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Socle-Determined Categories of Representations of Artinian Hereditary Tensor Algebras

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1. INTRODUCTION

Let R be a complete valuation ring with field of quotients K and residue field $\mathfrak{k}$. Let A be a finite-dimensional separable K-algebra. In these notes we consider the problem of constructing all indecomposable Λ-lattices of those R-orders Λ which satisfy the following conditions:

(1.1) (i) There exists a maximal R-order Γ in A such that $\operatorname{rad} \Gamma \subset \Lambda \subset \Gamma$. ($\operatorname{rad} \Gamma$ is the Jacobson radical of Γ.)

(ii) If G_i, $1 \leqslant i \leqslant s$, are the nonisomorphic indecomposable left Γ-lattices, then $\operatorname{Hom}_R(G_i, R)$ are projective right Λ-lattices.

(iii) $\mathfrak{A} = \Lambda/\operatorname{rad} \Gamma$ is a hereditary $\mathfrak{k}$-algebra.

It was shown in [3] that Λ satisfying (1.1) has global dimension of at most two. However, not all orders of global dimension two satisfy (1.1) [4].

It also should be noted that given any hereditary $\mathfrak{k}$-algebra $\mathfrak{A}$, it is possible to construct Λ satisfying (1.1) by embedding $\mathfrak{A}$ suitably into a product of full matrix algebras.

The main result in a subsequent paper [5] will be that in case Λ has finite lattice type, all indecomposable Λ-lattices are obtained from the projective ones via almost split sequences. The main device to prove this is a reduction to the Artinian case.

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Let $\mathfrak{A}$ be a hereditary finite-dimensional $\mathfrak{k}$-algebra, and let $S_1, \ldots, S_s$ be the nonisomorphic simple projective left $\mathfrak{A}$-modules. Then we denote by ${}_{\mathfrak{A}}\mathfrak{M}(S)$ the full subcategory of all finitely generated left $\mathfrak{A}$-modules, ${}_{\mathfrak{A}}\mathfrak{M}$, defined as follows

$$ {}_{\mathfrak{A}}\mathfrak{M}(S) = \left\{ U \in {}_{\mathfrak{A}}\mathfrak{M} \colon \operatorname{Soc}(U) \simeq \bigoplus_{i=1}^{s} S_i^{(n_i)} \right\}, $$

where $\operatorname{Soc}(U)$ denotes the socle of U, and $X^{(n)}$ denotes the direct sum of n copies of X. That means that ${}_{\mathfrak{A}}\mathfrak{M}(S)$ is the category of finitely generated left $\mathfrak{A}$-modules with projective socle. Let Λ satisfy (1.1); the following result will be shown in [5].

(1.2) THEOREM. *Let $\mathfrak{A} = \Lambda/\mathrm{rad}\,\Gamma$. The functor*

$$ F\colon {}_{\Lambda}\mathfrak{M}^0 \to {}_{\mathfrak{A}}\mathfrak{M}(S) \qquad \textit{with} \qquad M \mapsto M/(\mathrm{rad}\,\Gamma)M $$

is exact, and it is a representation equivalence. Here ${}_{\Lambda}\mathfrak{M}^0$ denotes the category of left Λ-lattices; i.e., left Λ-modules which are R-free of finite rank.

Thus, in order to study the representation theory of Λ it is enough to study the representation theory of ${}_{\mathfrak{A}}\mathfrak{M}(S)$. This will be done here.

We shall give a characterization of those hereditary $\mathfrak{k}$-tensor algebras $\mathfrak{A}$ such that ${}_{\mathfrak{A}}\mathfrak{M}(S)$ has only finitely many indecomposable objects. In order to state the result we have to introduce some *notation:*

Let γ be an oriented graph with valuation and without oriented loops; let $(F_i, {}_iM_j)$ be a $\mathfrak{k}$-species for γ [2, 1]. If $\mathfrak{A}$ is the tensor algebra of $(F_i, {}_iM_j)$, then $\mathfrak{A}$ is a hereditary $\mathfrak{k}$-algebra.

(1.3) DEFINITION. We say that γ is *reducible* if there exists an edge $a \to^{(1,1)} b$ such that the graph which is obtained from γ by removing the edge $a \to^{(1,1)} b$ is the disjoint union $\gamma' \cup A_m$ of some graph γ' and a second graph of type A_m, and a is a sink in γ' and b is a sink in A_m.

We then denote by $\gamma_{a=b}$ the graph obtained from γ by identifying a and b and omitting the edge between them, and say that $\gamma_{a=b}$ is obtained from γ by reduction. For example,

$$ \gamma\colon \begin{array}{c} \\ \to \\ \\ \end{array} \begin{array}{c} \downarrow \\ \cdot \\ \uparrow \end{array} \begin{array}{c} \\ \to\ \to \\ \\ \end{array} $$

can be reduced to

$$ \gamma_0\colon \begin{array}{c} \\ \to \\ \\ \end{array} \begin{array}{c} \downarrow \\ \cdot \\ \uparrow \end{array}. $$

γ is said to be irreducible if it cannot be reduced.

If $(F_i, {}_iM_j)$ is a $\mathfrak{k}$-species for γ, and $\gamma_0 = \gamma_{a=b}$ then the corresponding reduced $\mathfrak{k}$-species—observe ${}_aM_b \simeq F_a \simeq F_b$—is $(F_i', {}_iM_j')$, where ${}_aM_b$ is omitted and F_a identified with F_b.

(1.4) THEOREM I. (i) *Let $\mathfrak{A}$ be the tensor algebra of a $\mathfrak{k}$-species for γ. Then ${}_\mathfrak{A}\mathfrak{M}(S)$ has finitely many indecomposable objects if and only if ${}_{\mathfrak{A}_0}\mathfrak{M}(S)$ has finitely many indecomposable objects, where $\mathfrak{A}_0$ is the tensor algebra of the reduced $\mathfrak{k}$-species for γ_0, γ_0 is irreducible, and obtained from γ by a finite number of reductions.*

(ii) *If γ is irreducible, then ${}_\mathfrak{A}\mathfrak{M}(S)$ has finitely many indecomposable objects if and only if γ is the disjoint union of Dynkin diagrams.*

(iii) *If $\mathfrak{A}$ is of finite representation type and $M \in {}_\mathfrak{A}\mathfrak{M}(S)$ is indecomposable then, M is uniquely determined by its composition factors, moreover,* $\mathrm{End}_\mathfrak{A}(M)$ *is a skewfield.*

A complete list of the connected oriented valued graphs which reduce to Dynkin diagrams can be obtained from the authors upon request.

(1.5) THEOREM II. *Let $\mathfrak{A}$ be the tensor algebra of a $\mathfrak{k}$-species for an irreducible Dynkin diagram γ. Then all indecomposable objects of ${}_\mathfrak{A}\mathfrak{M}(S)$, with the possible exception of some in which each composition factor occurs at most once, are obtained from the indecomposable projective ones by applying iteratively the ordinary Coxeter transformation and remaining inside ${}_\mathfrak{A}\mathfrak{M}(S)$.*

Note that by means of this result, it is usually possible to compute the composition factors of the indecomposable representations in ${}_\mathfrak{A}\mathfrak{M}(S)$ explicitly, since one knows that the action of the ordinary Coxeter functor can be computed via the Coxeter transformation on the dimension type [1]. It will be shown in [5] that there exists a relative Coxeter transformation for ${}_\mathfrak{A}\mathfrak{M}(S)$; however, we do not know a closed formula for the change of the dimension type for this relative Coxeter transformation.

The paper is organized as follows:

In Section 2 we give an interpretation of ${}_\mathfrak{A}\mathfrak{M}(S)$ in terms of the representations of the underlying species. Section 3 is devoted to the proof of Theorem I, Section 4 to the proof of Theorem II. Finally in Section 5 we give some examples.

2. SOCLE REPRESENTATIONS OF SPECIES

For the terminology we refer to [1]. Let γ be an oriented connected graph with valuation and without oriented loops and $\mathfrak{I} = \langle F_i, {}_iM_j \rangle$ a species, where for each vertex i of γ, F_i is a skewfield, finite dimensional over the same field $\mathfrak{k}$, and for each edge $i \to j$, ${}_iM_j$ is an (F_i, F_j)-bimodule with $\dim_{F_i}({}_iM_j) = d_{i_j}$, and $\dim_{F_j}({}_iM_j) = d_{i_j'}$, where $\{d_{i_j}, d_{i_j'}\}$ is the valuation of γ. If $\mathfrak{A}$ is the tensor algebra of $\mathfrak{J}$, then $\mathfrak{A}$ is a graded algebra, defined as follows:

$$\mathfrak{A}^0 = \bigoplus_{i=1}^{n} F_i, \qquad \text{where } n \text{ is the number of vertices of } \gamma,$$

$$\mathfrak{A}^1 = \bigoplus_{i\to j} {}_iM_j,$$

$$\mathfrak{A}^i = \mathfrak{A}^{i-1} \otimes_{\mathfrak{A}^0} \mathfrak{A}^1, \qquad i \geqslant 2.$$

Since γ has no oriented loops $\mathfrak{A}^i = 0$ for i sufficiently large. Moreover,

$$\operatorname{rad}^i \mathfrak{A}/\operatorname{rad}^{i+1} \mathfrak{A} \cong \mathfrak{A}^i.$$

In particular, we have a splitting

$$\mathfrak{A}^1 = \operatorname{rad} \mathfrak{A}/\operatorname{rad}^2 \mathfrak{A} \hookrightarrow \mathfrak{A}.$$

Thus a right $\mathfrak{A}$-module U is nothing but a representation of $\mathfrak{J}$, namely $U \cong_{\mathfrak{A}^0} \bigoplus_{i=1}^n U_i$, where U_i are right F_i-vector spaces together with F_j-homomorphism

$${}_j\varphi_i\colon U_i \otimes_{F_i} {}_iM_j \to U_j,$$

coming from the right multiplication with $\mathfrak{A}^1$.

We shall now describe the category ${}_{\mathfrak{A}}\mathfrak{M}(S)$ consisting of those finitely generated right $\mathfrak{A}$-modules U such that the socle of U is projective. If $U \in \mathfrak{M}_{\mathfrak{A}}$ is a right $\mathfrak{A}$-module, then

$$\operatorname{Soc}(U) = \{u \in U\colon u \cdot \mathfrak{A}^1 = 0\}.$$

If we look at the corresponding representation

$$U \cong_{\mathfrak{A}^0} \bigoplus_{i=1}^{n} U_i,$$

then $\operatorname{Soc}(U) \cong \bigoplus_{i=1}^n \operatorname{Soc}(U)_i$. Now if i is a sink, i.e., F_i is a simple projective $\mathfrak{A}$-module, then there are no edges $i \to j$ and thus $U_i \cdot \mathfrak{A}^1 = 0$. Hence if i is a sink, then $U_i \subset \operatorname{Soc}(U)$. If i is not a sink, then there exists an edge $i \to j$ and so

$$\operatorname{Soc}(U)_i = \{u_i \in U_i\colon {}_j\varphi_i(u_i \otimes {}_im_j) = 0 \text{ for all } {}_im_j \in {}_iM_j \text{ and for all edges } i \to j\}. \tag{1}$$

Hence in terms of representations of $\mathfrak{J}$ we are interested in the category

$${}_{\mathfrak{J}}\mathfrak{M}(S) = \{U = (U_i, {}_i\varphi_j)\colon \operatorname{Soc}(U)_i = 0 \text{ if } i \text{ is not a sink}\}. \tag{2}$$

Under the natural isomorphism

$$\mathrm{Hom}_{F_j}(U_i \otimes_{F_i} {}_iM_j, U_j) \xrightarrow{\cong} \mathrm{Hom}_{F_i}(U_i, \mathrm{Hom}_{F_j}({}_iM_j, U_j))$$

$$\alpha \to (\bar{\alpha}: u_i \mapsto ({}_im_j \to \alpha(u_i \otimes {}_im_j))),$$

the F_j-homomorphism ${}_j\varphi_i$ corresponds to F_i-homomorphisms ${}_j\bar{\varphi}_i: U_i \to \mathrm{Hom}_{F_j}({}_iM_j, U_j)$, and from (1) it is clear that

$$_{\mathfrak{J}}\mathfrak{M}(S) = \Big\{U = (U_i, {}_i\varphi_j): \bigcap_{\substack{j \\ i \to j}} \mathrm{Ker}\, {}_j\bar{\varphi}_i = 0\Big\}. \tag{3}$$

Hence we have

(2.1) Proposition. *There is a natural equivalence*

$$_{\mathfrak{A}}\mathfrak{M}(S) \xrightarrow{\cong} {}_{\mathfrak{J}}\mathfrak{M}(S).$$

Therefore in the sequel we shall only work with the category $_{\mathfrak{J}}\mathfrak{M}(S)$, and we denote by $_{\mathfrak{J}}\mathfrak{M}$ the category of all finite-dimensional representations of $\mathfrak{J}$. To simplify the notation we shall write $_{\mathfrak{J}}\mathfrak{N}$ and $_{\mathfrak{J}}\mathfrak{N}(S)$, resp. for a complete set of nonisomorphic indecomposable objects in $_{\mathfrak{J}}\mathfrak{M}$ and $_{\mathfrak{J}}\mathfrak{M}(S)$, resp.

3. Classifications of the Graphs γ with $|\gamma\mathfrak{N}(S)| < \infty$

In this section we shall prove Theorem I from the Introduction. The definition of reducible is given in (1.3). We assume that γ is not the disjoint union of proper subgraphs.

(3.1) Reduction Lemma. *Let γ be reducible with species $\mathfrak{J}$ and let $\gamma_0 = \gamma_{a=b}$ with species $\mathfrak{J}_0$ be obtained from γ by one reduction. Then $|{}_{\mathfrak{J}}\mathfrak{N}(S)| < \infty$ iff $|{}_{\mathfrak{J}_0}\mathfrak{N}(S)| < \infty$. Moreover, if $|{}_{\mathfrak{J}}\mathfrak{N}(S)| < \infty$, then*

$$|{}_{\mathfrak{J}_0}\mathfrak{N}(S)| + 1 \leqslant |{}_{\mathfrak{J}}\mathfrak{N}(S)| \leqslant |{}_{\mathfrak{J}_0}\mathfrak{N}(S)| + m,$$

where $\gamma\backslash\langle a \xrightarrow{(1,1)} b\rangle = \gamma' \cup A_m$.

Proof. The definition of reducible implies that b is a sink in γ, and this sink has the form $a \to b \leftarrow c$ (we omit the valuation if it is trivial).

Given now an indecomposable representation $(U_i, {}_j\varphi_i)$ in $_{\mathfrak{J}}\mathfrak{N}(S)$. Since a is a sink in γ' we must have (cf. Section 2, (3)) $\mathrm{Ker}\, {}_b\bar{\varphi}_a = 0$. But ${}_aM_b = F_a = F_b$ and so $\mathrm{Ker}\, {}_b\varphi_a = 0$. We have to distinguish two cases:

Case 1. ${}_b\varphi_a$ is not an epimorphism.

Then we decompose $U_b = \operatorname{Im}\, {}_b\varphi_a \oplus X_b$. Since all indecomposable representations of A_m are one dimensional and since $(U_i, {}_j\varphi_i)$ was indecomposable, we must have ${}_b\varphi_a = 0$, $X_b \neq 0$, and so $(U_i, {}_j\varphi_i)$ is an indecomposable representation of A_m. There is at least one indecomposable representation of A_m lying in ${}_{\mathfrak{J}}\mathfrak{N}(S)$, namely, $0 \to F_b \leftarrow 0$, and there are at most m which are faithful at b. Hence we have obtained

$$|\mathfrak{U} \in {}_{\mathfrak{J}}\mathfrak{N}(S)\colon {}_b\varphi_a \text{ is an isomorphism}| + 1$$
$$\leqslant |{}_{\mathfrak{J}}\mathfrak{N}(S)| \leqslant |\mathfrak{U} \in {}_{\mathfrak{J}}\mathfrak{N}(S)\colon {}_b\varphi_a \text{ is an isomorphism}| + m.$$

Case 2. ${}_b\varphi_a$ is an isomorphism.

Since all indecomposable representations of A_m are one dimensional, we can find an isomorphism from $(U_i, {}_j\varphi_i)$ to a representation $(U_i, {}_j\varphi_i)$ such that ${}_b\varphi_a$ is the identity. But these representations are exactly those of ${}_{\mathfrak{J}_0}\mathfrak{N}(S)$. This proves the lemma. ▮

The next result will be useful to decide when $|{}_{\mathfrak{J}}\mathfrak{N}(S)| = \infty$.

(3.2) Test Lemma. *Assume γ contains a subgraph γ_0 with the same valuation on the edges, such that the species $\mathfrak{J}_0$ of γ_0 is a subspecies of $\mathfrak{J}$ of γ. If $|{}_{\mathfrak{J}_0}\mathfrak{N}(S)| = \infty$, then $|{}_{\mathfrak{J}}\mathfrak{N}(S)| = \infty$.*

Proof. We use induction with respect to the number of vertices outside of γ_0. Hence it suffices to prove the lemma in case there is exactly one vertex a_0 outside γ_0.

Let $s_1, \ldots, s_t$ be the sinks in γ_0 such that there exists an edge $s_i \xrightarrow{(\,\cdot\,)} a_0$; i.e., s_i is not a sink in γ.

Case 1. a_0 is a sink in γ.

Let $\mathfrak{U} = (U_i, {}_j\varphi_i) \in {}_{\mathfrak{J}_0}\mathfrak{N}(S)$. We define a representation $\mathfrak{V} = (V_i, {}_j\varphi_i)$ of $\mathfrak{J}$ as follows:

$$V_i = U_i \quad \text{if} \quad i \in \gamma_0 \qquad \text{and} \qquad {}_j\varphi_i = {}_j\psi_i \quad \text{if} \quad i, j \in \gamma_0,$$

and

$$V_{a_0} = \bigoplus_{k=1}^{t} U_{s_k} \otimes_{F_k} {}_{s_k}M_{a_0},$$

$${}_{a_0}\psi_{s_1} =: U_{s_1} \otimes_{F_{s_1}} {}_{s_1}M_{a_0} \to \bigoplus_{k=1}^{t} U_{s_k} \otimes_{F_{s_k}} {}_{s_k}M_{a_0},$$

the canonical injection into the direct sum. Then $\mathfrak{V}$ is a representation of γ, and since $\mathfrak{U} \in {}_{\mathfrak{J}_0}\mathfrak{M}(S)$ the representation $\mathfrak{V}$ lies in ${}_{\mathfrak{J}}\mathfrak{M}(S)$. Moreover, since $\mathfrak{U}$ is indecomposable the same holds for $\mathfrak{V}$. Also it is clear that this construction preserves and reflects isomorphism. Hence if a_0 is a sink in γ, then $|{}_{\mathfrak{J}_0}\mathfrak{N}(S)| = \infty$ implies $|{}_{\mathfrak{J}}\mathfrak{N}(S)| = \infty$.

Case 2. If a_0 is not a sink in γ, then there exists a directed path

$$a_0 \xrightarrow{(\,,\,)} b_1 \xrightarrow{(\,,\,)} \cdots \xrightarrow{(\,,\,)} b_m$$

with $b_i \in \gamma_0$ and b_m is a sink in γ_0. (Observe that there are no oriented loops, and hence from every point there exists a directed path to a sink.) In addition —for the same reason— $b_i \neq s_j$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant t$.

Let $\mathfrak{U} = (U_i, {}_j\varphi_i) \in {}_{\mathfrak{J}_0}\mathfrak{R}(S)$.

We define the representation $\mathfrak{V} = (V_i, {}_j\psi_i)$ of $\mathfrak{J}$ as follows:

$$V_c = U_c \quad \text{if} \quad c \in \gamma_0 \quad \text{and} \quad c \neq b_j, \quad 1 = j = m,$$

$${}_{c_1}\psi_{c_2} = {}_{c_1}\varphi_{c_2} \quad \text{if} \quad c_1, c_2 \in \gamma_0, \quad c_1, c_2 \neq b_j, \quad 1 \leqslant j \leqslant m,$$

$$V_{a_0} = V'_{a_0} = \bigoplus_{k=1}^{t} U_{s_k} \otimes_{F_{s_k}} {}_{s_k}M_{a_0},$$

$${}_{a_0}\psi_{s_1}: U_{s_1} \otimes_{F_{s_1}} {}_{s_1}M_{a_0} \to V'_{a_0},$$

the canonical injection into the direct sum.

Then we define recursively

$$V'_{b_1} = V_{a_0} \otimes_{F_0} {}_{a_0}M_{b_1},$$

$$V_{b_1} = V'_{b_1} \oplus U_{b_1},$$

$$V'_{b_i} = V'_{b_{i-1}} \otimes_{F_{b_{i-1}}} {}_{b_{i-1}}M_{b_i}, \qquad 2 \leqslant i \leqslant m,$$

$$V_{b_i} = V'_{b_i} \oplus U_{b_i},$$

and put

$${}_{b_1}\psi_{a_0}: V'_{a_0} \otimes_{F_{a_0}} {}_{a_0}M_{b_1} \xrightarrow{(\mathrm{id},0)} V'_{b_1} \oplus U_{b_1},$$

$${}_{b_{i+1}}\psi_{b_i}: (V'_{b_i} \otimes_{F_{b_i}} {}_{b_i}M_{b_{i+1}}) \oplus (U_{b_i} \otimes_{F_{b_i}} {}_{b_i}M_{b_{i+1}}) \xrightarrow{(\mathrm{id},\,{}_{b_{i+1}}\varphi_{b_i})} V_{b_{i+1}} \oplus U_{b_{i+1}}, \qquad 1 \leqslant i \leqslant m-1,$$

$${}_l\psi_{b_i}: (V'_{b_i} \otimes_{F_{b_i}} {}_{b_i}M_l) \oplus (U_{b_i} \otimes_{F_{b_i}} {}_{b_i}M_l) \xrightarrow{(0,\,{}_l\varphi_{b_i})} U_l, \qquad l \neq b_i, \ 1 \leqslant i \leqslant m,$$

$${}_{b_i}\psi_l: U_l \otimes_{F_l} {}_lM_{b_i} \xrightarrow{\binom{0}{{}_{b_i}\varphi_l}} V'_{b_i} \oplus U_{b_i}, \qquad l \neq b_i, \quad 1 \leqslant i \leqslant m.$$

Then $\mathfrak{V}$ is an indecomposable representation in ${}_{\mathfrak{J}}\mathfrak{R}(S)$; moreover, this construction reflects and preserves isomorphisms. This proves the lemma. ∎

We shall show next that every irreducible Euclidian graph γ with species $\mathfrak{J}$ has $|{}_{\mathfrak{J}}\mathfrak{N}(S)| = \infty$. To do so we shall use the list in [1, p. 39 ff.].

The homogeneous representations of the Euclidian diagrams are induced from the homogeneous representations of $\tilde{A}_{11}$ or $\tilde{A}_{12}$ with bimodule ${}_FM_G$, and the indecomposable representations are of the form (U_F, U_G, φ). We shall use this notation to construct indecomposable homogeneous representations in the socle category of the Euclidian diagrams.

(1) $\gamma: F \to^{(i,j)} G$, $i \cdot j \geqslant 4$. If $\varphi: U_F \otimes {}_FM_G \to U_G$ is an indecomposable representation. Then it is easily seen that

$$\operatorname{Ker} \bar{\varphi} \otimes {}_FM_G \to 0$$

is a direct summand of this representation. Hence if (U_F, U_G, φ) is indecomposable not injective, it lies in ${}_{\mathfrak{J}}\mathfrak{N}(S)$. So $|{}_{\mathfrak{J}}\mathfrak{N}(S)| = \infty$.

(2) $\tilde{A}_n$ is irreducible for every orientation; there are infinitely many indecomposable homogeneous representations in ${}_{\mathfrak{J}}\mathfrak{M}(S)$. However, not all simple homogeneous representations lie in ${}_{\mathfrak{J}}\mathfrak{M}(S)$.

(3) For $\tilde{B}_n$, $\tilde{C}_n$, $\widetilde{BC}_n$, $\widetilde{BD}_n$, $\widetilde{CD}_n$, and $\tilde{D}_n$ it suffices to consider irreducible graphs of type

(*) $\tilde{B}_2$, $\tilde{C}_2$, $\widetilde{BC}_2$, $\widetilde{BD}_3$, $\widetilde{CD}_3$, $\tilde{D}_4$.

In each of these graphs there exists a unique center z. We first consider the case where z is a sink. $\tilde{B}_2$: $G_1 \to F \leftarrow G_2$. In [1, p. 41] the homogeneous representations are listed for the orientation

$$G_1 \to F \to G_2$$

as

$$U_{G_1} \xrightarrow{\text{id}} U_{G_1} \otimes_{G_1} F \xrightarrow{\varphi} V_{G_2},$$

where $(U_{G_1}, V_{G_2}, \varphi)$ is an indecomposable representation of $G_1 \to^{{}_{G_1}F_{G_2}} G_2$. Except if $\dim_{G_1}(U_{G_1}) = 1$, φ cannot be injective. Hence for the remainding ones we obtain indecomposable representations in the above orientation as follows:

$$U_{G_1} \xrightarrow{\text{id}} U_{G_1} \otimes_{G_1} F \xleftarrow{\psi} \operatorname{Ker} \varphi,$$

where

$$\psi: \operatorname{Ker} \varphi \otimes_{G_2} F \to U_{G_1} \otimes_{G_1} F$$

$$x \otimes y \mapsto xy.$$

We consider the corresponding homomorphism (cf. Section 2(2))

$$\bar{\psi}\colon \operatorname{Hom}_{G_2}(\operatorname{Ker}\varphi, \operatorname{Hom}_F({}_{G_2}F, U_{G_1} \otimes_{G_1} F))$$
$$\alpha \mapsto (f \to \alpha f),$$

which is surely injective. So the socle category is of infinite type.

For $\tilde{C}_2$ and $\widetilde{BC}_2$ the same argument as above shows that the socle category is of infinite type.

$$\widetilde{BD}_3\colon \quad \begin{array}{ccc} F & & \\ & \searrow & \\ & & F \leftarrow G, \qquad |F : G| = 2. \\ & \nearrow & \\ F & & \end{array}$$

The bimodule for $\tilde{A}_{22}$ is $M = {}_F F_G \otimes_G {}_G F_F$. The indecomposable noninjective representation

$$(U_F, V_F, \varphi) \qquad \text{of} \quad F \xrightarrow{M} F$$

corresponds to the indecomposable representation of $\widetilde{BD}_3$

$$\begin{array}{ccc} V_F & & \\ & \searrow & \\ & & V_F \oplus U_F \leftarrow \Gamma_\varphi, \\ & \nearrow & \\ U_F & & \end{array}$$

where Γ_φ is injective and

$$\psi\colon \Gamma_\varphi \otimes_G F_F \to V_F \oplus U_F$$
$$x \otimes y \to xy.$$

It follows from (1) that $\bar{\psi}$ is injective. The same argument works for $\widetilde{CD}_3$.

$$\tilde{D}_4\colon \quad \begin{array}{ccccc} F & & & & F \\ & \searrow & & \swarrow & \\ & & F & & \\ & \nearrow & & \nwarrow & \\ F & & & & F \end{array} \quad .$$

The bimodule for $\tilde{A}_{22}$ is $M = {}_F(F \oplus F)_F$, and the indecomposable non-injective representation (U_F, V_F, φ) of $F \xrightarrow{M} F$ corresponds to

$$\begin{array}{ccccc} U_F & & & & \Gamma_1 \\ & \searrow & & \swarrow & \\ & & U_F \oplus V_F & & \\ & \nearrow & & \nwarrow & \\ V_F & & & & \Gamma_2 \end{array},$$

where $\varphi = (\varphi_1, \varphi_2)$ and Γ_i = graph of φ_i. As above this shows that we have infinite type.

If z is not a sink, we can use the representations in [1]—with a slight modification for $\widetilde{BC}_n$—to conclude that the socle category is of infinite type.

(4) We next consider the graphs $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$, which all have a unique center z. Let γ be one of these. Let $a \to b$ be an edge in γ. If $a \neq z$, and $a \to b$ is directed towards the center, then ${}_b\varphi_a = {}_b\bar{\varphi}_a$ must be injective, since $\gamma \backslash \{z\}$ is a disjoint union of graphs of type A_m, and all indecomposable representations of A_m have local dimension at most one. If $a \neq z$ and $a \to b$ directed away from the center, then because of the irreducibility of γ, a is a source $c \leftarrow a \to b$ and by the first part, ${}_c\varphi_a$ is injective and so $\operatorname{Ker} {}_c\varphi_a \cap \operatorname{Ker} {}_b\varphi_a = 0$. This argument shows that for indecomposable representation, we only have to verify the kernel condition for the central point z. For an irreducible γ we have three possible orientations at the center,

$$(\alpha) \quad \begin{array}{c} \downarrow \\ \to z \leftarrow, \end{array}$$

$$(\beta) \quad \begin{array}{c} \downarrow \\ \leftarrow z \to, \end{array}$$

$$(\gamma) \quad \begin{array}{c} \uparrow \\ \leftarrow z \to. \end{array}$$

Orientation (α) is treated in [1] and one sees that the socle category is of infinite type. Observe that a change of orientations not neighboring z does not influence the module structure at the neighbors of the center.

Using the lists and orientation in [1] we use the following notation for the indecomposable homogeneous representations of $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$:

$$\begin{array}{ccc} & C & \\ & \downarrow & \\ B \longrightarrow \cdots \longrightarrow & M_0 & \longleftarrow \cdots \longleftarrow A. \end{array}$$

Then inspection of the list in [1] shows $A \cap B = 0$, where A and B are viewed as submodules of M_0.

For (β) we obtain thus indecomposable representations which have the following form at the center

$$\begin{array}{ccccc} & & C & & \\ & & \downarrow & & \\ \cdots M_0/B & \xleftarrow{\varphi_1} & M_0 & \xrightarrow{\varphi_2} & M_0/A \cdots. \end{array}$$

Since $A \cap B = 0$, $\operatorname{Ker} \varphi_1 \cap \operatorname{Ker} \varphi_2 = 0$ and so the socle condition is satisfied and we have infinite type.

For (γ) we get

$$\begin{array}{ccccc} & & M_0/C & & \\ & & \uparrow & & \\ \cdots M_0/B & \longleftarrow & M_0 & \longrightarrow & M_0/A \cdots \end{array}$$

and with (β) we conclude that also here we have infinite type.

(5) $\tilde{F}_{41}$ and $\tilde{F}_{42}$. As in the arguments for the $\tilde{E}_i$ we only need to consider the edge $0 \overset{(1,2)}{—} 0$ in the various irreducible situations. For $\tilde{F}_{41}$ we have the graph

$$G - G - G - F - F.$$

Given an indecomposable representation

$$\cdots U_G \to V_F \cdots$$

then $\operatorname{Ker} \bar{\varphi}$ induces a direct summand which is a representation of an A_m. Hence for almost all indecomposable representations, $\bar{\varphi}$ is injective. The same argument works for the other orientations of $\tilde{F}_{41}$ and for $\tilde{F}_{42}$.

(6) $\tilde{G}_{21}$ and $\tilde{G}_{22}$ are treated exactly as $\tilde{F}_{41}$ and $\tilde{F}_{14}$, resp.

Combining the results we have shown

(3.3) PROPOSITION. *Let γ be an irreducible graph corresponding to an Euclidian diagram with species $\mathfrak{I}$. Then $|{}_{\mathfrak{I}}\mathfrak{R}(S)| = \infty$.*

We are now in the position to prove

(3.4) THEOREM I. *Let γ be a connected oriented graph without loops and $\mathfrak{I}$ a species for γ. Let γ_0 be an irreducible graph with induced species $\mathfrak{I}_0$, where γ_0 is obtained from γ by a finite number of reductions.*

(i) $|{}_{\mathfrak{J}}\mathfrak{R}(S)| < \infty$ *if and only if* $|{}_{\mathfrak{J}_0}\mathfrak{R}(S)| < \infty$.

(ii) $|{}_{\mathfrak{J}_0}\mathfrak{R}(S)| < \infty$ *if and only if* $\mathfrak{J}_0$ *is the irreducible graph of a Dynkin diagram.*

(iii) *In case* ${}_{\mathfrak{J}}\mathfrak{R}(S)$ *is finite, each representation in* ${}_{\mathfrak{J}}\mathfrak{R}(S)$ *is uniquely determined by its dimension type.*

Proof. Part (i) follows by successive applications of the reduction lemma (3.1).

(ii) In view of (3.3) it only remains to show the following. Assume that γ_0 is not the graph of an irreducible Dynkin diagram. Then it contains as a subgraph an irreducible Euclidian diagram. To show this we assume that γ_0 is not a Dynkin diagram but contains only reducible Euclidian diagrams. We observe that in an irreducible graph γ_0 at each sink s of γ_0 there exists a subgraph of one of the following types

$$(\alpha)\quad \leftarrow \cdot \rightarrow s \qquad (\beta)\quad \xrightarrow{(i,j)} s,\ \ \downarrow \qquad i \cdot j \geqslant 2,$$

$$(\gamma)\quad \rightarrow s \leftarrow.$$

If now γ_1 is a reducible Euclidian diagram contained in γ_0, say, γ_1 can be reduced at $a \rightarrow b$. Then b is a sink in γ_1. Since γ_0 cannot contain any $\tilde{A}_n$, there exists a unique path in γ_0 from b to a sink s. Repeating this arguments for all edges of γ_1 which can be reduced, and observing that the sinks in γ_0 have the form (α) (β) (γ) one finds that γ must also contain an irreducible Euclidian diagram.

(iii) Assume now that γ is a graph with species $\mathfrak{J}$ such that $|{}_{\mathfrak{J}}\mathfrak{R}(S)| < \infty$. Every indecomposable representation omitted by means of the reduction lemma (3.1) is uniquely determined by its dimension type. Hence we may assume that γ is a Dynkin diagram, but here it is known that every indecomposable representation is uniquely determined by its dimension type [1].

(3.5) Corollary. *Let* γ *be an oriented graph and* $\mathfrak{J}$ *a species for* γ. *If* $|{}_{\mathfrak{J}}\mathfrak{R}(S)| < \infty$, *then for every indecomposable representation* $\mathfrak{U}$ *in* ${}_{\mathfrak{J}}\mathfrak{R}(S)$, $\mathrm{End}_{\mathfrak{J}}(\mathfrak{U})$ *is a skewfield.*

Proof. We may restrict ourselves to a connected graph, and we shall use induction on the number of vertices of γ. If γ is irreducible, then γ is a Dynkin diagram and here the result is known for all indecomposable representations. Hence we may assume that γ is reducible. According to the proof of (3.1) —using the notation of (3.1)—there are two types of indecomposable representation of γ:

In the first case, $\mathfrak{U} \in {}_{\mathfrak{J}}\mathfrak{R}(S)$ is an indecomposable representation of an A_m, and hence $\mathrm{End}_{\mathfrak{J}}(\mathfrak{U}) = \mathrm{End}_{A_m}(\mathfrak{U})$ is a skewfield. The other case is that for

$\mathfrak{U} = (U_i, {}_j\varphi_j)$, ${}_b\varphi_a = id$, and $\mathfrak{U}$ can be viewed a representation of γ_0, which has fewer vertices than γ; with species $\mathfrak{J}_0$, where $|{}_{\mathfrak{J}_0}\mathfrak{N}(S)| < \infty$ and $\mathrm{End}_{\mathfrak{J}}(\mathfrak{U}) = \mathrm{End}_{\mathfrak{J}_0}(\mathfrak{U})$ is a skewfield by induction. ▌

4. Determination of the Dimension Type of the Indecomposable Objects in ${}_{\mathfrak{J}}\mathfrak{M}(S)$

In this section we assume that $\mathfrak{J}$ is a species of an irreducible graph for a Dynkin diagram and prove

(4.1) Theorem II. *All indecomposable objects of ${}_{\mathfrak{J}}\mathfrak{N}(S)$—except possibly representations $\mathfrak{U} = (U_i, {}_j\varphi_i)$ with $\dim_{F_i}(U_i) \leqslant 1$—are of the form $C^{-s}(\mathfrak{R})$, where $\mathfrak{R}$ is an indecomposable projective object in ${}_{\mathfrak{J}}\mathfrak{N}(S)$, and $C^{-}(\mathfrak{B})$ is the Coxeter transformation of $\mathfrak{B}$.*

The exceptional representations will turn out to be indecomposable representations of an A_n, which is a subgraph of γ. A typical example is the graph

$$\gamma\colon \leftarrow \quad \overset{\uparrow}{\leftarrow} \rightarrow \leftarrow.$$

If $\mathfrak{X}$ is the indecomposable representation of dimension type

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \uparrow & & & & & & \\ 0 & \leftarrow & 1 & \leftarrow & 1 & \rightarrow & 1 & \leftarrow & 0, \end{array}$$

then $\mathfrak{X} \notin {}_{\mathfrak{J}}\mathfrak{N}(S)$, but $C^{-}(\mathfrak{X})$ has dimension type

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \uparrow & & & & & & \\ 0 & \leftarrow & 0 & \leftarrow & 1 & \rightarrow & 1 & \leftarrow & 1 \end{array}$$

and so it lies in ${}_{\mathfrak{J}}\mathfrak{N}(S)$. However, this will be essentially the only exception. In practice, this is not a big handicap, since the representations $\mathfrak{U} = (U_i, {}_j\varphi_i)$ with $\dim_{F_i} U_i \leqslant 1$ can easily be constructed anyway.

It should be observed that it is not always possible to apply C^{-}, namely, in case $\mathfrak{X}$ is an indecomposable injective representation of $\mathfrak{J}$, then $C^{-}(\mathfrak{X})$ is not a representation, since the dimension vector of $C^{-}(\mathfrak{X})$ is not positive.

In all the various cases that occur we shall use the following general argument: In order to prove the result, we take $\mathfrak{X} \in {}_{\mathfrak{J}}\mathfrak{N} \backslash {}_{\mathfrak{J}}\mathfrak{N}(S)$ and we show either $C^{-}(\mathfrak{X}) \in {}_{\mathfrak{J}}\mathfrak{N} \backslash {}_{\mathfrak{J}}\mathfrak{N}(S)$ or $C^{-}(\mathfrak{X}) \in {}_{\mathfrak{J}}\mathfrak{N}(S)$ has dimension at most one at each vertex, or $\mathfrak{X}$ is injective. This will give the desired result, since in case of Dynkin diagrams all $\mathfrak{X} \in {}_{\mathfrak{J}}\mathfrak{N}$ are of the form $C^{-s}(\mathfrak{R})$, where $\mathfrak{R}$ is indecomposable projective and thus lies in ${}_{\mathfrak{J}}\mathfrak{N}(S)$.

(4.2) We need some more ***notation***: If k is a source in γ we denote by $s_k^-(\gamma)$ the following graph: The vertices and valuations are the same as those of γ; the orientations of all edges at k are reversed; the others left invariant. We denote by $S_k^-(\mathfrak{I})$ the corresponding species. For $\mathfrak{X} \in {}_{\mathfrak{I}}\mathfrak{N}$, $S_k^-(\mathfrak{X}) \in {}_{S_k^-(\mathfrak{I})}\mathfrak{N}$ and for a suitable numbering of the vertices in γ, we have $C^-(\mathfrak{X}) = S_{k_n}^- S_{k_{n-1}}^- \cdots S_{k_1}^-(\mathfrak{X})$ [1].

(4.3) LEMMA. *If γ is an irreducible A_m, then* (4.1) *is true for all* $\mathfrak{U} \in {}_{\mathfrak{I}}\mathfrak{N}(S)$.

Proof. An irreducible A_m has the form

$$\leftarrow \rightarrow \leftarrow \rightarrow \cdots \leftarrow \rightarrow;$$

in particular m is odd. Hence if $\mathfrak{X} \in {}_{\mathfrak{I}}\mathfrak{N} \backslash {}_{\mathfrak{I}}\mathfrak{N}(S)$, then $\mathfrak{X}$ is injective, and so $C^-(\mathfrak{X})$ is not defined. ∎

(4.4) LEMMA. *If γ is an irreducible F_4 or a G_2 then* (4.1) *is true, for all* $\mathfrak{U} \in {}_{\mathfrak{I}}\mathfrak{N}(S)$.

Proof. If $\mathfrak{X} \in {}_{\mathfrak{I}}\mathfrak{N} \backslash {}_{\mathfrak{I}}\mathfrak{N}(S)$, for $\mathfrak{I}$ a species for G_2, then $\mathfrak{X}$ is injective, and so $C^-(\mathfrak{X})$ is not defined.

For F_4, the only orientations for an irreducible graph are

$$\text{(a)} \qquad \longrightarrow \xrightarrow{(1,2)} \longleftarrow; \quad \longrightarrow \xrightarrow{(2,1)} \longleftarrow.$$

$$\text{(b)} \qquad \longleftarrow \xrightarrow{(1,2)} \longleftarrow; \quad \longleftarrow \xrightarrow{(2,1)} \longrightarrow.$$

In case (a), if $\mathfrak{X} \in {}_{\mathfrak{I}}\mathfrak{N} \backslash {}_{\mathfrak{I}}\mathfrak{N}(S)$, then $\mathfrak{X}$ is either injective, or $\mathfrak{X}$ has dimension type

$$0 \rightarrow 1 \rightarrow 0 \leftarrow 0,$$

an application of the Coxeter transformation C^- yields the dimension type $1 \rightarrow 0 \rightarrow 0 \leftarrow 0$ and so $C^-(\mathfrak{X}) \in {}_{\mathfrak{I}}\mathfrak{N} \backslash {}_{\mathfrak{I}}\mathfrak{N}(S)$.

In case (b) any $\mathfrak{X} \in {}_{\mathfrak{I}}\mathfrak{N} \backslash {}_{\mathfrak{I}}\mathfrak{N}(S)$ is injective and so $C^-(\mathfrak{X})$ can not be formed. This proves (4.4). ∎

Next we turn to B_n, C_n, D_n, E_6, E_7, E_8.

In all these diagrams there is a unique center c, which is obvious for D_n, E_6, E_7, E_8; for B_n and C_n we define it as

$$B_n\colon \cdot \overset{(1,2)}{\text{——}} c \text{——} \cdot \text{——} \cdot \cdots \cdot \text{——} \cdot,$$

$$C_n\colon \cdot \overset{(2,1)}{\text{——}} c \text{——} \cdot \text{——} \cdot \cdots \cdot \text{——} \cdot.$$

(4.5) LEMMA. *Let γ be an irreducible graph of type B_n, C_n, D_n, E_6, E_7, E_8, and let $\mathfrak{X} \in {}_3\mathfrak{N}\backslash{}_3\mathfrak{N}(S)$ be such that*

$$0 = V_c = \bigcap_{\substack{i\\ c\to i}} \operatorname{Ker}(X_c \xrightarrow{{}_i\Phi_c} \operatorname{Hom}_{F_i}({}_cM_i , X_i)),$$

then either $\mathfrak{X}$ is injective or $C^{-}(\mathfrak{X}) \in {}_3\mathfrak{N}\backslash{}_3\mathfrak{N}(S)$, or $C^{-}(\mathfrak{X})$ is at most one dimensional at each vertex.

Proof. If c is a source, then $\mathfrak{X}$ is injective. Hence we may assume that c is not a source. In order that $V_c \neq 0$, we have at least one oriented edge $c \to i$. Hence the center must have the form (observe that γ is irreducible):

$$\begin{array}{ccc} & \vdots & \\ & a & \\ (1,1) & \downarrow & \\ & c & \\ (1,2) & \downarrow & \\ & b & \end{array} \qquad \text{or} \qquad \begin{array}{ccccc} & & \vdots & & \\ & & a & & \\ & & \downarrow & & \\ \cdot & & c & & \cdot \\ & \swarrow & & \searrow & \\ b_1 & & & & b_2 . \end{array}$$

Since $V_c \neq 0$, and since the branch of γ containing a is an A_m, $\mathfrak{X}$ must have the following form:

$$\begin{array}{c} V_a \\ \downarrow \\ V_c \\ \downarrow \\ 0 \end{array} \qquad \text{or} \qquad \begin{array}{ccccc} & & V_a & & \\ & & \downarrow & & \\ & & V_c & & \\ & \swarrow & & \searrow & \\ 0 & & & & 0, \end{array}$$

where V_c is one dimensional and $V_a = 0$ (case (b)) or it is one dimensional (case (a)). We now apply the partial Coxeter transformations to the branch containing a until we reach V_c. Then we have one of the following dimension types:

$$\begin{array}{c} \vdots \\ 1 \\ \uparrow \\ 1 \\ \downarrow \\ 0 \end{array} \qquad \begin{array}{ccccc} & & \vdots & & \\ & & 1 & & \\ & & \uparrow & & \\ & & 1 & & \\ & \swarrow & & \searrow & \\ 0 & & & & 0 \end{array} \qquad \text{(a)}$$

$$\begin{array}{c} 0 \\ \uparrow \\ 1 \\ \downarrow \\ 0 \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ & & 1 & & \\ & \swarrow & & \searrow & \\ 0 & & & & 0 \end{array} \qquad \text{(b)}$$

Applying the partial Coxeter transformation S_c^- to the dimension type, we obtain

$$\begin{array}{ccccc} 1 & & 1 & & \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \\ \uparrow & \nearrow & & \nwarrow & \\ 0 & 0 & & & 0 \end{array} \qquad \text{(a')}$$

$$\begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ -1 & & -1 & & \\ \uparrow & \nearrow & & \nwarrow & \\ 0 & 0 & & & 0. \end{array} \qquad \text{(b')}$$

Since the remaining partial Coxeter transformations do not influence the central position, we see that either $C^-(\mathfrak{X})$ is not defined or $C^-(\mathfrak{X}) \in {}_{\mathfrak{J}}\mathfrak{N}\backslash{}_{\mathfrak{J}}\mathfrak{N}(S)$, unless a is a source and if $C^-(\mathfrak{X}) = (\varphi_i, {}_j\varphi_i)$, then $\varphi_c = 0$. If a is a source, then it may happen that $C^-(\mathfrak{X}) \in {}_{\mathfrak{J}}\mathfrak{N}(S)$; but then $C^-(\mathfrak{X}) = (\varphi_i, {}_j\varphi_i)$ has the property that $\varphi_c = 0$, and $\dim_{F_i}(\varphi_i) = 1$. This proves the lemma. ∎

(4.6) LEMMA. *If γ is an irreducible B_n, C_n, D_n, E_6, E_7, E_8, then* (4.1) *holds.*

Proof. Let $\mathfrak{X} \in {}_{\mathfrak{J}}\mathfrak{N}\backslash{}_{\mathfrak{J}}\mathfrak{N}(S)$, $\mathfrak{X} = (X_i, {}_j\varphi_i)$. In view of (4.5) we may assume that there exists a vertex a such that $0 = V_a = \bigcap_{b, a \to b} \operatorname{Ker}(X_a \xrightarrow{{}_b\varphi_a} X_b)$ and $a \neq c$.

If a is a source, then $\mathfrak{X} = (V_a, 0)$ and $\mathfrak{X}$ is injective, so $C^-(\mathfrak{X})$ cannot be formed.

Hence we may assume that a is not a source, i.e., we have a unique chain

$$d \to a \to b.$$

In γ there is a unique (unoriented) path π from a to the center c.

Case 1. The path π goes via b. In this case the branch of γ containing $d \to a$ and ending in a is an A_m, and so $\mathfrak{X}$ is one dimensional at each vertex and the same argument as in the proof of (4.5) holds.

Case 2. The path π goes via d. In this case γ would not be irreducible, so that this situation cannot occur. This proves the lemma and also (4.1). ∎

5. Examples

Let γ:

$$\begin{array}{ccccc} & & & g & \\ & & \swarrow & & \searrow \\ a & b & c & & h \\ \searrow & \downarrow & \swarrow & & \\ & d & & & \\ & \downarrow & & & \\ & e & & & \\ & \downarrow & & & \\ & f & & & \end{array}$$

with F-trivial species. Then γ is neither a Dynkin diagram nor an extended Dynkin diagram, and so $\mathfrak{I}$ is of wild representation type; however, ${}_{\mathfrak{I}}\mathfrak{N}(S)$ is finite. We shall list all indecomposable representations explicitly: Applying the reduction lemma (3.1) to $e \to f$ we obtain the representation

$$(1)\qquad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 0 & 0 & 0 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 0 & & & \\ & \downarrow & & & \\ & 0 & & & \\ & \downarrow & & & \\ & 1 & & & \end{array}$$

and the remaining representations are all of the form

$$\begin{array}{ccccc} & & & U_g & \\ & & \swarrow & & \searrow \\ U_a & U_b & U_c & & U_h \\ \searrow & \downarrow & \swarrow & & \\ & U_d & & & \\ & \downarrow & & & \\ & U_e & & & \\ & \| & & & \\ & U_e & & & \end{array}$$

We now apply the reduction lemma to $d \to e$, and obtain the representation

$$(2)\qquad \begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & \swarrow & & \searrow & \\ 0 & & 0 & & 0 & & & & 0 \\ & \searrow & \downarrow & \swarrow & & & & & \\ & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & 1 & & & & & & \\ & & \downarrow & & & & & & \\ & & 1 & & & & & & \end{array}$$

and the remainding representations are all of the form

$$\begin{array}{ccccccccc} & & & & & & U_g & & \\ & & & & & \swarrow & & \searrow & \\ U_a & & U_b & & U_c & & & & U_h \\ & \searrow & \downarrow & \swarrow & & & & & \\ & & U_d & & & & & & \\ & & \parallel & & & & & & \\ & & U_d & & & & & & \\ & & \parallel & & & & & & \\ & & U_d & & & & & & \end{array}$$

So by the reduction lemma (3.1) it suffices to consider the irreducible graph

$$\begin{array}{ccccccccc} & & & & & & g & & \\ & & & & & \swarrow & & \searrow & \\ a & & b & & c & & & & h \\ & \searrow & \downarrow & \swarrow & & & & & \\ & & d & & & & & & \end{array}$$

which comes from a Dynkin diagram. We have to apply the Coxeter transformations to the dimension type of the indecomposable projective representations:

$$(3)\qquad \begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & \swarrow & & \searrow & \\ 0 & & 0 & & 0 & & & & 0 \Rightarrow \\ & \searrow & \downarrow & \swarrow & & & & & \\ & & 1 & & & & & & \end{array} \qquad (4)\qquad \begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & \swarrow & & \searrow & \\ 1 & & 1 & & 1 & & & & 0 \Rightarrow \\ & \searrow & \downarrow & \swarrow & & & & & \\ & & 2 & & & & & & \end{array}$$

$$(5)\qquad \begin{array}{ccccccccc} & & & & & & 1 & & \\ & & & & & \swarrow & & \searrow & \\ 1 & & 1 & & 2 & & & & 1 \Rightarrow \\ & \searrow & \downarrow & \swarrow & & & & & \\ & & 2 & & & & & & \end{array} \qquad (6)\qquad \begin{array}{ccccccccc} & & & & & & 2 & & \\ & & & & & \swarrow & & \searrow & \\ 1 & & 1 & & 2 & & & & 1 \Rightarrow \\ & \searrow & \downarrow & \swarrow & & & & & \\ & & 2 & & & & & & \end{array}$$

$$(7)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 1 & 1 & 1 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow \quad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 0 & 0 & 0 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & -1 & & & \end{array}$$

It should be observed that (7) is injective.

$$(8)\quad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 1 & 0 & 0 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow \quad (9)\quad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 0 & 1 & 1 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow$$

$$(10)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 1 & 0 & 1 & & 1 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow \quad (11)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 0 & 1 & 1 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow$$

$$\begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 1 & 0 & 0 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 0 & & & \end{array}$$

The last representation does not lie in ${}_{\mathfrak{z}}\mathfrak{N}(S)$ anymore.

$$(12)\quad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 0 & 1 & 0 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow \quad (13)\quad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 1 & 0 & 1 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow$$

$$(14)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 0 & 1 & 1 & & 1 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow \quad (15)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 1 & 0 & 1 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow$$

$$\begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 0 & 1 & 0 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 0 & & & \end{array}$$

The last representation does not lie in ${}_3\mathfrak{N}(S)$ anymore.

$$(16)\quad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 0 & 0 & 1 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow \qquad (17)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 1 & 1 & 1 & & 1 \\ \searrow & \downarrow & \swarrow & & \\ & 2 & & & \end{array} \Rightarrow$$

$$(18)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 1 & 1 & 2 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 2 & & & \end{array} \Rightarrow \qquad (19)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 1 & 1 & 1 & & 1 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow$$

$$\begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 0 & 0 & 1 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 0 & & & \end{array}$$

This representation does not lie in ${}_3\mathfrak{N}(S)$.

$$(20)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 0 & 0 & 1 & & 1 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow \qquad (21)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 1 & 1 & 1 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 2 & & & \end{array} \Rightarrow$$

$$(22)\quad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 1 & 1 & 1 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow \qquad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 0 & 0 & 1 & & 1 \\ \searrow & \downarrow & \swarrow & & \\ & 0 & & & \end{array}$$

This representation does not lie in ${}_3\mathfrak{N}(S)$.

$$(23)\quad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 0 & 0 & 0 & & 1 \\ \searrow & \downarrow & \swarrow & & \\ & 0 & & & \end{array} \Rightarrow \qquad (24)\quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 0 & 0 & 1 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow$$

$$(25)\quad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 1 & 1 & 0 & & 0 \\ \searrow & \downarrow & \swarrow & & \\ & 1 & & & \end{array} \Rightarrow \qquad \begin{array}{ccccc} & & & 0 & \\ & & \swarrow & & \searrow \\ 0 & 0 & 1 & & \\ \searrow & \downarrow & \swarrow & & \\ & 0 & & & \end{array}$$

This representation does not lie in ${}_{\mathfrak{Z}}\mathfrak{N}(S)$. But this is one of the exceptional representations $\mathfrak{X} \in {}_{\mathfrak{Z}}\mathfrak{N}(S)$ such that $C^{-}(\mathfrak{X}) \in {}_{\mathfrak{Z}}\mathfrak{N}(S)$. Since this dimension type gives under the Coxeter transformation

$$\text{(26)} \quad \begin{array}{ccccc} & & & 1 & \\ & & \swarrow & & \searrow \\ 0 & 0 & 0 & & 1 \\ \searrow & \downarrow & \swarrow & & \\ & 0 & & & \end{array}$$

which again lies in ${}_{\mathfrak{Z}}\mathfrak{N}(S)$, and this is the only representation which one has not obtained as Coxeter transformation of a representation in ${}_{\mathfrak{Z}}\mathfrak{N}(S)$. Altogether, $\mathfrak{I}$ has 26 indecomposable representations in ${}_{\mathfrak{Z}}\mathfrak{N}(S)$.

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