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Diagrammatic Methods in the Representation Theory of Orders

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In the last years, certain diagrammatic methods have been used very successfully in the representation theory of Artin algebras. In 1972, P. Gabriel introduced the notion of a quiver in order to deal with finite dimensional algebras over an algebraically closed field which are either hereditary or have radical square zero, and he proved that such an algebra is of finite representation type if and only if the underlying diagram is a Dynkin diagram [8]. If the base field is not necessarily algebraically closed, one has to use the notion of a species, also introduced by Gabriel [9], and obtains a similar result [6]. Further investigations of representations of quivers and species have been carried out by Bernstein, Gelfand, and Ponomarev [3, 10], Donovan and Freislich [5], Nazarova [15, 16], and Dlab and Ringel [7, 19], in particular one obtains a complete classification of all representations in case the underlying diagram is Euclidean. Other diagrammatic methods were introduced by Nazarova and Roiter in connection with their positive solution of the second Brauer–Thrall conjecture for finite dimensional algebras over algebraically closed fields [13, 17, 18], and they can be adapted in order to deal with arbitrary base fields [20].

In the present paper we shall consider lattices over R-orders, where R is a complete valuation ring, instead of representations of artinian rings, and show that we can use similar diagrammatic methods in order to obtain analogues of the above mentioned results.

More precisely, R is a complete valuation ring with field of quotients K, A is a finite dimensional separable K-algebra and Λ an R-order in A. By ${}_{\Lambda}\mathfrak{M}^0$ we denote the category of left Λ-lattices. We choose a fixed hereditary R-order Γ containing Λ and a two-sided Γ-ideal I in Λ such that $I \subset \operatorname{rad}(\Gamma)$. Since $M \in {}_{\Lambda}\mathfrak{M}^0$ is R-torsion-free, we have an embedding $M \hookrightarrow K \otimes_R M$; the latter

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is an Λ-module, and so we can form ΓM, namely the Γ-lattice generated by M. With $M \in {}_\Lambda\mathfrak{M}^0$ we associate the pair

$$M/IM \xrightarrow{\sigma} \Gamma M/IM,$$

where σ is induced by the inclusion $M \hookrightarrow \Gamma M$. This construction induces a functor $\mathbb{F}$ from ${}_\Lambda\mathfrak{M}^0$ to the category $\mathfrak{C}$ defined as follows: $\mathfrak{A} = \Lambda/I$ and $\mathfrak{B} = \Gamma/I$ are artinian algebras; the objects in $\mathfrak{C}$ will be pairs $X \rightarrow^{\sigma} Y$, where X is a finitely generated left $\mathfrak{A}$-module, Y is a finitely generated projective left $\mathfrak{B}$-module and σ is an $\mathfrak{A}$-monomorphism such that $\mathfrak{B}(\mathrm{Im}(\sigma)) = Y$. Morphisms in $\mathfrak{C}$ are commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "\sigma'"] & Y' \end{tikzcd},$$

where α is an $\mathfrak{A}$-homomorphism and β is a $\mathfrak{B}$-homorphism.

The central observation, which has been made independently by Green and Reiner [11], is the following:

THEOREM A. *The functor* $\mathbb{F}$: ${}_\Lambda\mathfrak{M}^0 \rightarrow \mathfrak{C}$ *is a representation equivalence.*

This result allows us to use freely the results from artinian algebras and translate them to Λ-lattices.

Obviously, the category $\mathfrak{C}$ has its simplest form if both $\mathfrak{A}$ and $\mathfrak{B}$ are semisimple. This is equivalent to studying orders Λ such that the radical of Λ is at the same time the radical of a hereditary order Γ. It should be noted that every order can be embedded in such an order. These orders have been studied by Bäckström [1] under some restrictive conditions. He has classified those orders among these, which are of finite lattice type; without, however, listing the indecomposable lattices explicitly. We shall call such orders *Bäckström-orders* (i.e., orders Λ, such that the radical of Λ is at the same time the radical of a hereditary order Γ). With every Bäckström-order one can associate a valued graph in the following way: Since the property of being a Bäckström-order is invariant under Morita-equivalence, we may assume Λ to be basic. Hence,

$$\Lambda/\mathrm{rad}(\Lambda) \cong \prod_{i=1}^{s} \mathfrak{k}_i ,$$

$$\Gamma/\mathrm{rad}(\Gamma) = \prod_{j=s+1}^{t} (\mathfrak{k}_j)_{n_j} ,$$

where $\mathfrak{f}_i$, $1 \leqslant i \leqslant t$ are finite dimensional skewfields over $R/\mathrm{rad}(R)$, and $(\mathfrak{f})_n$ denotes the full ring of $n \times n$ matrices over $\mathfrak{f}$. Let S_j, $s+1 \leqslant j \leqslant t$ be a full set of simple $\Gamma/\mathrm{rad}(\Gamma)$-modules with $\mathrm{End}_\Gamma(S_j) \simeq \mathfrak{f}_j$. Then

$$ {}_iS_j = \mathfrak{f}_i \otimes_\Lambda S_j, \qquad 1 \leqslant i \leqslant s, \quad s+1 \leqslant j \leqslant t, $$

are $(\mathfrak{f}_i, \mathfrak{f}_j)$-bimodules. Moreover, we put

$$ d_{ij} = \begin{cases} \dim_{\mathfrak{f}_i}({}_iS_j), & 1 \leqslant i \leqslant s, \quad s+1 \leqslant j \leqslant t, \\ 0 \text{ else} \end{cases} $$

$$ d'_{ji} = \begin{cases} \dim_{\mathfrak{f}_j}({}_iS_j), & 1 \leqslant i \leqslant s, \quad s+1 \leqslant j \leqslant t, \\ 0 \text{ else.} \end{cases} $$

Let (G, δ) be the valued graph with t vertices v_i, $1 \leqslant i \leqslant t$ and valuation (d_{ij}, d'_{ji}).

Then we say that the Bäckström-order Λ is of *species* $(\mathfrak{f}_i, {}_iS_j)$, and that (G, δ) is its ***valued graph.***

Using Theorem A, one obtains:

THEOREM B. *Let Λ be a Bäckström-order of species $(\mathfrak{f}_i, {}_iS_j)$. Then there is a representation equivalence between the category ${}_\Lambda\mathfrak{M}^0$ of Λ-lattices, and the full subcategory of the category of representations of the species $(\mathfrak{f}_i, {}_iS_j)$ consisting of those representations which have no simple direct summand.*

Combining this with known results from the representation theory of species, one obtains for Bäckström-orders Λ with valued graph (G, δ) the following classification:

(i) Λ has only a finite number of non-isomorphic indecomposable lattices if and only if (G, δ) is a finite union of Dynkin diagrams. In this case, the isomorphism classes of indecomposable Λ-lattices correspond bijectively to the non-simple positive roots of (G, δ).

(ii) If (G, δ) is the union of Dynkin diagrams and Euclidean diagrams, then the isomorphism classes of indecomposable Λ-lattices can be classified in the following way: To every non-simple positive Weyl root, there exists a unique indecomposable Λ-lattice. The remaining indecomposable Λ-lattices correspond to the null roots.

Theorem A seems to be of interest not only for the investigation of such special classes of Λ-orders as the Bäckström-orders, but should also give some possibility to transfer general results on module categories over Artin algebras to categories of lattices over orders. Note that Theorem A gives a representation equivalence $\mathbb{F}$ between the category ${}_\Lambda\mathfrak{M}^0$ of Λ-lattices, and a full subcategory $\mathfrak{C}$ of the category $\mathfrak{C}$ of all finitely generated modules over a certain Artin algebra $\mathfrak{D}$.

It seems plausible that certain general results which are known to be true for module categories, thus for $\tilde{\mathfrak{C}}$, are, more generally, true for well-behaved full subcategories such as our category $\mathfrak{C}$. In this way, one should be able to use the representation equivalence $\mathbb{F}$ in order to derive similar results for ${}_\Lambda\mathfrak{M}^0$. In particular, we show in the last section that $\mathbb{F}$ reflects certain types of chains of indecomposable objects the existence of which established for $\tilde{\mathfrak{C}}$ by the recent proof of the second Brauer–Thrall conjecture [18, 20].

In order to make the paper rather self contained, we have included the definitions and the statement of the theorems concerning representations of species which are needed, and give full references to the proofs. Moreover, we have included several examples to demonstrate how these methods can be used to construct indecomposable lattices explicitly.

The results of this paper were presented at the Oberwolfach-meeting in February 1977. The second author wants to express his gratitude to the Deutsche Forschungsgemeinschaft for support. We shall use the following notations:

${}_\Lambda\mathfrak{M}^0$ = category of left Λ-lattices,
${}_\mathfrak{A}\mathfrak{M}^f$ = category of finitely generated left $\mathfrak{A}$-modules,
${}_\mathfrak{A}\mathfrak{P}^f$ = category of finitely generated projective left $\mathfrak{A}$-modules.

1. DESCRIPTION OF LATTICES VIA PAIR CATEGORIES

In this section we shall reduce the description of lattices for orders to a problem in the representation theory of modules over an Artin algebra where we then can use known diagrammatic methods.

Notation. R is a complete valuation ring with parameter π, residue field $\mathfrak{k}$ and field of quotients K. Λ is an R-order in the semi-simple finite dimensional K-algebra A. (Note that some of the results will only be valid if A is separable.)

We choose Γ to be an R-order in A containing Λ and assume that I is a twosided Γ-ideal contained in rad(Λ), the Jacobson radical of Λ. We observe that then automatically $I \subset \operatorname{rad}(\Gamma)$; in fact, $I \subset \operatorname{rad}(\Lambda)$, and so I is nilpotent modulo $\pi\Lambda$; however, $\pi\Lambda \subset \pi\Gamma$, and so I is nilpotent modulo $\pi\Gamma$; i.e., $I \subset \operatorname{rad}(\Gamma)$.

As special situation we have the following in mind:

(i) Γ is hereditary, and I is a Γ-ideal;

(ii) Γ is a twosided ring of multipliers of rad(Λ), where Λ is assumed to be non-hereditary; i.e.,

$$\Gamma = \{a \in A\colon a \operatorname{rad}(\Lambda) \subset \operatorname{rad}(\Lambda)\} \cap \{a \in A\colon \operatorname{rad}(\Lambda)a \subset \operatorname{rad}(\Lambda)\},$$

and $I = \operatorname{rad}(\Lambda)$. Γ is then a proper over-order of Λ [12].

Every Λ-lattice M is canonically embedded in the A-module $K \otimes_R M$ and so we have a canonical injection $\iota\colon M \to \Gamma M$.

In the sequel we shall be concerned with the full subcategory of those Λ-lattices M, such that ΓM is a projective Γ-lattice; i.e.,

$${}_\Lambda\mathfrak{M}^0(\Gamma) = \{M \in {}_\Lambda\mathfrak{M}^0 \colon \Gamma M \in {}_\Gamma\mathfrak{P}^f\}.$$

This is an additive subcategory of ${}_\Lambda\mathfrak{M}^0$, and ${}_\Lambda\mathfrak{M}^0(\Gamma) = {}_\Lambda\mathfrak{M}^0$ if and only if Γ is hereditary.

We put

$$\mathfrak{A} = \Lambda/I \qquad \text{and} \qquad \mathfrak{B} = \Gamma/I,$$

then $\mathfrak{A}$ and $\mathfrak{B}$ are finitely generated algebras over the commutative local artinian ring $\bar{R} = R/(R \cap I)$, moreover, the inclusion $\Lambda \hookrightarrow \Gamma$ induces an $\bar{R}$-algebra-homomorphism $\mathfrak{A} \hookrightarrow \mathfrak{B}$.

(1.1) We construct the *pair category* $\mathfrak{C}$ as follows: The *objects* consist of a finitely generated left $\mathfrak{A}$-module U and a finitely generated projective left $\mathfrak{B}$-module V, together with an $\mathfrak{A}$-monomorphism $\sigma\colon U \to V$, subject to the condition $\mathfrak{B}(\mathrm{Im}(\sigma)) = V$. This object is denoted by $U \to^{\sigma} V$. A *morphism in* $\mathfrak{C}$ is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & V \arrow[d, "\beta"] \\ U' \arrow[r, "\sigma'"] & V' \end{tikzcd}$$

where α is an $\mathfrak{A}$-homomorphism and β a $\mathfrak{B}$-homomorphism.

We can construct in a natural way a functor as follows:

$$\begin{aligned} \mathbb{F}\colon {}_\Lambda\mathfrak{M}^0(\Gamma) &\to \mathfrak{C}, \\ M \quad &\mapsto (M/IM \xrightarrow{\sigma} \Gamma M/IM), \end{aligned}$$

where σ is induced by the inclusion $\iota\colon M \to \Gamma M$.

Moreover, if $\alpha_1\colon M \to M'$ is a homomorphism of Λ-lattices, then it induces a Γ-homomorphism $\beta_1\colon \Gamma M \to \Gamma M'$ such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "\alpha_1"'] & \Gamma M \arrow[d, "\beta_1"] \\ M' \arrow[r, "\iota'"] & \Gamma M' \end{tikzcd} .$$

But $\alpha_1 |_{IM} = \beta_1 |_{IM}$, and so we obtain a morphism in $\mathfrak{C}$:

$$\begin{tikzcd}
M/IM \arrow[r] \arrow[d, "\alpha"'] & \Gamma M/IM \arrow[d, "\beta"] \\
M'/IM' \arrow[r] & \Gamma M'/IM'
\end{tikzcd}$$

It should be noted that $\Gamma M/IM \in {}_{\mathfrak{B}}\mathfrak{P}^f$, since $\Gamma M \in {}_{\Gamma}\mathfrak{P}^f$, and $\mathfrak{B}(M/IM) = (\Gamma/I)(M/IM) = \Gamma M/IM$. Hence, $\mathbb{F}$ is a functor from ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ to the category $\mathfrak{C}$.

(1.2) THEOREM I. *The functor* $\mathbb{F}$ *induces a representation equivalence between* ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ *and* $\mathfrak{C}$.

We remark that a similar result was independently obtained in [11].

The *proof* is done in several steps:

(1.3) LEMMA. $\mathbb{F}$ *is—up to isomorphisms—surjective on objects.*

Proof. Let $U \rightarrow^{\sigma} V$ be an object in $\mathfrak{C}$. We may assume that σ is a settheoretic inclusion. Since V is a projective $\mathfrak{B}$-module, there exists—by the method of lifting idempotents—a projective Γ-lattice Q with $Q/IQ \cong V$ as $\mathfrak{B}$-modules. Let $\kappa\colon Q \rightarrow V$ be the induced epimorphism and put $M = \kappa^{-1}(U)$.

Claim. $M \in {}_{\Lambda}\mathfrak{M}^0(\Gamma)$ and $\mathbb{F}\colon M \rightarrow (U \rightarrow^{\sigma} V)$ (up to the isomorphism). Since $\mathfrak{B}U = V$, we have $\Gamma M + IQ = Q$, and so by Nakayama's lemma $\Gamma M = Q$; consequently $IM = I\Gamma M = IQ$, and we have an isomorphism:

$$\begin{tikzcd}
M/IM \arrow[r] \arrow[d, phantom, "\wr\|"] & \Gamma M/IM \arrow[d, phantom, "\wr\|"] \\
U \arrow[r, "\sigma"] & V
\end{tikzcd}$$

It is clear that M is a Λ-module, and so (1.3) is proved.

It should be noted that $\mathbb{F}$ is additive. If $M = M_1 \oplus M_2$, $M_i \neq 0$, $i = 1, 2$ and $M \in {}_{\Lambda}\mathfrak{M}^0(\Gamma)$, then $M_i \in {}_{\Lambda}\mathfrak{M}^0(\Gamma)$ and $M_i/IM_i \neq 0$, $i = 1, 2$, by Nakayama's lemma.

(1.4) LEMMA. $\mathbb{F}$ *is surjective on morphisms.*

Proof. Given a morphism (α, β) in $\mathfrak{C}$. Because of (1.3) we may assume, that we have the following commutative diagram

$$\begin{tikzcd}
M/IM \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & \Gamma M/IM \arrow[d, "\beta"] \\
M'/IM' \arrow[r, "\sigma'"] & \Gamma M'/IM'
\end{tikzcd}, \quad M, M' \in {}_{\Lambda}\mathfrak{M}^0(\Gamma). \tag{1.5}$$

Now, $\Gamma M \in {}_\Gamma\mathfrak{P}^f$ and so we can complete the following diagram commutatively by Γ-homomorphisms

$$\begin{tikzcd}
\Gamma M \arrow[r, "\kappa"] \arrow[d, "\beta_1"'] & \Gamma M/IM \arrow[r] \arrow[d, "\beta"] & 0 \\
\Gamma M' \arrow[r, "\kappa'"] & \Gamma M'/IM' \arrow[r] & 0.
\end{tikzcd}$$

We have the following Λ-isomorphisms:

$$\Gamma M/M \simeq (\Gamma M/IM)/(M/IM) =: C,$$
$$\Gamma M'/M' \simeq (\Gamma M'/IM')/(M'/IM') =: C',$$

and so the commutative diagram (1.5) induces a Λ-homomorphism $\gamma\colon C \to C'$. Hence we can find a Λ-homomorphism α_1, making the following diagram commute:

$$\begin{tikzcd}
 & 0 \arrow[rr] & & M/IM \arrow[rr] \arrow[dd, "\alpha"] & & \Gamma M/IM \arrow[rr] \arrow[dd, "\beta"] & & C \arrow[rr] \arrow[dd, "\gamma"] & & 0 \\
0 \arrow[rr] & & M \arrow[ur] \arrow[rr] \arrow[dd, "\alpha_1"] & & \Gamma M \arrow[ur, "\kappa"] \arrow[rr] \arrow[dd, "\beta_1"] & & C \arrow[ur, equal] \arrow[rr] \arrow[dd, "\gamma"] & & 0 & \\
 & 0 \arrow[rr] & & M'/IM' \arrow[rr] & & \Gamma M'/IM' \arrow[rr] & & C' \arrow[rr] & & 0 \\
0 \arrow[rr] & & M' \arrow[ur] \arrow[rr] & & \Gamma M' \arrow[ur, "\kappa'"] \arrow[rr] & & C' \arrow[ur, equal] \arrow[rr] & & 0 &
\end{tikzcd}$$

Since β is uniquely determined by β_1, we conclude that $\mathbb{F}$ is surjective on morphisms. This proves (1.4).

(1.6) LEMMA. *$\mathbb{F}$ recovers decompositions.*

Proof. In view of (1.4) it suffices to show that $\mathbb{F}$ recovers isomorphisms. So let—with the notation of the proof of (1.4)—(α, β) be an isomorphism. We have to show that α_1—cf. above—is an isomorphism. From Nakayama's lemma it follows that α_1 is an epimorphism, and so the R-rank of M is at least as large as the R-rank of M'. By using the inverse of (α, β) we conclude that M and M' have the same R-rank; hence, α_1 is an isomorphism.

This completes the proof of (1.2).

(1.7) *Remark.* (1.2) holds in the following more general situation—the proof being verbatim the same. Let $N_1, \ldots, N_t$ be a finite set of indecomposable Γ-lattices, and let $\mathfrak{K}$ be the full additive subcategory of ${}_\Gamma\mathfrak{M}^0$ generated by $\{N_1, \ldots, N_t\}$ and put $\mathfrak{L} = \{M \in {}_\Lambda\mathfrak{M}^0\colon \Gamma M \in \mathfrak{K}\}$. The canonical projection $\kappa\colon N \to N/IN$, $N \in \mathfrak{L}$ induces a homomorphism

$$\mathrm{Hom}_\Gamma(N', N) \xrightarrow{\kappa_*} \mathrm{Hom}_\Gamma(N', N/IN) = \mathrm{Hom}_\Gamma(N'/IN', N/IN).$$

Put ${}_{N'}\Omega_N = \mathrm{Im}(\kappa_*)$ and define the category $\overline{\mathfrak{R}}$ to have objects $\bar{N} = N/IN$, $N \in \mathfrak{R}$ and morphisms $\overline{\mathfrak{R}}(N', N) = {}_{N'}\Omega_N$. The category $\mathfrak{C}(\mathfrak{R})$ has as objects pairs $U \rightarrow^{\sigma} V$, where $U \in {}_{\mathfrak{A}}\mathfrak{M}^f$, $V \in \mathfrak{R}$, σ is an $\mathfrak{A}$-monomorphism with $\mathfrak{B}\,\mathrm{Im}(\sigma) = V$. Morphisms are commutative diagrams

$$\begin{tikzcd}
U \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & V \arrow[d, "\beta"] \\
U' \arrow[r, "\sigma'"] & V'
\end{tikzcd},$$

where α is an $\mathfrak{A}$-homomorphism, and β a morphism in $\overline{\mathfrak{R}}$.

(1.8) Corollary. *If Γ is hereditary, then ${}_\Lambda\mathfrak{M}^0$ is representation equivalent to $\mathfrak{C}$.*

We remark, that in case A is separable, there always exists such a hereditary order.

(1.9) Example 1.

$$\Lambda = \begin{pmatrix} R & R \\ \pi^2 R & R \end{pmatrix} \subset \Gamma = \begin{pmatrix} R & R \\ R & R \end{pmatrix}$$

and

$$I = \begin{pmatrix} \pi^2 R & \pi^2 R \\ \pi^2 R & \pi^2 R \end{pmatrix},$$

all these are viewed as matrix rings. Then

$$\mathfrak{A} = \begin{pmatrix} \bar R & \bar R \\ 0 & \bar R \end{pmatrix} \hookrightarrow \mathfrak{B} = \begin{pmatrix} \bar R & \bar R \\ \bar R & \bar R \end{pmatrix},$$

where $\bar R = R/\pi^2 R$. $\mathfrak{B}$ has—up to isomorphism—only one indecomposable projective left module $\bar G = \binom{\bar R}{\bar R}$. The Loewy series of $\bar G$ as $\mathfrak{B}$-module is

$$\begin{pmatrix} \bar R \\ \bar R \end{pmatrix} \supset \begin{pmatrix} \pi\bar R \\ \pi\bar R \end{pmatrix} \supset 0;$$

the lattice of maximal submodules of $\bar G$ as $\mathfrak{A}$-module is

$$\begin{array}{ccccccc}
 & & & \overset{\epsilon_1}{\supset} \begin{pmatrix} \bar R \\ 0 \end{pmatrix} \overset{\epsilon_2}{\supset} & & & \\
\begin{pmatrix} \bar R \\ \bar R \end{pmatrix} & \underset{\epsilon_1}{\supset} & \begin{pmatrix} \bar R \\ \pi\bar R \end{pmatrix} & & \begin{pmatrix} \pi\bar R \\ 0 \end{pmatrix} & \supset & 0 \\
 & & & \underset{\epsilon_2}{\supset} \begin{pmatrix} \pi\bar R \\ \pi\bar R \end{pmatrix} \underset{\epsilon_1}{\supset} & & &
\end{array}$$

where ϵ_1 and ϵ_2 indicate the extension classes.

As indecomposable objects one notes at once the following objects in $\mathfrak{C}$:

$$\text{(i)}\ \begin{pmatrix}\bar R\\ 0\end{pmatrix} \hookrightarrow \begin{pmatrix}\bar R\\ \bar R\end{pmatrix}; \qquad \text{(ii)}\ \begin{pmatrix}\bar R\\ \pi\bar R\end{pmatrix} \hookrightarrow \begin{pmatrix}\bar R\\ \bar R\end{pmatrix}; \qquad \text{(iii)}\ \begin{pmatrix}\bar R\\ \bar R\end{pmatrix} \hookrightarrow \begin{pmatrix}\bar R\\ \bar R\end{pmatrix}.$$

However, $\binom{\pi\bar R}{0} \hookrightarrow \binom{\bar R}{\bar R}$ is not an object in our category $\mathfrak{C}$, since

$$\mathfrak{B}\begin{pmatrix}\pi\bar R\\ 0\end{pmatrix} = \begin{pmatrix}\pi\bar R\\ \pi\bar R\end{pmatrix} \neq \begin{pmatrix}\bar R\\ \bar R\end{pmatrix}.$$

Since Λ has exactly three non-isomorphic indecomposable lattices (cf. Section 3), the above listed are all indecomposable objects in $\mathfrak{C}$.

Example 2. Let

$$\Lambda = \left(\begin{pmatrix} a & \pi b\\ \pi c & a + \pi d\end{pmatrix}, a, b, c, d, \in R\right).$$

We indicate this by writing

$$\Lambda = \begin{pmatrix} R & & \pi R\\ & \diagdown & \\ \pi R & & R\end{pmatrix}.$$

We choose $\Gamma = \begin{pmatrix} R & R\\ R & R\end{pmatrix}$ and $I = \begin{pmatrix} \pi R & \pi R\\ \pi R & \pi R\end{pmatrix}$. Then we have

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{k} & & 0\\ & \diagdown & \\ 0 & & \mathfrak{k}\end{pmatrix} \cong \mathfrak{k} \hookrightarrow \mathfrak{B} = \begin{pmatrix} \mathfrak{k} & \mathfrak{k}\\ \mathfrak{k} & \mathfrak{k}\end{pmatrix}.$$

$\mathfrak{B}$ has one indecomposable projective module $\bar G = \binom{\mathfrak{k}}{\mathfrak{k}}$, and this module is $\mathfrak{B}$-simple. As $\mathfrak{A}$-module, however, $\bar G \cong \mathfrak{k} \oplus \mathfrak{k}$. The objects

$$\underbrace{\mathfrak{k} \oplus \mathfrak{k} \oplus \cdots \oplus \mathfrak{k}}_{n} \hookrightarrow \underbrace{\begin{pmatrix}\mathfrak{k}\\ \mathfrak{k}\end{pmatrix} \oplus \begin{pmatrix}\mathfrak{k}\\ \mathfrak{k}\end{pmatrix} \oplus \cdots \oplus \begin{pmatrix}\mathfrak{k}\\ \mathfrak{k}\end{pmatrix}}_{n+1},$$

where the inclusion is induced by

$$(0,\dots, 0, \underset{i}{1}, 0,\dots, 0) \to \left(\begin{pmatrix}0\\ 0\end{pmatrix},\dots, \begin{pmatrix}0\\ 0\end{pmatrix}, \begin{pmatrix}1\\ 0\end{pmatrix}_{i}, \begin{pmatrix}0\\ 1\end{pmatrix}_{i+1}, \begin{pmatrix}0\\ 0\end{pmatrix},\dots, \begin{pmatrix}0\\ 0\end{pmatrix}\right)$$

surely are indecomposable, and for different n they are not isomorphic.

This shows that $\mathfrak{C}$ and hence also ${}_{\Lambda}\mathfrak{M}^0$ have infinitely many non-isomorphic indecomposable objects, though $\mathfrak{A}$ and $\mathfrak{B}$ have only one indecomposable module each.

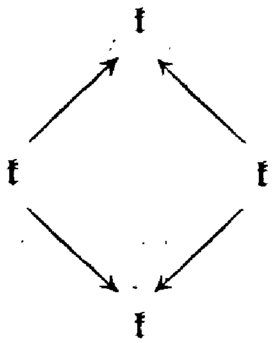
EXAMPLE 3. Let

$$\Lambda = \begin{pmatrix} R & \pi R & R & R \\ \pi R & R & R & R \\ \pi R & \pi R & R & \pi R \\ \pi R & \pi R & \pi R & R \end{pmatrix} \quad \text{and} \quad \Gamma = (R)_4\,, I = (\pi R)_4\,.$$

Then

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{k} & 0 & \mathfrak{k} & \mathfrak{k} \\ 0 & \mathfrak{k} & \mathfrak{k} & \mathfrak{k} \\ 0 & 0 & \mathfrak{k} & 0 \\ 0 & 0 & 0 & \mathfrak{k} \end{pmatrix} \hookrightarrow \mathfrak{B} = (\mathfrak{k})_4\,,$$

and $\mathfrak{A}$ is the tensoralgebra over $\mathfrak{k}$ of the graph



and hence, it has infinitely many non-isomorphic indecomposable finitely generated modules (Donovan–Freislich [5], Nazarova [16], for a proof see also [7]).

On the other hand, Λ has only six non-isomorphic indecomposable lattices (cf. Section 2), represented in matrixform as follows:

$$\begin{pmatrix} R \\ R \\ R \\ R \end{pmatrix}, \begin{pmatrix} R \\ R \\ R \\ \pi R \end{pmatrix}, \begin{pmatrix} R \\ R \\ \pi R \\ \pi R \end{pmatrix}, \begin{pmatrix} R \\ R \\ \pi R \\ R \end{pmatrix}, \begin{pmatrix} R \\ \pi R \\ \pi R \\ \pi R \end{pmatrix}, \begin{pmatrix} \pi R \\ R \\ \pi R \\ \pi R \end{pmatrix},$$

and so $\mathfrak{C}$ is of finite representation type.

2. BÄCKSTRÖM-ORDERS AND VALUED GRAPHS

(2.1) DEFINITION. An R-order Λ in a semisimple K-algebra A is said to be:

(i) a *B-order* if there exists an R-order $\Gamma \supsetneq \Lambda$ with $\operatorname{rad}(\Gamma) = \operatorname{rad}(\Lambda)$,

(ii) a *Bäckström-order*, if there exists a hereditary R-order $\Gamma \supsetneq \Lambda$ with $\operatorname{rad}(\Gamma) = \operatorname{rad}(\Lambda)$.

Remark. In his thesis [1] Bäckström has characterized all Bäckström-orders of finite lattice type under the following hypotheses:

(i) K is a local algebraic number field,

(ii) $\mathrm{End}_\Lambda(P_i)/\mathrm{rad}_\Lambda(P_i) = \mathfrak{k}$ is the same for all indecomposable projective Λ-lattices.

(2.2) Lemma. *The class of B-orders and Bäckström-orders resp. is invariant under Morita equivalence.*

Proof. Let P be a progenerator for Λ, then ΓP is a progenerator for Γ. We put $\Delta = \mathrm{End}_\Lambda(P)$ and $\Omega = \mathrm{End}_\Gamma(\Gamma P)$. By [21, I, Ch. IV, 3.8] we have

$$\begin{aligned}\mathrm{rad}(\Delta) &= \mathrm{Hom}_\Lambda(P, \mathrm{rad}(\Lambda)P) = \mathrm{Hom}_\Lambda(P, \mathrm{rad}(\Gamma)\,\Gamma P)\\ &= \mathrm{Hom}_\Gamma(\Gamma P, \mathrm{rad}(\Gamma)\,\Gamma P) = \mathrm{rad}(\Omega).\end{aligned}$$

Moreover, if Γ is hereditary so is Ω.

Let Λ be a non-hereditary R-order. Then there exists a unique order Λ_0 different from Λ and maximal with respect to $\mathrm{rad}(\Lambda)$ being an ideal over Λ_0, provided A is separable.

(2.3) Proposition. *Let Λ be an R-order and Λ_0 be as defined above.*

(i) *If Λ is a B-order, then* $\Lambda/\mathrm{rad}(\Lambda) \neq \Lambda_0/\mathrm{rad}(\Lambda_0)$.

(ii) *Assume that* $\Lambda/\mathrm{rad}(\Lambda) \neq \Lambda_0/\mathrm{rad}(\Lambda_0)$ *and that*

$$\Lambda_0/\mathrm{rad}(\Lambda) \cong \Lambda_0/\mathrm{rad}(\Lambda_0) \oplus X,$$

then Λ is a B-order. (This latter condition is satisfied if $\Lambda_0/\mathrm{rad}(\Lambda)$ is a $\mathfrak{k}$-algebra.)

(iii) *There exists a B-order Λ_1 with $\Lambda \subset \Lambda_1 \subset \Lambda_0$ such that* $\Lambda/\mathrm{rad}(\Lambda) = \Lambda_1/\mathrm{rad}(\Lambda_1)$, *if* $\Lambda/\mathrm{rad}(\Lambda) \neq \Lambda_0/\mathrm{rad}(\Lambda_0)$.

(iv) *Λ is contained in a Bäckström-order Λ_1 with* $\Lambda/\mathrm{rad}(\Lambda) \cong \Lambda_1/\mathrm{rad}(\Lambda_1)$.

Proof. (i) Assume that Λ is a B-order, then $\Lambda \subsetneq \Gamma \subset \Lambda_0$, and so $\Lambda/\mathrm{rad}(\Lambda) \subsetneq \Gamma/\mathrm{rad}(\Lambda) \subset \Lambda_0/\mathrm{rad}(\Lambda_0)$, since $\mathrm{rad}(\Lambda_0) \cap \Gamma = \mathrm{rad}(\Gamma) = \mathrm{rad}(\Lambda)$, and so $\Lambda/\mathrm{rad}(\Lambda) \neq \Lambda_0/\mathrm{rad}(\Lambda_0)$.

(ii) Assume that the condition of (ii) is satisfied. We then have an injection

$$\Lambda/\mathrm{rad}(\Lambda) \to \Lambda_0/\mathrm{rad}(\Lambda) \cong \Lambda_0/\mathrm{rad}(\Lambda_0) \oplus X.$$

If Λ_1 is the inverse image of $\Lambda_0/\mathrm{rad}(\Lambda_0)$ under the canonical homomorphism $\Lambda_0 \to \Lambda_0/\mathrm{rad}(\Lambda)$, then surely $\mathrm{rad}(\Lambda) = \mathrm{rad}(\Lambda_1)$, and hence, Λ is a B-order.

(iii) We have a natural injection

$$\Lambda/\mathrm{rad}(\Lambda) \to \Lambda_0/(\mathrm{rad}(\Lambda) + \pi\Lambda_0),$$

and we let Λ_1 be the inverse image of $\Lambda/\mathrm{rad}(\Lambda)$ under the natural homomorphism $\Lambda_0 \to \Lambda_0/(\mathrm{rad}(\Lambda) + \pi\Lambda_0)$, then Λ_1 is a B-order by (ii) and $\Lambda \subset \Lambda_1 \subset \Lambda_0$.

(iv) Let Λ' be the pullback of the following diagram of natural homomorphisms

$$\begin{tikzcd}
\Lambda/\mathrm{rad}(\Lambda) \arrow[r] & \Lambda_0/\mathrm{rad}(\Lambda_0) \\
\Lambda' \arrow[u] \arrow[r] & \Lambda_0 \arrow[u]
\end{tikzcd}$$

Then $\Lambda/\mathrm{rad}(\Lambda) = \Lambda'/\mathrm{rad}(\Lambda')$. If $\Lambda = \Lambda'$, then $\mathrm{rad}(\Lambda) = \mathrm{rad}(\Lambda_0)$ and so the ring of multipliers of $\mathrm{rad}(\Lambda_0)$ coincides with Λ_0; i.e., Λ_0 is hereditary [12]. Repeating the same construction with Λ' we eventually will reach a Bäckström-order Λ_1 with $\Lambda/\mathrm{rad}(\Lambda) \cong \Lambda_1/\mathrm{rad}(\Lambda_1)$.

EXAMPLE. Let

$$\Lambda = \left\{\begin{pmatrix} \alpha & \pi^2 R \\ \pi^2 R & \alpha + \pi^2 R \end{pmatrix}, \alpha \in R\right\}.$$

Then

$$\Lambda_0 = \begin{pmatrix} R & & \pi R \\ & \diagdown\!\!\diagdown & \\ \pi R & & R \end{pmatrix},$$

and there is no Bäckström-order between Λ and Λ_0.

We shall next associate with every B-order Λ with corresponding order Γ such that $\mathrm{rad}(\Lambda) = \mathrm{rad}(\Gamma)$, a valued graph in such a way that the non-simple indecomposable representations of that graph over $\mathfrak{k}$ are in bijection to the indecomposable lattices in the category

$${}_\Lambda\mathfrak{M}^0(\Gamma) = \{M \in {}_\Lambda\mathfrak{M}^0 : \Gamma M \in {}_\Lambda\mathfrak{P}\}.$$

(2.4) *Notation.* Let Λ be a basic B-order contained in Γ with $\mathrm{rad}(\Lambda) = \mathrm{rad}(\Gamma)$. (A basic order Λ is one for which $\Lambda/\mathrm{rad}(\Lambda)$ is a product of skewfields.) In view of (2.2) the above assumption is no restriction. Then

$$\Lambda/\mathrm{rad}(\Lambda) \cong \prod_{i=1}^{s} \mathfrak{f}_i,$$

$$\Gamma/\mathrm{rad}(\Gamma) \cong \prod_{j=s+1}^{t} (\mathfrak{f}_j)_{n_j},$$

where $\mathfrak{f}_i$, $1 \leqslant i \leqslant t$, are finite dimensional skewfields over $\mathfrak{k}$, and $(\mathfrak{f})_n$ denotes the full ring of $(n \times n)$-matrices over $\mathfrak{f}$. Let S_j, $s+1 \leqslant j \leqslant t$, be a full set of simple $\Gamma/\mathrm{rad}(\Gamma)$-modules with $\mathrm{End}_\Gamma(S_j) \cong \mathfrak{f}_j$.

Since Λ is basic, $\mathfrak{f}_i$, $1 \leqslant i \leqslant s$, is a twosided simple Λ-module, and so

$$ {}_iS_j = \mathfrak{f}_i \otimes_\Lambda S_j, \qquad 1 \leqslant i \leqslant s, \quad s+1 \leqslant j \leqslant t, $$

are $(\mathfrak{f}_i, \mathfrak{f}_j)$-bimodules. Moreover, we put

$$ \begin{cases} d_{ij} & \begin{array}{lll} \dim_{\mathfrak{f}_i}({}_iS_j), & 1 \leqslant i \leqslant s, & s+1 \leqslant j \leqslant t, \\ 0 \quad \text{else} & & \end{array} \end{cases} $$

$$ \begin{cases} d'_{ij} = & \begin{array}{lll} \dim_{\mathfrak{f}_j}({}_iS_j), & 1 \leqslant i \leqslant s, & s+1 \leqslant j \leqslant t, \\ 0 \quad \text{else} & . & \end{array} \end{cases} $$

(2.5) Definition. Let (G, δ) be the valued graph with vertices v_i, $1 \leqslant i \leqslant t$ and valuation (d_{ij}, d'_{ij}). Then (G, δ) is said to be the *valued graph of the B-order Λ with corresponding Γ*.

Example 4.

$$ \Lambda = \begin{pmatrix} R & \pi R & R \\ \pi R & R & R \\ \pi R & \pi R & R \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} R & R & R \\ R & R & R \\ \pi R & \pi R & R \end{pmatrix}, $$

then

$$ \mathrm{rad}(\Lambda) = \mathrm{rad}(\Gamma) = \begin{pmatrix} \pi R & \pi R & R \\ \pi R & \pi R & \pi R \\ \pi R & \pi R & \pi R \end{pmatrix}, $$

$$ \Lambda/\mathrm{rad}(\Lambda) \simeq \prod_{i=1}^{3} \mathfrak{k}_i, \quad \mathfrak{k}_i \simeq \mathfrak{k}, \quad 1 \leqslant i \leqslant 3, $$

$$ \Gamma/\mathrm{rad}(\Gamma) = (\mathfrak{k}_4)_2 \prod \mathfrak{k}_5, \quad \mathfrak{k}_4 \simeq \mathfrak{k}_5 \simeq \mathfrak{k}. $$

Then

$$ S_4 = \begin{pmatrix} \mathfrak{k}_4 \\ \mathfrak{k}_4 \end{pmatrix}, \qquad S_5 = \mathfrak{k}_5, $$

and so

$$ {}_1S_4 = \mathfrak{k}_1 \otimes_\Lambda \begin{pmatrix} \mathfrak{k}_4 \\ \mathfrak{k}_4 \end{pmatrix} \simeq \begin{pmatrix} \mathfrak{k} \\ 0 \end{pmatrix} = {}_1\mathfrak{k}_4; \qquad {}_1S_5 = 0, $$

$$ {}_2S_4 = \mathfrak{k}_2 \otimes_\Lambda \begin{pmatrix} \mathfrak{k}_4 \\ \mathfrak{k}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathfrak{k} \end{pmatrix} = {}_2\mathfrak{k}_4; \qquad {}_2S_5 = 0, $$

$$ {}_3S_4 = \mathfrak{k}_3 \otimes_\Lambda \begin{pmatrix} \mathfrak{k}_4 \\ \mathfrak{k}_4 \end{pmatrix} = 0; \qquad {}_3S_5 = {}_3\mathfrak{k}_5. $$

Hence, the associated graph is

$$1 \xrightarrow{(1,1)} 4 \xleftarrow{(1,1)} 2 \quad 3 \xrightarrow{(1,1)} 5.$$

It should be noted that the graph is not connected, though Λ has no central idempotents.

EXAMPLE 5.

$$\Lambda = \begin{pmatrix} R & \pi R \\ \pi R & R \end{pmatrix}, \quad \Gamma = \begin{pmatrix} R & R \\ R & R \end{pmatrix}, \quad \mathrm{rad}(\Lambda) = \mathrm{rad}(\Gamma) = \begin{pmatrix} \pi R & \pi R \\ \pi R & \pi R \end{pmatrix},$$

$$\Lambda/\mathrm{rad}(\Lambda) = \mathfrak{k}_1, \qquad \Gamma/\mathrm{rad}(\Gamma) = (\mathfrak{k}_2)_2, \qquad \mathfrak{k}_1 \cong \mathfrak{k}_2 = \mathfrak{k}.$$

$$S_2 = \begin{pmatrix} \mathfrak{k}_2 \\ \mathfrak{k}_2 \end{pmatrix}, \quad {}_1S_2 = \mathfrak{k}_1 \otimes_\Lambda \begin{pmatrix} \mathfrak{k}_2 \\ \mathfrak{k}_2 \end{pmatrix} = {}_1\mathfrak{k}_2 \oplus {}_1\mathfrak{k}_2,$$

and hence the associated graph is

$${}_1\cdot \xrightarrow{(2,2)} \cdot_2 .$$

EXAMPLE 6.

$$\Lambda = \begin{pmatrix} R & R & R & R \\ \pi R & R & \pi R & \pi R \\ \pi R & \pi R & R & \pi R \\ \pi R & \pi R & \pi R & R \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} R & R & R & R \\ \pi R & R & R & R \\ \pi R & R & R & R \\ \pi R & R & R & R \end{pmatrix},$$

$$\mathrm{rad}(\Lambda) = \mathrm{rad}(\Gamma) = \begin{pmatrix} \pi R & R & R & R \\ \pi R & \pi R & \pi R & \pi R \\ \pi R & \pi R & \pi R & \pi R \\ \pi R & \pi R & \pi R & \pi R \end{pmatrix} \quad ,$$

and so

$$\Lambda/\mathrm{rad}(\Lambda) = \mathfrak{k}_1 \textstyle\prod \mathfrak{k}_2 \prod \mathfrak{k}_3, \qquad \mathfrak{k}_1 = \mathfrak{k}, \quad i = 1, 2, 3.$$

$$\Gamma/\mathrm{rad}(\Gamma) = \mathfrak{k}_4 \textstyle\prod (\mathfrak{k}_5)_3,$$

$$S_4 = \mathfrak{k}_4, \qquad S_5 = \begin{pmatrix} \mathfrak{k} \\ \mathfrak{k} \\ \mathfrak{k} \end{pmatrix}.$$

$$\begin{aligned} {}_1S_4 &= {}_1\mathfrak{k}_4, & {}_1S_5 &= {}_1\mathfrak{k}_5, \\ {}_2S_4 &= 0, & {}_2S_5 &= {}_2\mathfrak{k}_5, \\ {}_3S_4 &= 0, & {}_3S_5 &= {}_3\mathfrak{k}_5. \end{aligned}$$

Hence, the valued graph is

$$\begin{tikzcd} & & 2 \arrow[d, "{(1,1)}"] & \\ 4 & 1 \arrow[l, "{(1,1)}"'] \arrow[r, "{(1,1)}"] & 5 & 3. \arrow[l, "{(1,1)}"'] \end{tikzcd}$$

EXAMPLE 7. Let L be a quadratic extension field of K with ring of integers S and assume that $\operatorname{rad}(S) = \pi S$. Then $[S/\operatorname{rad}(S): R/\pi R] = 2$.

We put

$$\Lambda = \left(\begin{tikzcd} R + \pi S \arrow[dr, dash] & S \\ \pi S & R + \pi S \end{tikzcd}\right), \qquad \Gamma = \begin{pmatrix} S & S \\ \pi S & S \end{pmatrix},$$

then

$$\operatorname{rad}(\Lambda) = \operatorname{rad}(\Gamma) = \begin{pmatrix} \pi S & S \\ \pi S & \pi S \end{pmatrix},$$

and

$$\Lambda/\operatorname{rad}(\Lambda) \cong \mathfrak{k}_1, \qquad \Gamma/\operatorname{rad}(\Gamma) = \mathfrak{K}_2 \prod \mathfrak{K}_3, \qquad \mathfrak{K}_2 \cong \mathfrak{K}_3 \cong S/\operatorname{rad}(S).$$

Then the associated graph is

$$3 \xleftarrow{(2,1)} 1 \xrightarrow{(1,2)} 2.$$

(2.6) We have to recall briefly some *concepts from the representation theory of valued graphs* (for the general theory we refer to [7].

A *valued graph* (G, δ) is a finite set G of vertices together with non-negative integers d_{ij} and d'_{ij} for all pairs $i, j \in G$ such that $d_{ii} = d'_{ii} = 0$, and subject to the condition that there exist non-zero natural numbers f_i satisfying

$$d_{ij} f_j = d'_{ji} f_i \qquad \text{for all} \quad i, j \in G.$$

(Note that there is a one-to-one correspondence between valued graphs and symmetrizable Cartan matrices [14].)

An *orientation* Ω of a valued graph (G, δ) is given by prescribing for each edge $\{i, j\}$ with $d_{ij} \neq 0$ an order—indicated by an arrow $i \Rightarrow j$. An orientation is said to be *admissible* if there exist no oriented loops, i.e., no circuits with orientation $i_1 \Rightarrow i_2 \Rightarrow \cdots \Rightarrow i_{S-1} \Rightarrow i_S = i_1$.

A *modulation* $\mathfrak{M}$ of a valued graph (G, δ) is a set of skewfields $\mathfrak{k}_i$, $i \in G$, which are finite dimensional over $\mathfrak{k}$, together with $(\mathfrak{k}_i, \mathfrak{k}_j)$-bimodules ${}_iM_j$ for all edges $\{i, j\}$ of G, such that

$$\dim_{\mathfrak{k}_i}({}_iM_j) = d_{ij}, \qquad \dim_{\mathfrak{k}_j}({}_iM_j) = d'_{ji},$$

and with $(\mathfrak{f}_i, \mathfrak{f}_j)$-bimodule homomorphisms

$$\mathrm{Hom}_{\mathfrak{f}_i}({}_iM_j, \mathfrak{f}_i) \simeq \mathrm{Hom}_{\mathfrak{f}_j}({}_iM_j, \mathfrak{f}_j).$$

This is certainly satisfied if $\mathfrak{f}_i$ and $\mathfrak{f}_j$ are finite dimensional over a common central subfield. We put

$${}_jM_i' = \mathrm{Hom}_{\mathfrak{f}_i}({}_iM_j, \mathfrak{f}_i) \simeq \mathrm{Hom}_{\mathfrak{f}_i}({}_iM_j, \mathfrak{f}_j).$$

A ***realization*** $(\mathfrak{M}, \Omega)$ of a valued graph (G, δ) is a modulation $\mathfrak{M}$ together with an admissible orientation Ω.

A ***representation*** $\mathfrak{X} = \{X_i, {}_j\varphi_i\}$ of a realization $(\mathfrak{M}, \Omega)$ of (G, δ) is a set of finite dimensional left $\mathfrak{f}_i$-vectorspaces X_i, $i \in G$, together with $\mathfrak{f}_j$-linear mappings

$${}_j\varphi_i : {}_jM_i \otimes_{\mathfrak{f}_i} X_i \to X_j \qquad \text{for all oritented edges} \quad i \to j.$$

A morphism $\alpha: \mathfrak{X} \Rightarrow \mathfrak{X}'$ between representations $\mathfrak{X} = \{X_i, {}_j\varphi_i\}$ and $\mathfrak{X}' = \{X_i', {}_j\varphi_i'\}$ is defined as a set $\alpha = (\alpha_i)$ of $\mathfrak{f}_i$-homomorphisms making the following diagrams commute

$$\begin{tikzcd}
{}_jM_i \otimes_{\mathfrak{f}_i} X_i \arrow[r, "{}_j\varphi_i"] \arrow[d, "1\otimes\alpha_i"'] & X_j \arrow[d, "\alpha_j"] \\
{}_jM_i \otimes_{\mathfrak{f}_i} X_i' \arrow[r, "{}_j\varphi_i'"] & X_j'
\end{tikzcd} , \qquad \text{for each edge} \quad i \to j.$$

These representations form an abelian category, denoted by $\mathfrak{L}(\mathfrak{M}, \Omega)$.

Given a representation $\mathfrak{X} = \{X_i, {}_j\varphi_i\}$ of a realization we define the dimension-map:

$$\dim: \mathfrak{L}(\mathfrak{M}, \Omega) \to \mathfrak{Q}^G,$$

where $\mathfrak{Q}^G$ is the rational vectorspace of dimension $|G|$, by

$$\dim: \mathfrak{X} \to (x_i)_{i \in G}, \quad x_i = \dim_{\mathfrak{f}_i}(X_i).$$

The vector $\dim(\mathfrak{X})$ is called the *dimensiontype* of the representation $\mathfrak{X}$.

For each $k \in G$ let $\mathfrak{x}_k \in \mathfrak{Q}^G$ denote the vector with $x_k = 1$, $x_i = 0$ for $i \neq k$, and for each $k \in G$ define linear transformations

$$\sigma_k: \mathfrak{Q}^G \to \mathfrak{Q}^G,$$
$$\mathfrak{x} \to \mathfrak{y},$$

with $x_i = y_i$ for $i \neq k$ and $y_k = -x_k + \sum_{i \in G} d_{ki} x_i$.

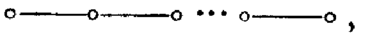
W_G will denote the *Weyl group of* G, i.e., the group of all linear transformations of $\mathbb{Q}^G$ generated by the reflections σ_k, $k \in G$. A vector $\mathfrak{r} \in \mathbb{Q}^G$ satisfying $\mathfrak{r}w = \mathfrak{r}$ for every $w \in W_G$ is said to be *stable*. A vector $\mathfrak{r} \in \mathbb{Q}^G$ is called a *root of* (G, δ), if $\mathfrak{r} = \mathfrak{r}_k w$ for some $k \in G$ and $w \in W_G$. A root $\mathfrak{r}$ is said to be *positive* if $x_i \geqslant 0$ for all $i \in G(\mathfrak{r} = (x_i))$ [2].

For the representations of a realization $(\mathfrak{M}, \Omega)$ of a connected valued graph (G, δ) we have as main result:

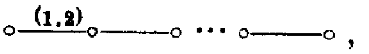
(2.7) THEOREM (Dlab–Ringel [7]):

(i) $\mathfrak{L}(\mathfrak{M}, \Omega)$ *is of finite representation type if and only if* (G, δ) *is a* Dynkin diagram, *i.e., a valued graph of one of the following forms*:

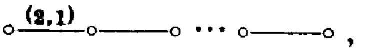
A_n :



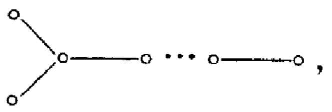
B_n :



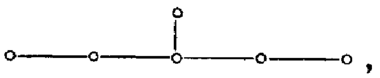
C_n :



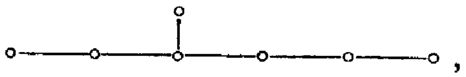
D_n :



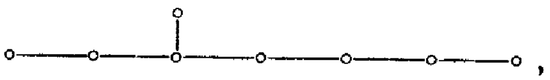
E_6 :



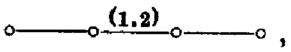
E_7 :



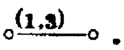
E_8 :



F_4 :



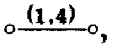
G_2 :



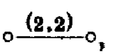
If there is no valuation written, then this shall mean that the valuation is (1, 1). *Moreover, the mapping* dim: $\mathfrak{L}(\mathfrak{M}, \Omega) \to \mathbb{Q}$ *induces a bijection between the isomorphism classes of indecomposable representations of* $(\mathfrak{M}, \Omega)$ *and the positive roots of* (G, δ).

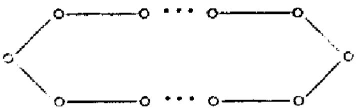
(ii) *If* (G, δ) *is an* Euclidean diagram, *i.e., a valued graph of one of the following forms*

$\tilde{A}_{11}$:



$\tilde{A}_{12}$:

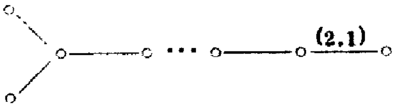


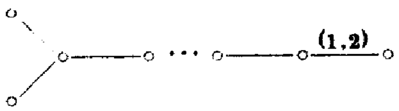
$\tilde{A}_n$:  ,

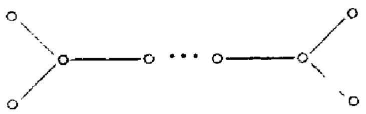
$\tilde{B}_n$:  ,

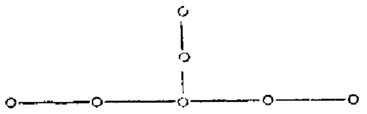
$\tilde{C}_n$:  ,

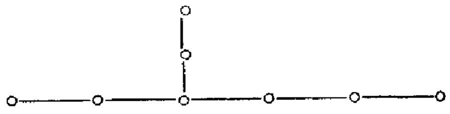
$\widetilde{BC}_n$:  ,

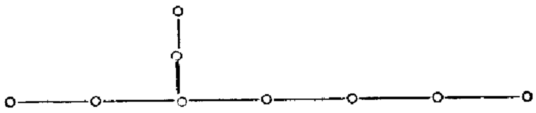
$\widetilde{CD}_n$:  ,

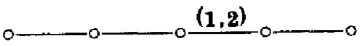
$\widetilde{DD}_n$:  ,

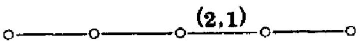
$\tilde{D}_n$:  ,

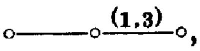
$\tilde{E}_6$:  ,

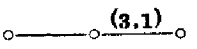
$\tilde{E}_7$:  ,

$\tilde{E}_8$:  ,

$\tilde{F}_{41}$:  ,

$\tilde{F}_{42}$:  ,

$\tilde{G}_{21}$:  ,

$\tilde{G}_{22}$:  ,

then the category $\mathfrak{L}(\mathfrak{M}, \Omega)$ *has two kinds of indecomposable representations: Those of discrete dimension types and those of continuous dimension types. The mapping* dim: $\mathfrak{L}(\mathfrak{M}, \Omega) \to \mathbb{Q}^G$ *induces a bijection between the isomorphism classes of indecomposable representations of discrete dimension type and the positive roots of* (G, δ). *The continuous dimension types are the positive integral multiples of the least stable positive integral vector of* $\mathbb{Q}^G$.

The explicit data for constructing the representations can be found in [7].

The aim of this section is to show that the indecomposable Λ-lattices in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ of a B-order are in bijection to the non-simple indecomposable representations of the graph (G, δ) of (2.5), (2.6), with the following orientation and modulation: There are arrows $i \to j$, if $d_{ij} \neq 0$ and $i < j$. This surely is an admissible orientation. The modulation is given as follows: The skewfields are the $\mathfrak{k}_i$, $1 \leqslant i \leqslant t$, with bimodules ${}_iS_j$, as defined in (2.4).

Hence, in the examples we have the following situation:

EXAMPLE 4.

$$\mathfrak{k} \xrightarrow{\mathfrak{k}} \mathfrak{k} \xrightarrow{\mathfrak{k}} \mathfrak{k} \quad \mathfrak{k} \xrightarrow{\mathfrak{k}} \mathfrak{k} \quad \text{is of type } A_3 \overset{\circ}{\cup} A_2\,,$$

EXAMPLE 5.

$$\mathfrak{k} \xrightarrow{\mathfrak{k}^{(2)}} \mathfrak{k} \quad \text{is of type} \quad A_{12}\,,$$

EXAMPLE 6.

$$\begin{tikzcd}
 & & \mathfrak{k} \arrow[d, "\mathfrak{k}"] & & \\
\mathfrak{k} \arrow[r, "\mathfrak{k}"] & \mathfrak{k} \arrow[r, "\mathfrak{k}"] & \mathfrak{k} \arrow[r, "\mathfrak{k}"] & \mathfrak{k} & \text{is of type} \quad D_5\,,
\end{tikzcd}$$

EXAMPLE 7.

$$\mathfrak{K} \xleftarrow{{}_{\mathfrak{K}}\mathfrak{K}_{\mathfrak{k}}} \mathfrak{k} \xrightarrow{{}_{\mathfrak{k}}\mathfrak{K}_{\mathfrak{K}}} \mathfrak{K}, \quad \text{is the Euclidean diagram } \tilde{C}_2\,.$$

We shall next construct—for a B-order Λ—a functor from ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ to $\mathfrak{L}(\mathfrak{M}, \Omega)$, where $(\mathfrak{M}, \Omega)$ is the above realization of the graph (G, δ) of the B-order Λ. We shall demonstrate this first with Example 4:

EXAMPLE 4. It is easily seen that the following are all indecomposable Λ-lattices—up to isomorphism:

$$P_1 = \begin{pmatrix} R \\ \pi R \\ \pi R \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \pi R \\ R \\ \pi R \end{pmatrix}, \qquad P_3 = \begin{pmatrix} R \\ R \\ R \end{pmatrix}, \qquad M = \begin{pmatrix} R \\ R \\ \pi R \end{pmatrix}.$$

Passing to the category $\mathfrak{C}$, we obtain the following indecomposable objects:

$$\begin{array}{cccc}
\mathfrak{k}_1 \xrightarrow{{}_4\tilde{\varphi}_1} \begin{pmatrix} \mathfrak{k}_4 \\ \mathfrak{k}_4 \end{pmatrix}; & \mathfrak{k}_2 \xrightarrow{{}_4\tilde{\varphi}_2} \begin{pmatrix} \mathfrak{k}_4 \\ \mathfrak{k}_4 \end{pmatrix}; & \mathfrak{k}_3 \xrightarrow{{}_5\tilde{\varphi}_3} \mathfrak{k}_5\,; & \mathfrak{k}_1 \oplus \mathfrak{k}_2 \longrightarrow \begin{pmatrix} \mathfrak{k}_4 \\ \mathfrak{k}_4 \end{pmatrix}; \\
1 \longmapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & 1 \longmapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & 1 \longmapsto 1, & (1, 1) \longmapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{array}$$

Before we can go on with this example, we have to prove a general result:

(2.8) LEMMA. *With the notation of* (2.4), *and for* X_i *and* X_j *finite dimensional* $\mathfrak{k}_i$*- and* $\mathfrak{k}_j$*-vectorspaces resp., there is a natural isomorphism*

$$\Phi\colon \operatorname{Hom}_{\mathfrak{k}_i}(X_i\,,\,{}_iS_j \otimes_{\mathfrak{k}_j} X_j) = \operatorname{Hom}_{\mathfrak{k}_j}({}_jS_i \otimes_{\mathfrak{k}_i} X_i\,,\,X_j),$$

where as above

$${}_jS_i \simeq \operatorname{Hom}_{\mathfrak{k}_j}({}_iS_j\,,\,\mathfrak{k}_j).$$

Proof. For brevity we write

$${}_iS_j^* = \operatorname{Hom}_{\mathfrak{k}_j}({}_iS_j\,,\,\mathfrak{k}_j).$$

Then we have isomorphisms

$$\operatorname{Hom}_{\mathfrak{k}_i}(X_i\,,\,S_j \otimes_{\mathfrak{k}_j} X_j) \simeq \operatorname{Hom}_{\mathfrak{k}_i}(X_i\,,\,{}_iS_j \otimes_{\mathfrak{k}_j} X_j) \simeq$$

$$\operatorname{Hom}_{\mathfrak{k}_i}(X_i\,,\,\operatorname{Hom}_{\mathfrak{k}_j}({}_iS_j^*,\,X_j)) = \operatorname{Hom}_{\mathfrak{k}_j}({}_iS_i^* \otimes_{\mathfrak{k}_i} X_i\,,\,X_j).$$

Under this isomorphism, the above indecomposable objects in Example 4 correspond to the following representations of $(\mathfrak{M}, \Omega)$:

$$\mathfrak{k}_1 \xrightarrow{\text{id}} \mathfrak{k}_4\,; \qquad \mathfrak{k}_2 \xrightarrow{\text{id}} \mathfrak{k}_4\,; \qquad \mathfrak{k}_3 \xrightarrow{\text{id}} \mathfrak{k}_5\,; \qquad \mathfrak{k}_1 \xrightarrow{\text{id}} \mathfrak{k}_4 \xleftarrow{\text{id}} \mathfrak{k}_2\,.$$

The above are indeed all the indecomposable non-simple representations of $A_3 \mathbin{\dot\cup} A_2$.

We shall next define a functor

$$\mathfrak{G}\colon \mathfrak{C} \to \mathfrak{L}(\mathfrak{M}, \Omega).$$

To do so, let $X \to^{\sigma} Y$ be an object in $\mathfrak{C}$. For $1 \leqslant i \leqslant s$ we put

$$X_i = \operatorname{Hom}_{\mathfrak{A}}(\mathfrak{k}_i\,,\,X);$$

then $X = \bigoplus_{i=1}^{s} X_i$, and for $s + 1 \leqslant j \leqslant t$ we put

$$X_j = \operatorname{Hom}_{\mathfrak{B}}(S_j\,,\,Y);$$

then with $V_j = S_j \otimes_{\mathfrak{k}_j} X_j$ we have

$$Y = \bigoplus_{j=s+1}^{t} V_j\,.$$

Moreover, σ is an $\mathfrak{A}$-monomorphism $\sigma\colon \bigoplus_{i=1}^{s} X_i \to \bigoplus_{j=s+1}^{t} V_j$, and hence it decomposes uniquely—observe that the X_i and V_j, $1 \leqslant i \leqslant s$, $s+1 \leqslant j \leqslant t$, are uniquely determined, not just up to isomorphism—into $\mathfrak{f}_i$-homomorphisms

$${}_j\tilde{\varphi}_i : X_i \to V_j = S_j \otimes_{\mathfrak{f}_j} X_j .$$

By (2.8) this determines a unique $\mathfrak{f}_j$-homomorphism

$${}_j\varphi_i : {}_jS_i \otimes_{\mathfrak{f}_i} X_i \to X_j .$$

Hence, we have constructed a representation in $\mathfrak{L}(\mathfrak{M}, \Omega)$.

(2.9) THEOREM II. $\mathfrak{G}$ *is a categorical equivalence between* $\mathfrak{C}$ *and the full subcategory of all objects in* $\mathfrak{L}(\mathfrak{M}, \Omega)$ *without simple direct summands.*

Proof. Recall that the simple objects in $\mathfrak{L}(\mathfrak{M}, \Omega)$ are exactly those

$$\mathfrak{X} = \{X_i, {}_j\varphi_i\} \text{ with } X_{i_0} = \mathfrak{f}_{i_0} \text{ and } X_i = 0 \text{ for } i \neq i_0$$

—i.e., the objects of dimensiontype $\mathfrak{r}_{i_0}$. Thus the image of $\mathfrak{G}$ does not contain any simple representation in $\mathfrak{L}(\mathfrak{M}, \Omega)$.

The remainder of the proof will consist of several lemmata.

(2.10) LEMMA. $\mathfrak{G}$ *is a functor.*

Proof. Given a morphism in $\mathfrak{C}$

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "\sigma'"] & Y' \end{tikzcd}$$

,

With the above decompositions we get commutative diagrams

$$\begin{tikzcd} X_i \arrow[r, "{}_j\tilde{\varphi}_i"] \arrow[d, "\alpha_i"'] & V_j \arrow[d, "\beta_j"] \\ X_i' \arrow[r, "{}_j\tilde{\varphi}_i'"] & V_j' \end{tikzcd}$$

Recall that $V_j = S_j \otimes_{\mathfrak{f}_j} X_j$ and that V_j is a left module over $(\mathfrak{f}_j)_{n_j}$, thus

$$\begin{aligned} \mathrm{Hom}_\Gamma(S_j \otimes_{\mathfrak{f}_j} X_j, S_j \otimes_{\mathfrak{f}_j} X_j') &= \mathrm{Hom}_{\mathfrak{f}_j}(X_j, \mathrm{Hom}_\Gamma(S_j, S_j \otimes_{\mathfrak{f}_j} X_j')) \\ &= \mathrm{Hom}_{\mathfrak{f}_j}(X_j, X_j'), \end{aligned}$$

hence, β_j is of the form

$$1 \otimes \bar{\beta}_j : S_j \otimes_{\mathfrak{f}_j} X_j \to S_j \otimes_{\mathfrak{f}_j} X'_j .$$

Because of the isomorphism Φ in (2.8), we get the commutative diagrams

$$\begin{tikzcd}
{}_jS_i \otimes_{\mathfrak{f}_i} X_i \arrow[r] \arrow[d, "1 \otimes \alpha_i"'] & X_j \arrow[d, "\beta_j"] \\
{}_jS_i \otimes_{\mathfrak{f}_i} X'_i \arrow[r] & X'_j
\end{tikzcd}$$

and therefore $\mathbb{G}$ is a functor.

(2.11) LEMMA. *If $X \to^{\sigma} Y$ in $\mathfrak{C}$ is indecomposable, so is its image under* $\mathbb{G}$.

Proof. The proof of (2.10) shows that

$$\mathrm{Morph}_{\mathfrak{C}}(X \xrightarrow{\sigma} Y, X' \xrightarrow{\sigma'} Y') = \mathrm{Morph}_{\mathfrak{L}(\mathfrak{M},\Omega)}(\mathbb{G}(X \xrightarrow{\sigma} Y), \mathbb{G}(X' \xrightarrow{\sigma'} Y')),$$

and clearly isomorphisms and split morphisms are preserved and recovered; whence the statement of (2.11).

(2.12) LEMMA. *Let $\mathfrak{X} = \{X_i , {}_j\varphi_i\}$ be a non-simple indecomposable object in $\mathfrak{L}(\mathfrak{M}, \Omega)$, then it lies—up to isomorphism—in the image of* $\mathbb{G}$.

Proof. We put $V_j = S_j \otimes_{\mathfrak{f}_j} X_j$, $s + 1 \leqslant j \leqslant t$; the morphisms

$${}_j\varphi_i : {}_jS_i \otimes_{\mathfrak{f}_i} X_i \to X_j$$

induce—because of (2.8)—$\mathfrak{f}_i$-homomorphisms

$${}_j\bar{\varphi}_i : X_i \to V_j .$$

We now put

$$X = \bigoplus_{i=1}^{s} X_i , \quad Y = \bigoplus_{j=s+1}^{t} V_j , \quad \sigma = \bigoplus_{i,j} {}_j\tilde{\varphi}_i .$$

Then X is an $\mathfrak{A}$-module, Y is a $\mathfrak{B}$-module and σ is an $\mathfrak{A}$-homomorphism. If we can show that σ is monic and $\mathfrak{B}\,\mathrm{Im}(\sigma) = Y$, then $X \to^{\sigma} Y \in \mathfrak{C}$, and surely $\mathbb{G} : (X \to^{\sigma} Y) \rightsquigarrow \mathfrak{X}$.

Claim. σ is monic.

Proof. Assume to the contrary that $\mathrm{Ker}(\sigma) \neq 0$, then

$$\mathrm{Ker}(\sigma) = \bigoplus_i \left(\mathrm{Ker}\left(\bigoplus_j {}_j\tilde{\varphi}_i\right)\right);$$

i.e., $\mathrm{Ker}(\sigma) = \oplus_i X_i'$ with $X_i'\mathfrak{f}_i$-submodules of X_i, not all zero. Now, $\mathfrak{X}' = \{X_i', 0\}$ is a representation in $\mathfrak{L}(\mathfrak{M}, \Omega)$, which is different from zero, and which is a direct summand of $\mathfrak{X}$, since the following diagram is commutative

$$\begin{tikzcd} {}_jS_i \otimes_{\mathfrak{f}_i} X_i \arrow[r, "{}_j\varphi_i"] & X_j \\ {}_jS_i \otimes_{\mathfrak{f}_i} X_i' \arrow[r] \arrow[u, "i \otimes \iota_i"'] & 0 \arrow[u] \end{tikzcd},$$

where $\iota_i\colon X_i' \to X_i$ are the inclusions; the splitting $X_i \to X_i'$ obviously also makes the diagram commute. Hence, by the indecomposability of $\mathfrak{X}$, either $\mathfrak{X} = \mathfrak{X}'$ or $\mathfrak{X}' = 0$. If $\mathfrak{X} = \mathfrak{X}'$, it is easily seen, that $\mathfrak{X} = \mathfrak{X}'$ is a direct sum of simple representations, a contradiction to our assumption. Whence $\mathrm{Ker}(\sigma) = 0$.

We now assume that $\mathfrak{B}\,\mathrm{Im}(\sigma) \neq Y$. Let $C = Y/(\mathfrak{B}\,\mathrm{Im}(\sigma))$. Then a similar argument as above shows that $\mathfrak{X}$ must be simple. This completes the proof of (2.12) and also that of Theorem II.

We can now state the main result in this section, which follows easily from Theorems I and II.

(2.13) THEOREM III. *Let Λ be a B-order with valued graph (G, δ).*

(i) *${}_\Lambda\mathfrak{M}^0(\Gamma)$ is of finite lattice type if and only if (G, δ) is a finite union of Dynkin diagrams. In this case the isomorphism classes of indecomposable Λ-lattices correspond bijectively to the non-simple positive roots of (G, δ).*

(ii) *If (G, δ) is the union of Dynkin diagrams and Euclidean diagrams, then the isomorphism classes of indecomposable Λ-lattices in ${}_\Lambda\mathfrak{M}^0(\Gamma)$ can be classified according to the classification of the indecomposable representations of the diagrams in* (2.7).

(2.14) *Remarks.* (i) It should be noted, that in both cases the indecomposable representations of ${}_\Lambda\mathfrak{M}^0(\Gamma)$ can be listed explicitly (cf. examples below).

(ii) It is remarkable that Bäckström [1] has proved the first part of (2.13) for some Bäckström-orders by computation without reference to Dynkin diagrams, in case $\mathfrak{f}_i = \mathfrak{f}_j = R/\mathrm{rad}(R)$ is finite; he has even given a bound on the number of generators for the indecomposable representations, though he does not indicate how to classify them.

(iii) To every oriented admissible valued graph with $d_{ij} \leqslant d_{ij}'$ for an arrow $i \to j$, one can construct a Bäckström-order which has this graph.

(iv) In case of Dynkin diagrams, the indecomposable representations are determined by their dimension types (2.7). In terms of lattices this means that the indecomposable lattices M in ${}_{\Lambda}\mathfrak{M}^0(\Gamma)$ are uniquely determined by ΓM and $M/\mathrm{rad}(M)$.

(v) It is interesting to note that the finiteness of the representation type does not depend on the choice of R, if one considers Bäckström-orders. In general this depends heavily on the ground ring, e.g., for integral group rings of finite groups.

(vi) There may be more than one Bäckström-order corresponding to a fixed valued graph: e.g.,

$$\Lambda = \{(r, r \cdot \pi R), r \in R\} \qquad \text{and} \quad \Lambda' = \begin{pmatrix} R & R \\ \pi R & R \end{pmatrix}$$

both have graph $\cdot \leftarrow \cdot \rightarrow \cdot$: i.e., A_3.

There are some immediate consequences for general orders.

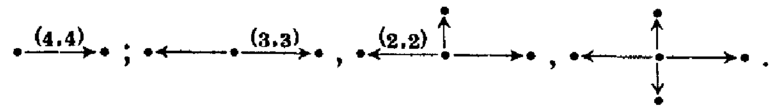
(2.15) PROPOSITION. *Let Λ be an arbitrary R-order, and choose a hereditary order Γ such that* $\mathrm{rad}(\Lambda) \subset \mathrm{rad}(\Gamma)$. *Then* $\Gamma/\mathrm{rad}(\Gamma)$ *is a* $\Lambda/\mathrm{rad}(\Lambda)$*-module, and as in* (2.5) *we may associate a graph $(G, \delta)_\Lambda$ with Λ. If $(G, \delta)_\Lambda$ is not a disjoint union of Dynkin diagrams, then Λ is of infinite lattice type.*

Proof. Since $\mathrm{rad}(\Lambda) \subset \mathrm{rad}(\Gamma)$, we conclude that $\Lambda/\mathrm{rad}(\Lambda) \simeq (\Lambda + \mathrm{rad}(\Gamma))/\mathrm{rad}(\Gamma)$. Now $\Lambda_0 = \Lambda + \mathrm{rad}(\Gamma)$ is a Bäckström-order with graph $(G, \delta)_\Lambda$. If Λ is of finite type, the same must hold for Λ_0, and so $(G, \delta)_\Lambda$ must be a union of Dynkin diagrams.

From this follow many of the well known necessary criteria for Λ to be of finite lattice type, e.g.

(2.16) PROPOSITION (Dade [4], [22]). *Let Λ be an R-order in A, and let e be a primitive idempotent of Λ. If Ae is the direct sum of t simple A-modules with $t \geqslant 4$, then Λ is of infinite lattice type.*

Proof. Because of (2.15) we may assume that Λ is a Bäckström-order. If $t \geqslant 4$, then (G, δ) has a subgraph of one of the following types:



In each of these cases (G, δ) can not be a union of Dynkin diagrams.

Remark. From the above one can easily derive the dual statement: There can not be t, $t \geqslant 4$, non-isomorphic projective Λ-modules, which contribute to one projective module over the hereditary order.

We shall conclude this paper by giving some *explicit examples.*

EXAMPLE 4. $(G, \delta) = A_3 \cup A_2$, and so there are 4 non-simple positive roots; i.e., the representations listed above form a complete set of indecomposable ones.

EXAMPLE 5.

$$\Lambda = \begin{pmatrix} R & & \pi R \\ & \diagdown & \\ \pi R & & R \end{pmatrix} \text{ has graph } \circ\overset{(2,2)}{\text{———}}\circ,$$

and so it corresponds to the Euclidean diagram $\tilde{A}_{12}$. For the sake of simplicity we assume that $\mathfrak{k}$ is algebraically closed. Then there are five types of indecomposable Λ-lattices:

(i) $$M_n = \begin{pmatrix} R & & R & & R & & \cdots & & R & & R & & \pi R \\ & \diagdown & & \diagdown & & \diagdown & & \diagdown & & \diagdown & & \diagdown & \\ \pi R & & R & & R & & \cdots & & R & & R & & R \end{pmatrix}$$

(ii) $$M_n' = \begin{pmatrix} R & & R & & R & & \cdots & & R & & R & & R \\ & \diagdown & & \diagdown & & \diagdown & & \diagdown & & \diagdown & & \diagdown & \\ R & & R & & R & & \cdots & & R & & R & & R \end{pmatrix}$$

(iii) $$M_{n,0} = \begin{pmatrix} R & & R & & R & & \cdots & & R & & R & & R \\ & \diagdown & & \diagdown & & \diagdown & & \diagdown & & \diagdown & & \diagdown & \\ \pi R & & R & & R & & \cdots & & R & & R & & R \end{pmatrix}$$

(iv) $$M_{n,\infty} = \begin{pmatrix} R & & R & & R & & \cdots & & R & & R & & \pi R \\ & \diagdown & & \diagdown & & \diagdown & & \diagdown & & \diagdown & & \diagdown & \\ R & & R & & R & & & & R & & R & & R \end{pmatrix}$$

(v) $$M_{n,\alpha} = \left\{ \begin{pmatrix} r_1 & r_2 & r_3 & \cdots & r_n \\ \alpha r_1 & r_1 + \alpha r_2 & r_2 + \alpha r_3 & \cdots & r_{n-1} + \alpha r_n \end{pmatrix} \middle| \, r_i \in R \right\},$$

where in (v) the element α runs through a complete set of representatives of the non-zero residue classes in R/π. (If $\mathfrak{k}$ is not algebraically closed, then the eigenvalues α have to be replaced by the compagnion matrix of the corresponding irreducible polynomial.)

EXAMPLE 6.

$$\Lambda = \begin{pmatrix} R\diagdown & R & R & R \\ \pi R & R & \pi R & \pi R \\ \pi R & \pi R & R & \pi R \\ \pi R & \pi R & \pi R & R \end{pmatrix}, \qquad (G, \delta) = \begin{tikzcd} & & 2 \arrow[d] & \\ 4 & 1 \arrow[l] \arrow[r] & 5 & 3. \arrow[l] \end{tikzcd}$$

The graph is D_5, and it has 20 indecomposable representations, among them five simple ones. Hence from these representations we obtain the indecomposable Λ-lattices as follows:

1.)
$$\begin{tikzcd} & & 0 \arrow[d] & \\ \mathfrak{k} & \mathfrak{k} \arrow[l, "\mathrm{id}"'] \arrow[r] & 0 & 0; \arrow[l] \end{tikzcd} \qquad \begin{pmatrix} R \\ \pi R \\ \pi R \\ \pi R \end{pmatrix}$$

2.)
$$\begin{tikzcd} & & 0 \arrow[d] & \\ \mathfrak{k} & \mathfrak{k} \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & \mathfrak{k} & 0; \arrow[l] \end{tikzcd} \qquad \left(\begin{tikzcd} R \arrow[dr, equal] & R \\ \pi R & R \\ \pi R & \pi R \\ \pi R & \pi R \end{tikzcd}\right)$$

3.)
$$\begin{tikzcd} & & \mathfrak{k} \arrow[d, "\mathrm{id}"] & \\ \mathfrak{k} & \mathfrak{k} \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & \mathfrak{k} & 0; \arrow[l] \end{tikzcd} \qquad \left(\begin{tikzcd} R \arrow[dr, equal] & R \\ \pi R & R \\ \pi R & R \\ \pi R & \pi R \end{tikzcd}\right)$$

4.)
$$\begin{tikzcd} & & \mathfrak{k} \arrow[d, "\mathrm{id}"] & \\ \mathfrak{k} & \mathfrak{k} \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & \mathfrak{k} & \mathfrak{k}; \arrow[l, "\mathrm{id}"'] \end{tikzcd} \qquad \left(\begin{tikzcd} R \arrow[dr, equal] & R \\ \pi R & R \\ \pi R & R \\ \pi R & R \end{tikzcd}\right)$$

5.)
$$\begin{tikzcd} & & 0 \arrow[d] & \\ \mathfrak{k} & \mathfrak{k} \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & \mathfrak{k} & \mathfrak{k}; \arrow[l, "\mathrm{id}"'] \end{tikzcd} \qquad \left(\begin{tikzcd} R \arrow[dr, equal] & R \\ \pi R & R \\ \pi R & \pi R \\ \pi R & R \end{tikzcd}\right)$$

6.)
$$\begin{tikzcd} & & 0 \arrow[d] & \\ 0 & \mathfrak{k} \arrow[l] \arrow[r, "\mathrm{id}"] & \mathfrak{k} & 0; \arrow[l] \end{tikzcd} \qquad \begin{pmatrix} R \\ R \\ \pi R \\ \pi R \end{pmatrix}$$

7.)
$$\begin{tikzcd} & & \mathfrak{k} \arrow[d, "\mathrm{id}"] & \\ 0 & \mathfrak{k} \arrow[l] \arrow[r, "\mathrm{id}"] & \mathfrak{k} & 0; \arrow[l] \end{tikzcd} \qquad \begin{pmatrix} R \\ R \\ R \\ \pi R \end{pmatrix}$$

8.)
$$\begin{tikzcd}
 & & \mathfrak{k} \arrow[d] & \\
0 & \mathfrak{k} \arrow[l] \arrow[r, "\mathrm{id}"] & \mathfrak{k} & \mathfrak{k}; \arrow[l, "\mathrm{id}"']
\end{tikzcd}
\qquad
\begin{pmatrix} R \\ R \\ R \\ R \end{pmatrix}$$

9.)
$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
0 & \mathfrak{k} \arrow[l] \arrow[r, "\mathrm{id}"] & \mathfrak{k} & \mathfrak{k}; \arrow[l, "\mathrm{id}"']
\end{tikzcd}
\qquad
\begin{pmatrix} R \\ R \\ \pi R \\ R \end{pmatrix}$$

10.)
$$\begin{tikzcd}
 & & \mathfrak{k} \arrow[d, "\mathrm{id}"] & \\
0 & 0 \arrow[l] \arrow[r] & \mathfrak{k} & \mathfrak{k}; \arrow[l, "\mathrm{id}"']
\end{tikzcd}
\qquad
\begin{pmatrix} R \\ \pi R \\ R \\ R \end{pmatrix}$$

11.)
$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
0 & 0 \arrow[l] \arrow[r] & \mathfrak{k} & \mathfrak{k}; \arrow[l, "\mathrm{id}"']
\end{tikzcd}
\qquad
\begin{pmatrix} R \\ \pi R \\ \pi R \\ R \end{pmatrix}$$

12.)
$$\begin{tikzcd}
 & & \mathfrak{k} \arrow[d, "\mathrm{id}"] & \\
0 & 0 \arrow[l] \arrow[r] & \mathfrak{k} & 0; \arrow[l, "\mathrm{id}"']
\end{tikzcd}
\qquad
\begin{pmatrix} R \\ \pi R \\ R \\ \pi R \end{pmatrix}$$

13.)
$$\begin{tikzcd}
 & & \mathfrak{k} \arrow[d, "\Delta"] & \\
0 & \mathfrak{k} \arrow[l] \arrow[r, "\iota_1"] & \mathfrak{k}^{(2)} & \mathfrak{k}; \arrow[l, "\iota_2"']
\end{tikzcd}
\qquad
\begin{pmatrix} R & & R \\ R & & \pi R \\ R & = & R \\ \pi R & & R \end{pmatrix}$$

14.)
$$\begin{tikzcd}
 & & \mathfrak{k} \arrow[d, "\Delta"] & \\
\mathfrak{k} & \mathfrak{k} \arrow[l, "\mathrm{id}"'] \arrow[r, "\iota_1"] & \mathfrak{k}^{(2)} & \mathfrak{k}; \arrow[l, "\iota_2"']
\end{tikzcd}
\qquad
\begin{pmatrix} R & \diagdown & R & & R \\ \pi R & & R & & \pi R \\ \pi R & & R & = & R \\ \pi R & & \pi R & & R \end{pmatrix}$$

$$\begin{tikzcd} & & \mathfrak{k} \arrow[d, "\Delta"] & \\ 15.) \quad \mathfrak{k} & \mathfrak{k}^{(2)} \arrow[l, "\pi_2"'] \arrow[r, "\mathrm{id}"] & \mathfrak{k}^{(2)} & \mathfrak{k}; \arrow[l, "\iota_2"'] \end{tikzcd} \qquad \begin{pmatrix} R & \diagdown\!\!\diagdown & R & & R \\ \pi R & & R & & R \\ \pi R & & R & = & R \\ \pi R & & R & & \pi R \end{pmatrix}$$

Here id stands for the identity map, ι_1 and ι_2 indicate injections and π_1, π_2 projections onto the indicated components: Δ is the diagonal map.

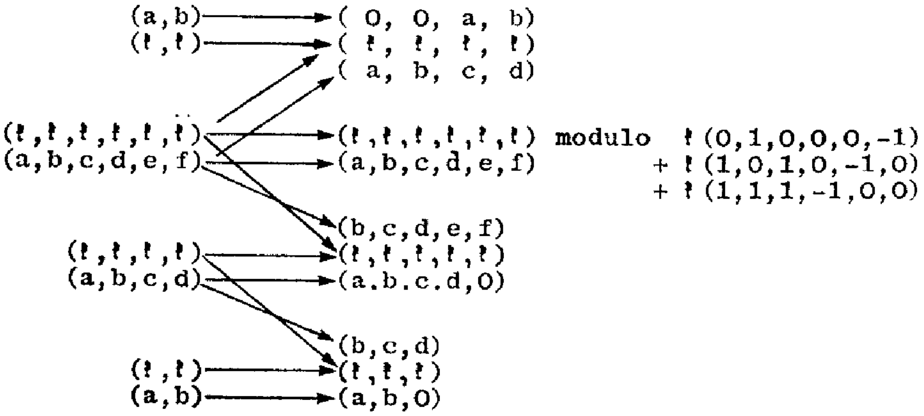
EXAMPLE.

$$\Lambda = \left\{ \begin{bmatrix} R & \pi R & R & R & R & R & R \\ \pi R & \alpha & R & R & R & R & R \\ \pi R & \pi R & \alpha' & R & R & R & R \\ \pi R & \pi R & \pi R & \alpha'' & \pi R & R & R \\ \pi R & \pi R & \pi R & \pi R & \beta & R & R \\ \pi R & \pi R & \pi R & \pi R & \pi R & \beta' & \pi R \\ \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & R \end{bmatrix}, \alpha \equiv \alpha' \equiv \alpha'', \beta \equiv \beta' \bmod(\pi R) \right\}$$

is a Bäckström-order with graph E_8:

$$1 \rightarrow 5 \leftarrow 2 \rightarrow 7 \leftarrow 3 \rightarrow 8 \leftarrow 4.$$

The highest dimensional indecomposable representation is given as follows:



The corresponding indecomposable Λ-lattice has R-rank 105, and is given as follows

$$M = \left\{ \begin{bmatrix} \pi R & \pi R & R & R & \pi R & \pi R & \pi R & R & R & R & R & R & R & R & R \\ \alpha & \alpha' & \alpha'' & \alpha''' & \pi R & \pi R & \pi R & R & R & R & R & R & R & R & R \\ \pi R & \pi R & \pi R & \pi R & \beta & \beta' & \beta'' & R & R & R & R & R & R & R & R \\ \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \gamma & \gamma' & \gamma'' & \gamma''' & \gamma'''' & \pi R & \pi R & \pi R \\ \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & R & \delta & \epsilon & \iota & \pi R & \pi R & \pi R & \pi R \\ \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \delta' & \epsilon' & \iota' \\ \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & \pi R & R & R & R \end{bmatrix}, \right\}$$

subject to the following conditions, which are to be interpreted as congruences modulo πR:

$$\begin{aligned}
\beta &\equiv \alpha + \alpha''' + \gamma''';\\
\beta' &\equiv \alpha' + \alpha''' + \gamma'''';\\
\beta'' &\equiv \alpha'' + \alpha''' + \gamma''';\\
\alpha' &\equiv \gamma;\\
\alpha'' &\equiv \gamma';\\
\alpha''' &\equiv \gamma'';\\
\delta &\equiv \delta';\\
\epsilon &\equiv \epsilon';\\
\iota &\equiv \iota'.
\end{aligned}$$

3. The Brauer–Thrall Conjectures

One of the reasons for considering the representation equivalence stated in Theorem A, was the question, whether the well-known distinction between the various representation types of Artin algebras carries over to the categories of lattices over orders.

(3.1) First, let us note that the category $\mathfrak{C}$ considered in Theorem A is, in fact, a full subcategory of the category of representations over a convenient Artin algebra. As in Section 1, choose a hereditary R-order Γ containing Λ and a two-sided Γ-ideal I with $I \subseteq \operatorname{rad}(\Lambda)$ and define $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ as in Section 1. Denote by $\tilde{\mathfrak{C}}$ the category with objects $\sigma\colon U \to V$, where U is a finitely generated $\mathfrak{A}$-module, V is a finitely generated $\mathfrak{B}$-module, and σ is an $\mathfrak{A}$-homomorphism. Morphisms are the obvious commutative diagrams. Then it is clear that $\tilde{\mathfrak{C}}$ is equivalent to the category of finitely generated left modules over the triangular matrix ring

$$\mathfrak{D} = \begin{pmatrix} \mathfrak{B} & {}_{\mathfrak{B}}\mathfrak{B}_{\mathfrak{A}} \\ 0 & \mathfrak{A} \end{pmatrix},$$

which is an Artin algebra. By definition, $\mathfrak{C}$ is a full subcategory of $\tilde{\mathfrak{C}}$. In this way, Theorem A relates Λ-lattices to (suitable) $\mathfrak{D}$-modules. Note however, that it may happen that $\mathfrak{C}$ (and therefore ${}_{\Lambda}\mathfrak{M}^0$) has only a finite number of isomorphism classes of indecomposable objects, whereas in $\tilde{\mathfrak{C}}$, there is an infinite number of such isomorphism classes (see Example 3). Since $\tilde{\mathfrak{C}}$ is a module category, there are some rather general results available, as the solution of the second Brauer–Thrall conjecture [18]. It seems plausible that similar results will hold for certain full subcategories of module categories, as for example the category $\mathfrak{C}$, considering both the known examples and the methods of proof. In this last section, we will show that the expected properties of $\mathfrak{C}$

carry over to corresponding properties of ${}_{\Lambda}\mathfrak{M}^0$. In this way, we would like to initiate a study of subcategories of module categories similar to $\mathfrak{C}$.

In the proof of the second Brauer–Thrall conjecture for an Artin algebra, a one-parameter family of indecomposable modules X^t is constructed, and to every t a chain of indecomposable modules

$$X_1{}^t \subset X_2{}^t \subset \cdots \subset X_1{}^t \subset X_{i+1}^t \subset \cdots$$

such that $X_i{}^t/X_{i-1}^t \approx X_1{}^t$ (here, $X_0{}^t = 0$).

(3.2) LEMMA. *Let X_1 be an object in $\mathfrak{C}$, and let*

$$X_1 \subset X_2 \subset \cdots \subset X_i \subset X_{i+1} \subset \cdots$$

be a chain of indecomposable modules in $\tilde{\mathfrak{C}}$ such that for all i,

$$X_i/X_{i-1} \approx X_1 \qquad \text{for all} \quad i \geqslant 2.$$

Then there exists a chain of indecomposable Λ-lattices

$$M_1 \subset M_2 \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots$$

such that $M_i/M_{i-1} \approx M_1$ for all $i \geqslant 2$, and $X_i = (M_i/IM_i \hookrightarrow \Gamma M_i/IM_i)$.

Proof. Note that the category $\mathfrak{C}$ is closed under extensions in $\tilde{\mathfrak{C}}$, thus with X_1 also all X_i belong to $\mathfrak{C}$. According to Theorem A, there exist indecomposable Λ-lattices M_i such that

$$(M_i \xrightarrow{\sigma_i} \Gamma M_i/IM_i) = X_i\,.$$

We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
M_i/IM_i \arrow[r, "\sigma_i"] \arrow[d, "\alpha"] & \Gamma M_i/IM_i \arrow[d, "\beta"] \\
M_{i+1}/IM_{i+1} \arrow[r, "\sigma_{i+1}"] \arrow[d, "\alpha'"] & \Gamma M_{i+1}/IM_{i+1} \arrow[d, "\beta"] \\
M_1/IM_1 \arrow[r, "\sigma_1"] \arrow[d] & \Gamma M_1/IM_1 \arrow[d] \\
0 & 0
\end{tikzcd}$$

with exact columns. Since the $\mathfrak{B}$-modules in the right column are projective, this column is split exact; moreover, β and β' are induced from homomorphisms of Γ-lattices, say β_1 and β_1', such that the sequence

$$0 \longrightarrow \Gamma M_i \xrightarrow{\beta_1} \Gamma M_{i+1} \xrightarrow{\beta_1'} \Gamma M_1 \longrightarrow 0$$

is split exact. Hence, we have the following situation:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & M_i \arrow[r, "\tau_i"] \arrow[d, "\alpha_1"'] & \Gamma M_i \arrow[d, "\beta_1"] \\
0 \arrow[r] & M_{i+1} \arrow[r, "\tau_{i+1}"] \arrow[d, "\alpha_1'"'] & \Gamma M_{i+1} . \arrow[d, "\beta_1'"] \\
0 \arrow[r] & M_1 \arrow[r, "\tau"] & \Gamma M_1
\end{tikzcd}$$

Here we have put $\alpha_1 = \beta_1 |_{M_i}$ and $\alpha_1' = \beta_1' |_{M_{i+1}}$.

It should be noted that we view M_i as submodule of ΓM_i etc. Then α_1 and α_1' are Λ-homomorphisms,

$$\alpha_1 : M_i \rightarrow M_{i+1}, \qquad \alpha_1' : M_{i+1} \rightarrow M_1,$$

which reduce to α and α' resp., and which make the above diagram commute. Moreover, α_1 is injective and α_1' is surjective, since $I \subset \operatorname{rad}(\Lambda)$. Because of $\beta_1\beta_1' = 0$ we have $\alpha_1\alpha_1' = 0$. Let now $x\alpha_1' = 0$, then $x\tau_{i+1} = y\beta_1$ with $y \in \Gamma M_i$. On the other hand there exists $y' \in M_i$ such that $y'\alpha_1 - x = z \in IM_{i+1}$. Hence, $y - y' \in IM_i$, and so $y \in M_i$; i.e., the left hand column is exact.

(3.3) Note that in case there exists a chain of indecomposable Λ-lattices

$$M_1 \subset M_2 \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots,$$

with $M_{i+1}/M_i \approx M_1$ for all $i \geqslant 2$, then the union $M = \varinjlim M_i$ is R-free but does not split off any Λ-lattice.

Proof. Since M_{i+1}/M_i is a Λ-lattice, the embedding $M_i \subset M_{i+1}$ is pure, thus M is R-free.

Assume that there exists $N \in {}_{\Lambda}\mathfrak{M}^0$ such that N is a direct summand of L, say $\pi: L \rightarrow N$ is a splitting with injection $\iota: N \rightarrow L$. Then there exists an index i_0 such that for all $i > i_0$, $\operatorname{Im}(\iota) \subset M_i$. Thus N is a direct summand of M_i for all $i > i_0$, a contradiction.

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