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Finite Dimensional Hereditary Algebras of Wild Representation Type

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Let R be a finite dimensional hereditary algebra. We are concerned with the problem of determining the indecomposable R-modules of finite length. This problem completely has been solved in the case when R is of finite or of tame representation type, but seems to be rather hopeless in the case of wild representation type. In this situation, the only known classes of modules are the so-called pre-projective and the pre-injective ones. The remaining indecomposable modules are called regular. In this paper, we want to initiate the study of the regular modules. The result we obtain seems to be rather surprising: we will show that the regular modules behave rather similar to modules over a serial algebra.

In order to state the main theorem, we need the notion of an irreducible homomorphism, which was introduced by Auslander and Reiten [3]. Let X and Y be two non-zero R-modules. A homomorphism $f: X \to Y$ is said to be *irreducible*, if it is neither a split monomorphism, nor a split epimorphism, and if for any factorisation $X \xrightarrow{f'} I \xrightarrow{f''} Y$ of f, either f' is a split monomorphism or f'' is a split epimorphism. Note that an irreducible homomorphism is always either a monomorphism or an epimorphism. A non-zero R-module S will be called *quasi-simple*, if S is regular, and there is no irreducible monomorphism of the form $U \to S$ with U non-zero. In this case, we will call the map $0 \to S$ *irreducible*.

Theorem. *Let R be a finite dimensional hereditary algebra. Let M and N be regular R-modules, and let*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$$

and

$$0 = N_0 \subset N_1 \subset \cdots \subset N_{t-1} \subset N_t = N$$

be chains of irreducible monomorphisms. Then we have:

(1) *If M and N are isomorphic, then $s = t$ and there is an isomorphism $f: M \to N$ with $M_i f = N_i$ for all i.*

(2) *On the other hand, if $s = t$, and, for some i, M_i/M_{i-1} is isomorphic to N_i/N_{i-1}, then M and N are isomorphic.*

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(3) *The factors M_i/M_{i-1} are quasi-simple.*

Conversely, given a quasi-simple module S, a natural number s, and a natural number i with $1 \leqq i \leqq s$, *then there exists a regular R-module M with a chain* $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ *of irreducible monomorphisms such that* M_i/M_{i-1} *is isomorphic to S.*

One should be aware in (1) that in general not every isomorphism $f\colon M \to N$ will satisfy $M_i f = N_i$ for all i. In the last section, we will give examples of this situation. This means that the analogy to serial modules is not complete. However, similar to the case of modules over a serial algebra, we see that the classification of regular modules completely is reduced to the classification of quasi-simple modules, as the following corollary shows.

Corollary. *Let* $\mathscr{R}$ *be the set of regular R-modules, let* $\mathscr{S}$ *be the set of quasi-simple R-modules. Then there is a bijection* $\mathscr{R} \to \mathscr{S} \times \mathbb{N}$, *where a regular module M with a chain of irreducible monomorphisms* $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ *is mapped onto the pair* $(M/M_{s-1}, s)$.

Instead of considering only modules over finite dimensional hereditary algebras, we will work most of the time in a more general setting. Let R be an Artin algebra. Define the *Auslander graph* of R in the following way: its points are the isomorphism classes of indecomposable R-modules (of finite length), and there is an arrow $[X] \to [Y]$ (where $[X]$ denotes the isomorphism class containing the module X) if and only if there exists an irreducible homomorphism $X \to Y$. This is a locally finite graph, and we will consider its connected components. In Section 2, we will show that for R hereditary, any component of the Auslander graph which does not contain a projective or an injective module is "quasi-serial". In the rest of the paper, we will consider modules belonging to quasi-serial components, and we will prove the theorem above in this more general setting.

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1. Preliminaries

Throughout the paper, R will be an Artin algebra, thus, the center C of R is an Artinian ring, and R is finitely generated as a C-module. Usually, we will consider left R-modules of finite length, and we will call them just modules. Module homomorphisms will be written on the opposite side of the scalars, thus, for (left) R-modules X, Y, Z and homomorphisms $f\colon X \to Y$, $g\colon Y \to Z$, the composition of f and g will be denoted by fg. If f is a homomorphism, we denote its kernel by $\operatorname{Ker} f$, its cokernel by $\operatorname{Cok} f$. The length of a module M is denoted by $|M|$. If M and N are isomorphic modules, we write $M \approx N$. If $X \subseteq Y$, then we denote the inclusion map usually by $m\colon X \to Y$, and the projection map by $p\colon Y \to Y/X$.

Also, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the integers, and $\mathbb{R}$ the real numbers.

1.1. *The Auslander Construction.* Let M be an R-module, let $P_1 \xrightarrow{f} P_0 \to M$ be the first two terms of a minimal projective resolution of M. Applying the functor *

$=\mathrm{Hom}_R(\ ,R)$, we obtain a map $f^*: P_0^* \to P_1^*$ of right R-modules, whose cokernel (a right R-module) will be denoted by $\mathrm{Tr}\, M$. Let ${}_C I$ be a C-module which is a minimal injective cogenerator. Then $D=\mathrm{Hom}_C(\ ,I)$ is a duality functor from (left) R-modules to right R-modules, or from right R-modules to (left) R-modules. For any R-module M, we have constructed in this way an R-module $AM = D\mathrm{Tr}\, M$, and an R-module $A^{-1}M = \mathrm{Tr}\, DM$.

Let M be indecomposable. Then AM and $A^{-1}M$ are indecomposable or zero. And, $AM \neq 0$ iff M is not projective, and then $M \approx A^{-1}AM$. Similarly, $A^{-1}M \neq 0$ iff M is not injective, and then $M \approx AA^{-1}M$.

An indecomposable module M is called *pre-projective* iff $A^n M = 0$ for some $n \in \mathbb{N}$, iff $M \approx A^{-k}P$ for some indecomposable projective module P and some $k \in \mathbb{N}_0$. Dually, an indecomposable module M is called *pre-injective* iff $A^{-n}M = 0$ for some $n \in \mathbb{N}$, iff $M \approx A^k I$ for some indecomposable injective module I and some $k \in \mathbb{N}_0$.

1.2. *The Auslander Reiten Sequences.* The link between the Auslander construction on the one hand, and the irreducible homomorphisms on the other hand, is given by a special type of short exact sequences which were introduced by Auslander and Reiten (who called them almost split exact sequences).

Definition. $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called an *Auslander Reiten sequence,* iff $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence, and both f and g are irreducible homomorphisms ([3], Prop. 2.15, and Prop. 2.6b).

We collect the main properties which we will need in the sequel. Assume $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an Auslander Reiten sequence. Then X and Z both are indecomposable, and the sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ does not split. In particular, X is not injective, and Z is not projective. Given a map $h: X \to M$ which is not a split monomorphism, it can be extended to Y, thus there is $h': Y \to M$ with $h = fh'$. Dually, given a map $h: M \to Z$ which is not a split epimorphism, we can lift h to Y, thus there is $h': M \to Y$ with $h = h'g$.

Existence and unicity: Given an indecomposable module X which is not injective, there is an Auslander Reiten sequence $X \to Y \to Z$. Given an indecomposable module Z which is not projective, there is an Auslander Reiten sequence $X \to Y \to Z$. If $X \to Y \to Z$ and $X' \to Y' \to Z'$ are two Auslander Reiten sequences, then the sequences are isomorphic iff X, X' are isomorphic, iff Z, Z' are isomorphic ([2], Prop. 4.3, a, b, c).

Relation between Auslander Reiten sequences and irreducible homomorphisms: Let $f: X \to Y$ be a homomorphism with X indecomposable and not injective, and Y non-zero. Then f is irreducible iff there exists $f': X \to Y'$ such that

$$X \xrightarrow{(f,f')} Y \oplus Y' \to \mathrm{Cok}(f,f')$$

is Auslander Reiten sequence. Dually, let $g: Y \to Z$ be a homomorphism with Y non-zero and Z indecomposable and not projective, then g is irreducible iff there exists $g': Y' \to Z$ such that

$$\mathrm{Ker}\begin{pmatrix} g \\ g' \end{pmatrix} \to Y \oplus Y' \xrightarrow{\binom{g}{g'}} Z$$

is an Auslander Reiten sequence ([3], Th. 2.4).

Relation between Auslander Reiten sequences and the Auslander construction: Let $X \to Y \to Z$ be an Auslander Reiten sequence. Then $X \approx AZ$, and $Z \approx A^{-1}X$ ([2], Prop. 4.3). On the other hand, if $X \to Y \to Z$ is an Auslander Reiten sequence, and X is not projective, then there exists an injective module I and an Auslander Reiten sequence $AX \to AY \oplus I \to AZ$ ([4], Prop. 2.2b.i). Dually, if $X \to Y \to Z$ is an Auslander Reiten sequence, and Z is not injective, then there exists a projective module P and an Auslander Reiten sequence $A^{-1}X \to A^{-1}Y \oplus P \to A^{-1}Z$ ([4], Prop. 2.2a.ii).

1.3. *The Hereditary Case.* We assume now that R is, in addition, hereditary (submodules of projective modules are projective). Also, we may assume that R is twosided indecomposable, thus the center C of R is a field.

Let $S_1, \ldots, S_n$ be a maximal set of pairwise non-isomorphic simple modules. We assume that the indices are ordered in such a way that $\mathrm{Ext}^1(S_i, S_j) \neq 0$ implies $i > j$. For any R-module M, let $\dim M$ be the *dimension vector* of M in $\mathbb{N}_0^n \subset \mathbb{R}^n$ with i-th component $(\dim M)_i$ being the number of composition factors of the form S_i in a composition series of M. The elements $b_i = \dim S_i$ form the canonical basis of $\mathbb{R}^n$.

We define on $\mathbb{R}^n$ a symmetric bilinear form B as follows: Let

$$B(b_i, b_i) = \dim_C \mathrm{End}(S_i),$$

and

$$B(b_i, b_j) = -\tfrac{1}{2}\dim_C \mathrm{Ext}^1(S_i, S_j) \qquad \text{for } i > j.$$

Let s_i be the reflection of $\mathbb{R}^n$ on the hyperplane orthogonal to b_i with respect to B. The linear transformation $c = s_1 \ldots s_n$ of $\mathbb{R}^n$ is called the *Coxeter transformation* for R.

The bilinear form B can be used to define the representation type of R. Namely, R is of finite representation type if and only if B is positive definite [7], and we say that R is of *tame* type if and only if B is semidefinite, but not definite. In the remaining case that B is indefinite, we call R *wild*. This is the case we are mainly interested in. However, our results will also apply to the regular modules in the tame case. For, in case R is tame, the full subcategory **R** of all regular modules is abelian, and it is a serial category which has no non-zero objects which are injective or projective in this subcategory. (For tensor algebras, this has been established in [7], Th. 3.5, and [9], Th. 1; there is only one remaining case, namely a ring of type $\tilde{A}_n$ with a non-split bimodule extension, and this case has been treated in [8].) As a consequence, we see a monomorphism $X \to Y$ with X, Y indecomposable regular, is irreducible if and only if Y/X is simple when considered as an object of **R**. Thus, in this case, the "quasi composition series" asserted by the theorem is given by an ordinary composition series inside the category **R**.

An element $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ will be called *positive* iff $x_i \geqq 0$ for all i. The following criterion due to Berman, Moody and Wonenberger [5] will be very useful: The bilinear form B is positive semidefinite, but not definite, if and only if c has a non-zero positive fix vector. For, our bilinear form B is given by a Cartan matrix, and, using Lemma 1.3 of [7], an element $x \in \mathbb{R}^n$ is a fix vector for c if and only if it is a fix vector for all the reflections s_i, and this is equivalent to x being a null root in the sense of [5].

Since R is hereditary, the construction Tr is in fact functorial, and therefore A and A^{-1} are functors. In this case, the functor A is left exact, whereas the functor A^{-1} is right exact; in particular, A preserves monomorphisms, and A^{-1} preserves epimorphisms.

Also, in this case, the effect of A and A^{-1} on indecomposable modules can be measured rather easily using the Coxeter transformation. Namely, if M is indecomposable and not projective (so that $AM \neq 0$), then

$$\dim AM = (\dim M)c,$$

whereas if M is indecomposable and not injective (so that $A^{-1}M \neq 0$), then

$$\dim A^{-1}M = (\dim M)c^{-1}.$$

In the case of a tensor algebra, this has been shown in [7] for the Coxeter functors, and Brenner and Butler have proved in [6] that A is a Coxeter functor. The general case has been considered in [1].

2. The Regular Modules over a Finite Dimensional Hereditary Algebra

The main result of this section will be the theorem that every regular module over a finite dimensional hereditary algebra belongs to a quasi-serial component of the Auslander graph. Here, a component $\mathscr{C}$ of the Auslander graph of an Artin algebra is called *quasi-serial* if it does not contain any projective or any injective module, and if for any Auslander Reiten sequence $X \to \bigoplus_{i=1}^{k} Y_i \to Z$ in $\mathscr{C}$, we have $k \leqq 2$, and, in case $k=2$ and $|Y_1| \leqq |Y_2|$, then $|Y_1| < |X| < |Y_2|$ and $|Y_1| < |Z| < |Y_2|$. The reason for calling such a component quasi-serial will be seen in the next sections.

2.1. **Lemma.** *Let X, Y be indecomposable modules over a finite dimensional hereditary algebra, and let $X \to Y$ be an irreducible homomorphism. If one of the modules X, Y is pre-projective, both are. If one of the modules X, Y is pre-injective, both are.*

Proof. First, assume Y is pre-projective, say $Y \approx A^{-k}P$ for some $k \in \mathbb{N}_0$ and some indecomposable projective module P. Now either $A^k X = 0$, and then X is pre-projective, or, using the results of 1.2, there is an irreducible homomorphism $A^k X \to A^k Y = P$. Since P is projective, this cannot be an epimorphism, thus it is a monomorphism, and therefore (since the algebra is hereditary), $A^k X$ is projective. This shows that X has to be pre-projective.

On the other hand, assume X is pre-projective. If Y is not projective, there exists an Auslander Reiten sequence of the form $U \to X \oplus X' \to Y$ for some modules U and X'. Thus, we have an irreducible homomorphism $U \to X$, and by the first part of the proof, U is pre-projective, say $U \approx A^{-l} P'$ for some $l \in \mathbb{N}_0$ and some indecomposable projective module P'. As a consequence, $Y \approx A^{-1} U \approx A^{-l-1} P'$ is preprojective.

The pre-injective part of the statement follows by duality.

In order to show that certain components of the Auslander graph of an Artin algebra are quasi-serial, one may use induction on the length of the modules. The following lemma can be used in this situation.

2.2. Lemma. *Let $\mathscr{C}$ be a component of the Auslander graph of some Artin algebra, such that $\mathscr{C}$ does not contain any projective or injective module. Assume there is some $t \in \mathbb{N}$ with the following property:*

(S_t) *If $X \to \bigoplus_{i=1}^{k} Y_i \to Z$ is an Auslander Reiten sequence in $\mathscr{C}$ with Y_i indecomposable for all i, and $|A^z Z| < t$ for some $z \in \mathbb{Z}$, then $k \leqq 2$, and, in case $k=2$ and $|Y_1| \leqq |Y_2|$, then $|Y_1| < |X| < |Y_2|$, $|Y_1| < |Z| < |Y_2|$.*

Let M be an indecomposable module in $\mathscr{C}$ with $|M| \leqq t$, and let

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$$

be a chain of irreducible monomorphisms with $s \geqq 2$, and M_i indecomposable for all i.

Then $M_i/M_{i-1} \approx A^{s-i}(M/M_{s-1})$ for $1 \leqq i \leqq s$, and, for $1 \leqq j \leqq i \leqq s-1$,

$$M_i/M_{j-1} \xrightarrow{(p,m)} M_i/M_j \oplus M_{i+1}/M_{j-1} \xrightarrow{\binom{m}{-p}} M_{i+1}/M_j$$

is an Auslander Reiten sequence.

Proof. We use induction on s. The first case we have to consider is $s=2$. Consider the Auslander Reiten sequence ending with M_1, say $AM_1 \to \bigoplus_{i=1}^{k} Y_i \to M_1$ with Y_i indecomposable. Since $|A^{-1}AM_1| = |M_1| < |M| \leqq t$, we can apply ($S_t$) in order to conclude that $k=1$, the case $k=2$ being impossible since otherwise we would have an irreducible monomorphism, say $Y_1 \to M_1$, contradicting the assumptions. Applying A^{-1}, we see that the Auslander Reiten sequence starting with M_1 has the form $M_1 \to A^{-1}Y_1 \to A^{-1}M_1$, in particular, the middle term is indecomposable. Since we know that the inclusion $M_1 \to M_2$ is irreducible, we can conclude that $M_1 \to M_2 \to M_2/M_1$ is the Auslander Reiten sequence.

Now assume the result is true for $s-1$. In particular, we have for $2 \leqq i \leqq s-1$ the Auslander Reiten sequences

$$M_{i-1} \xrightarrow{(p,m)} M_{i-1}/M_1 \oplus M_i \xrightarrow{\binom{m}{-p}} M_i/M_1,$$

from which it follows that all the inclusions $M_{i-1}/M_1 \to M_i/M_1$ ($2 \leqq i \leqq s-1$) are irreducible. Also, we will use that the projection $M_{s-1} \to M_{s-1}/M_1$ is irreducible. Since $|M_{s-1}| < |M| \leqq t$, we can apply (S_t) in order to conclude that the middle term of the Auslander Reiten sequence starting with M_{s-1} is the direct sum of at most two indecomposable modules. Since we know the two irreducible maps $m\colon M_{s-1} \to M_s$ and $p\colon M_{s-1} \to M_{s-1}/M_1$, we can calculate the cokernel of (p,m) and see that

$$M_{s-1} \xrightarrow{(p,m)} M_{s-1}/M_1 \oplus M_s \xrightarrow{\binom{m}{-p}} M_s/M_1$$

is an Auslander Reiten sequence. As a consequence, also the inclusion $M_{s-1}/M_1 \to M_s/M_1$ is irreducible. The induction hypothesis now applied to the chain $0 = M_1/M_1 \subset M_2/M_1 \subset \cdots \subset M_{s-1}/M_1 \subset M_s/M_1$ of irreducible monomorphisms shows that all the remaining sequences

$$M_i/M_{j-1} \xrightarrow{(p,m)} M_i/M_j \oplus M_{i+1}/M_{j-1} \xrightarrow{\binom{m}{-p}} M_{i+1}/M_j$$

with $j \geqq 2$ are Auslander Reiten sequences. Also, by induction we know that $M_i/M_{i-1} \approx A^{s-i-1}(M_{s-1}/M_{s-2})$ for $1 \leqq i \leqq s-1$. The Auslander Reiten sequence

$$M_{s-1}/M_{s-2} \to M_s/M_{s-2} \to M_s/M_{s-1}$$

shows that $M_{s-1}/M_{s-2} \approx A(M_s/M_{s-1})$. This concludes the proof.

2.3. **Theorem.** *Let R be a finite dimensional hereditary algebra. Let M be an indecomposable regular R-module. Then the component of the Auslander graph which contains M is quasi-serial.*

Proof. By 2.1, if two modules belong to the same component of the Auslander graph, and one of them is regular, both are. Thus, if $X \to Y \to Z$ is an Auslander Reiten sequence, and one of the modules is regular, then $A^z X \to A^z Y \to A^z Z$ is an Auslander Reiten sequence for any $z \in \mathbb{Z}$, and all the modules $A^z X$, $A^z Y$, $A^z Z$ are regular.

Let $X \to Y \to Z$ be an Auslander Reiten sequence with regular modules. Let $Y = \bigoplus_{i=1}^{k} Y_i$ be a decomposition of Y into indecomposable modules Y_i. Note that $Z = A^{-1}X$. We want to show that $k \leqq 2$, and that in case $k=2$ the modules Y_1 and Y_2 satisfy the appropriate length conditions. This will be shown in several steps.

(1) *If $Y = Y' \oplus Y''$, and $|X| \leqq |Y'|$, then $|X| > |Y''|$.* For, assume we have both $|X| \leqq |Y'|$ and $|X| \leqq |Y''|$. The map $X \to Y' \oplus Y''$ breaks up into two irreducible maps $X \to Y'$ and $X \to Y''$ which have to be monomorphisms. Let m be a natural number with $|A^m X| \leqq |A^{m+1} X|$. Since X is regular, there is the exact sequence

$$0 \to A^{m+1}X \to A^{m+1}Y' \oplus A^{m+1}Y'' \to A^m X \to 0.$$

Now A preserves monomorphisms, thus the maps $A^{m+1}X \to A^{m+1}Y'$ and $A^{m+1}X \to A^{m+1}Y''$ have to be monomorphisms. Therefore

$$\begin{aligned}|A^{m+1}Y'| + |A^{m+1}Y''| &= |A^{m+1}X| + |A^m X| \leqq 2 \cdot |A^{m+1}X| \\ &< |A^{m+1}Y'| + |A^{m+1}Y''|\end{aligned}$$

leads to a contradiction.

(2) *For $i \neq j$, we have $|X| < |Y_i| + |Y_j|$.* For, assume $|X| \geqq |Y_i \oplus Y_j|$, then the given irreducible map $X \to Y_i \oplus Y_j$ has to be an epimorphism. Choose a natural k such that $|A^{-k}X| \leqq |A^{-k-1}X|$. Since A^{-1} preserves epimorphisms, the corresponding maps $A^{-k}X \to A^{-k}Y_i \oplus A^{-k}Y_j$ and $A^{-k-1}X \to A^{-k-1}Y_i \oplus A^{-k-1}Y_j$ are again epimorphisms. With X also Y_i is regular, and the Auslander Reiten sequence ending with Y_i has the form

$$AY_i \to X \oplus C_i \to Y_i$$

for some module C_i. Applying A^{-k-1} we get the exact sequence

$$0 \to A^{-k}Y_i \to A^{-k-1}X \oplus A^{-k-1}C_i \to A^{-k-1}Y_i \to 0,$$

and similarly

$$0 \to A^{-k}Y_j \to A^{-k-1}X \oplus A^{-k-1}C_j \to A^{-k-1}Y_j \to 0.$$

Thus

$$2\cdot|A^{-k-1}X|\leqq|A^{-k}Y_i|+|A^{-k-1}Y_i|+|A^{-k}Y_j|+|A^{-k-1}Y_j|$$
$$<|A^{-k}X|+|A^{-k-1}X|,$$

a contradiction.

(3) *$k\leqq 3$, and, in case $k=3$, $|Y_i|<|X|$ for all i.* For, if $k\geqq 4$, then $|X|<|Y_1\oplus Y_2|$, and $|X|<|Y_3\oplus Y_4|$ by (2), contradicting (1). Similarly, if $k=3$ and $|X|\leqq|Y_i|$ for some i, say $i=1$, then $|X|<|Y_2\oplus Y_3|$ by (2), and thus again we get a contradiction to (1).

(4) *If $k=2$, and $|Y_1|\leqq|Y_2|$, then $|Y_1|<|X|<|Y_2|$, and $|Y_1|<|Z|<|Y_2|$.* For, by (1) and the fact that $X\to Y_i$ is either a proper epimorphism or a proper monomorphism, in particular $|X|\neq|Y_i|$, we see that either $|Y_1|\leqq|Y_2|<|X|$, or $|Y_1|<|X|<|Y_2|$. By duality, either $|Y_1|\leqq|Y_2|<|Z|$, or $|Y_1|<|Z|<|Y_2|$. Since $|X|+|Z|=|Y_1|+|Y_2|$, we conclude that it is impossible that at the same time $|Y_1|\leqq|Y_2|<|X|$ and $|Y_1|\leqq|Y_2|<|Z|$, and we see that $|Y_1|<|X|<|Y_2|$ iff $|Y_1|<|Z|<|Y_2|$.

(5) It remains to exclude the case $k=3$. There is an epimorphism $X\to Y_i$, for all $1\leqq i\leqq 3$, and therefore we obtain an epimorphism

$$\bigoplus_{i=1}^{3} AZ=\bigoplus_{i=1}^{3} X\to\bigoplus_{i=1}^{3} Y_i\to Z.$$

The same argument for A^tZ instead of Z yields an epimorphism $\bigoplus_{i=1}^{3} A^{t+1}Z\to A^tZ$. Using König's graph theorem, we obtain a chain of maps

$$\cdots\to A^{t+1}Z\xrightarrow{f_{t+1}}A^tZ\to\cdots\xrightarrow{f_2}AZ\xrightarrow{f_1}Z$$

such that the composition $g_t=f_t\cdots f_1$ is non-zero for all $t\in\mathbb{N}$. The images I_t of g_t form a decreasing chain of submodules of Z, thus $I_s=I_t$ for all $s,t\geqq N$, for some $N\in\mathbb{N}$. Let $I=I_N$. Then, for $t\geqq N$, the map g_t maps A^tZ onto I, thus $A^{-t}g_t$ maps Z onto $A^{-t}I$, thus $\dim A^{-t}I\leqq\dim Z$. Since there is only a finite number of different positive dimension vectors $\leqq\dim Z$, we conclude that $\dim A^{-s}I=\dim A^{-t}I$ for some $s\neq t$, say $s<t$. Since I is an epimorphic image of A^NZ, it has no preprojective direct summand, and since I is a submodule of Z, it has no preinjective direct summand, thus I is regular, and therefore $\dim A^iI=(\dim I)c^i$ for all $i\in\mathbb{Z}$. Let

$$q=\sum_{i=s+1}^{t}\dim A^{-i}I=\sum_{i=s+1}^{t}(\dim I)c^{-i}.$$

Then q is a positive vector in $\mathbb{R}^n$ fixed by the Coxeter transformation c. However, this is only possible in case the quadratic form belonging to R is semidefinite, by the criterion of Berman, Moody and Wonenberger. But then R is tame, and the category of regular modules is abelian and serial, and therefore the Auslander Reiten sequences of regular modules are known: they can be constructed in this subcategory, and the middle term has at most two indecomposable summands. This shows that the case $k=3$ cannot occur.

2.4. Again, let R be hereditary. We want to add the precise conditions for the existence of an R-module M with $A^k M \approx M$ for some $k \in \mathbb{N}$.

Theorem. *Let R be a twosided indecomposable finite dimensional hereditary algebra, and M a non-zero R-module. Then $A^k M \approx M$ for some $k \in \mathbb{N}$ if and only if R is tame and M is a direct sum of regular modules.*

Proof. Assume we have $A^k M \approx M$ for some $k \in \mathbb{N}$. Then $\sum_{i=0}^{k-1} \dim A^i M = \sum_{i=0}^{k-1} (\dim M) c^i$ is a non-zero positive fix vector for the Coxeter transformation c, and therefore, by the criterion of Berman, Moody and Wonenberger, R has to be tame. Assume now that R is tame. We know from [7, 9, 8] that for any regular module M, there exists $k \in \mathbb{N}$ with $A^k M \approx M$. If M is arbitrary, say $M = \bigoplus_{i=1}^{d} M_i$ with M_i indecomposable for all i, and one of the summands, say M_d, is pre-projective or pre-injective, then $A^z M_d = 0$ for some $z \in \mathbb{Z}$, and therefore $A^z M$ is the direct some of less than d indecomposable summands. This shows that we cannot have $A^k M \approx M$ for any $k \in \mathbb{N}$.

2.5. We end this section by listing some other Artin algebras which have quasi-serial components.

First, let R be an artin algebra with $(\operatorname{rad} R)^2 = 0$. It is well-known how to analyse the R-modules: one constructs the hereditary Artin algebra

$$R' = \begin{pmatrix} R/\operatorname{rad} R & 0 \\ \operatorname{rad} R & R/\operatorname{rad} R \end{pmatrix}$$

and a functor α from R-modules to R'-modules with $\alpha(M) = M/\operatorname{rad} M \oplus \operatorname{rad} M$. Then α preserves the length of modules, and, if $X \to Y \to Z$ is an Auslander Reiten sequence of R-modules with X not simple, then $\alpha(X) \to \alpha(Y) \to \alpha(Z)$ is an Auslander Reiten sequence of R'-modules [4]. We call an indecomposable R-module M regular provided the component of the Auslander graph which contains M does not contain a projective or an injective module. (In case R is hereditary, this definition coincides with the previous one according to 2.1.) It follows easily that the image of a regular R-module under α is regular, and that the components of the Auslander graph of R which contain regular modules are quasi-serial.

Let G be a dihedral 2-group, thus G is generated by two elements g_1, g_2 such that $g_1^2 = g_2^2 = 1$, and $(g_1 g_2)^q = 1$, for some power q of 2. Let F be a field of characteristic 2. The indecomposable FG-modules have been determined in [10], and M.C.R. Butler has calculated the Auslander Reiten sequences. In particular, he has found that all components which contain modules of the second kind are quasi-serial, and that there are, in addition two other components which are quasi-serial, namely the components which contain the modules $FG/FG(g_1 - 1)$ and $FG/FG(g_2 - 1)$.

3. Quasi-Serial Components

Let R be an Artin algebra. We will consider quasi-serial components of the Auslander graph of R. For brevity, we will call an indecomposable module *quasi-serial* if it belongs to a quasi-serial component. Note however – that in contrast to the property of being serial — the property of being quasi-serial depends strongly on the whole category of R-modules. In particular, if the module M is annihilated by some ideal I of R, then M may be quasi-serial as an R-module without being quasi-serial as an R/I-module. The notion of a quasi-serial module is self-dual. Thus, with any assertion which is valid for quasi-serial modules, also its dual assertion is valid.

3.1. Let M be quasi-serial. We call M *quasi-simple* in case there does not exist an irreducible monomorphism $U \to M$ with U non-zero. Now, if M is quasi-simple, and $AM \to Y \to M$ is an Auslander Reiten sequence, then Y has to be indecomposable. Applying A^{-1}, we see that the middle term $A^{-1}Y$ of the Auslander Reiten sequence $M \to A^{-1}Y \to A^{-1}M$ also is indecomposable, thus there does not exist an irreducible epimorphism $M \to V$ with V non-zero (and conversely). If M is quasi-simple, we will call the two maps $0 \to M$ and $M \to 0$ *irreducible*.

3.2. *Let M be quasi-serial.*

(a) *If $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ is a chain of irreducible monomorphisms, then the modules M_i/M_j with $0 \leqq j < i \leqq s$ are indecomposable; the modules M_i/M_{i-1} with $1 \leqq i \leqq s$ are quasi-simple and $M_i/M_{i-1} \approx A^{s-i}(M/M_{s-1})$, and, for $1 \leqq j \leqq i \leqq s-1$, the sequences*

$$M_i/M_{j-1} \xrightarrow{(p,m)} M_i/M_j \oplus M_{i+1}/M_{j-1} \xrightarrow{\binom{m}{-p}} M_{i+1}/M_j$$

are Auslander Reiten sequences.

(b) *If $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ is a chain of conclusions, then this is a chain of irreducible monomorphisms if and only if*

$$M = M/M_0 \to M/M_1 \to \cdots \to M/M_{s-1} \to M/M_s = 0$$

is a chain of irreducible epimorphisms.

(c) If $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ *and* $0 = M'_0 \subset M'_1 \subset \cdots \subset M'_{t-1} \subset M'_t = M$ *are two chains of irreducible monomorphisms, then $s = t$, and, for all $0 \leqq j < i \leqq s$, we have $M_i/M_j \approx M'_i/M'_j$.*

Proof. First, assume $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ is a chain of irreducible monomorphisms. Consider an Auslander Reiten sequence $AM \to Y \to M$. Since $M_{s-1} \hookrightarrow M_s$ is irreducible, M_{s-1} is a direct summand of Y. However, any direct decomposition of Y has at most one indecomposable summand Y_1 with $|Y_1| < |M|$. This shows that M_{s-1} (if $s \geqq 2$) is indecomposable. By induction, all M_i with $1 \leqq i \leqq s$ are indecomposable. Since the condition (S_t) is satisfied for all $t \in \mathbb{N}$, we can apply Lemma 2.2, and obtain that $M_i/M_{i-1} \approx A^{s-i}(M/M_{s-1})$ and the form of the Auslander Reiten sequences as stated in (a). From the form of the Auslander Reiten sequences, it follows that the modules M_i/M_{i-1} are quasisimple and that the canonical epimorphisms $M/M_{i-1} \to M/M_i$ are irreducible. This proves one direction of (b), the other follows by duality. The duality arguement also shows that the modules M/M_j, with $0 \leqq j < s$, are indecomposable. Consequently, all modules M_i/M_j, with $0 \leqq j < i \leqq s$, are indecomposable.

It remains to prove (c). We will show by induction on s that $s = t$, and $M_i \approx M'_i$ for all i. If $s = 1$, then M does not contain a non-zero submodule with irreducible inclusion, thus also $t=1$. Now assume $s>1$, $t>1$. The inclusions $m: M_{s-1} \to M$ and $m': M'_{t-1} \to M$ are irreducible, thus there are homomorphisms $f: X \to M$, $f': X' \to M$ such that

$$\operatorname{Ker}\binom{m}{f} \to M_{s-1} \oplus X \xrightarrow{(m\ f)} M, \qquad \operatorname{Ker}\binom{m'}{f'} \to M'_{t-1} \oplus X' \xrightarrow{(m'\ f')} M$$

both are Auslander Reiten sequences. By the uniqueness, we see that $M_{s-1} \oplus X \approx M'_{t-1} \oplus X'$. Now this module is the direct sum of at most indecomposable modules $Y_1 \oplus Y_2$ with $|Y_1| < |M| < Y_2|$. Since $|M_{s-1}| < |M|$, and $|M'_{t-1}| < |M|$, we conclude that $M_{s-1} \approx Y_1 \approx M'_{t-1}$. By induction, $s-1=t-1$, and $M_i \approx M'_i$ for $1 \leqq i \leqq s-1$. Using again duality, we also conclude that $M/M_i \approx M/M'_i$ for all i, and this combines to the assertion of (c).

3.3. As a consequence, we see that the following concepts are of importance: Let M be a quasi-serial module with a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ of irreducible monomorphisms. We will call s the *quasi-length* of M, and denote it by $l(M)$. By the previous result, this is an invariant of the module M, and it also coincides with the length of a chain of irreducible epimorphisms starting with M and ending with 0. The quasi-simple modules are just those quasi-regular modules which have quasi-length 1. The quotients M_i/M_{i-1} in the chain above are quasi-simple, and independent of the choice of the chain, we call them the *quasi composition factors* of M. Since $M_i/M_{i-1} \approx A^{s-i}(M/M_{s-1})$, any one of them determines uniquely the other ones. In particular, we will concentrate our attention to the top factor M/M_{s-1}, which we will denote by $t(M)$. Note that for an irreducible epimorphsim $M \to M'$ with $M' \neq 0$, we have $t(M) = t(M')$, whereas for an irreducible monomorphism $M' \to M$ with $M' \neq 0$, we have $t(M') = A\,t(M)$.

Let $\mathscr{R}$ be the set of isomorphism classes of quasi-serial R-modules, and $\mathscr{S}$ the set of isomorphism classes of quasi-simple R-modules. By the previous remarks, there is a well-defined map

$$\mathscr{R} \to \mathscr{S} \times \mathbb{N}, \qquad \text{defined by} \quad M \to (t(M), l(M)).$$

Our aim is to show that this is a bijection. First, we will prove the surjectivity.

3.4. *Let S be a quasi-simple module, and $s \in \mathbb{N}$. Then there exists a quasi-serial module M with $t(M) = S$ and $l(M) = s$.*

Proof. Induction on s. For $s=1$, nothing has to be shown. In case $s=2$, consider an Auslander Reiten sequence $AS \to M \to S$. Since S is quasi-simple, M is indecomposable. Since $0 \to AS \to M$ is a chain of irreducible monomorphism, $l(M) = 2$, and $t(M) = S$.

Now, let $s > 2$. With S also $A^{-1}S$ is quasi-simple, since the Auslander Reiten sequence starting with S and ending with $A^{-1}S$ has indecomposable middle term. By induction, there is a quasi-serial module X with $t(X) = A^{-1}S$ and $l(X) = s-1$. Let $X \to Y \to A^{-1}X$ be an Auslander Reiten sequence starting with X. Since X is not quasi-simple, Y decomposes as the direct sum of two indecomposable modules, $Y = Y_1 \oplus Y_2$ with $|Y_1| < |X| < |Y_2|$. Let $M = Y_2$. Since there is an irreducible monomorphism $X \to Y_2$, we know that $l(M) = l(Y_2) = l(X) + 1 = s$, and $t(M) = t(Y_2) = A\,t(X) = AA^{-1}S = S$.

The injectivity of the map $\mathscr{R} \to \mathscr{S} \times \mathbb{N}$ will be considered in the next section; it will follow from a general extension result for homomorphisms. We end this section by pointing out the effect of the construction A.

3.5. *Let M be quasi-serial. Then $t(AM)=A\,t(M)$ and $l(AM)=l(M)$.*

Proof, by induction on $l(M)$. If $l(M)=1$, then M is quasi-simple, and the Auslander Reiten sequence ending with M is of the form $AM \to X \to M$ for some indecomposable module M. Therefore also AM is quasi-simple, and $t(AM)=AM=A\,t(M)$.

Now, let $l(M)\geqq 2$. Let $M \to M'$ be an irreducible epimorphism. Let $AM' \to M \oplus Y \to M'$ be an Auslander Reiten sequence, with Y indecomposable or zero. Applying A, we get an Auslander Reiten sequence $A^2M' \to AM \oplus AY \to AM'$. If $l(M)=2$, then M' is quasi-simple, and therefore $Y=0$, thus $AY=0$, thus $AM \to AM'$ is an irreducible epimorphism. If $l(M)>2$, then $|M|>|M'|$ shows that $|Y|<|M'|$, thus there is an irreducible monomorphism $Y \to M'$, and therefore $l(Y)=s-2$. By induction, $l(AY)=l(Y)=s-2$, $l(AM')=l(M')=s-1$, thus the given irreducible homomorphism $AY \to AM'$ cannot be an epimorphism, and therefore has to be a monomorphism. This shows $|AY|<|AM'|$, and therefore $|AM'|<|AM|$. Consequently, the given irreducible homomorphism $AM \to AM'$ has to be an epimorphism. Thus, in both cases we have shown the existence of an irreducible epimorphism $AM \to AM'$. As a consequence, $l(AM)=l(AM')+1=l(M')+1=l(M)$, and $t(AM)=t(AM')=A\,t(M')=A\,t(M)$.

4. Extensions of Homomorphisms

We consider the following problem: Let M be a quasi-serial R-module, let $M' \subset M$ be a submodule with irreducible inclusion. Under what conditions is it possible to extend a given R-homomorphism $f'\colon M' \to X$ to M? Thus, we are looking for an R-homomorphism $f\colon M \to X$ with $f|M'=f'$. It will be seen that this is possible under rather weak conditions on X, and we will later use this in order to extend endomorphisms of submodules of M to endomorphisms of M.

4.1. *Let M be a quasi-serial R-module with a chain $0=M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ of irreducible monomorphisms. Let X be an indecomposable R-module with $X \not\approx M_{s-1}/M_i$ for $0 \leqq i \leqq s-2$. Then, any homomorphism $M_{s-1} \to X$ can be extended to a homomorphism $M \to X$.*

Proof, by induction on s. If $s=1$, then nothing has to be shown, since $M_{s-1}=0$. Now, let $s>1$. By 3.2, there is the Auslander Reiten sequence

$$M_{s-1} \xrightarrow{(p,m)} M_{s-1}/M_1 \oplus M \xrightarrow{\binom{m''}{-p''}} M/M_1,$$

where m, m'' are the canonical inclusions, and p, p'' are the canonical projections. Let $f\colon M_{s-1} \to X$ be a given homomorphism. Since $X \not\approx M_{s-1}$, f cannot be a split monomorphism, thus f can be extended to $M_{s-1}/M_1 \oplus M$ along (p,m), thus there are homomorphisms $g\colon M_{s-1}/M_1 \to X$ and $h\colon M \to X$ with

$$f = pg + mh.$$

We will use the fact that $f-mh=pg$ factors over M_{s-1}/M_1. By 3.2,

$$0=M_1/M_1\subset M_2/M_1\subset\cdots\subset M_{s-1}/M_1\subset M_s/M_1$$

is a chain of irreducible monomorphisms, and, for

$$1\leqq i\leqq s-2,\ (M_{s-1}/M_1)/(M_i/M_1)\approx M_{s-1}/M_i\approx X,$$

thus, by induction, the homomorphism $g\colon M_{s-1}/M_1\to X$ can be extended to a homomorphism $g'\colon M_s/M_1\to X$, that is $g=m'g'$, where $m'\colon M_{s-1}/M_1\to M_s/M_1$ denotes the canonical inclusion. Let $p'\colon M_s\to M_s/M_1$ denote the canonical projection, thus $mp'=pm'$. Therefore

$$pg=pm'g'=mp'g'.$$

This shows that

$$f=pg+mh=mp'g'+mh=m(p'g'+h)$$

factors through m, and that $p'g'+h$ is the extension of f to M we were looking for.

4.2. **Corollary.** *Let M, N be quasi-serial with $l(M)\leqq l(N)$. Let $M'\subset M$ be an irreducible monomorphism. Then any homomorphism $f'\colon M'\to N$ can be extended to a homomorphism $f\colon M\to N$.*

Proof. Let $0=M_0\subset M_1\subset\cdots\subset M_{s-1}\subset M_s=M$ be a chain of irreducible monomorphisms such that $M'=M_{s-1}$. Then, for $0\leqq i\leqq s-2$,

$$l(M_{s-1}/M_i)=s-1-i<s=l(M)\leqq l(N),$$

thus M_{s-1}/M_i and N cannot be isomorphic.

4.2*. The dual statement: *Let M, N be quasi-serial with $l(M)\geqq l(N)$. Let $N\to N'$ be an irreducible epimorphism. Then any homomorphism $f'\colon M\to N'$ can be lifted to a homomorphism $f\colon M\to N$.*

4.3. *Let M, N be quasi-serial, let*

$$0=M_0\subset M_1\subset\cdots\subset M_{s-1}\subset M_s=M,$$
$$0=N_0\subset N_1\subset\cdots\subset N_{s-1}\subset N_s=N$$

be chains of irreducible monomorphisms. Let $0\leqq j<i\leqq s$. Then, given any homomorphism $f_i^j\colon M_i/M_j\to N_i/N_j$, there exists $f\colon M\to N$ with $M_jf\subseteq N_j$, $M_if\subseteq N_i$ such that f induces f_i^j. And, in this situation, f_i^j is an isomorphism if and only if f is an isomorphism.

Proof. Given $f_i^j\colon M_i/M_j\to N_i/N_j$, we may consider it as a homomorphism into N/N_j, and by 4.2, we may extend it to a homomorphism $f^j\colon M/M_j\to N/N_j$. Considering this as a homomorphism from M to N/N_j, we may use 4.2* in order to find a lifting $f\colon M\to N$. Thus, we have a commutative diagram

$$\begin{array}{ccccc} M_i/M_j & \xrightarrow{m} & M/M_j & \xleftarrow{p} & M \\ \downarrow{\scriptstyle f_i^j} & & \downarrow{\scriptstyle f^j} & & \downarrow{\scriptstyle f} \\ N_i/N_j & \xrightarrow{m} & N/N_j & \xleftarrow{p} & N \end{array}$$

with canonical inclusions m, and canonical projections p. If f is an isomorphism, then clearly also f_i^j has to be an isomorphism. On the other hand, if f_i^j is an isomorphism, say with inverse g_i^j, then we get similarly $g: N \to M$ with $N_i g \subseteq M_i$ and $N_j g \subseteq M_j$ such that g induces g_i^j. The endomorphisms fg of M and gf of N cannot be nilpotent, since otherwise also the endomorphisms $f_i^j g_i^j$ of M_i/M_j and $g_i^j f_i^j$ of N_i/N_j would be nilpotent. Thus fg and gf are automorphisms, and therefore f and g are isomorphisms.

4.4. *Let M, N be quasi-serial with $t(M) \approx t(N)$ and $l(M) = l(N)$. Then $M \approx N$.*

Proof. Since $l(M) = l(N)$, we can apply the previous lemma. Let $M' \subset M$ and $N' \subset N$ be irreducible monomorphisms. By assumption, there is an isomorphism f': $M/M' \to N/N'$, and the previous lemma tells us that f' is induced by an isomorphism f: $M \to N$.

Combining this assertion with 3.6 we get the following reduction theorem for quasi-serial modules:

4.5. **Theorem.** *Let R be an Artin algebra. Let $\mathscr{R}$ be the set of isomorphism classes of quasi-serial R-modules, $\mathscr{S}$ the set of isomorphism classes of quasi-simple R-modules. Then there is a bijection $\mathscr{R} \to \mathscr{S} \times \mathbb{N}$ which is given as follows: a quasi-serial module M with a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ of irreducible monomorphisms, is mapped onto the pair $(M/M_{s-1}, s)$.*

We note another consequence of 4.1.

4.6. *Let M be a quasi-serial module with a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ of irreducible monomorphisms. Let S be a quasi-simple module with $S \not\approx M_i/M_{i-1}$ for $1 \leqq i \leqq s-1$. Then any homomorphism $M_1 \to S$ can be extended to a homomorphism $M \to S$.*

Proof. By assumption, $S \not\approx M_{s-1}/M_{s-2}$, and also $S \not\approx M_{s-1}/M_i$ for $0 \leqq i \leqq s-3$, since S is quasi-simple. Thus, we know from 4.1 that any homomorphism $M_{s-1} \to S$ can be extended to a homomorphism $M \to S$. The result now follows by induction on s.

4.6*. *Let M be a quasi-serial module with a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ of irreducible monomorphisms. Let S be a quasi-simple module with $S \not\approx M_i/M_{i-1}$ for $2 \leqq i \leqq s$. Then any homomorphism $S \to M/M_{s-1}$ can be lifted to a homomorphism $S \to M$.*

5. The Endomorphism Ring of a Quasi-Serial Module

Our first aim is to show that every endomorphism of a quasi-serial module M is the sum of a nilpotent endomorphism and an endomorphism which preserves a given chain of irreducible monomorphisms.

5.1. *Let M be a quasi-serial module, and let* $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ *be a chain of irreducible monomorphisms. Let E be the set of endomorphisms f of M such that $M_i f \subseteq M_i$ for all i. Then* $\mathrm{End}(M) = E + \mathrm{rad}\,\mathrm{End}(M)$.

Proof, by induction on s. For $s=1$, nothing has to be shown. Let $s>1$. Define $E' = \{f \in \mathrm{End}(M_{s-1}) \mid M_i f \subseteq M_i\}$ for $1 \leqq i \leqq s-1$. By 4.3, the restriction map $\rho: E \to E'$, defined by $\rho(f) = f \mid M_{s-1}$, is surjective. Denote by $\mu: E \to \mathrm{End}(M)$, $\mu': E' \to \mathrm{End}(M_{s-1})$ the inclusions, by $\pi: \mathrm{End}(M) \to F(M) = \mathrm{End}(M)/\mathrm{rad}\,\mathrm{End}(M)$, and $\pi': \mathrm{End}(M_{s-1}) \to F(M_{s-1})$ the projections. Consider the $\mathrm{End}(M_{s-1}) - \mathrm{End}(M)$-bimodule $N(M_{s-1}, M) = \mathrm{rad}(M_{s-1}, M)/\mathrm{rad}^2(M_{s-1}, M)$, which is, in fact, an $F(M_{s-1}) - F(M)$-bimodule. Since we are working in a quasi-serial component, we know that both the left vector space ${}_{F(M_{s-1})}N(M_{s-1}, M)$ and the right vector space $N(M_{s-1}, M)_{F(M)}$ are one-dimensional. If we denote the canonical inclusion $M_{s-1} \subset M$ by m, then the residue class $\bar{m}$ of m in $N(M_{s-1}, M)$ is non-zero, and therefore we can define an isomorphism $\gamma: F(M) \to F(M_{s-1})$ by $\bar{m} g = \gamma(g) \bar{m}$ for $g \in F(M)$. We claim that the diagram

$$\begin{array}{ccccc} E & \xrightarrow{\mu} & \mathrm{End}(M) & \xrightarrow{\pi} & F(M) \\ \downarrow{\scriptstyle \rho} & & & & \downarrow{\scriptstyle \gamma} \\ E' & \xrightarrow{\mu'} & \mathrm{End}(M_{s-1}) & \xrightarrow{\pi'} & F(M_{s-1}) \end{array}$$

commutes. For, let $f \in E$, then $mf = \rho(f) m$ by definition of ρ, thus

$$\pi' \mu' \rho(f) \cdot \bar{m} = \rho(f) \cdot \bar{m} = \bar{m} \cdot f = \bar{m} \cdot \pi \mu(f) = \gamma \pi \mu(f) \cdot \bar{m},$$

and therefore $\pi' \mu' \rho = \gamma \pi \mu$.

Now, by induction, $\pi' \mu'$ is surjective, thus also $\pi' \mu' \rho$ is surjective, and therefore $\pi \mu$ is surjective. This proves the lemma.

5.2. Let M be quasi-serial with a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ of irreducible monomorphisms. We recall from 4.3 that any endomorphism of some quasi-composition factor M_i/M_{i-1} is induced by some endomorphism of M.

We consider now the corresponding question of extending homomorphisms between different quasi-composition factors of a quasi-serial module M to endomorphisms of M.

5.3. **Theorem.** *Let M be a quasi-serial module, let* $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ *be a chain of irreducible monomorphisms. Assume that the quasi composition factors M_i/M_{i-1} are pairwise non-isomorphic. Let $1 \leqq i, j \leqq s$, and $i \neq j$. Then, given any homomorphism $f_{ij}: M_i/M_{i-1} \to M_j/M_{j-1}$, there exists a nilpotent endomorphism f of M with $M_i f \subseteq M_j$, $M_{i-1} f \subseteq M_{j-1}$, such that f induces f_{ij}.*

Proof. First, we consider the case $j < i$. Using 4.3, we see that the homomorphism

$$M_i/M_{j-1} \xrightarrow{p} M_i/M_{i-1} \xrightarrow{f_{ij}} M_j/M_{j-1} \xrightarrow{m} M_i/M_{j-1},$$

with p the canonical projection, m the canonical inclusion, is induced by some endomorphism $f: M \to M$ satisfying $M_i f \subseteq M_i$, $M_{j-1} f \subseteq M_{j-1}$. Since the non-zero module M_{i-1}/M_{j-1} lies in the kernel of $pf_{ij}m$, the endomorphism f cannot be an automorphism, and therefore is nilpotent.

Next, consider the case $i<j$. With a similar argument as above, we may assume $i=1, j=s$. For, if we can find to f_{ij}: $M_i/M_{i-1} \to M_j/M_{j-1}$ a nilpotent endomorphism f': $M_j/M_{i-1} \to M_j/M_{i-1}$ which induces f_{ij}, then we can use 4.3 in order to see that f' is induced by an endomorphism f of M. Thus, assume there is given a homomorphism f_{1s}: $M_1 \to M/M_{s-1}$, with $1 \leqq s-1$. By 4.6*, we find a lifting $f_1: M_1 \to M$, such that $f_1 p = f_{1s}$, where $p: M \to M/M_{s-1}$ is the canonical projection. Now, by 4.2, we can extend the homomorphism $f_1: M_1 \to M$ to a homomorphism f: $M \to M$. We do not know whether the so constructed f is nilpotent. However, using 5.1, we can write $f = g + h$ where $M_i g \subseteq M_i$ for all i, and h is nilpotent. Since $M_1 g \subseteq M_1 \subseteq M_{s-1}$, we see that the induced homomorphism g_{1s}: $M_1 \to M/M_{s-1}$ is zero. Thus $h_{1s} = f_{1s} - g_{1s} = f_{1s}$.

Remark. The assumption that the quasi composition factors of M are pairwise non-isomorphic, cannot be deleted. This is shown by the example of a tame hereditary algebra. There are quasi-simple modules S with $AS \approx S$. Taking a quasi-serial module M of quasi-length >1, and with $t(M) = S$, all quasi-composition factors of M are isomorphic to S. However, M has only one chain of submodules with irreducible inclusions, so this chain has to be preserved by the endomorphisms of M.

As a consequence of the theorem above, we get in this case the precise conditions for the uniqueness of a quasi composition chain:

5.4. **Corollary.** *Let M be quasi-serial, and assume its quasi-composition factors are pairwise non-isomorphic. Let $S = t(M)$. Then the following conditions are equivalent:*

(i) *If* $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ *and*

$$0 = M'_0 \subset M'_1 \subset \cdots \subset M'_{s-1} \subset M'_s = M$$

are two chains of submodules of M with irreducible inclusions, then $M_i = M'_i$ for all i.

(ii) $\mathrm{Hom}(A^k S, S) = 0$ *for all* $1 \leqq k \leqq l(M) - 1$.

Proof. We can identify $\mathrm{Hom}(A^k S, S)$ and $\mathrm{Hom}(M_{s-k}/M_{s-k-1}, M/M_{s-1})$. Thus, assume there is $0 \neq g$: $M_{s-k}/M_{s-k-1} \to M/M_{s-1}$. By the theorem, g is induced by some endomorphism f of M, and therefore $M_{s-k} f \nsubseteq M_{s-1}$, thus $M_{s-k} f \nsubseteq M_{s-k}$, for $k \geqq 1$. If f is an automorphism, let $M'_i = M_i f$, otherwise let $M'_i = M_i(1+f)$. Then we have the different chains M_i and M'_i.

Conversely, suppose there are two different chains $\{M_i\}$ and $\{M'_i\}$. We show that $M_i \subseteq M'_i$ for all i. Suppose there is some i with $M_i \nsubseteq M'_i$. Choose i minimal. Choose j minimal with $M_i \subseteq M_j$. Thus, $M_i \nsubseteq M_{j-1}$, and therefore the composition of the canonical maps

$$M_i \subset M \to M/M_{j-1}$$

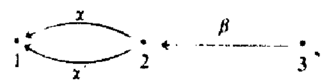
is non-zero. It maps M_i into M_{j-1}, and M_{i-1} into 0, thus it induces a non-zero map

$$A^{s-i} S \approx M_i/M_{i-1} \to M_j/M_{j-1} \approx A^{s-j} S.$$

If we apply A^{-s+j}, we get a non-zero map $A^{j-i} S \to S$, and $1 \leqq j - i \leqq s - 1$.

6. Examples

Let F be a fixed (commutative) field. We consider the algebra R given by the quiver



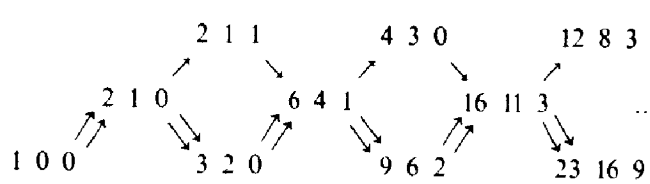
over F, thus R is the matrix ring

$$R=\left\{\begin{pmatrix} x_1 & & & 0 \\ 0 & x_1 & & \\ y_1 & y_2 & x_2 & \\ y_3 & y_4 & y_5 & x_3 \end{pmatrix} \middle| \, x_i, y_i \in F \right\}.$$

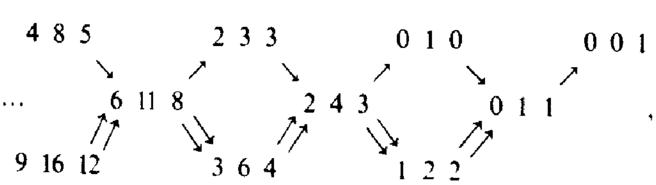
An R-module is given by $V=(V_1, V_2, V_3, \alpha, \alpha': V_2 \to V_1, \beta: V_3 \to V_2)$ where V_1, V_2, V_3 are F-vector spaces, and α, α', β are linear transformations. Note that $\dim V = (v_1, v_2, v_3)$ with v_i the dimension of the vector space V_i. The quadratic form associated to R is

$$X_1^2+X_2^2+X_3^2-2X_1X_2-X_2X_3.$$

6.1. We determined in this paper the structure of the Auslander graph of all components which contain regular modules. There are two remaining components, one consists of the pre-projective modules, the other of the pre-injective ones. The pre-projective component has the form



the preinjective component has the form

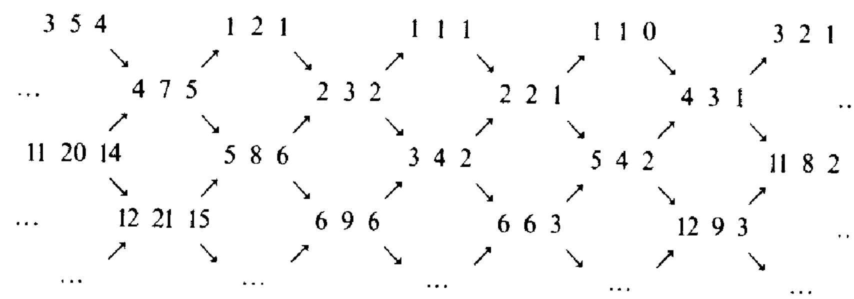


Here, the double arrow $X \rightrightarrows Y$ indicates that the space

$$N(X, Y)=\operatorname{Hom}(X, Y)/\operatorname{rad}^2(X, Y)$$

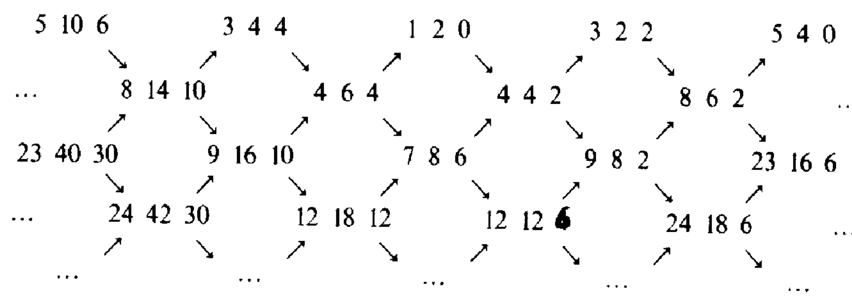
of irreducible maps is twodimensional over k. Note that the pre-projective and the pre-injective modules are uniquely determined by their dimension vector.

We also want to list some of the dimension vectors in four types of components which contain regular modules: There is a one-parameter family of indecomposable modules of dimension type (1 1 0), indexed by the projective line $\mathbb{P}_1(F)$ over F, namely $S_\lambda=(F, F, 0, \lambda_1: F \to F, \lambda_2: F \to F, 0)$ where $\lambda=(\lambda_1:\lambda_2) \in \mathbb{P}_1(F)$, and were for $\lambda_1 \in F$ we also denote by λ_i the multiplication by λ_i. The dimension vectors in the component containing S_λ are as follows:



To repeat: for any element $\lambda \in \mathbb{P}_1(F)$ we get a component of this type.

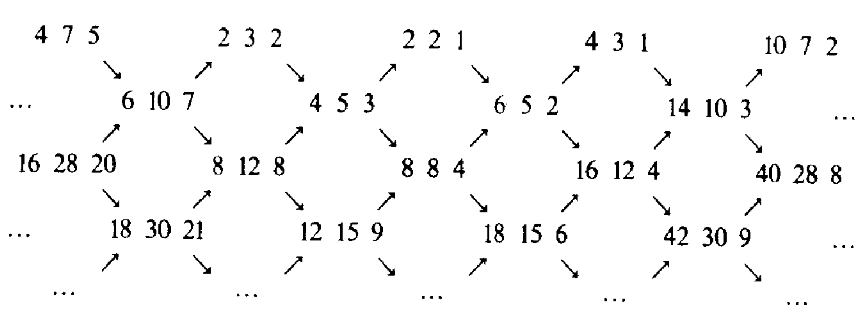
There is only one indecomposable module of dimension type (1 2 0), thus there is only one component of the following form (note that the indecomposable module with dimension vector (1 2 0) is injective when considered as a module over $\cdot_1 \leftleftarrows \cdot_2$, but regular, when considered as module over R).



Next, we consider the components containing quasi-simple modules of type (2 3 2). Note that there are also quasi-serial modules of type (2 3 2) with quasi length 2. In fact, if M is indecomposable, and $\dim M = (2\ 3\ 2)$, then $l(M) = 2$ if and only if the restriction $\tilde{M}$ of M to the subquiver $\cdot_1 \leftleftarrows \cdot_2$ decomposes (it decomposes then into the sum of the injective module (1 2) and a module of type (1 1)). The modules with indecomposable restriction are of the form

$$F^2 \underset{\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}}{\overset{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}}{\leftleftarrows}} F^3 \xleftarrow{\ \beta\ } F^2$$

where β is a monomorphism. Thus, the set of isomorphism classes of quasi-simple modules of type (2 3 2) can be indexed by the set of two-dimensional subspaces of F^3, that is, by the projective plane $\mathbb{P}_2(F)$. A component containing a quasi-simple module of type (2 3 2) has the form:



Finally, there is only one indecomposable module of type (2 3 0), and it is quasi-simple. The component containing it has the following form:

$$
\begin{array}{ccccccccccccccccc}
6\;12\;7 & & & & 4\;5\;5 & & & & 2\;3\;0 & & & & 6\;4\;3 & & & & 12\;9\;1 \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\cdots & & 10\;17\;12 & & & & 6\;8\;5 & & & & 8\;7\;3 & & & & 18\;13\;4 & & \cdots \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
28\;48\;36 & & & & 12\;20\;12 & & & & 12\;12\;8 & & & & 20\;16\;4 & & & & 52\;36\;12 \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\cdots & & 30\;51\;36 & & & & 18\;24\;15 & & & & 24\;21\;9 & & & & 54\;39\;12 & & \cdots \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\cdots & & & & \cdots & & & & \cdots & & & & \cdots & & & & \cdots
\end{array}
$$

It is easy to check that the only regular modules of length $\leqq 5$ are of dimension type (1 1 0), (1 1 1), (1 2 0), (1 2 1), (2 2 0), (2 2 1) or (2 3 0), and all but (2 2 0) belong to components described above.

As a consequence, *any indecomposable module M of dimension type* (3 3 1) *or* (4 5 2) *is quasi-simple.* For, (3 3 1) and (4 5 2) are not dimension types of pre-projective or pre-injective modules, thus M has to be regular. If we assume that M is not quasi-simple, then one of its quasi composition factors would have length $\leqq 5$, thus either M belongs to one of the components described above, or the components of $\dim M$ would be divisible by 2, both being impossible.

6.2. We are going to construct *a module M with two different submodules M' and M'' such that $0 \to M' \to M$ and $0 \to M'' \to M$ are chains of irreducible monomorphisms.*

Let $S = S_\lambda$ for some $\lambda \in \mathbb{P}_1(F)$, say $\lambda = (\lambda_1 : \lambda_2) = (1:0)$, be one of the indecomposable modules of type (1 1 0). Then $AS = (F, F, F; \lambda_1 : F \to F, \lambda_2 : F \to F, 1 : F \to F)$. For, $\dim AS = (1\ 1\ 1)$, and the described module is the only indecomposable module T with $\dim T = (1\ 1\ 1)$ and $\mathrm{Ext}^1(S, T) \neq 0$.

Let V be the following module

$$
F^3 \underset{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\longleftleftarrows}} F^3 \xleftarrow{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} F.
$$

Then V is indecomposable, of dimension type (3 3 1), and it has unique submodules $V_1 \subset V_2$ such that $V_1 \approx AS$, $V_2/V_1 \approx S$, and $V/V_2 \approx S$. If we apply A, we get the module AV with submodules $AV_1 \subset AV_2$ such that $AV_1 \approx A^2S$, $AV_2/AV_1 \approx AS$, and $AV/AV_2 \approx AS$. In particular, there is a non-zero homomorphism $AV \to V$ with kernel AV_2 and image V_1.

Let M be the quasi-serial module with $t(M) = V$ and $l(M) = 2$. Thus, $\dim M = (6\ 7\ 4)$. According to 5.4, M has two different submodules M' and M'' with irreducible inclusions. This can be seen, in this example, directly. By construction, M contains a submodule M' with irreducible inclusion, and $M' \approx AV$, $M/M' \approx V$. As a consequence, there are submodules X, Y of M with $X \subset M' \subset Y$ such that $M'/X \approx AS \approx Y/M'$. The fact that $M' \to M \to M/M'$ is an Auslander Reiten sequence, implies that $Y = M' \oplus Y'$ for some submodule Y'. Let $f' : M' \to Y'$ be a fixed epimorphism with kernel X, and let $M'' \subset M' \oplus Y' = Y$ be the graph of f'. We claim that the inclusion $M'' \subset M$ is an irreducible monomorphism. For, using again that $M' \to M \to M/M'$ is an Auslander Reiten sequence, we see that M'/X has a complement, say X'/X, in M/X. Let f be the composition of f' and canonical maps

$$M \to M/X' \approx M'/X \xrightarrow{f'} Y' \subset M.$$

Then $f|M' = f'$, and therefore M'' is the image of M' under the automorphism $1+f$.

6.3. Next, we want to construct *a module M with two different chains $0 \subset M_1 \subset M_2 \subset M$, $0 \subset M'_1 \subset M'_2 \subset M$ of submodules with irreducible inclusions such that $M_1 \nsubseteq M'_2$.*

Again, let $S = S_{(1:0)}$, and let V be the regular module of dimension type (3 3 1) having submodules $V_1 \subset V_2$ such that $V_1 \approx AS$, $V_2/V_1 \approx V/V_2 \approx S$ which was constructed in 6.2. We claim that $\mathrm{Hom}(A^2S, V) = 0$. In order to see this, it is enough to show that $\mathrm{Hom}(A^2S, AS) = 0 = \mathrm{Hom}(A^2S, S)$. Now clearly $\mathrm{Hom}(T, S) = 0$ for any indecomposable module T of dimension type (1 1 1) or (0 1 0), thus $\mathrm{Hom}(A^2S, AS) \approx \mathrm{Hom}(AS, S) = 0$, and also $\mathrm{Hom}(A^2S, S) = 0$, since A^2S is the extension of an indecomposable module of type (1 1 1) by the module of type (0 1 0).

Since $\mathrm{Ext}^1(V, A^2S)$ contains $\mathrm{Ext}^1(AS, A^2S)$ as a submodule, we see that $\mathrm{Ext}^1(V, A^2S)$ is non-zero. Using the fact that $\mathrm{Hom}(A^2S, V) = 0$, we see that any non-zero element in $\mathrm{Ext}^1(V, A^2S)$ gives rise to an indecomposable module W with a submodule $W_1 \approx A^2S$, and $W/W_1 \approx V$. Thus, W has submodules $W_1 \subset W_2 \subset W_3$ such that $W_1 \approx A^2S$, $W_2/W_1 \approx AS$, $W_3/W_2 \approx W/W_3 \approx S$. The dimension type of W is (4 5 2), thus W is quasi-simple.

We fix such a quasi-simple module W. Applying A^2, we get the module A^2W with a factor module $A^2W/A^2W_3 \approx A^2S$, thus $\mathrm{Hom}(A^2W, W) \neq 0$.

Now, let M be the quasi-serial module with $t(M) = W$ and $l(W) = 3$. Its dimension type is (22 36 24). Let $0 \subset M_1 \subset M_2 \subset M$ be a chain of submodules with irreducible inclusions. Given a non-zero homomorphism f': $M_1 = A^2W \to W = M/M_2$, there exists $f \in \mathrm{End}(M)$ such that f' is the composition $M_1 \subset M \xrightarrow{f} M \to M/M_2$. In particular, $M_1 f \nsubseteq M_2$. If f is an automorphism, let $M'_i = M_i f$, otherwise, let $M'_i = M_i(1+f)$.

6.4. We use this opportunity to point out that for the projective module P with dimension vector (2 1 1), no simple submodule A is a direct summand of the radical $\mathrm{rad}\, P$ of P, whereas all the quotients P/A are quasi-simple, since they are indecomposable and of type (1 1 1). This answers a question raised by Auslander in [3].

Note Added in Proof

The author was informed that on the basis of a partial result presented by the author June 1977 in Oberwolfach, M. Auslander, R. Bautista, M.I. Platzek, I. Reiten and S.O. Smalø were able to give an independent proof of Theorem 2.3. This proof will appear in a paper entitled "Almost split exact sequences whose middle term has at most two indecomposable summands".

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