]). The theorem above gives at least the general conditions for which pairs (x, y) every set of n $x \times y$-matrices is decomposable: namely, every set of n $x \times y$-matrices can be decomposed if and only if (x, y) is neither a Weyl root nor an imaginary root. And, if (x, y) is a Weyl root, there is just one indecomposable set of n $x \times y$-matrices—every other set is either equivalent to this or can be decomposed.

As second application, assume that $G \subseteq F$ is a finite field extension, and let ${}_FM_G = {}_FF_G$, canonically. The objects (X_F, Y_G, φ) in $\mathfrak{l}({}_FM_G)$ with φ surjective, correspond just to the G-subspaces of F-vector spaces: consider $\ker \varphi$ as G-subspace of X_F. Again, Theorem 3 gives the precise condition for what dimensions every pair consisting of an F-space together with a G-subspace, can be decomposed. Also, it is quite interesting to note here a consequence for the endomorphism rings $\mathrm{End}(X_F; U_G) = \{\varphi \in \mathrm{End}(X_F) \mid \varphi(U) \subseteq U\}$ of the pairs $U_G \subseteq X_F$. In case $\dim F_G \leqslant 4$, these endomorphism rings are of quite restricted type. Namely, if $U_G \subseteq X_F$ is indecomposable, or, what is the same, if $E = \mathrm{End}(X_F; U_G)$ is a local ring, then E is a division ring, or, at least, a uniserial ring. This follows from [7] together with our Theorem 1. On the other hand, in case $\dim F_G \geqslant 5$, any finite dimensional G-algebra is of the form $\mathrm{End}(X_F; U_G)$ for suitable $U_G \subseteq X_F$. This is an obvious consequence of Theorem 2. Similar results hold of course in case $G \subseteq F$ is an inclusion of division rings, provided there is a central subfield K of F with $K \subseteq G$ and $\dim_K F$ finite.

In the nonalgebraic case, we have to restrict the investigation to bimodules which are not simple, the other case seems to be harder to attack. Of course, this implies immediately that $\dim {}_FM \geqslant 2$ and $\dim M_G \geqslant 2$. We begin with the case $\dim {}_FM = \dim M_G = 2$.

THEOREM 4. *Let* ${}_FM_G$ *be a nonsimple bimodule with* $\dim {}_FM = \dim M_G = 2$. *Then, in* $\mathfrak{l}({}_FM_G)$, *there is precisely one indecomposable representation with dimension type* (x, y), *where* x, y *are natural numbers with* $|x - y| = 1$. *All the other indecomposable representations have dimension type* (x, x), *with* x *a positive integer, and their direct sums form a full, exact,* *abelian subcategory* $\mathfrak{r}({}_FM_G)$. *This category* $\mathfrak{r}({}_FM_G)$ *is equivalent to the product category* $\mathfrak{m}_R \times \mathfrak{u}$, *where* $R = F[T; \epsilon, \delta]$ *is a skew polynomial ring in one variable, for some automorphism* ϵ *and some* $(1, \epsilon)$-*derivation* δ *of* F, *and* $\mathfrak{m}_R$ *is the category of all right* R-*modules of finite length, whereas* $\mathfrak{u}$ *is a uniserial category of global dimension* 1 *and with only one simple object. Conversely, given a division ring* F, *an automorphism* ϵ, *and an* $(1, \epsilon)$-*derivation* δ *of* F, *there exists a bimodule* ${}_FM_F$ *with* $\mathfrak{r}({}_FM_F) = \mathfrak{m}_R \times \mathfrak{u}$ *for the corresponding* $R = F[T; \epsilon, \delta]$.

Using results of Cozzens [5] and of MacConnell and Robson [10], we get the following corollaries.

COROLLARY 1. *There exists a bimodule* ${}_FM_G$ *with* $\dim {}_FM = \dim M_G = 2$, *such that, for every positive integer* n, *the number of indecomposable representation of* ${}_FM_G$ *of length* n *is equal to* 1 *or* 2.

COROLLARY 2. *There exists a bimodule* ${}_FM_G$ *with* $\dim_F M = \dim M_G = 2$, *such that there is a full exact embedding* $\mathfrak{w}(K) \to \mathfrak{l}({}_FM_G)$, *for some commutative field* K.

The two corollaries above show that the behavior of the category $\mathfrak{l}({}_FM_G)$ in the case $\dim_F M = \dim M_G = 2$ can be rather different. Also, the first corollary shows that the second Brauer–Thrall conjecture, which was stated for finite dimensional algebras over an infinite field (and recently proved over algebraically closed fields [12]), cannot be generalized to arbitrary artinian rings, say with infinite center. In fact, the matrix ring $R = \left[\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right]$ constructed with the bimodule of Corollary 1, can have arbitrarily large center, and the category $\mathfrak{m}_R$ of right R-modules of finite length coincides with the category $\mathfrak{l}({}_FM_G)$.

Finally, we consider the case $(\dim_F M)(\dim M_G) > 4$. In this case, the behavior of ${}_FM_G$ is always wild, provided we do not restrict to representations of finite length. The center Z of ${}_FM_G$ is defined to be the center of the matrix ring $\left[\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right]$. For technical reasons, we have to assume that the dimension of M over Z is not too large, namely $< \aleph_\iota$, the first strongly inaccessible cardinal number.

THEOREM 5. *Let* ${}_FM_G$ *be a nonsimple bimodule with* $(\dim_F M)(\dim M_G) > 4$. *Let* Z *be the center of* M, *and assume* $\dim_Z M < \aleph_\iota$. *Then, there is a full and exact embedding* $\mathfrak{W}(K) \to \mathfrak{L}({}_FM_G)$ *for some commutative field* K.

Of course, K will always contain the center Z. However, we do not know, whether we always can choose $K = Z$.

1. Finite Localization, That Is: Simplification

Our final aim is to construct indecomposable objects in an abelian category $\mathfrak{C}$. Always, there will be certain obvious indecomposable objects, and we want to use them in order to build larger ones. Namely, given indecomposable objects X, and Y, we will look for nonsplit exact sequences

$$0 \to X \to E \to Y \to 0,$$

hoping that E is indecomposable, again. But of course, this is quite rarely the case. There is, however, one situation, where for trivial reasons every nonsplit exact sequence gives rise to an indecomposable extension, namely when both X and Y are simple. Now, if X and Y are not necessarily simple in $\mathfrak{C}$, but belong to a full, exact, extension closed abelian subcategory $\mathfrak{U}$ of $\mathfrak{C}$, such that X and Y are simple, when considered as objects of $\mathfrak{U}$, then again, every nonsplit extension E is indecomposable (E is indecomposable as object of $\mathfrak{U}$, and therefore as object of $\mathfrak{C}$). Thus, given X and Y, we try to find such a subcategory $\mathfrak{U}$ where X and Y are simple, and we will call this the process of *simplification*. Now, necessary conditions obviously are that the endomorphism rings $\mathrm{End}(X)$ and $\mathrm{End}(Y)$ both be division rings, and that either X and Y be isomorphic, or that $\mathrm{Hom}(X, Y) = \mathrm{Hom}(Y, X) = 0$. As we will see, these conditions are also sufficient for such a subcategory $\mathfrak{U}$ to exist. We have reserved this section for considering the process of simplification in more detail.

1.1. Let $\mathfrak{C}$ be an abelian category. An object X with $\mathrm{End}(X)$ a division ring (or, the isomorphic class of X) will be called a *point* of $\mathfrak{C}$. Two points X and Y are called *orthogonal*, provided $\mathrm{Hom}(X, Y) = \mathrm{Hom}(Y, X) = 0$.

Example 1. Let A be a commutative ring, and $\mathfrak{C} = {}_A\mathfrak{M}$ the category of all A-modules. *There is a canonical bijection between the points of ${}_A\mathfrak{M}$ and the prime ideals of A.* Namely, if I is a prime ideal of A, let $Q(A/I)$ be the quotient field of A/I. Then $\mathrm{End}({}_AQ(A/I)) = Q(A/I)$, thus ${}_AQ(A/I)$ is a point, and I is its annihilator. Conversely, let ${}_AX$ be a point, say with endomorphism ring $D = \mathrm{End}({}_AX)$, and let I be the annihilator of ${}_AX$. Now A/I embeds into the center Z of D, and since Z is a field, also $Q(A/I)$ can be embedded into Z. Therefore, X may be considered as a $Q(A/I)$-module, and $D = \mathrm{End}({}_AX) = \mathrm{End}({}_{Q(A/I)}X)$ (note that the inclusion $\mathrm{End}({}_AX) \supseteq \mathrm{End}({}_{Q(A/I)}X)$ is trivial, whereas any $\varphi \in \mathrm{End}({}_AX)$ commutes with all elements of Z, thus with the action of $Q(A/I)$ on X). But $Q(A/I)$ is a field, and the full endomorphism ring $\mathrm{End}({}_{Q(A/I)}X)$ of a vector space is without zero divisors only in case $\dim_{Q(A/I)}X = 1$. Thus ${}_{Q(A/I)}X = {}_{Q(A/I)}Q(A/I)$, and therefore ${}_AX = {}_AQ(A/I)$.

EXAMPLE 2. Let R be a (not necessarily commutative) ring, and $\mathfrak{C} = {}_R\mathfrak{M}$ the category of all left R-modules. A point X in ${}_R\mathfrak{M}$ with $D = \mathrm{End}(X)$ can be considered as a right D-vector space X_D, and X is said to be *finite* point in case X_D is finite dimensional. *There is a canonical bijection between the finite points of* ${}_R\mathfrak{M}$ *and the ring epimorphisms* $R \to S$ *with* S *a simple artinian ring* (where we identify $R \to S$ with $R \to S \xrightarrow{\alpha} S'$ for any isomorphism α). Namely, it is well known that a ring homomorphism $R \to S$ is an epimorphism if and only if the forget functor ${}_S\mathfrak{M} \to {}_R\mathfrak{M}$ is full. Thus, if $R \to S$ is a ring epimorphism, and ${}_SX$ is the unique simple module over the simple artinian ring S, then $D = \mathrm{End}({}_SX) = \mathrm{End}({}_RX)$ is a division ring, and X_D is finite dimensional, thus ${}_RX$ is a finite point. Conversely, let ${}_RX$ be a finite point, and $S = \mathrm{End}(X_D)$. We claim that the canonical map $R \to S$ is an epimorphism. Since S is simple artinian, there is only one simple S-module, namely ${}_SX$, and every S-module is a direct sum of copies of ${}_SX$. Let ${}_SA$, ${}_SB$ be S-modules, and $f\colon {}_RA \to {}_RB$ be an R-homomorphism. Now ${}_SA = \bigoplus_I X_i$, and ${}_SB = \bigoplus_J X_j$, with $X_i = X_j = X$, and f is given by its components (f_{ij}), where all $f_{ij} : X_i \to X_j$ belong to D. Thus, the maps f_{ij} are not only R-linear, but even S-linear, and therefore f itself is S-linear.

As we have seen in the previous example, every point in ${}_A\mathfrak{M}$, for A commutative, is finite, and P. M. Cohn has proposed to call the set of finite points of ${}_R\mathfrak{M}$ (with a suitable topology) the spectrum of R.

1.2. Let $\mathscr{X}$ be a set (or class) of pairwise orthogonal points in $\mathfrak{C}$. If A is an object of $\mathfrak{C}$, then an *$\mathscr{X}$-filtration* of A is given by a sequence of subobjects

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = A,$$

with $A_i/A_{i-1} \in \mathscr{X}$, for $1 \leqslant i \leqslant n$.

THEOREM. *Let $\mathscr{X}$ be a class of pairwise orthogonal points in the abelian category $\mathfrak{C}$. Let $\mathfrak{U}(\mathscr{X})$ be the full subcategory of all objects of $\mathfrak{C}$ with an $\mathscr{X}$-filtration. Then $\mathfrak{U}(\mathscr{X})$ is an exact abelian subcategory which is closed under extensions, and the set $\mathscr{X}$ is the set of all simple objects in $\mathfrak{U}(\mathscr{X})$.*

Proof. It is obvious that $\mathfrak{U}(\mathscr{X})$ is closed under extensions. Let

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$$

and

$$0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_m = B$$

be two $\mathscr{X}$-filtrations, with $n, m \geqslant 1$. Let $f\colon A \to B$ be a morphism. We prove by induction on n that the image of f has an $\mathscr{X}$-filtration.

By induction, we can assume that $f(A_1) \neq 0$. We even may assume that $f(A_1) = B_1$. Namely, there is some i with $f(A_1) \subseteq B_i$ and $f(A_1) \not\subseteq B_{i-1}$. Then

$$A_1 \xrightarrow{f} B_i \xrightarrow{\epsilon} B_i/B_{i-1}$$

(with the canonical projection ϵ) is a nonzero homomorphism, and therefore an isomorphism, since both A_1 and B_i/B_{i-1} belong to $\mathscr{X}$. As a consequence,

$$f(A_1) \oplus B_{i-1} = B_i .$$

If we replace B_j by $B_j' = f(A_1) + B_{j-1}$ for $1 \leqslant j < i$, and take $B_j' = B_j$ for $i \leqslant j \leqslant m$, then we get another $\mathscr{X}$-filtration of B and $f(A_1) = B_1'$. Now we apply induction to the induced map f: $A/A_1 \to B/B_1$, and note that the image of f is an extension of the image of f by B_1.

Also, the kernel of f has an $\mathscr{X}$-filtration. Namely, if A_1 is contained in $\ker(f)$, the result follows by induction. And, if $A_1 \not\subseteq \ker(f)$, then, as above, we have $A_1 \cap \ker(f) = 0$, and we may assume $f(A_1) = B_1$. But $\ker(f)$ is isomorphic to the kernel of the induced map f: $A/A_1 \to B/B_1$. Similarly, the cokernel of f has an $\mathscr{X}$-filtration.

As a consequence, $\mathfrak{U}(\mathscr{X})$ is an exact abelian subcategory. It is easy to see that the elements of $\mathscr{X}$ are simple objects in $\mathfrak{U}(\mathscr{X})$. Also, every object of $\mathfrak{U}(\mathscr{X})$ has a composition series with factors in $\mathscr{X}$, thus all simple objects of $\mathfrak{U}(\mathscr{X})$ belong to $\mathscr{X}$.

1.3. Perhaps we should give an indication why we consider the process of simplification as a type of localization, namely as the finite localization (as compared with the injective and the inversive method of localization; see [3]). *Finite localization* should mean that we try to construct the modules of *finite* length over the appropriate localized ring (provided such a ring can be defined). However, we do not even define a localization functor (that is, a left adjoint to the inclusion $\mathfrak{U}(\mathscr{X}) \subseteq \mathfrak{C}$), which, of course, could only be done in case $\mathfrak{C}$ itself is a length category, whereas otherwise we would have to replace $\mathfrak{U}(\mathscr{X})$ by a larger category, say the profinite closure of $\mathfrak{U}(\mathscr{X})$ in $\mathfrak{C}$.

1.4. The class of points in a given category usually will be rather large. An indication for this can be derived from a theorem of Corner's, which we state here also for later reference. A *Corner quintuple* $(A; A_i)$ of rank m is given by a free abelian group A and five subgroups A_i of A such that both A_i and A/A_i are free of rank m, and such that for all abelian groups B and B', the canonical map $\beta \to \beta \otimes 1$

$$\mathrm{Hom}(B, B') \to \mathrm{Hom}((B \otimes A; B \otimes A_i), (B' \otimes A; B' \otimes A_i))$$

is bijective. Here, of course, the right Hom-set consists of all homomorphisms $\gamma: B \otimes A \to B' \otimes A$, such that $\gamma(B \otimes A_i) \subseteq B' \otimes A_i$ for all $1 \leqslant i \leqslant 5$. And, two Corner quintuples $(A; A_i)$ and (A', A_i') are called *orthogonal* provided

$$\begin{aligned}&\mathrm{Hom}((B \otimes A; B \otimes A_i), (B' \otimes A'; B' \otimes A_i'))\\ &\quad = 0 = \mathrm{Hom}((B \otimes A'; B \otimes A_i'), (B' \otimes A; B' \otimes A_i))\end{aligned}$$

for all abelian groups B, B'.

THEOREM (Corner [4]). *Let m be a cardinal number, $m < \aleph_\iota$, the first strongly inaccessible cardinal number. Then there is a set of 2^m orthogonal Corner quintuples of rank m.*

Let K be a field, let $R = K\langle x_1, \ldots, x_5 \rangle$ be the free associative K-algebra with five generators, and let m be a cardinal number with $\max(\aleph_0, |K|) \leqslant m < \aleph_\iota$. If $(A; A_i)$ is a Corner quintuple, let ${}_RX$ be given by the K-vector space ${}_KK \otimes_{\mathbb{Z}} A$ such that the multiplication with x_i on X is a vector space endomorphism with image ${}_KK \otimes_{\mathbb{Z}} A_i$ (here, $\mathbb{Z}$ denotes the ring of all rational integers). Then, $\mathrm{End}({}_RX) = \mathrm{End}({}_KK) = K$, thus ${}_RX$ is a point. Starting with 2^m orthogonal Corner quintuples of rank m, we define in this way 2^m orthogonal points ${}_RX$ with $|X| = m$.

On the other hand, the class of finite points is always a set. Namely, as we have seen, the finite points correspond to the epimorphisms $R \to S$ with S simple artinian. However, for any epimorphism $R \to S$, we have $|S| \leqslant \max(\aleph_0, |R|)$ [9]. Therefore, there are in the category ${}_R\mathfrak{M}$ at most 2^{2^m} different finite points, with $m = \max(\aleph_0, |R|)$.

1.5. We are not only interested in constructing objects of finite length, but also larger ones, namely to build prescribed extensions of homogeneous semisimple objects of type Y by homogeneous semisimple objects of type X, where X and Y are nonisomorphic simple objects, or, more general, orthogonal points. The extension group $\mathrm{Ext}^1(X, Y)$ will usually be denoted just by $\mathrm{Ext}(X, Y)$. Note that $\mathrm{Ext}(X, Y)$ is, in a natural way, an $\mathrm{End}(Y)$–$\mathrm{End}(X)$-bimodule. In order to specify certain types of extensions, we will consider $\mathrm{End}(Y)$–$\mathrm{End}(X)$-submodules of $\mathrm{Ext}(X, Y)$.

Given division rings F and G, and a bimodule ${}_GN_F$, denote by $\mathfrak{L}^*({}_GN_F)$ the category of all triples (A_F, B_G, φ) where $\varphi: A_F \to B_G \otimes_G N_F$ is F-linear, and with morphisms $(\alpha, \beta): (A_F, B_G, \varphi) \to (A_F', B_G', \varphi')$ given by $\alpha: A_F \to A_F'$, $\beta: B_G \to B_G'$ satisfying $\varphi'\alpha = (\beta \otimes 1)\varphi$. Note, that this corresponds just to Gabriel's definition of the representations of a species [8]. In case $\dim_G N$ is finite, we have $\mathfrak{L}^*({}_GN_F) = \mathfrak{L}(\mathrm{Hom}_G({}_GN_F, {}_GG_G))$. The following lemma seems to be well known.

LEMMA. *Let $\mathfrak{C}$ be a Grothendieck category. Let X and Y be orthogonal points with* $\mathrm{End}(X) = F$, $\mathrm{End}(Y) = G$, *and let ${}_GN_F$ be a submodule of* $\mathrm{Ext}(X, Y)$. *Then there is a full exact embedding $\mathfrak{L}^*({}_GN_F) \to \mathfrak{C}$, such that* $(F_F, 0, 0)$ *goes to X and* $(0, G_G, 0)$ *goes to Y.*

Proof. (cf. [1, 8]). First, we want to construct the image of (N_F, G_G, id). In order to do so, consider "all possible" extensions of Y by X which belong to ${}_GN_F$. That is, assume the extensions

$$0 \to Y_i \to Z_i \to X_i \to 0, \qquad \text{with} \quad Y_i = Y, \qquad X_i = X, \text{ and } i \in I$$

form a basis of N_F. Let U be the kernel of the canonical map $\bigoplus_{i\in I} Y_i \to Y$, and let $Z = (\bigoplus_{i\in I} Z_i)/U$. Obviously, Z looks really similar to (N_F, G_G, id). Now, every object A of $\mathfrak{L}^*({}_GN_F)$ is the kernel of a map

$$\textstyle\bigoplus_J (N_F, G_G, id) \to \bigoplus_L (F_F, 0, 0),$$

and such a map is given by its first component which is of the form (f_{ijl}): $\bigoplus_J \bigoplus_I F_F \to \bigoplus_L F_F$. But this map can also be considered as a map

$$\textstyle\bigoplus_J Z \twoheadrightarrow \bigoplus_J Z/Y = \bigoplus_J \bigoplus_I X_i \to \bigoplus_L X,$$

since $F = \mathrm{End}(X)$. The image of A will therefore be just the kernel of this map $\bigoplus_J Z \to \bigoplus_L X$.

2. HOMOMORPHISMS AND EXTENSIONS FOR SPECIES

2.1. Let $S = (F_i, {}_iM_j)_{1\leqslant i,j\leqslant n}$ be a species. Given two representations $A = (A_i, {}_j\varphi_i)$ and $B = (B_i, {}_j\varphi_i)$[we denote the maps by the same letter φ], we define a map γ_{AB} as follows

$$\gamma_{AB}\colon \bigoplus_i \mathrm{Hom}_{F_i}(A_i, B_i) \to \bigoplus_{i,j} \mathrm{Hom}_{F_j}(A_i \otimes_i M_j, B_j),$$

with $\gamma_{AB}(\alpha) = \delta$, where ${}_j\delta_i = {}_j\varphi_i(\alpha_i \otimes 1) - \alpha_j {}_j\varphi_i$. The importance of the map γ_{AB} rests in the following fact.

LEMMA. $\ker \gamma_{AB} = \mathrm{Hom}(A, B)$, $\mathrm{cok}\, \gamma_{AB} = \mathrm{Ext}(A, B)$.

The first assertion is obvious. And, for any $\delta \in \bigoplus \mathrm{Hom}(A_i \otimes_i M_j, B_j)$, define an exact sequence $E(\delta)$ as

$$0 \to (B_i, {}_j\varphi_i) \xrightarrow{\mu} \left(B_i \oplus A_i, \begin{bmatrix} {}_j\varphi_i & {}_j\delta_i \\ 0 & {}_j\varphi_i \end{bmatrix}\right) \xrightarrow{\epsilon} (A_i, {}_j\varphi_i) \to 0$$

with the canonical inclusion μ and the canonical projection ϵ. It is easy to check that every extension of B by A is given by such an exact sequence $E(\delta)$. Now, $E(\delta)$ and $E(\delta')$ are equivalent (that is, they define the same element in $\operatorname{Ext}(A, B)$) if and only if there is a commutative diagram

$$(*)\qquad \begin{array}{ccccccccc} 0 & \to & B & \to & \left(B_i \oplus A_i, \begin{bmatrix} {}_j\varphi_i & {}_j\delta_i \\ 0 & {}_j\varphi_i \end{bmatrix}\right) & \to & A & \to & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \bar{\alpha}} & & \downarrow{\scriptstyle 1} & & \\ 0 & \to & B & \to & \left(B_i \oplus A_i, \begin{bmatrix} {}_j\varphi_i & {}_j\delta_i' \\ 0 & {}_j\varphi_i \end{bmatrix}\right) & \to & A & \to & 0. \end{array}$$

The map $\bar{\alpha}_i : B_i \oplus A_i \to B_i \oplus A_i$ can be written in matrix form, and taking into account that the squares in (*) commute, it follows that $\bar{\alpha}_i = \left[\begin{smallmatrix} 1 & \alpha_i \\ 0 & 1 \end{smallmatrix}\right]$ with $\alpha_i : A_i \to B_i$. The fact that $\bar{\alpha}$ is a map of representations is expressed by the equalities

$$\begin{bmatrix} {}_j\varphi_i & {}_j\delta_i' \\ 0 & {}_j\varphi_i \end{bmatrix}\left(\begin{bmatrix} 1 & \alpha_i \\ 0 & 1 \end{bmatrix} \otimes 1\right) = \begin{bmatrix} 1 & \alpha_j \\ 0 & 1 \end{bmatrix}\begin{bmatrix} {}_j\varphi_i & {}_j\delta_i \\ 0 & {}_j\varphi_i \end{bmatrix}$$

for all i, j. The only nontrivial term is

$${}_j\varphi_i(\alpha_i \otimes 1) + {}_j\delta_i' = {}_j\delta_i + \alpha_{j\,j}\varphi_i .$$

Thus, $E(\delta)$ and $E(\delta')$ are equivalent if and only if there exists a family $\alpha = (\alpha_i)_i$ of maps $\alpha_i : A_i \to B_i$ such that

$${}_j\varphi_i(\alpha_i \otimes 1) + {}_j\delta_i' = {}_j\delta_i + \alpha_{j\,j}\varphi_i .$$

Thus, $E(\delta)$ and $E(\delta')$ are equivalent if and only if there exists a family $\alpha = (\alpha_i)_i$ of maps $\alpha_i : A_i \to B_i$ such that

$${}_j\delta_i - {}_j\delta_i' = {}_j\varphi_i(\alpha_i \otimes 1) - \alpha_{j\,j}\varphi_i ,$$

that is, $\delta - \delta' = \gamma_{AB}(\alpha)$. This proves $\operatorname{cok} \gamma_{AB} = \operatorname{Ext}(A, B)$.

2.2. In case of a K-species, we want to use the previous lemma in order to give another interpretation to its quadratic form. Let $f_i = \dim_K F_i$ and $m_{ij} = \dim_K {}_iM_j$. Define a bilinear form on $\mathbb{Q}^n$ as

$$\tilde{q}(x, y) = \sum_i f_i x_i y_i - \sum_{i,j} m_{ij} x_i y_j \qquad \text{for} \quad x = (x_i), \quad y = (y_i) \in \mathbb{Q}^n.$$

Note that $\tilde{q}$ is usually not symmetric, and that the corresponding quadratic form is just q.

LEMMA. *Let A and B be representations of the K-species S with bilinear form $\tilde{q}$. Then*

$$\tilde{q}(\dim A, \dim B) = \dim_K \operatorname{Hom}(A, B) - \dim_K \operatorname{Ext}(A, B).$$

Proof. If $\dim A = (a_1, ..., a_n)$, and $\dim B = (b_1, ..., b_n)$, then the K-dimension of $\bigoplus_i \operatorname{Hom}_{F_i}(A_i, B_i)$ is $\sum_i f_i a_i b_i$, whereas the K-dimension of $\bigoplus_{i,j} \operatorname{Hom}_{F_j}(A_i \otimes_i M_j, B_j)$ is given by $\sum_{i,j} m_{ij} a_i b_j$. Therefore, the lemma follows from the previous one.

2.3. We want to use this interpretation of the quadratic form q of a K-species S, in order to distinguish the behavior of points with respect to extensions.

LEMMA. *Let X be a point in* $\mathrm{I}(S)$, *and let* $\mathfrak{U}(X)$ *be the local subcategory generated by X. Then we have:*

(a) $q(X) > 0$ iff $\mathfrak{U}(X)$ *is semisimple,*

(b) $q(X) = 0$ iff $\mathfrak{U}(X)$ *is uniserial and not semisimple,*

(c) $q(X) < 0$ iff *the objects of height* $\leqslant 2$ *in* $\mathfrak{U}(X)$ *form a subcategory which is not of finite representation type.*

Proof. We only have to note that $\operatorname{Ext}(X, X)$ is an $\operatorname{End}(X)$–$\operatorname{End}(X)$-bimodule. Thus, if $\dim_K \operatorname{End}(X) > \dim_K \operatorname{Ext}(X, X)$, then obviously $\operatorname{Ext}(X, X) = 0$, and consequently $\mathfrak{U}(X)$ is semisimple. And, if the dimensions are equal, then the vector spaces ${}_{\operatorname{End}(X)}\operatorname{Ext}(X, X)$ and $\operatorname{Ext}(X, X)_{\operatorname{End}(X)}$ are one-dimensional, which implies that $\mathfrak{U}(X)$ is a uniserial category. Finally, if $\dim_K \operatorname{End}(X) < \dim_K \operatorname{Ext}(X, X)$, then the vector spaces ${}_{\operatorname{End}(X)}\operatorname{Ext}(X, X)$ and $\operatorname{Ext}(X, X)_{\operatorname{End}(X)}$ both are m-dimensional, for some $m > 1$. But since this is also true inside the full subcategory of all objects of $\mathfrak{U}(X)$ of height $\leqslant 2$, it follows that this category cannot be of finite representation type.

2.4. In a special case, we want to give a precise formula for $\operatorname{Ext}(A, B)$. Let ${}_F M_G$ be a bimodule. For $0 \neq x$ in M, denote by (x) the representation $(F_F, M/xG, \pi)$ with $\pi = \pi_x$ the canonical projection.

LEMMA. *Let $0 \neq x, y$ in M. Then* $\operatorname{Ext}((x), (y)) = M/Fx + yG$.

Proof. Define $\epsilon\colon \operatorname{Hom}(F_F \otimes {}_F M_G, M/yG) \to M/Fx + yG$ by $\epsilon(\delta) = \delta(1 \otimes x) + Fx \in M/Fx + yG$. Obviously, ϵ is surjective. We have $\epsilon\gamma_{(x)(y)} = 0$, since for $\alpha\colon F_F \to F_F$ and $\beta\colon M/xG \to M/yG$, we have

$$\epsilon\gamma_{(x)(y)}(\alpha, \beta) = \epsilon(\pi_y(\alpha \otimes 1) - \beta\pi_x) = \pi_y(\alpha \otimes 1)(1 \otimes x) - \beta\pi_x(1 \otimes x),$$

and this last element is just $Fx + yG$. Finally, let δ be an element of $\mathrm{Hom}(F_F \otimes {}_FM_G, M/yG)$, then we may write $\delta = \pi_y \delta'$ for some $\delta' : F_F \otimes {}_FM_G \to M_G$. If $\epsilon(\delta) = 0$, then $\delta'(1 \otimes x) \in Fx + yG$, therefore, there is some $f \in F$ with $\delta'(1 \otimes x) - fx \in yG$. If we denote by $\alpha_f : F_F \to F_F$ the left multiplication with f, then $\delta - \pi_y(\alpha_f \otimes 1) : F_F \otimes {}_FM_G \to M/yG$ maps x to 0, therefore we may factor this map through π_x. That is, we find $\beta : M/xG \to M/yG$ with $\delta = \pi_y(\alpha_f \otimes 1) - \beta\pi_x = \gamma_{(x)(y)}(\alpha_f, \beta)$.

3. Algebraic Bimodules

3.1. We assume in this section that ${}_FM_G$ is a bimodule which is algebraic over K. Our aim is to prove Theorem 3.

Let $a = \dim M_G$, $b = \dim {}_FM$. The bilinear form $\tilde{q}$ is given (up to a scalar multiple) by the matrix

$$\begin{bmatrix} a & -ab \\ 0 & b \end{bmatrix},$$

the corresponding quadratic form again will be denoted by q. Note that in our case the Weyl group is generated by two involutions, so there are just two Coxeter transformations, namely

$$c = \begin{bmatrix} ab - 1 & a \\ -b & -1 \end{bmatrix}$$

and its inverse. The imaginary cone is defined to be the set of all positive elements (x, y) in $\mathbb{Q}^2$ with $q(x, y) \leqslant 0$, and it is easy to see that its boundary consists of the nonnegative eigenvectors of c. The positive imaginary roots are, by definition, the integral elements in the imaginary cone.

For a representation (X_F, Y_G, φ) of ${}_FM_G$, let $x = \dim X_F$ and $y = \dim Y_G$. We want to determine the possible pairs (x, y) for indecomposable representations.

3.2. *For any positive Weyl root there is precisely one indecomposable representation, and all others belong to imaginary roots.*

Namely, if (x, y) is positive, but does not belong to the imaginary cone, then either there is some n with $c^n(x, y) \geqslant 0$ and $c^{n+1}(x, y) \ngeqslant 0$, or there is some n with $c^{-n}(x, y) \geqslant 0$ and $c^{-n-1}(x, y) \ngeqslant 0$. Let C^+ and C^- be the Coxeter functors on $\mathrm{I}({}_FM_G)$ which correspond to c and c^{-1} [7]. If A is an indecomposable representation, then either $C^+(A)$ is again indecomposable, and then $\dim C^+(A) = c(\dim A)$ or else A is of the form $(0, G_G, 0)$ or

(F_F, M_G, id). In the first case, we stress the fact that $c(\dim A) > 0$. Thus, if $\dim A = (x, y)$, and $c^n(x, y) \geqslant 0$, but $c^{n+1}(x, y) \not\geqslant 0$, then $c^n(x, y)$ is equal either to $(0, 1)$ or to $(1, a)$, which implies that $(x, y) = c^{-n}(0, 1)$ or $= c^{-n}(1, a)$ is a Weyl root. And, in both cases, also A is uniquely determined, namely, either $A = C^{-n}(0, G_G, 0)$ or $A = C^{-n}(F_F, M_G, id)$. A similar argument of course works in case that $\dim A = (x, y)$ and $c^{-n}(x, y) \geqslant 0$, but $c^{-n-1}(x, y) \not\geqslant 0$.

3.3. It remains to construct enough indecomposable representations for any imaginary root $(x, y) > 0$. It is easy to see that *we may assume* $x \leqslant y$.

Namely, if R is a ring, denote by R^0 its opposite ring. We claim that the categories $\mathfrak{l}({}_FM_G)$ and $\mathfrak{l}({}_{G^0}M_{F^0})$ are dual to each other. In order to prove this, define a contravariant functor $\mathfrak{l}({}_FM_G) \to \mathfrak{l}^*(M^*)$ by $(X_F, Y_G, \varphi) \mapsto (Y^*, X^*, \varphi^*)$ where $*$ denotes the duality with respect to K. Note that Y^* is a right G^0-space, and X^* a right F^0-space. Also, M^* is an F^0–G^0-bimodule. Obviously, this functor has an inverse functor, and therefore defines a duality. But obviously, $\mathfrak{l}^*(M^*)$ is equivalent to $\mathfrak{l}({}_{G^0}M_{F^0})$.

3.4. In $\mathbb{Q}^2$, the *fundamental cone* is defined to be the set of all positive elements (x, y) with

$$1/(b-1) \leqslant x/y \leqslant a-1 \qquad \text{in case } a, b \geqslant 2,$$

$$2 \leqslant x/y \leqslant a-2 \qquad \text{in case } b = 1, \text{ and}$$

$$1/(b-2) \leqslant x/y \leqslant \tfrac{1}{2} \qquad \text{in case } a = 1.$$

LEMMA. *For $ab = 4$, the fundamental cone and the imaginary cone coincide. For $ab \geqslant 5$, the fundamental cone is the closure of a fundamental domain of the action of c and c^{-1} on the interior of the imaginary cone.*

For the proof, we only note that in case $ab \geqslant 5$, the fundamental cone is contained in the interior of the imaginary cone, and that the boundary lines of the fundamental cone are mapped into each other under c and c^{-1}.

3.5. The integral vectors in the fundamental cone which are of the form $(1, y)$ are called *fundamental vectors*; in the case $a = 3$, $b = 2$, also $(2, 3)$ is considered as a fundamental vector, and in the case $a = 5$, $b = 1$, also $(2, 5)$ is considered as fundamental vector.

Thus, the sequence of fundamental vectors is for $a \geqslant 3$, $b \geqslant 3$

$$(1, a-1), \quad (1, a-2), \ldots, \quad (1, 2), \quad (1, 1),$$

for $a = 3$, $b = 2$

$$(1, 2), \quad (2, 3), \quad (1, 1),$$

for $a \geqslant 6$, $b = 1$

$$(1, a-2), \quad (1, a-3), \ldots, \quad (1, 3), \quad (1, 2),$$

and for $a = 5$, $b = 1$

$$(1, 3), \quad (2, 5), \quad (1, 2).$$

We say that the fundamental vector u follows the fundamental vector v, provided in the above ordering, u comes directly after v. We note the following facts:

(1) *Every integral vector (x, y) in the fundamental cone and with $x \leqslant y$ is an integral linear combination of two fundamental vectors u and v, where u follows v.*

(2). *If u and v are fundamental vectors, where u follows v, then $\tilde{q}(u, v) < 0$.*

The proofs are rather obvious.

3.6. LEMMA *Let $(1, y)$ be a fundamental vector. Then, there are two orthogonal points A_1 and A_2 in $\mathrm{I}({}_F M_G)$ with* $\dim A_i = (1, y)$, *for $i = 1, 2$. In case K is infinite, there are infinitely many pairwise orthogonal points A_i with* $\dim A_i = (1, y)$.

Proof. The indecomposable representations A with dimension type $\dim A = (1, y)$ are, of course, just given by epimorphisms $\varphi: M_G \to Y_G$, with $\dim Y_G = y$, therefore they correspond to the $(a-y)$-dimensional G-subspaces of the a-dimensional G-space M_G [taking the kernel of φ]. Thus we have to look at the grassmanian $\mathrm{Gr}_{a-y,y}$ with respect to G. This is a variety of dimension $(a-y) \cdot y \cdot g$ over K, where $g = \dim_K G$. The multiplicative group $F^\times$ of F operates on it as an algebraic group, and $K^\times \subseteq F^\times$ operates trivially. Thus, if we denote $f = \dim_K F$, then the closure of any orbit has dimension $\leqslant f - 1$. But one easily checks the following inequality

$$f - 1 < (a-y) \cdot y \cdot g;$$

for $2 \leqslant y \leqslant a-2$, it is a consequence of $a < (a-y) \cdot y$, and otherwise we are in the situation $b \geqslant 2$, which means $2f \leqslant ag$. Therefore, if K is infinite, $\mathrm{Gr}_{a-y,y}$ cannot be covered by a finite number of such orbits, and this implies that there are infinitely many pairwise nonisomorphic indecomposable representations A_i with $\dim A_i = v$. Obviously, every such representation is a point, and nonisomorphic ones are orthogonal. In the case where K is finite, the number of elements of $\mathrm{Gr}_{a-y,y}$ is well known, and it is easy to check that the group $F^\times/K^\times$ cannot operate transitively on $\mathrm{Gr}_{a-y,y}$.

3.7. We can give now a *proof of Theorem* 3 *in case* $ab \leqslant 4$. The case $ab \leqslant 3$ is well-known [6], but follows also from 3.2, since in this case, there does not exist any imaginary root. In case $ab = 4$, all imaginary roots are integer multiples of the fundamental root $(1, y)$, with $y = 1$ in case $a = b = 2$, and $y = 2$ in case $a = 4$ and $b = 1$. By the previous result, we have enough points A_i with $\dim A_i = (1, y)$, and in the full subcategories $\mathfrak{U}(A_i)$ we find indecomposable representations for any positive multiple of $(1, y)$. Here, we use that $\mathfrak{U}(A_i)$ is not semisimple, but hereditary (in fact, $\mathfrak{U}(A_i)$ is a uniserial category of global dimension 1 with one simple object—see (2.3) or Section 4).

It remains to consider the case $ab \geqslant 5$. First, we restrict to the cases where all fundamental roots are of the form $(1, y)$, that is, we assume that $a + b + ab \geqslant 12$.

3.7. Lemma *Let* $a + b + ab \geqslant 12$. *Let* $(1, y)$ *and* $(1, y + 1)$ *be fundamental vectors. Let* A *be a point in* $\mathfrak{l}({}_F M_G)$ *with* $\dim A = (1, y + 1)$. *Then there is another point* B, *orthogonal to* A, *such that* $\dim B = (1, y)$.

We start proving the inequality

$$y(y + 1)\, b + a \leqslant yab. \qquad (*)$$

Consider first the case $b \geqslant 2$. Since $(1, y + 1)$ is a fundamental vector, we must have $y + 2 \leqslant a$. In case $a/b \leqslant y$, we have

$$y(y + 1)\, b + a \leqslant y(y + 1)\, b + yb \leqslant y(y + 2)\, b \leqslant yab.$$

Next, for $2/(b - 1) \leqslant y \leqslant a/b$, the inequalities $y + 2 \leqslant yb$ and $yb \leqslant a$ imply

$$y(y + 1)\, b + a \leqslant (y + 1)\, a + a = (y + 2)\, a \leqslant yab.$$

It remains the case $b = 2$, $y = 1$. In this case, $a \geqslant 4$ implies our inequality. In the case $b = 1$, we have even $y \leqslant a - 3$. Thus, if in addition $a/2 \leqslant y$, then

$$y(y + 1) + a \leqslant y(a - 2) + a \leqslant ya.$$

In case $3 \leqslant y \leqslant a/2$, then

$$y(y + 1) + a \leqslant (a/2)(y + 1) + a = (a/2)\, y + \tfrac{3}{2} \cdot a \leqslant ay.$$

Finally, for $y = 2$, we remind that $b = 1$ implies $a \geqslant 6$, and therefore our inequality. This proves (*).

Now, let $f = \dim_K F$, $g = \dim_K G$. Then $ag = bf$. Thus, multiplying (*) with g/b, we get

$$y(y+1)g + f \leqslant yag. \qquad (**)$$

The indecomposable representations B of the form $(1, y)$ correspond to (orbits of) elements $(u_1, \ldots, u_y)$ in $(M^*)^y = M^* \oplus \cdots \oplus M^*$, namely, we consider the representations $(F_F, (G_G)^y, \varphi)$ with $\varphi: F_F \otimes {}_F M_G \to (G_G)^y$ being determined by the map $\bar{\varphi}: F_F \to (G_G)^y \otimes M^* = (M^*)^y$ with $1 \mapsto (u_1, \ldots, u_y)$. In order to get an indecomposable representation, we have to assume that the elements $u_1, \ldots, u_y$ are independent in the G-vector space ${}_G M^*$. Note, that the last condition defines an open and dense subset of $(M^*)^y$, since it is the complement of some closed subvariety of lower dimension. Therefore, this subset has dimension yag. Now, given a representation A of dimension type $(1, y+1)$, say given by elements $v_1, \ldots, v_{y+1} \in M^*$, then there is a nonzero homomorphism $A \to B$ if and only if we have

$$\begin{bmatrix} u_1 \\ \vdots \\ u_y \end{bmatrix} \in M_{y+1,y}(G) \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_{y+1} \end{bmatrix} \cdot F,$$

where $M_{p,q}(G)$ denotes the set of $p \times q$-matrices over G. For fixed $(v_1, \ldots, v_{y+1})$, its image under the mappings defined by the elements of $M_{y+1,y}(G) F$ is of dimension $\leqslant y(y+1)g + f - 1$. Here, the last summand -1 comes from the fact that K operates centrally.

Now using the inequality (**), we conclude in the case where K is infinite that there exists an orbit which corresponds to some indecomposable representation B such that $\operatorname{Hom}(A, B) = 0$. Of course, we also have, for trivial reasons, $\operatorname{Hom}(B, A) = 0$. In the case where K is finite, a similar argument counting elements, gives the same result.

3.8. The *proof of Theorem* 3 *in case* $a + b + ab \geqslant 12$ is now quite easy. In fact, we show: ***If K is infinite and (x, y) a positive imaginary root, then there exist infinitely many pairwise orthogonal points A_i with*** $\dim A_i = (x, y)$.

Let (x, y) be an element of the fundamental cone, and $x \leqslant y$. First, assume that (x, y) is a multiple of some fundamental vector u, say $(x, y) = m \cdot u$, with $m \in \mathbb{N}$. Then, let A_1 and A_2 be representations with $\dim A_i = (x, y)$, and which are orthogonal points. We simplify A_1 and A_2, and construct in this subcategory a serial object with $m - 1$ composition factors of the form A_1 and the top composition factor of the form A_2

$$\begin{array}{l} \bullet\ A_2 \\ | \\ \bullet\ A_1 \\ | \\ \vdots \\ | \\ \bullet\ A_1 . \end{array}$$

Otherwise, (x, y) can be written as $(x, y) = m_1 u + m_2 v$, where $m_1, m_2 > 0$, and u and v are fundamental vectors such that u follows v. In this case, let A, B be orthogonal points with $\dim A = v$, $\dim B = u$. We simplify again, and form a serial object of length $m_1 + m_2$, with the first m_1 composition factors of the form B, the other ones of the form A

$$\begin{array}{l} \bullet\ A \\ | \\ \vdots \\ | \\ \bullet\ A \\ | \\ \bullet\ B \\ | \\ \vdots \\ | \\ \bullet\ B . \end{array}$$

Obviously, in both cases, we get a point with dimension (x, y). And, in case K is infinite, we can construct infinitely many isomorphism classes for any such (x, y). The result follows, using Coxeter functors [7] and (3.4).

3.9. Consider now *the case* $(a, b) = (3, 2)$. First, we look at the variety of all 3-dimensional subspaces U_G of $M_G \oplus M_G$, on which the group of all invertible 2×2-matrices over F operates. Also, we are interested in the variety of all 2-dimensional subspaces, and its images under the maps $M_G \to M_G \oplus M_G$ given by nonzero pairs (f_1, f_2) of elements of F. We call the subspace $U_G \subseteq M_G \oplus M_G$ proper, provided there is no nonzero homomorphism $(f_1, f_2)\colon M_G \to M_G \oplus M_G$ which maps a 2-dimensional subspace V_G into U_G. A dimension argument shows that there are two (and, if K is infinite, even infinitely many) orbits of 3-dimensional proper subspaces.

Now, the 3-dimensional proper subspaces $U_G \subseteq M_G \oplus M_G$ correspond just to representations $B = (F_F \oplus F_F, M_G \oplus M_G / U_G, \pi)$ with π the canonical projection, which have the properties

(1) $\dim B = (2, 3)$,

(2) if A is an indecomposable representation with $\dim A = (1, 1)$, then $\mathrm{Hom}(A, B) = 0$.

From this, it is easy to derive several other properties, namely,

(3) if C is an indecomposable representation with $\dim C = (1, 2)$, then $\mathrm{Hom}(B, C) = 0$.

For, any nontrivial homomorphism $B \to C$ has to be an epimorphism, and therefore the kernel has dimension type $(1, 1)$ which is impossible according to (2).

(4) If B_1 and B_2 both satisfy (1) and (2), then any nonzero map $B_1 \to B_2$ is an isomorphism.

For, the proper subobjects of B_2 are of dimension types $(1, 2)$, $(1, 3)$, or $(0, i)$, but B_1 has no such homomorphic image. We may formulate this also as:

(4′) If B_1 and B_2 both satisfy (1) and (2), and are nonisomorphic, then they are orthogonal points.

We construct now for every integral element in the fundamental cone an indecomposable representation using similar methods as in 3.8. However, in this case, we are not always able to simplify, and therefore have to use the following lemma: If X and Y are indecomposable, with $\mathrm{Hom}(X, Y) = 0$, and if $0 \to X \to E \to Y \to 0$ is exact, and defines a nonzero element in $\mathrm{Ext}^1(Y, X)$, then E is indecomposable. Note that in this case the fact that Y and X are points does not imply that E is a point, since we may have nonzero maps $Y \to X$.

3.10. In the *case* $(a, b) = (5, 1)$ we argue as in the previous case, with the only difference that now we want to construct representations B with

(1) $\dim B = (2, 5)$, and

(2) if A is an indecomposable representation with $\dim A = (1, 2)$, then $\mathrm{Hom}(A, B) = 0$.

We do not know whether it is possible in the cases $(a, b) = (3, 2)$ and $(5, 1)$ to construct enough *orthogonal points* for every imaginary root.

4. K-Species with Semidefinite Quadratic Form

4.1. Let S be a connected K-species without cycles, and with positive semidefinite quadratic form q. The elements $x \in \mathbb{Q}^n$ with $q(x) = 0$ form a one-dimensional subspace of $\mathbb{Q}^n$, and we denote by h a generator of this subspace. We want to consider the full subcategory $\mathfrak{h}(S)$ of $\mathfrak{l}(S)$ of all homogeneous representations of S. Recall from [7] that an indecomposable representation of S is said to be regular, provided $C^{+n}(X) \neq 0$ and $C^{-n}(X) \neq 0$, for all natural n, where C^+ and C^- denote the Coxeter functors for S. The direct sums of regular indecomposable representations form an abelian exact subcategory, denoted by $\mathfrak{r}(S)$. Then, a simple object X in $\mathfrak{r}(S)$ is called homogeneous in case $q(\dim X) = 0$. And an arbitrary object in $\mathfrak{r}(S)$ is called homogeneous, provided it has a composition series with simple homogeneous factors. In particular, for every homogeneous object X, $q(\dim X) = 0$, and therefore, there is some z_X in $\mathbb{Q}$ such that $\dim X = z_X \cdot h$.

Now let X and Y be homogeneous representations. We denote by $\tilde{q}$ the (nonsymmetric) bilinear form introduced in 2.2 and corresponding to q. The interpretation given there yields

$$\dim_K \mathrm{Hom}(X, Y) - \dim_K \mathrm{Ext}(X, Y) = \tilde{q}(\dim X, \dim Y) = z_X z_Y \cdot \tilde{q}(h,h) = 0.$$

Therefore, Theorem 1 follows from the following general result.

4.2. LEMMA *Let $\mathfrak{C}$ be an abelian K-category, where every object has finite length. Assume that for all objects A, B in $\mathfrak{C}$,* $\dim_K \mathrm{Hom}(A, B) = \dim_K \mathrm{Ext}(A, B)$. *Then $\mathfrak{C}$ is the direct sum of categories which are uniserial, have global dimension* 1, *and contain a unique simple object.*

Recall that a category is called a length category, provided every object has a composition series. And $\mathfrak{C}$ is called uniserial, provided every indecomposable object has a unique composition series, and all the simple factors in this composition series are isomorphic. Finally, we mention that a category is a K-category, provided K can be embedded into the center of $\mathfrak{C}$ (the center of $\mathfrak{C}$ is the endomorphism ring of the identity functor); in this case, we fix an embedding, and then all abelian groups $\mathrm{Hom}(A, B)$ and $\mathrm{Ext}(A, B)$ become K-vector spaces, and $\dim_K \mathrm{Hom}(A, B)$ is assumed to be finite.

Proof of the lemma. If A and B are nonisomorphic simple objects, then $\mathrm{Hom}(A, B) = 0$ implies $\mathrm{Ext}(A, B) = 0$. As a consequence, $\mathfrak{C}$ can be written as the direct sum of categories with a unique simple object in each of them. Therefore, we may assume that $\mathfrak{C}$ itself contains only one simple object, say C. Now we have $\dim_K \mathrm{End}(C) = \dim_K \mathrm{Ext}(C, C)$, in particular, $\mathrm{Ext}(C, C) \neq 0$. But $\mathrm{Ext}(C, C)$ can be considered as an $\mathrm{End}(C)$–$\mathrm{End}(C)$-bimodule, and the equality implies that both vector spaces ${}_{\mathrm{End}(C)}\mathrm{Ext}(C, C)$ and $\mathrm{Ext}(C, C)_{\mathrm{End}(C)}$ are one-dimensional. As a consequence, $\mathfrak{C}$ has to be uniserial. Finally, given an exact sequence

$$0 \to B' \to B \to B'' \to 0,$$

we get in any abelian category an exact sequence

$$0 \to \mathrm{Hom}(A, B') \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B'') \to$$
$$\mathrm{Ext}(A, B') \to \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B''),$$

for arbitrary A in $\mathfrak{C}$. But this is a sequence of K-vector spaces and K-linear maps, therefore our dimension condition implies that the last map is an epimorphism. As a consequence, gl.dim. $\mathfrak{C} \leq 1$. But since $\mathrm{Ext}(C, C) \neq 0$, we know gl.dim. $\mathfrak{C} \neq 0$. This concludes the proof.

5. K-Species with Indefinite Quadratic Form

5.1. Lemma *Let S be a K-species without cycles and with indefinite quadratic form. Then there exists a bimodule ${}_FM_G$ which is algebraic over K, and which satisfies* $(\dim_F M)(\dim M_G) \geq 5$, *such that there is a full exact embedding* $\mathfrak{l}({}_FM_G) \to \mathfrak{l}(S)$.

Proof. We may assume that S is connected. Let F_i denote (not only the field but also) the one-dimensional representation with $(F_i)_i$ the one-dimensional vector space $(F_i)_{F_i}$, and with $(F_i)_j = 0$ for $i \neq j$.

Assume that there exists a subset I of the index set of S, such that $S \mid I$ has semidefinite quadratic form (and is also connected). Let $i \in I$ and $j \in I$ be neighbors. Now there exists an indecomposable representation X on I such that (1) $q(\dim X) > 0$, (2) $\mathrm{End}(X) = F_i$, and (3) $\dim X_i$ is arbitrarily large. (Namely, a representation of the form $C^{-m}P_i$, with P_i the projective cover in $\mathfrak{l}(S \mid I)$ of F_i, and C^- the Coxeter functor for $S \mid I$, satisfies the first two conditions, and $\dim X_i$ increases with m [7]). Now X and F_j obviously are orthogonal points, and in case ${}_iM_j \neq 0$,

$$\dim_K \mathrm{Ext}(X, F_j) = -\tilde{q}(\dim X, \dim F_j) = m_{ij}x_i,$$

whereas in case ${}_jM_i \neq 0$

$$\dim_K \mathrm{Ext}(F_j, X) = m_{ji}x_j,$$

so these dimensions can be arbitrarily large, respectively. We consider only the first case. Let $F = F_i$, $G = F_j$, ${}_GN_F = \mathrm{Ext}(X, F_j)$, and ${}_FM_G = N^*$ the dual module (that is, the K-dual, or the G-dual, or the F-dual, all are isomorphic as bimodules). By 1.5 there is a full exact embedding $\mathfrak{l}^*({}_GN_F) \to \mathfrak{l}(S)$, however, $\mathfrak{l}({}_FM_G)$ and $\mathfrak{l}^*({}_GN_F)$ are isomorphic as categories. This proves the result in case such a restriction $S \mid I$ exists.

Now assume, for any subset I of the index set of S, the corresponding quadratic form is positive definite. Then either S itself is just a bimodule, or else the index set of S consists of three points $\{1, 2, 3\}$, and we may suppose ${}_1M_3 = 0 = {}_3M_1$. Instead of considering the species, we will look at the whole moduled graph: that is, if ${}_1M_2 \neq 0$, then we define ${}_2M_1 = ({}_1M_2)^*$, and so on. In this way, we deal with all possible cases at the same time. But in order to consider a specific species, we have to introduce an orientation. There are three different orientations

$\bullet \rightarrow \bullet \rightarrow \bullet$ denote the species by S',

$\bullet \leftarrow \bullet \rightarrow \bullet$ denote the species by S'',

$\bullet \rightarrow \bullet \leftarrow \bullet$ denote the species by S'''.

Also there are functors $s_1^-: \mathrm{I}(S') \rightarrow \mathrm{I}(S'')$, and $s_2^-: \mathrm{I}(S'') \rightarrow \mathrm{I}(S''')$ which kill just one indecomposable representation, namely F_1, and F_2, respectively. Now there is an obvious full embedding $\mathrm{I}({}_1M_2 \otimes {}_2M_3) \rightarrow \mathrm{I}(S')$, given by $(X, Y, \varphi) \rightarrow V$ with $V_1 = X$, $V_2 = X \otimes {}_1M_2$, $V_3 = Y$, and it is easy to see that also the compositions with s_1^- and $s_2^-s_1^-$ are embeddings (here, we use that ${}_1M_2$ is not at the same time one-dimensional as a left vector space and as a right vector space).

5.2. It remains to prove Theorem 2 in case S is a bimodule. Now given a bimodule ${}_FM_G \neq 0$, its *center* is defined as the set of all elements $(f, g) \in F \times G$ with $fm = mg$ for all $m \in M$. If (f, g) belongs to the center of M, then f belongs to the center of the division ring F, and g to the center of G. For, let $f' \in F$, and $0 \neq m \in M$. Then

$$f(f'm) = (f'm)g = f'(mg) = f'fm,$$

and therefore $ff' = f'f$. Also, if (f, g), (f', g) and (f, g') belong to the center of M, then $f = f'$ and $g = g'$. Therefore, often we will identify the center of M with its projection into F, or into G. Note that *in case* $\dim M_G \geqslant 2$, *the center of M is then just the set of elements $f \in F$ with $fm \in mG$ for all $m \in M$.* For, we claim that from $fm_1 = m_1 g_1$, $fm_2 = m_2 g_2$ $(0 \neq m_1, m_2 \in M)$ we may conclude $g_1 = g_2$. There is an element $m \in M$, such that m and m_1, and similarly m and m_2 are linearly independent in M_G. Let $fm = mg$. Then $f(m + m_1) = mg + m_1 g_1 \in (m + m_1)G$ implies $g = g_1$, thus also $g = g_2$.

Of course, if ${}_FM_G$ is algebraic over K, then K is canonically embedded into the center of M, and M is algebraic over its center.

5.3. We assume now that ${}_FM_G$ is an algebraic bimodule with center K, such that $(\dim {}_FM)(\dim M_G) \geqslant 5$. Let $f = \dim_K F$, and $g = \dim_K G$.

If $\max(f, g) > 1$, *we construct orthogonal points* A_1, A_2 *such that* $\mathrm{End}(A_1)$

has K-dimension $< \max(f, g)$ *and* $\operatorname{Ext}(A_i, A_j) \neq 0$ *for all choices* $1 \leqslant i, j \leqslant 2$.
Consider first the case that both $\dim {}_FM$ and $\dim M_G$ are $\geqslant 2$. Since $\max(f, g) > 1$, there is a_1 such that either $Fa_1 \not\subseteq a_1G$ or $a_1G \not\subseteq Fa_1$. Let $A_1 = (a_1) = (F_F, M/a_1G, \pi)$. Since $\operatorname{End}(A_1)$ can be identified both with $\{f \in F \mid fa_1 \in a_1G\}$ and with $\{g \in G \mid a_1 g \in Fa_1\}$, it follows that $\dim_K \operatorname{End}(A_1) < \max(f, g)$. In case K is infinite, we have seen previously in Section 3 that there is some other $a_2 \neq 0$ in M such that A_1 and $A_2 = (a_2)$ are nonisomorphic. In case K is finite, a counting argument shows that Fa_1G is a proper subset of M, and therefore there is again such an element a_2. The assertion about $\operatorname{Ext}(A_i, A_j)$ is a consequence of the fact that $\tilde{q}(\dim A_i, \dim A_j) < 0$.

Next, consider the case $\dim {}_FM = 1$, thus ${}_FM_G = {}_FF_G$. Let $U_i (i = 1, 2)$ be 2-dimensional subspaces of F_G such that $fU_1 \neq U_2$ for all $f \in F$, and $A_i = (F_F, F/U_i, \pi)$. Then $\operatorname{End}(A_i)$ can be embedded into $\operatorname{End}(U_{iG})$, therefore $\dim_K \operatorname{End}(A_i) \leqslant 4 \cdot g < f$. Again, the assertion about Ext follows from the value of the bilinear form $\tilde{q}$. A similar argument works in case $\dim M_G = 1$.

5.4. We show by induction on $\max(f, g)$ the existence of a full exact embedding $\mathfrak{w}(K') \to \mathfrak{l}({}_FM_G)$, where K' is an extension field of K. If $\max(f, g) = 1$, then $F = K = G$, and therefore ${}_FM_G$ is of the form K^m, with $m \geqslant 3$. Obviously, there is such an embedding of $\mathfrak{w}(K)$ into $\mathfrak{l}({}_FM_G)$.

If $\max(f, g) > 1$, then we have those two orthogonal points A_1, A_2. We construct two representations X and Y as extensions of the form

$$X \begin{array}{l} \bullet\, A_2 \\ | \\ \bullet\, A_1 \end{array}, \qquad Y \begin{array}{l} \bullet\, A_2 \\ | \\ \bullet\, A_2 \\ | \\ \bullet\, A_1 \\ | \\ \bullet\, A_1 \end{array}.$$

That is: if we simplify A_1 and A_2, then X and Y are both serial objects, and the composition factors are as indicated. Then X and Y are again orthogonal points, and their endomorphism rings can be embedded both into $\operatorname{End}(A_1)$. It remains to consider the $\operatorname{End}(Y)$-$\operatorname{End}(X)$-bimodule $\operatorname{Ext}(X, Y)$. Now, $\operatorname{Ext}(X, Y)$ maps as $\operatorname{End}(Y)$-module onto $\operatorname{Ext}(A_1, Y)$, and it is easily seen that the $\operatorname{End}(Y)$-vector space $\operatorname{Ext}(A_1, Y)$ is at least three-dimensional. Also, as $\operatorname{End}(X)$-vector space, $\operatorname{Ext}(X, Y)$ maps onto $\operatorname{Ext}(X, A_2)$ and this vector space is at least two-dimensional.

On the other hand, we have a full exact embedding

$$\mathfrak{l}(\operatorname{Ext}(X, Y)^*) \xrightarrow{\approx} \mathfrak{l}^*(\operatorname{Ext}(X, Y)) \hookrightarrow \mathfrak{l}({}_FM_G),$$

therefore, the result follows by induction. Note that the center of Ext(X, Y) contains K, but perhaps may be larger—this is the place where the extension field K' comes in.

6. Bimodules with Constant Dimension

6.1. Let ${}_FM_G$ be a bimodule. Denote by $M^L = \mathrm{Hom}_F({}_FM_G, {}_FF_F)$ and by $M^R = \mathrm{Hom}_G({}_FM_G, {}_GG_G)$ the dual modules with respect to F and G respectively. Both M^L and M^R are G–F-bimodules, and we may continue the process. Thus, let $M_{(0)} = M$, and define

$$M_{(i+1)} = M_{(i)}^{LR} \quad \text{for} \quad i \geqslant 0,$$

$$M_{(i-1)} = M_{(i)}^{RL} \quad \text{for} \quad i \leqslant 0.$$

In this way, we get a family $M_{(i)}$ of F–G-bimodules, indexed over the integers $\mathbb{Z}$. If all $M_{(i)}$ are finite dimensional on either side, then we have for all i, $M_{(i+1)} = M_{(i)}^{LR}$ and $M_{(i-1)} = M_{(i)}^{RL}$. Now ${}_FM_G$ is said to have *constant dimension* (with respect to dualization) provided $\dim {}_FM = \dim {}_FM_{(i)}$ and $\dim M_G = \dim M_{(i)G}$ for all i, and these dimensions are finite.

Note that $\dim(M^L)_F = \dim {}_FM$, and $\dim {}_G(M^R) = \dim(M^{LR})_G$, provided these dimensions are finite. In particular, if ${}_FM_G$ has constant dimension, then all the bimodules $M_{(i)}$ and $M_{(i)}^L$ have equal F-dimension, and equal G-dimension.

6.2. If ${}_FM_G$ is a bimodule with $\dim_F M = 1 = \dim M_G$, then ${}_FM_G$ has contant dimension. Namely, in this case, we may identify M with F, since ${}_FM$ is one-dimensional, and then we may identify G with F, since G is just $\mathrm{End}({}_FM) = \mathrm{End}({}_FF) = F$. Thus, we may identify ${}_FM_G$ with the canonical bimodule ${}_FF_F$.

6.3. Extensions of bimodules with constant dimension have constant dimension. For, let ${}_FM_G$ be a bimodule, and assume ${}_FX_G$ is a submodule such that both ${}_FX_G$ and ${}_F(M/X)_G$ have constant dimension. Consider M^L. Let $X^\perp = \{\varphi \mid \varphi \in M^L \text{ and } \varphi(X) = 0\}$; this is a G–F-submodule of M^L. But it is well known that $X^\perp$ is isomorphic to $(M/X)^L$, and that $M^L/X^\perp$ is isomorphic to X^L, both as G–F-bimodules. From this it follows that $\dim_F M = \dim(M^L)_F$ and $\dim M_G = \dim_G(M^L)$.

6.4. We will call ${}_FM_G$ *affine*, provided $(\dim {}_FM)(\dim M_G) = 4$ (the diagram of the species ${}_FM_G$ is just an "affine" diagram). *Every affine, nonsimple bimodule ${}_FM_G$ has constant dimension.* Namely, let ${}_FX_G$ be a nonzero proper submodule of ${}_FM_G$. Then $\dim_F X = \dim {}_F(M/X) = 1$, and

$\dim X_G = \dim(M/X)_G = 1$, and therefore, both ${}_F X_G$ and ${}_F(M/X)_G$ have constant dimension. As we will see, in this case it is quite easy to give an explicit formula for the dual M^L.

Let ${}_F M_G$ be affine and not simple. Let ${}_F X_G$ be a nonzero proper submodule of ${}_F M_G$. Let $0 \neq x_0 \in X$ and $t \in M \backslash X$. We may identify ${}_F X_G$ with ${}_F F_F$ using the element x_0 as $1 \in F$. In particular, we have identified F with G, and X is an F–F-bimodule such that $x_0 f = f x_0$ for all $f \in F$. Since x_0, t is a basis of ${}_F M$, we may define (set-)mappings δ, ϵ from F into F by

$$tf = f^{\delta} \cdot x_0 + f^{\epsilon} \cdot t, \qquad \text{for} \quad f \in F.$$

Obviously, δ and ϵ both are additive, and the equality

$$(f_1 f_2)^{\delta} x_0 + (f_1 f_2)^{\epsilon} t = t(f_1 f_2) = (t f_1) f_2 = (f_1^{\delta} f_2 + f_1^{\epsilon} f_2^{\delta}) x_0 + (f_1^{\epsilon} f_2^{\epsilon}) t$$

shows that ϵ is an endomorphism of F and that δ is an $(\epsilon, 1)$-derivation. Also, $F x_0 + F^{\epsilon} t$ is a right F-subspace of M_F, and therefore equal to M. Thus, it follows that ϵ is also surjective, thus an automorphism. Conversely, let F be a division ring, and ϵ an automorphism of F, δ an $(\epsilon, 1)$-derivation of F. Define an F–F-bimodule $M(\epsilon, \delta)$ in the following way. Let ${}_F M(\epsilon, \delta) = {}_F F \oplus {}_F F$, with right F-structure given by

$$(a, b) \cdot f = (af + bf^{\delta}, bf^{\epsilon}), \qquad \text{for} \quad a, b, f \in F.$$

Since ϵ is an automorphism, $\dim M(\epsilon, \delta)_F = 2$, thus $M(\epsilon, \delta)$ is affine. Also, $F \oplus 0$ is an F–F-submodule, so $M(\epsilon, \delta)$ is not simple. This shows: *the affine nonsimple bimodules are just those bimodules which are of the form $M(\epsilon, \delta)$.*

If we do not allow the identification of the two division rings operating on the bimodule via the operation itself, we have to consider another automorphism of F. Let α, ϵ be two automorphisms of the division ring F, and let δ be an (ϵ, α)-derivation of F. We define $M(\alpha, \epsilon, \delta)$ as ${}_F F \oplus {}_F F$ with right F-structure given by

$$(a, b) \cdot f = (af^{\alpha} + bf^{\delta}, bf^{\epsilon}), \qquad \text{for} \quad a, b, f \in F.$$

(If we allow the identification, then $M(\alpha, \epsilon, \delta)$ becomes just $M(\alpha^{-1}\epsilon, \alpha^{-1}\delta)$.) We claim that $M(\alpha, \epsilon, \delta)^L = M(\epsilon^{-1}, \alpha^{-1}, -\alpha^{-1}\delta\epsilon^{-1})$. In order to see this, consider the left F-subspace U of ${}_F V_F = M(\alpha, \epsilon, \delta) \otimes_F M(\epsilon^{-1}, \alpha^{-1}, -\alpha^{-1}\delta\epsilon^{-1})$ generated by the elements $(1, 0) \otimes (1, 0)$, $(0, 1) \otimes (0, 1)$, and $(1, 0) \otimes (0, 1) - (0, 1) \otimes (1, 0)$. Then U is in fact an F–F-submodule, and ${}_F(V/U)_F$ is isomorphic to the canonical one-dimensional bimodule ${}_F F_F$. The epimorphism ${}_F V_F \to {}_F F_F$ defines an isomorphism

$$M(\epsilon^{-1}, \alpha^{-1}, -\alpha^{-1}\delta\epsilon^{-1}) \to \operatorname{Hom}({}_F M(\alpha, \epsilon, \delta), {}_F F) = M(\alpha, \epsilon, \delta)^L.$$

Similarly, $M(\alpha, \epsilon, \delta)^R = M(\epsilon^{-1}, \alpha^{-1}, -\epsilon^{-1}\delta\alpha^{-1})$. Note that in this way, we may get bimodules with constant dimension such that M^L and M^R are non-isomorphic.

6.5. Let $S = (F_i, {}_iM_j)_{1\leqslant i,j\leqslant n}$ be a connected species without oriented cycles, such that all bimodules ${}_{F_i}({}_iM_j)_{F_j}$ have constant dimension. Then we may define the Coxeter functors C^+ and C^- as usual [7], except that C^+ and C^- now are no longer endofunctors of $\mathrm{l}(F_i, {}_iM_j)$, but

$$C^+ \colon \mathrm{l}(F_i, {}_iM_j) \to \mathrm{l}(F_i, {}_iM_j^{\epsilon_{ij}}),$$

with $\epsilon_{ij} = LR$ or $= RL$, and similarly (but with reversed ϵ_{ij}) for C^-. But nevertheless, for all representations $V = (V_i, {}_j\varphi_i)$ in $\mathrm{l}(F_i, {}_iM_j)$ and in $\mathrm{l}(F_i, {}_iM_j^{\epsilon_{ij}})$, we may suppose $\dim V = (\dim(V_i)_{F_i})_i$ to lie in one and the same vector space $\mathbb{Q}^n$. Then there is again the Coxeter transformation c on $\mathbb{Q}^n$ which satisfies

$$\dim C^+V = c \dim V,$$

for all representations V in $\mathrm{l}(F_i, {}_iM_j)$ which are indecomposable and for which $C^+V \neq 0$. Here, of course, we use *precisely* the condition that all ${}_iM_j$ have constant dimension. As usual, we may describe those indecomposable representations V of S, which satisfy either $C^{+m}V = 0$ or $C^{-m}V = 0$, for some natural m. Indeed, such a representation must be uniquely determined by its dimension type, thus, for certain dimension types $x \in \mathbb{Q}^n$, we get again with purely combinatorial arguments the existence and unicity of an indecomposable representation V with $\dim V = x$. Therefore, if the graph of S is a Dynkin diagram, then we have the usual bijection between the indecomposable representations of S and the positive roots of the Dynkin diagram. Similarly, if the graph of S is not a Dynkin diagram, then obviously we have infinitely many indecomposable representations, namely at least the $2n$ infinite series $C^{-m}P_i$ and $C^{+m}Q_i$, with $1 \leqslant i \leqslant n$, and $m \in \mathbb{N}$, where the P_i are the indecomposable projective representations, and the Q_i are the indecomposable injective representations.

6.6. Returning to the case of a single bimodule ${}_FM_G$ with constant dimension, and assuming that ${}_FM_G$ is affine, that is,

$$(\dim M_G) \cdot (\dim_F M) = 4,$$

we conclude that the four infinite series $C^{-m}P_i$, $C^{+m}Q_i$, $1 \leqslant i \leqslant 2$, exhaust the set of all indecomposable representations V with $\dim C^+V \neq \dim V$. Namely, in $\mathbb{Q}^2$, any Weyl root can be obtained from a base root by applying (positive or negative) powers of a fixed Coxeter transformation. In the case $\dim_F M = \dim M_G = 2$, we therefore have:

LEMMA. *Let ${}_FM_G$ be a bimodule with constant dimension and* $\dim_F M = \dim M_G = 2$. *Then, for every positive element (x, y) in $\mathbb{Z}^2$ with $|x - y| = 1$, there is precisely one indecomposable representation $V = (X_F, Y_G, \varphi)$ such that* $\dim X_F = x$ *and* $\dim Y_G = y$, *and all the other indecomposable representations of ${}_FM_G$ satisfy* $\dim X_F = \dim Y_G$.

7. The Category $\mathfrak{r}({}_FM_G)$ for a Nonsimple Affine Bimodule

7.1. Let ${}_FM_G$ be an affine bimodule with constant dimension. Given an indecomposable representation (X, Y, φ), its defect ∂V is defined by $\partial V = \dim X_F - \dim Y_G$, in case $\dim M_G = \dim_F M = 2$, and by $\partial V = 2 \cdot \dim X_F - \dim Y_G$, in case $\dim M_G = 4$ and $\dim_F M = 1$. A representation is called *regular* provided it is the direct sum of indecomposable representations with zero defect. We denote by $\mathfrak{r}({}_FM_G)$ the full subcategory of $\mathfrak{l}({}_FM_G)$ of all regular representations. As usual, we have the following equivalences.

7.2. *The following assertions are equivalent for a representation V*

(i) *V is regular,*

(ii) $\partial V = 0$, *and* $\partial V' \leqslant 0$, *for every monomorphism* $V' \to V$,

(iii) $\partial V = 0$, *and* $\partial V'' \geqslant 0$, *for every epimorphism* $V \to V''$.

The proof uses only that the Coxeter functor C^+ preserves monomorphisms, in order to get (i) $\Rightarrow$ (ii), and that C^- preserves epimorphisms, in order to get (i) $\Rightarrow$ (iii). The remaining implications are trivial.

7.3. COROLLARY $\mathfrak{r}({}_FM_G)$ *is an abelian exact subcategory of* $\mathfrak{l}({}_FM_G)$, *and closed under extensions.*

We show only the last statement. Let $0 \to V' \to V \to V'' \to 0$ be an exact sequence in $\mathfrak{l}({}_FM_G)$, and assume V' and V'' both are regular. Since ∂ is additive on extensions, $\partial V = 0$. Let U be a subobject of V. Then $U \cap V'$ is a subobject of V', thus $\partial(U \cap V') \leqslant 0$. Also $U/(U \cap V') = U + V'/V'$ is a subobject of V'', thus also $\partial U/(U \cap V') \leqslant 0$. Again, since ∂ is additive on extensions, $\partial U \leqslant 0$. Thus (ii) is satisfied for V.

7.4. If ϵ is an automorphism and δ an $(\epsilon, 1)$-derivation of some division ring F, the skew polynomial ring $F[T; \epsilon, \delta]$ has as elements the formal sums $\sum_{i=0}^{n} f_i T^i$, with $f_i \in F$ and $n \in \mathbb{N}$, and the multiplication is induced by the multiplication of F and the rule

$$Tf = f^{\epsilon} T + f^{\delta} \qquad \text{for} \quad f \in F.$$

THEOREM. *Let ϵ be an automorphism and δ an $(\epsilon, 1)$-derivation of the division ring F. Let $M = M(\epsilon, \delta)$ and $R = F[T; \epsilon, \delta]$. Then $\mathfrak{r}(M) = \mathfrak{m}_R \times \mathfrak{u}$, where $\mathfrak{u}$ is a uniserial category with one simple object and* gl.dim. $\mathfrak{u} = 1$.

Proof. Let $e = (1, 0)$, $t = (0, 1) \in M = F \oplus F$. Let $N = Fe = eF$, and $\pi: F_F \otimes_F M_F \to (M/N)_F$ the canonical projection. Then $S = (F_F, M/N, \pi)$ is a simple regular representation, and we define $\mathfrak{u}$ as the full subcategory of all objects in $\mathfrak{r}(M)$ with all composition factors isomorphic to S.

If (X, Y, φ) is a representation of M, let $A = \{a \in X \mid \varphi(a \otimes e) = 0\}$ and $B = \{b \in X \mid \text{there is } a \in X \text{ with } \varphi(a \otimes e + b \otimes t) = 0\}$. Note that A and B are F-subspaces of X_F. This is obvious for A, and with $a \otimes e + b \otimes t$ also $(a \otimes e + b \otimes t) f^{(\epsilon^{-1})} = (a + bf^{(\epsilon^{-1}\delta)}) \otimes e + bf \otimes t$ belongs to the kernel of φ.

If $V = (X, Y, \varphi)$ is now a simple representation in $\mathfrak{r}(M)$, then either $V = S$, or else $A = 0$, and $B = X$. Namely, assume there is $0 \neq a \in A$. Then $V' = (aF, \varphi(aF \otimes M), \varphi')$ with φ' the restriction of φ to aF, satisfies $\partial V' \geqslant 0$, so by 7.2, $\partial V' = 0$, and therefore V' is regular. Since V is simple, we conclude $V' = V$, and obviously V' is isomorphic to S. Thus, we may assume $A = 0$. But since $V = 0$, the condition $\ker \varphi \cap X \otimes e = 0$ implies $\ker \varphi + X \otimes e = X \otimes M$, thus $B = X$.

If T is simple in $\mathfrak{r}(M)$, and not isomorphic to S, then $\operatorname{Ext}(T, S) = 0$. Namely, let $0 \to S \hookrightarrow V \xrightarrow{\gamma} T \to 0$ be exact, with $V = (X, Y, \varphi)$, $S = (S_1, S_2, \pi)$ and $T = (T_1, T_2, \psi)$. Form $B(V)$, and we claim that $S_1 \cap B(V) = 0$. Otherwise, there is some $a \otimes e + b \otimes t$ in $\ker \varphi$ with $b \in S_1$, and under γ we get $a^{\gamma} \otimes e \in \ker \psi$. Now, since $A(T) = 0$, $a^{\gamma} = 0$, and therefore $a \in S_1$, thus $b \in B(S_1) = 0$. Also, we claim $S_1 + B(V) = X$. For, $B(V)$ maps under γ onto $B(T) = T_1$. Let b_i, $1 \leqslant i \leqslant n$, be a basis of $B(V)$. Since $S_1 \otimes e \subseteq \ker \varphi$, and $S_1 \oplus B(V) = X$, there are elements $a_i \in B(V)$ such that $a_i \otimes e + b_i \otimes t \in \ker \varphi$, for all i. It then follows that $\ker \varphi = (S_1 \otimes M \cap \ker \varphi) \oplus (B(V) \otimes M \cap \ker \varphi)$ and therefore V is the direct sum of S and T.

Also, if T is simple and nonisomorphic to S, then $\operatorname{Ext}(S, T) = 0$. Again, let $S = (S_1, S_2, \pi)$, $V = (X, Y, \varphi)$ and $T = (T_1, T_2, \psi)$ and assume there is given an exact sequence $0 \to T \to V \xrightarrow{\gamma} S \to 0$. We need in X an element $a \neq 0$ with $\varphi(a \otimes e) = 0$. Let $a \otimes e + b \otimes t$ be an element of $\ker \varphi$ which does not belong to $T_1 \otimes M$. Then, under γ, we get $a^{\gamma} \otimes e + b^{\gamma} \otimes t$ in $\ker \pi = S_1 \otimes e$, and therefore $b^{\gamma} = 0$, which means $b \in T_1 = B(T_1)$. As a consequence, there is $a' \otimes e + b \otimes t$ in $\ker \psi$, with $a' \in T_1$. It follows that $(a - a') \otimes e$ belongs to $\ker \varphi$, and $a - a' \neq 0$. The last inequality is a consequence of the fact that $a' \in T_1$, whereas $a \notin T_1$, since $A(T) = 0$.

Next, we will show that $\mathrm{Ext}(S, S)$ is one-dimensional both as left $\mathrm{End}(S)$-vector space, as well as right $\mathrm{End}(S)$-vector space. The endomorphism ring $\mathrm{End}(S)$ of S consists just of the left multiplications by elements of F, since the canonical projection $\pi: F_F \otimes_F M_F \to (M/N)_F$ is, in fact, an F–F-homomorphism. On the other hand, using the notation of 2.4, we have $S = (e)$, and therefore $\mathrm{Ext}(S, S) = \mathrm{Ext}((e), (e)) \simeq M/Fe + eF = M/N$. Obviously, this isomorphism is an isomorphism of bimodules, namely of the $\mathrm{End}(S)$–$\mathrm{End}(S)$-bimodule $\mathrm{Ext}(S, S)$ and the F–F-bimodule ${}_F(M/N)_F$, with respect to the identification of $\mathrm{End}(S)$ and F mentioned above. As a consequence, $\mathfrak{u}$ is a uniserial category. It is well known that the category $\mathfrak{l}(M)$ is hereditary. Using Corollary 7.3, we see that the same is true for $\mathfrak{r}(M)$, and therefore also for the subcategory $\mathfrak{u}$, since $\mathfrak{r}(M)$ is the direct product of $\mathfrak{u}$ and some other subcategory $\mathfrak{m}$. Since $\mathrm{Ext}(S, S) \neq 0$, we conclude gl.dim. $\mathfrak{u} = 1$.

It remains to determine the category $\mathfrak{m}$. Now $\mathfrak{m}$ is the full subcategory of $\mathfrak{r}(M)$ of all objects with composition factors of the form $V = (X, Y, \varphi)$ such that $A(V) = 0$ and $B(V) = X$, and we want to show that $\mathfrak{m}$ is equivalent to the category $\mathfrak{m}_R$ of all R-modules of finite length over the skew polynomial ring $R = F[T; \epsilon, \delta]$. Given such an R-module A_R, define $D(A_R) = (A_F, A_F, \varphi)$ where $\varphi: A_F \otimes_F M_F \to A_F$ is given by $\varphi(a \otimes e) = a$ and $\varphi(a \otimes t) = aT$, for $a \in A$. It is an easy calculation that φ is indeed F-linear, and that D is a functor from $\mathfrak{m}_R$ into $\mathfrak{m}$. Conversely, given an object $V = (X, Y, \varphi)$ in $\mathfrak{m}$, then $\varphi(\,?\otimes e)$ is an isomorphism of X_F onto Y_F, and X_F becomes an R-module, if we set $aT = b$, provided $\varphi(a \otimes t) = \varphi(b \otimes e)$, that is provided the element $a \otimes t - b \otimes e$ belongs to the kernel of φ. This then shows that D is an equivalence of categories. Therefore, $\mathfrak{r}(M)$ is the product of the categories $\mathfrak{m}_R$ and $\mathfrak{u}$.

7.5. We have proved Theorem 4 of the introduction. Let us derive from this the corollaries mentioned there. Let F be a differentially closed field with derivation δ. Then, as Cozzens [5] has shown, $R = F[T; \delta]$ has just one simple module, namely ${}_RF$, and this module is injective. Therefore, ${}_FM_F = M(1, \delta)$ furnishes an example for Corollary 1. On the other hand, let $F = K(Y)$ be the field of rational functions in one variable Y over a field K, and let δ be the usual derivation in $K(Y)$. Again, we consider the ring $R = F[T; \delta]$. If K is algebraically closed, then for any simple R-module, A, $\mathrm{End}(A) = K$, whereas $\mathrm{Ext}(A, B)$ is infinite dimensional as K-vector space, for any two simple R-modules A and B. This was shown by MacConnell and Robson [10]. But we know that this implies that $\mathfrak{w}(K)$ can be embedded into $\mathfrak{r}({}_FM_F)$ where $M = M(1, \delta)$.

8. Nonsimple, Affine Bimodules

In this final section, we give a proof of Theorem 5. Thus, let ${}_FM_G$ be a nonaffine bimodule with center Z, and assume $\dim_Z M < \aleph_\iota$. Note that Z operates centrally on every Hom-set and every Ext-set of the category $\mathfrak{L}({}_FM_G)$. Let $0 \neq U \neq M$ be a proper F–G-submodule of ${}_FM_G$.

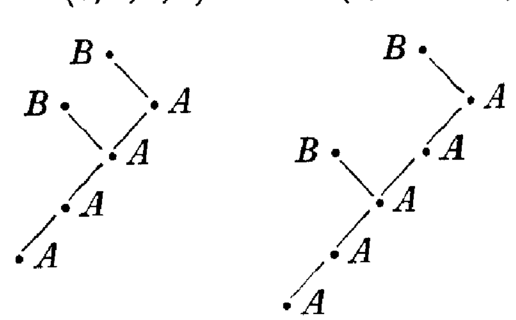
8.1. *There are two orthogonal points A, B in $\mathfrak{L}({}_FM_G)$ with* $\operatorname{Ext}(A, A) \neq 0$ *and either* $\operatorname{Ext}(A, B) \neq 0$ *or else* $\operatorname{Ext}(B, A) \neq 0$.

Proof. Let $0 \neq a \in U$, and $b \in M \backslash U$, and consider the representations $A = (a) = (F_F, M/aG)$ and $B = (b) = (F_F, M/bG)$ introduced in Section 2. Then both A and B are points, and $\operatorname{Ext}(A, A) = M/Fa + aG \neq 0$, since $Fa + aG \subseteq U$. If $\dim_F U \geqslant 2$, then $Fa + bG \neq M$, since Fa is properly contained in U, and $bG \cap U = 0$. If $\dim(M/U)_G \geqslant 2$, then again $Fa + bG \neq M$, this time since $Fa + bG + U/U = bG + U/U$ is a proper subspace of M/U. In both cases, therefore, $\operatorname{Ext}(A, B) = M/Fa + bG \neq 0$. Similarly, if $\dim U_G \geqslant 2$, or if $\dim {}_F(M/U) \geqslant 2$, then $\operatorname{Ext}(B, A) \neq 0$. It is obvious that A and B are orthogonal.

8.2. *There are countably many pairwise orthogonal points A_i in $\mathfrak{L}({}_FM_G)$ such that* $\operatorname{Ext}(A_i, A_j) \neq 0$ *for all i, j.*

Proof. We assume, the points A and B of 8.1. satisfy $\operatorname{Ext}(B, A) \neq 0$, the other case can be proved dually. Simplify A and B. Let $e = (e_1, \ldots, e_n)$ be a sequence with $e_i = 0$, or 1. Given such a sequence e, we define a representation $X(e)$ of length $n + \sum e_i$ as follows. $X(e)$ contains a uniserial subobject Y_n of length n with composition factors of the form A, such that $X(e)/Y_n$ is a direct sum of copies of B. Obviously, Y_n is uniquely determined as subobject of $X(e)$. Moreover, if $e_i = 1$, then $X(e)$ shall contain a serial subobject of length $i + 1$, such that its hat (the upper composition factor) is of the form B, and all the other composition factors are of the form A. The structure of $X(e)$ can be illustrated by the following pictures.

$e = (0, 0, 1, 1)$ $\quad$ $e = (0, 0, 1, 0, 1)$



Of course, such an object $X(e)$ usually is not uniquely determined. However, objects with the prescribed structure do exist, since we are working in a global dimension 1 category with A and B simple.

Now, assume there is a homomorphism $X(e) \to X(f)$ given, with $e = (e_1, ..., e_n)$ and $f = (f_1, ..., f_m)$ 0, 1-sequences. Then $Y(e)$ is mapped into $Y(f)$, say onto a subobject of length k. It follows that $e_{n-k+i} \leqslant f_i$ for $1 \leqslant i \leqslant k$. Having this in mind, it is rather easy to see that for $e = (0, 1)$, $(0, 0, 1, 1), (0, 0, 1, 0, 1, 1), \ldots$, we get an infinite set of orthogonal points. Here, the nth term will be the sequence $(0, e_2, ..., e_{2n-1}, 1)$ with $e_{2i} = 0$, and $e_{2i+1} = 1$ for $1 \leqslant i < n$. Also, since $\operatorname{Ext}(B, A) \neq 0$, it follows that for arbitrary sequences e and f, we have $\operatorname{Ext}(X(e), X(f)) \neq 0$.

8.3. *There are two orthogonal points X, Y such that the* $\operatorname{End}(Y)$–$\operatorname{End}(X)$*-bimodule* $\operatorname{Ext}(X, Y)$ *contains a submodule of the form* $\bigoplus_{i=1}^{3} N_i$ *with* $N_i \neq 0$ *for all* i.

Proof. Let $A_0, ..., A_4$ be five pairwise orthogonal points with $\operatorname{Ext}(A_i, A_j) \neq 0$ for all i, j. Simplify those A_i's. Let $X = A_0$, and construct an indecomposable Y with an exact sequence

$$0 \to \bigoplus_{i=1}^{3} A_i \to Y \to A_4 \to 0.$$

Of course, Y is again a point, and any automorphism of Y leaves invariant the subobjects A_i, $1 \leqslant i \leqslant 3$, of Y. We consider the corresponding long exact sequence with respect to $\operatorname{Hom}(X, -)$. Since $\operatorname{Hom}(X, A_4) = 0$, we get an exact sequence

$$0 \to \operatorname{Ext}\left(X, \bigoplus_{i=1}^{3} A_i\right) \xrightarrow{\gamma} \operatorname{Ext}(X, Y) \to \operatorname{Ext}(X, A_4)$$

and the decomposition

$$\operatorname{Ext}\left(X, \bigoplus_{i=1}^{3} A_i\right) = \bigoplus_{i=1}^{3} \operatorname{Ext}(X, A_i)$$

is a decomposition of $\operatorname{End}(Y)$–$\operatorname{End}(X)$-bimodules. Of course, by assumption, $\operatorname{Ext}(X, A_i) \neq 0$ for all i. Thus, let N_i be the image of $\operatorname{Ext}(X, A_i)$ in $\operatorname{Ext}(X, Y)$ under γ.

8.4. Let $D = \operatorname{End}(X)$, $E = \operatorname{End}(Y)$ and ${}_EN_D = \bigoplus_{i=1}^{3} N_i$. According to 1.5, $\mathfrak{L}^*({}_EN_D)$ *can be considered as a full exact subcategory of* $\mathfrak{L}({}_FM_G)$. Let K be the center of ${}_EN_D$, then Z can be considered as a subfield of K. It is easy to see that with $\dim_Z M < \aleph_0$ also $\dim_Z N < \aleph_0$, and therefore, a fortiori, $\dim_K N < \aleph_\iota$. Again, we will use that K operates centrally on all Hom-sets and all Ext-sets, now of the category $\mathfrak{L}^*({}_E N_D)$.

8.5. *There exist two orthogonal points A, B in $\mathfrak{L}^*({}_E N_D)$ with* $\mathrm{End}(A) = K = \mathrm{End}(B)$ *and* $\dim_K \mathrm{Ext}(A, B) \geqslant 3$.

Proof. Let $(C_0 ; C_i)$ be a Corner quintuple of rank $\geqslant \max(\aleph_0, \dim_K N)$. Let I_i be a base of C_0/C_i and J_i be a base of C_i. Since both sets have the same cardinality, we may choose a bijection $\pi_i : I_i \to J_i$, for $1 \leqslant i \leqslant 5$.

Let P_i be a basis of N_i over K. The cardinality of P_i is $\leqslant \dim_K N$, therefore there is a surjection $\epsilon_i : I_i \to P_i$, for $1 \leqslant i \leqslant 5$.

We want to construct a representation $A = (C_0 \otimes_{\mathbb{Z}} D_D, C_0 \otimes_{\mathbb{Z}} E_E, \varphi)$, where φ: $C_0 \otimes_{\mathbb{Z}} D_D \to C_0 \otimes_{\mathbb{Z}} E_E \otimes N$. The ith component $\varphi_i : C_0 \otimes_{\mathbb{Z}} D_D \to C_0 \otimes_{\mathbb{Z}} E_E \otimes N_i$ of φ will be defined as the composition of

$$C_0 \otimes_{\mathbb{Z}} D_D \to (C_0/C_i) \otimes_{\mathbb{Z}} D_D \to C_i \otimes_{\mathbb{Z}} N_i \to C_0 \otimes_{\mathbb{Z}} E_E \otimes N_i ,$$

where the first map is the canonical projection, the last map is given by the inclusion $C_i \subseteq C_0$ and the canonical isomorphism $N_i \to E_E \otimes_E N_i$, and the middle map is defined by

$$k \otimes d \to \pi_i(k) \otimes \epsilon_i(k)d, \qquad \text{for} \quad k \in I_i, \qquad d \in D.$$

Note that the kernel of this map $\varphi_i : C_0 \otimes_{\mathbb{Z}} D_D \to C_0 \otimes_{\mathbb{Z}} E_E \otimes N_i$ is precisely $C_i \otimes_{\mathbb{Z}} D_D$, and that the set $\{x \in C_0 \otimes_{\mathbb{Z}} E_E \mid \text{there is } k \in I_i \text{ with } (x \otimes N_i) \cap \varphi_i(k \otimes D) \neq 0\}$ generates just $C_i \otimes_{\mathbb{Z}} E_E$.

Now, let (α, β) be an endomorphism of $A = (C_0 \otimes_{\mathbb{Z}} D_D, C_0 \otimes_{\mathbb{Z}} E_E, \varphi)$. Then $\alpha \in \mathrm{End}(C_0 \otimes_{\mathbb{Z}} D_D)$ has the property $\alpha(C_i \otimes_{\mathbb{Z}} D_D) \subseteq C_i \otimes_{\mathbb{Z}} D_D$ for all i, and since $(C_0 ; C_i)$ is a Corner quintuple, α is just scalar multiplication by some $d \in D$. If $\alpha = 0$, then obviously also $\beta = 0$, thus we may assume $d \neq 0$. We want to show that $\beta(C_i \otimes_{\mathbb{Z}} E_E) \subseteq C_i \otimes_{\mathbb{Z}} E_E$. Thus, let $x \in C_0 \otimes_{\mathbb{Z}} E_E$, and $0 \neq y \in N_i$ such that $x \otimes y$ belongs to $\varphi_i(k \otimes D)$, for some $k \in I_i$, say $x \otimes y = \varphi_i(k \otimes d')$, with $d' \in D$. Then

$$\beta x \otimes y = (\beta \otimes 1)\,\varphi_i(k \otimes d') = \varphi_i\alpha(k \otimes d') = \varphi_i(k \otimes dd'),$$

and therefore also $\beta x \in C_i \otimes_{\mathbb{Z}} E_E$. As a consequence, β is given by scalar multiplication by some $e \in E$. We claim that $xd = ex$ for all $x \in N$. First, let $x \in P_i$, so there is $k \in I_i$ with $\epsilon_i(k) = x$. Then

$$\begin{aligned}\pi_i(x) \otimes xd &= \varphi_i(k \otimes d) = \varphi_i\alpha(k \otimes 1) = (\beta \otimes 1)\,\varphi_i(k \otimes 1) = \\ &= \pi_i(k) \otimes ex,\end{aligned}$$

and therefore $xd = ex$, for $x \in P_i$. But P_i is a basis of N_i over the field K

which centrally operates on N_i, and consequently $xd = ex$ for all $x \in N_i$. But this, of course, implies that the pair (e, d) belongs to the center K of the bimodule ${}_EN_D$. Namely, e belongs to the center of E, since for $e' \in E$ and any $0 \neq x \in N$, we have

$$(ee')x = e(e'x) = (e'x)d = e'(xd) = e'(ex) = (e'e)x,$$

and similarly, d belongs to the center of D.

If $(C_0; C_i)$ and $(C_0'; C_i')$ are orthogonal Corner quintuples, and if we construct representations $A = (C_0 \otimes_{\mathbf{Z}} D_D, C_0 \otimes_{\mathbf{Z}} E_E, \varphi)$ and $B = (C_0' \otimes_{\mathbf{Z}} D_D, C_0' \otimes_{\mathbf{Z}} E_E, \varphi')$ as above, then A and B are orthogonal points. For, if $(\alpha, \beta): A \to B$, then $\alpha: C_0 \otimes_{\mathbf{Z}} D_D \to C_0' \otimes_{\mathbf{Z}} D_D$ satisfies $\alpha(C_i \otimes_{\mathbf{Z}} D_D) \subseteq C_i' \otimes_{\mathbf{Z}} D_D$, since $C_i \otimes_{\mathbf{Z}} D_D$ is the kernel of φ_i, and $C_i' \otimes_{\mathbf{Z}} D_D$ is the kernel of φ_i'. Therefore $\alpha = 0$, and then also $\beta = 0$.

In order to show that $\operatorname{Ext}(A, B)$ is sufficiently large, we use that $\operatorname{Ext}(A, B) = H/W$, where

$$H = \operatorname{Hom}_D(C_0 \otimes_{\mathbf{Z}} D_D, C_0' \otimes_{\mathbf{Z}} E_E \otimes {}_EN_D),$$

and

$$W = \{(1 \otimes e \otimes 1\varphi - \varphi'(1 \otimes d) \mid e \in E, d \in D\},$$

which is proved in a similar way as the corresponding assertion for $\mathfrak{L}({}_FM_G)$, see 2.1. We introduce

$$H_i = \operatorname{Hom}_D(C_0 \otimes_{\mathbf{Z}} D_D, C_0' \otimes_{\mathbf{Z}} E_E \otimes N_i),$$

$$W_i = \{\gamma \in H_i \mid \gamma(C_i \otimes_{\mathbf{Z}} D) \subseteq C_i' \otimes_{\mathbf{Z}} E_E \otimes N_i\}.$$

Then $H = \bigoplus_{i=1}^3 H_i$, and W_i is a proper K-subspace of H_i. But it is easy to see that $W \subseteq \bigoplus_{i=1}^3 W_i$, since φ vanishes on $C_i \otimes_{\mathbf{Z}} D$, and $\varphi'(C_0 \otimes_{\mathbf{Z}} D) \subseteq C_i' \otimes_{\mathbf{Z}} E_E \otimes N_i$. As a consequence, $\operatorname{Ext}(A, B)$ has $\bigoplus_{i=1}^3 H_i/W_i$ as an epimorphic image, and thus $\dim_K \operatorname{Ext}(A, B) \geqslant 3$.

8.6. *There is a full and exact embedding of* $\mathfrak{W}(K)$ *into* $\mathfrak{L}({}_FM_G)$.

Proof. Recall that $\mathfrak{W}(K)$ is defined to be the category of all right R-modules over the free K-algebra $K\langle x, y\rangle$ in the two variables x and y. Let U be a three-dimensional K-subspace of ${}_K\operatorname{Ext}(A, B)_K$ with basis u_1, u_2, u_3. There is a full and exact embedding of $\mathfrak{W}(K)$ into $\mathfrak{L}^*({}_KU_K)$ mapping the R-module M_R onto the representation (M_K, M_K, φ) with $\varphi: M_K \to M_K \otimes_K U_K$ defined by

$$\varphi(m) = m \otimes u_1 + mx \otimes u_2 + my \otimes u_3.$$

Also, by 1.5, there is a full and exact embedding of $\mathfrak{L}^*({}_KU_K)$ into $\mathfrak{L}^*({}_EN_D)$, so all together, we have the following functors.

$$\mathfrak{W}(K) \to \mathfrak{L}^*({}_KU_K) \to \mathfrak{L}^*({}_EN_D) \to \mathfrak{L}({}_FM_G),$$

and all three are full and exact embeddings.

Note added in proof (June 17, 1976). In this note, we want to comment on some of the results of the paper and to add several remarks on further developments.

The homological interpretation of the quadratic form q and its corresponding bilinear form $\tilde{q}$ given in Lemma 2.2 is crucial for the paper. It is possible to give another (and even easier) proof of the equality

$$\tilde{q}(\dim A, \dim B) = \dim_K \operatorname{Hom}(A, B) - \dim_K \operatorname{Ext}^1(A, B).$$

Both sides obviously are additive with respect to extensions (the right side, since we are working in a hereditary category), therefore one may assume that both representations A and B are simple, say $A = F_i$ and $B = F_j$. But then either $i = j$, and then $\dim_K \operatorname{Hom}(A, B) = f_i$, $\operatorname{Ext}^1(A, B) = 0$, or else $i \neq j$ and then $\operatorname{Hom}(A, B) = 0$ and $\dim_K \operatorname{Ext}^1(A, B) = m_{ij}$. In categories which are nonhereditary, but which have finite global dimension, the corresponding bilinear form

$$\sum_{k \geqslant 0} (-1)^k \dim_K \operatorname{Ext}^k(A, B)$$

should turn out to be of equal importance.

For two affine algebraic bimodules, namely, for $\mathbb{R}^{\mathbb{H}}\mathbb{H}$ and $\mathbb{C}^{\mathbb{C}}\mathbb{R} \otimes \mathbb{R}^{\mathbb{C}}\mathbb{C}$ (where $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ denote the fields of real, complex, and quaternion numbers, respectively) an explicit description of all finite dimensional indecomposable representations has been worked out in detail by V. Dlab and the author ("Real subspaces of a vector space over the quaternions," to appear, and "Normal forms of real matrices with respect to complex similarity," to appear in *Linear Algebra and its Applications*), using Theorems 1 and 4 of this paper.

M. Auslander has introduced the notion of an almost split exact sequence, and this concept turned out to be very fruitful. For algebraic bimodules ${}_FM_G$, it is possible to describe completely the almost split exact sequences: There are the obvious ones for the preprojective representations, namely,

$$0 \to C^{-k}P_1 \to \bigoplus_a C^{-k-1}P_2 \to C^{-k-1}P_1 \to 0,$$

$$0 \to C^{-k}P_2 \to \bigoplus_b C^{-k}P_1 \to C^{-k-1}P_2 \to 0,$$

where $P_1 = (F_F, M_G, id)$, $P_2 = (0, G_G, o)$ are the two indecomposable projective representations, $k \geqslant o$, and $a = \dim M_G$, $b = \dim {}_FM$. There are similar ones for the preinjective representations $C^{+k}I$ (where I is indecomposable injective and $k \geqslant o$), the middle term is again a direct sum of a or b indecomposable representations. The remaining almost split exact sequences

$$0 \to A \to B \to C \to 0$$

have the property that B is the direct sum of *at most two* indecomposable representations. This shows the amazing fact that the regular representations of an algebraic bimodule (i.e., those corresponding to imaginary roots) behave (in some respect) like serial objects. An account of this will be given in a forthcoming paper.

It has been shown in the paper that there is a clear distinction between three types of K-species: The species may be of finite, tame, or wild representation type. This distinction is, however, of relevance only for finite dimensional representations; the difference between "tame" and "wild" disappears if we consider infinite dimensional representations. Namely, it can be shown that for any K-species S with nondefinite quadratic form, there is a full and exact embedding $\mathfrak{W}(K') \subsetneq \mathfrak{L}(S)$, where K' is an extension field of K contained in one of the fields F_i. (This is only true provided we exclude, as we have done in the definition of a species, oriented cycles: in fact, there is no such embedding in case of a K-species of type $\tilde{\mathbb{A}}_n$ with cyclic orientation.) This embedding shows that the infinite dimensional representations also of a K-species with semidefinite quadratic form are rather awkward. On the other hand, in this case, there is one particular class of infinite dimensional representations which can be described completely: The locally indecomposable representations (a representation is called locally indecomposable provided every finite dimensional subobject is contained in a finite dimensional indecomposable subobject). For a discussion of infinite dimensional representations of K-species we refer to "Unions of chains of indecomposable modules," *Communications in Algebra*, **3** (1975), 1121–1144, and a forthcoming paper.

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