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Perfect Elements in the Free Modular Lattices

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Introduction

Recently, Gelfand and Ponomarev have investigated the structure of the free modular lattices $\mathfrak{D}^r$ with r generators $e_1, e_2, \ldots, e_r$. Recall that for $r \leqq 3$, the lattice $\mathfrak{D}^r$ is finite (R. Dedekind has shown that $\mathfrak{D}^3$ has 28 elements [3]) and that for $r \geqq 4$, the lattice $\mathfrak{D}^r$ is infinite.

The central concept introduced by Gelfand and Ponomarev in [7] and [8] is that of a perfect element of $\mathfrak{D}^r$; it is defined in terms of representations of $\mathfrak{D}^r$ as follows. A *representation* of $\mathfrak{D}^r$ (over a division ring F) is a lattice homomorphism ϱ of $\mathfrak{D}^r$ into the lattice of subspaces of a (finite dimensional) vector space over F. Thus, a representation ϱ may be interpreted as being given as $\mathbf{X} = (X_0; X_i)_{1 \leqq i \leqq r}$ where the X_i are subspaces of the vector space X_0, namely

$$\varrho(e_i) = X_i, \varrho(1) = X_0,$$

and for every element $p = p(e_1, \ldots, e_r)$ of $\mathfrak{D}^r$,

$$\varrho(p) = p(X_1, \ldots, X_r) = p(\mathbf{X}).$$

An element $p \in \mathfrak{D}^r$ is said to be *perfect* if $p(\mathbf{X}) = 0$ or X_0 for every indecomposable representation $\mathbf{X}$. Obviously, the perfect elements form a sublattice of $\mathfrak{D}^r$. For $r \geqq 4$, Gelfand and Ponomarev construct explicitely a countable number of upper and lower cubicles (Boolean lattices of 2^r elements)

$$\ldots \leqq \mathfrak{B}^+(l) \leqq \mathfrak{B}^+(l-1) \leqq \ldots \leqq \mathfrak{B}^+(1) \quad \text{and} \quad \ldots \geqq \mathfrak{B}^-(l) \geqq \mathfrak{B}^-(l-1) \geqq \ldots \geqq \mathfrak{B}^-(1),$$

show that all their elements are perfect and conjecture that these are the only perfect elements in $\mathfrak{D}^r$ (p. 5 of [7]; p. 100 of [8]).

In fact, the papers [7] and [8] consist largely in a study of a quotient of the lattice $\mathfrak{D}^r$. Call two elements $a, b \in \mathfrak{D}^r$ *q-linearly equivalent* if $a(\mathbf{X}) = b(\mathbf{X})$ for all representations $\mathbf{X}$ of $\mathfrak{D}^r$ over any division ring of characteristic q, and write $\mathfrak{D}^r_q$ for the corresponding quotient lattice, the *free q-linear lattice*. Let us call an element p

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of $\mathfrak{D}_q^r$ *perfect* if for any indecomposable representation $\mathbf{X}=(X_0;X_i)$ over a division ring of characteristic q, we have $p(\mathbf{X})=0$ or X_0. Clearly, the q-linear equivalence class of a perfect element of $\mathfrak{D}^r$ is perfect in $\mathfrak{D}_q^r$. Thus, the problem on the existence of perfect elements in $\mathfrak{D}^r$ comprizes two questions:

(i) determination of the sublattice of all perfect elements of the free q-linear lattice $\mathfrak{D}_q^r$ and

(ii) determination of the q-linear equivalence classes in $\mathfrak{D}^r$.

In this paper, we shall address ourselves to the first question. Let us recall that Gelfand and Ponomarev have proved that in $\mathfrak{D}_q^r$, any element of the lower cubicles $\mathfrak{B}_q^-(l)$ [i.e. of the cubicle of the q-linear equivalence classes of the elements of $\mathfrak{B}^-(l)$] is smaller than any element of the upper cubicles $\mathfrak{B}_q^+(l)$. In addition, they have shown that every perfect element p of $\mathfrak{D}_q^r$ which does not belong to any such cubicle must satisfy

$$\mathfrak{B}_q^-(l)\leqq p\leqq \mathfrak{B}_q^+(l) \quad \text{for all } l.$$

Here, we shall establish the following theorem.

Theorem 1. *For $r=4$, there are at most 16 perfect elements, and for $r\geqq 5$, there are at most 2 perfect elements in $\mathfrak{D}_q^r$ which do not belong to the cubicles.*

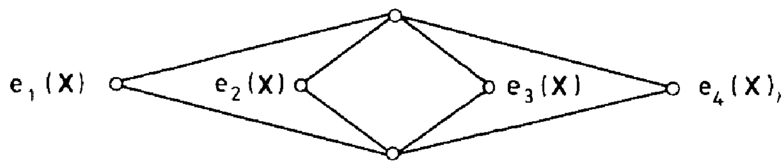
For the proof of the theorem, it is sufficient to work with representations over a fixed division ring F of characteristic q. For, assume $G\subseteq F$ is a division ring inclusion, and $(Y_0;Y_i)$ is a representation of $\mathfrak{D}^r$ over G, then $(Y_0\otimes_G F_F;Y_i\otimes_G F_F)$ is a representation of $\mathfrak{D}^r$ over F and it is easy to see that $Y_i\to Y_i\otimes_G F_F$ induces a lattice isomorphism of the corresponding sublattices generated by the Y_i, or $Y_i\otimes_G F_F$, respectively. Now we use the fact that any two division rings of the same characteristic can be embedded into a common division ring [2].

The proof of the theorem will depend on investigating the existence of non-zero homomorphism between the "regular" representations. (Note that all indecomposable representations are divided into the preprojective, preinjective, and regular ones; the definitions will be recalled at the beginning of the next section.) For, the fact that every perfect element p of $\mathfrak{D}_q^r$ which does not belong to any cubicle $\mathfrak{B}^+(l)$ or $\mathfrak{B}^-(l)$ satisfies $\mathfrak{B}_q^-(l)\leqq p\leqq \mathfrak{B}_q^+(l)$ for all l, means that $p(\mathbf{X})=0$ for the preprojective representations $\mathbf{X}$, and $p(\mathbf{X})=X_0$ for the preinjective representations $\mathbf{X}$. Thus, we only have to be concerned with the values $p(\mathbf{X})$ for indecomposable regular representations $\mathbf{X}$. We will use the following simple criterion:

If $\mathbf{X}$, $\mathbf{Y}$ are indecomposable representations of $\mathfrak{D}_q^r$ and if there is a non-zero homomorphisms $\varphi:\mathbf{X}\to\mathbf{Y}$, then, for any perfect element p in $\mathfrak{D}_q^r$, $p(\mathbf{Y})=0$ implies $p(\mathbf{X})=0$.

In case when $r=4$, the result now follows from the well-known structure of the subcategory of the regular representations (see [4]). Indeed, this subcategory is abelian, and is a direct sum of three categories $\mathscr{R}_j$ and the subcategory $\mathscr{H}$ of the homogeneous representations. For any two representations $\mathbf{X}$, $\mathbf{Y}$ of the same $\mathscr{R}_j$, there is a representations $\mathbf{Z}$ and non-zero maps $\mathbf{X}\to\mathbf{Z}$ and $\mathbf{Z}\to\mathbf{Y}$, so that for any perfect element p, $p(\mathbf{X})$ has the same value (0 or X_0) for all $\mathbf{X}\in\mathscr{R}_j$.

Moreover, the images of $\mathfrak{D}^r_q$ in the subspace lattice of X_0 of an indecomposable representation $\mathbf{X}=(X_0;X_i)$ in $\mathscr{H}$ are all of the form



and thus, for a perfect element p, $p(\mathbf{X})$ has the same value (0 or X_0) for all $\mathbf{X}\in\mathscr{H}$. In summary, there are at most 16 distinct perfect elements in $\mathfrak{D}^r_q$ in addition to those belonging to the cubicles.

In the case $r\geqq 5$, the statement of Theorem 1 is an immediate consequence (using the above criterion) of Theorem 2 on the existence of chains of non-zero homomorphisms between regular representations of $\mathfrak{D}^r_q$ which seems to be of some interest in itself.

Homomorphisms Between Regular Representations of $\mathfrak{D}^r_q$

Recall that a representation of $\mathfrak{D}^r$ is called *regular* in case it does not have an indecomposable preprojective or preinjective direct summand. Here, the indecomposable preprojective or preinjective representations are those obtained from the projective or injective ones by a successive application of one of the two Coxeter functors (see below). Alternatively, an intrinsic definition may be given as follows: Call $\mathbf{P}$ preprojective provided there is only a finite number of indecomposable representations $\mathbf{X}$ with $\mathrm{Hom}(\mathbf{X},\mathbf{P})\neq 0$, and call $\mathbf{I}$ preinjective provided there is only a finite number of indecomposable representations $\mathbf{Y}$ with $\mathrm{Hom}(\mathbf{I},\mathbf{Y})\neq 0$. (It is clear that this definition coincides with the usual one for all finite dimensional hereditary algebras.)

Now, the main theorem asserts that for $r\geqq 5$, any two non-zero regular representations can be connected by a sequence of non-zero maps (the composition of these maps may, of course, be zero!).

Theorem 2. *Let* $\mathbf{X}$, $\mathbf{Y}$ *be indecomposable regular representations of* $\mathfrak{D}^r_q$, $r\geqq 5$. *Then there exist indecomposable regular representations* $\mathbf{X}=\mathbf{X}_0, \mathbf{X}_1,\ldots,\mathbf{X}_t,\ldots,\mathbf{X}_l=\mathbf{Y}$ *with* $\mathrm{Hom}(\mathbf{X}_{t-1},\mathbf{X}_t)\neq 0$ *for all* $1\leqq t\leqq l$.

Before proving Theorem 2, let us return to the proof of Theorem 1. As mentioned earlier, Gelfand and Ponomarev have shown that any perfect element $p\in\mathfrak{D}^r_q$ which does not belong to a cubicle, must satisfy $p(\mathbf{X})=0$ for all preprojective $\mathbf{X}$, and $p(\mathbf{X})=X_0$ for all preinjective $\mathbf{X}$. Since the value of $p(\mathbf{X})$ for regular $\mathbf{X}$ is, for a fixed p, either always $=0$, or always the total space X_0, it is clear that there are at most two such elements in $\mathfrak{D}^r_q$. This completes the proof of Theorem 1.

In what follows, we shall always consider representations of $\mathfrak{D}^r$ over a fixed division ring F of characteristic q, or equivalently, representations of $\mathfrak{D}^r_q$ over F.

Recall the concept of the Coxeter functors C^+ and C^-. Given a representation $\mathbf{X}=(X_0;X_i)$ of $\mathfrak{D}^r_q$, denote by Y_0 the kernel of the summation map $\bigoplus_i X_i\to X_0$ and by Y_i $(1\leqq i\leqq r)$ the kernels of the respective component maps $Y_0\to X_i$. The resulting representation $\mathbf{Y}=(Y_0;Y_i)$ is, by definition, $C^+\mathbf{X}$. The functor C^- is defined dually; for the basic properties of these functors we refer to [6, 4]. In particular, denoting by $\mathbf{dim}\,\mathbf{X}$ the (integral) rational vector $(x_i)_{0\leq i\leq r}\in\mathbb{Q}^{r+1}$, where $x_i=\dim X_i$, we have, for every indecomposable $\mathbf{X}$ either $C^+\mathbf{X}=0$ or $\mathbf{dim}\,C^+\mathbf{X}=c(\mathbf{dim}\,\mathbf{X})$; here, the Coxeter transformation c on $\mathbb{Q}^{r+1}$ is defined by

$$c(x_0,\ldots,x_i,\ldots)=\left(\sum_{t=1}^{r}x_t-x_0,\ldots,\sum_{\substack{t=1\\t\neq i}}^{r}x_t-x_0,\ldots\right).$$

An indecomposable representation $\mathbf{X}$ is preprojective (or preinjective) if there exists an exponent m such that $C^{+m}\mathbf{X}=\mathbf{0}$ (or $C^{-m}\mathbf{X}=\mathbf{0}$).

In what follows, always $r\geqq 5$. Note that, with respect to the transformation c, the rational space $\mathbb{Q}^{r+1}$ decomposes as follows:

$$\mathbb{Q}^{r+1}=U\oplus\langle\mathbf{a}^+\rangle\oplus\langle\mathbf{a}^-\rangle$$

with the $(r-1)$-dimensional eigenspace U corresponding to -1, and with the (one-dimensional) eigenspaces $\langle\mathbf{a}^+\rangle$ and $\langle\mathbf{a}^-\rangle$ corresponding to $\lambda=\frac{1}{2}(r-2+\sqrt{r(r-4)})>1$ and λ^{-1}. Here, $U=\{(0,x_1,\ldots,x_r)\mid\sum_{t=1}^{r}x_t=0\}$, $\mathbf{a}^+=(r-\sqrt{r(r-4)},2,\ldots,2)$ and $\mathbf{a}^-=(r+\sqrt{r(r-4)},2,\ldots,2)$.

Lemma 1. *Let* $\mathbf{X}$ *be a non-zero regular representation of* $\mathfrak{D}_q^r$, $r\geqq 5$, *and*

$$\mathbf{dim}\,\mathbf{X}=\mathbf{u}+\xi^+\mathbf{a}^++\xi^-\mathbf{a}^-.$$

Then $\xi^+>0$ *and* $\xi^->0$.

Proof. First, both ξ^+ and ξ^- are non-zero. For, if both $\xi^+=0$, $\xi^-=0$, then $\mathbf{dim}\,C^+\mathbf{X}=-\mathbf{dim}\,\mathbf{X}$ gives a contradiction to the fact that both $\mathbf{dim}\,\mathbf{X}$ and $\mathbf{dim}\,C^+\mathbf{X}$ are positive. Thus, assume $\xi^+=0$ and $\xi^-\neq 0$. Then for all even natural m,

$$\mathbf{dim}\,C^{+m}\mathbf{X}=\mathbf{u}+\lambda^{-m}\xi^-\mathbf{a}^-.$$

But this cannot be integral for all such m, since $\lambda>1$ and therefore λ^{-m} is arbitrarily small.

Similarly, in case $\xi^+\neq 0$ and $\xi^-=0$, we use C^{-m}.

Second, from the fact that, for all even m, both

$$\mathbf{dim}\,C^{+m}\mathbf{X}=\mathbf{u}+\lambda^m\xi^+\mathbf{a}^++\lambda^{-m}\xi^-\mathbf{a}^-$$

and

$$\mathbf{dim}\,C^{-m}\mathbf{X}=\mathbf{u}+\lambda^{-m}\xi^+\mathbf{a}^++\lambda^m\xi^-\mathbf{a}^-$$

have non-negative components, we infer that both $\xi^+>0$ and $\xi^->0$.

Recall the definition of the (non-symmetric) bilinear form B on $\mathbb{Q}^{r+1}$

$$B(\mathbf{x},\mathbf{y})=\sum_{i=0}^{r}x_iy_i-\sum_{i=1}^{r}x_iy_0=x_0y_0+\sum_{i=1}^{r}x_i(y_i-y_0)$$

and the formula

$$B(\mathbf{dim}\,\mathbf{X},\mathbf{dim}\,\mathbf{Y})=\dim\operatorname{Hom}(\mathbf{X},\mathbf{Y})-\dim\operatorname{Ext}^1(\mathbf{X},\mathbf{Y})$$

of [9].

Lemma 2. *For* $r \geqq 5$, $B(\mathbf{a}^-, \mathbf{a}^+) > 0$.

Proof. Clearly, $B(\mathbf{a}^-, \mathbf{a}^+) = 2r(4 - r + \sqrt{r(r-4)}) > 0$ for all $r \geqq 5$.

Proposition. *Let* **X**, **Y** *be non-zero regular representations of* $\mathfrak{D}^r_q$, $r \geqq 5$. *Then there exists a natural* m_0 *such that* $\mathrm{Hom}(\mathbf{X}, C^{+m}\mathbf{Y}) \neq 0$ *for all* $m \geqq m_0$.

Proof. Let $\mathbf{dim}\,\mathbf{X} = \mathbf{u} + \xi^+\mathbf{a}^+ + \xi^-\mathbf{a}^-$ and $\mathbf{dim}\,\mathbf{Y} = \mathbf{v} + \eta^+\mathbf{a}^+ + \eta^-\mathbf{a}^-$ with $\mathbf{u}, \mathbf{v} \in U$ and real $\xi^+, \xi^-, \eta^+, \eta^-$. According to Lemma 1, $\xi^- > 0$ and $\eta^+ > 0$. For even natural n,

$$
\begin{aligned}
&B(\mathbf{dim}\,C^{-n}\mathbf{X}, \mathbf{dim}\,C^{+n}\mathbf{Y}) \\
&\quad = B(\mathbf{u} + \lambda^{-n}\xi^+\mathbf{a}^+ + \lambda^n\xi^-\mathbf{a}^-, \mathbf{v} + \lambda^n\eta^+\mathbf{a}^+ + \lambda^{-n}\eta^-\mathbf{a}^-) \\
&\quad = \xi^-\eta^+ B(\mathbf{a}^-, \mathbf{a}^+)\lambda^{2n} + c_1\lambda^n + c_2 + c_3\lambda^{-n} + c_4\lambda^{-2n}
\end{aligned}
$$

with real constants c_k, $1 \leqq k \leqq 4$. Consequently, in view of Lemmas 1 and 2,

$$B(\mathbf{dim}\,C^{-n}\mathbf{X}, \mathbf{dim}\,C^{+n}\mathbf{Y}) > 0$$

for large even n. Thus, applying the above mentioned formula of [9].

$$0 < \dim \mathrm{Hom}(C^{-n}\mathbf{X}, C^{+n}\mathbf{Y}) = \dim \mathrm{Hom}(\mathbf{X}, C^{+2n}\mathbf{Y}).$$

Substituting $C^{+t}\mathbf{Y}$, with $1 \leqq t \leqq 3$, for **Y**, we see that for large n, also $0 < \dim \mathrm{Hom}(\mathbf{X}, C^{+(2n+t)}\mathbf{Y})$, as required.

Proof of Theorem 2. Let **X**, **Y** be indecomposable representations of $\mathfrak{D}^r$ over the fixed division ring F. Let $\mathbf{Z}'$ be some regular representation of $\mathfrak{D}^r$ over the prime field K of F with $\mathrm{End}(\mathbf{Z}') = K$, and let $\mathbf{Z} = \mathbf{Z}' \otimes_K F_F$. Then $\mathrm{End}(\mathbf{Z}) = F$; in particular, **Z** is indecomposable and also regular. By the Proposition, there exists m with $\mathrm{Hom}(\mathbf{X}, C^{+m}\mathbf{Z}) \neq 0$ and $\mathrm{Hom}(\mathbf{Z}, C^{+m}\mathbf{Y}) \neq 0$, thus also $\mathrm{Hom}(C^{-m}\mathbf{Z}, \mathbf{Y}) \neq 0$. The indecomposable representation $\mathbf{Z}'$ is a module over a finite dimensional algebra over a communicative field, thus $\mathrm{Ext}^1(\mathbf{Z}', C^+\mathbf{Z}') \neq 0$. Namely, $C^+\mathbf{Z}'$ is just the dual of the transpose of $\mathbf{Z}'$ (see [5]), thus there exists the almost split sequence (see [1])

$$0 \to C^+\mathbf{Z}' \to \mathbf{E}' \to \mathbf{Z}' \to 0.$$

Tensoring with F we obtain a non-split exact sequence

$$0 \to C^+\mathbf{Z} \to \mathbf{E} \to \mathbf{Z} \to 0,$$

where $\mathbf{E} = \mathbf{E}' \otimes_K F_F$ and where we use $C^+\mathbf{Z} = C^+(\mathbf{Z}' \otimes_K F_F) = (C^+\mathbf{Z}) \otimes_K F_F$. Now with **Z** and $C^+\mathbf{Z}$, also **E** is regular. Applying C^{+i} and C^{-i}, for $1 \leqq i \leqq m$, we obtain a chain of non-zero maps

$$C^{+m}\mathbf{Z} = \mathbf{Z}_1 \to \mathbf{Z}_2 \to \ldots \to \mathbf{Z}_l = C^{-m}\mathbf{Z}$$

with regular representations $\mathbf{Z}_t$, $1 \leqq t \leqq l = 4m$. This finishes the proof of theorem 2.

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