manuscripta math. 31, 317 - 329 (1980)

manuscripta mathematica

BINARY POLYHEDRAL GROUPS AND EUCLIDEAN DIAGRAMS

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Recently, J.McKay [7] has observed that the irreducible complex representations of the binary polyhedral groups can be arranged in order to form the vertices of a Euclidean diagram in such a way that the tensor product of any irreducible representation M with the standard two-dimensional representation is the direct sum of the irreducible representations which are the neighbors of M in the diagram, and he asked for an explanation. In this note, we will show that any self-dual two-dimensional representation gives rise to a generalized Euclidean diagram, and that this in fact can be used to give a proof of the classification theorem of the binary polyhedral groups which at the same time furnishes a list of the irreducible representations and also gives the minimal splitting field.

Of course, we know of many different proofs for the classification of groups G with a faithful two-dimensional representation T (see [6],[9],[3]), however in case T also is self-dual it seems to be of interest to work directly with the corresponding generalized Euclidean diagram. In particular, we will see that the diagram gives immediately both the order of G and of G/G'.

It seems that the existence of additive, or subadditive functions for Euclidean, or Dynkin diagrams respectively, is one of the reasons for the frequent occurance (see for example [1]) of these types of Cartan matrices. Whereas in this paper we deal with the binary polyhedral groups, a similar approach has proved successful also for the representations theory of non-semisimple finite dimensional algebras [4].

1. Selfdual two-dimensional representations

Let k be a field of characteristic $p \geqslant o$, and G a group of order not divisible by p. Assume there exists a selfdual two-dimensional representation T of G over k. We will see that T determines a generalized Euclidean diagram (for the definition and the list of all possible cases, we refer to the next section). Let I be the set of all isomorphism classes of irreducible kG-modules,

0025-2611/80/0031/0317/$02.60

and, for any $i \in I$, fix a representative M_i. Let $E_i = \mathrm{End}_{kG}(M_i)$ be the endomorphism ring, $e_i = \dim_k E_i$, and $d_i = \frac{1}{e_i} \dim_k M_i$, thus d_i is just the dimension of M_i considered as an E_i-vectorspace. For any $j \in I$, decompose $T \otimes M_j$ as direct sum of irreducible modules, say

$$T \otimes M_j = \bigoplus_{i \in I} \bigoplus_{c_{ij}} M_i .$$

The k-dimension of $T \otimes M_j$ is $2d_je_j$, the k-dimension of the right side is $\sum_{i \in I} c_{ij}d_ie_i$, thus

$$(*) \qquad 2d_j = \sum_{i \in I} d_i(c_{ij} \frac{e_i}{e_j})$$

Also, c_{ij} is equal to the dimension of $\mathrm{Hom}(M_i, T\otimes M_j)$ as E_i-vectorspace, thus $c_{ij}e_i$ is its k-dimension. Similarly, c_{ji} equals the dimension of $\mathrm{Hom}(T\otimes M_i, M_j)$ as E_j-vectorspace, thus $c_{ji}e_j$ is its k-dimension. Since T is selfdual, the vectorspace $\mathrm{Hom}(M_i, T\otimes M_j)$ and $\mathrm{Hom}(T\otimes M_i, M_j)$ have the same k-dimension, thus

$$(**) \qquad c_{ij}e_i = c_{ji}e_j .$$

As a consequence, if we define $C = (C_{ij})_{ij}$ by

$$C_{ij} = \begin{cases} 2-c_{ii} & \\ & \text{for} \\ -c_{ij}\frac{e_i}{e_j} & \end{cases} \begin{matrix} i=j \\ \\ i \neq j \end{matrix} ,$$

then C <u>is a generalized Cartan matrix</u>, $e = (e_i)_i$ <u>is a symmetrization</u>, and $d = (d_i)_i$ <u>is an additive function for</u> C (again, for the definition of symmetrization and additive function, we refer to the next section). The classification of all connected generalized Cartan matrices with additive functions given in the next section determines the possible structure of C, and we want to apply this to the connected component I_o of I containing the trivial one-dimensional representation (as well as T).

LEMMA. An irreducible representation M belongs to I_o if and only if $\operatorname{Ker} M \supseteq \operatorname{Ker} T$.

Proof: If N is a neighbour of M in I, and $\operatorname{Ker} M \supseteq \operatorname{Ker} T$, then also $\operatorname{Ker} N \supseteq \operatorname{Ker} T$, since N is a direct summand of $T \otimes M$. Thus, any M in I_o satisfies $\operatorname{Ker} M \supseteq \operatorname{Ker} T$. The converse is a little more difficult, but also well-known, see [5], theorem V.10.8.

The generalized Euclidean diagram given by the generalized Cartan matrix $C|I_o$ will be called the diagram of T. In order to determine the possible diagrams of selfdual two-dimensional representations T, we may assume T to be faithful.

Now the trivial one-dimensional representation shows that there exists i with $d_i = e_i = 1$, thus both d and e have to be the functions given in the table of section 2, not just multiples. Also, we see that the cases $\tilde{\mathbb{A}}_{11}, \tilde{\mathbb{BC}}_n, \tilde{\mathbb{BD}}_n, \tilde{\mathbb{C}}_n, \tilde{\mathbb{CL}}_n, \tilde{\mathbb{F}}_{42}, \tilde{\mathbb{G}}_{22}$ cannot occur.

From d and e we can calculate both the order $|G|$ of G, as well as the order $|G/G'|$ of the commutator factor group G/G', namely

$$|G| = \sum_i d_i^2 e_i \qquad |G/G'| = \sum_{d_i=1} e_i .$$

(The first follows directly from the Wedderburn structure theorem for the semisimple algebra kG, the second from the fact that G' is contained in $\operatorname{Ker} M_i$ if and only if $d_i = 1$. Namely, since $\tilde{\mathbb{A}}_{11}, \tilde{\mathbb{BC}}_n$ does not occur, we always have $e_i \leqslant 3$, thus all possible endomorphism rings E_i are commutative.)

We denote by $\mathcal{C}_n, \mathcal{D}_n, \mathcal{Q}_n$ the cyclic, or dihedral, or binary dihedral group of order n. Thus, $\mathcal{D}_{2n}$ and $\mathcal{Q}_{4n}$ are defined by the following generators and relations

$$\mathcal{D}_{2n} = \langle x,y \mid x^n = 1,\ y^2 = 1,\ y^{-1}xy = x^{-1}\rangle$$
$$\mathcal{Q}_{4n} = \langle x,y \mid x^{2n} = 1,\ y^2 = x^n,\ y^{-1}xy = x^{-1}\rangle.$$

(Note that $\mathcal{D}_2 = C_2$, $\mathcal{D}_4 = C_2 \times C_2$, and $\mathcal{Q}_4 = C_4$; also, recall that the binary dihedral 2-groups are also called generalized quaternion). Also, $\mathcal{T}$ will denote the binary tetrahedral group (thus $\mathcal{T} = SL_2(F_3)$), $\mathcal{O}$ the binary octahedral groups, and $\mathcal{J}$ the binary icosahedral group (thus, $\mathcal{J} = SL_2(F_5)$), see [6],[9] or [8].

We denote by ζ_n primitive n-th root of 1, and $\rho_n = \zeta_n + \zeta_n^{-1}$. With k we denote the trivial (one-dimensional) representation. With these notations, we can now formulate the main result.

<u>THEOREM 1</u> <u>Let</u> G <u>be a finite group</u>, <u>let</u> k <u>be a field of characteristic</u> $p \geq 0$ <u>with</u> $p \nmid |G|$. <u>Let</u> T <u>be a faithful selfdual two-dimensional representation of</u> G <u>over</u> k. <u>Then we deal with one of the following cases</u>:

diagram	$\lvert G\rvert$	$\lvert G/G'\rvert$	G	k	T
$\tilde{\mathbb{A}}_n$	n+1	n+1	C_{n+1}	$\zeta_{n+1} \in k$	$M \oplus M^*$
$\tilde{\mathbb{A}}_3$	4	4	$C_2 \times C_2$		$M \oplus N$, $k \neq M \neq N \neq k$
$\tilde{\mathbb{A}}_{12}$	2	2	C_2		$M \oplus M$
$\tilde{\mathbb{B}}_n$	2n	2n	C_{2n}	$\rho_{2n} \in k, \zeta_{2n} \notin k$	
$\tilde{\mathbb{BL}}_n$	2n-1	2n-1	C_{2n-1}	$\rho_{2n-1} \in k, \zeta_{2n-1} \notin k$	
$\tilde{\mathbb{CD}}_{2n}$	4(2n-1)	4	$\mathcal{Q}_{4(2n-1)}$	$\zeta_{2(n-2)} \in k, \zeta_4 \notin k$	
$\tilde{\mathbb{D}}_n$	4(n-2)	4	$\mathcal{D}_{4(n-2)}$	$\rho_{2(n-2)} \in k$	
$\tilde{\mathbb{D}}_n$	4(n-2)	4	$\mathcal{Q}_{4(n-2)}$	$\zeta_{2(n-2)} \in k$ $n \equiv 0 (2)$ $\zeta_{4(n-2)} \in k$ $n \equiv 1 (2)$	
$\tilde{\mathbb{DL}}_n$	2(2n-1)	2	$\mathcal{D}_{2(2n-1)}$	$\rho_{2n-1} \in k$	

$\tilde{\mathbb{E}}_6$	24	3	T	$\zeta_{12} \in k$	
$\tilde{\mathbb{E}}_7$	48	2	O	$\zeta_8 \in k$	
$\tilde{\mathbb{E}}_8$	120	1	J	$\zeta_5 \in k$	
$\tilde{\mathbb{F}}_{41}$	24	3	T	$\zeta_4 \in k$, $\zeta_3 \notin k$	
$\tilde{\mathbb{L}}_o$	1	1	1		
$\tilde{\mathbb{L}}_1$	2	2	C_2		$k \oplus M$

<u>Proof</u>: Consider first the case where G is commutative. Recall that this just means $d_i = 1$ for all i, thus we deal with the cases $\tilde{\mathbb{A}}_n$, $\tilde{\mathbb{A}}_{12}$, $\tilde{\mathbb{B}}_n$, $\tilde{\mathbb{BL}}_n$, $\tilde{\mathbb{L}}_n$. Now, since G has a selfdual two-dimensional faithful representation T, it has to be cyclic or $C_2 \times C_2$. Namely, we may suppose that k is algebraically closed, thus $T = T_1 \oplus T_2$ with T_i one-dimensional. If $T_2 = T_1^*$, then T_1 is a faithful one-dimensional representation, thus G is a subgroup of the multiplicative group of a field, thus cyclic. Otherwise $T_1 = T_1^*$, $T_2 = T_2^*$, and $G = C_2 \times C_2$. In case $\tilde{\mathbb{L}}_n$, the diagram shows that there exists an irreducible module M which is a direct summand of $T \otimes M$. Now since $d_i = e_i = 1$ for all i, we see that T decomposes, say $T = T_1 \oplus T_2$, thus $T \otimes M = (T_1 \otimes M) \oplus (T_2 \otimes M)$, and we may suppose $M \approx T_1 \otimes M$. Since M is one-dimensional, this implies that T_1 is the trivial representation, and therefore any N is a direct summand of $T \otimes N$. Thus the diagram must contain for every vertex a loop. This shows that the only possible $\tilde{\mathbb{L}}_n$ are $\tilde{\mathbb{L}}_o$ and $\tilde{\mathbb{L}}_1$.

Next, consider the cases $\tilde{\mathbb{CD}}_n$, $\tilde{\mathbb{D}}_n$, $\tilde{\mathbb{DL}}_n$. Denote by $\underline{\underline{G}}$ the class of groups with only one-or two-dimensional irreducible representations having a two-dimensional faithful selfdual representation. Note that $\underline{\underline{G}}$ is closed under subgroups. If $G \in \underline{\underline{G}}$ and non-commutative, then we see that the order of G/G' is 2 or 4. If G is a non-commutative 2-group in $\underline{\underline{G}}$, then $|G/G'| = 4$ shows that G is either dihedral, semi-dihedral a generalized quaternion. However, it is easy to check that a semidihedral group cannot be embedded into $GL_2(k)$. By induction on $|G|$, we see that a group $G \in \underline{\underline{G}}$ of order $|G| = 2^s t$ with $(2,t) = 1$ contains a subnormal (thus normal) cyclic subgroup of order t. Let $t \neq 1$. Then G is a proper semidirect product of a cyclic group N of order t by a dihedral or generalized quaternion group H. Let $H_1 = C_H(N)$, the centralizer of N in H. Then H/H_1 embeds into the automorphism group of N, thus is cyclic, and of order 2 or 4. In fact, only for $H = \mathcal{Q}_4 (= C_4)$, there is a cyclic factor group of H of order 4, however one checks immediately that for a cyclic subgroup N of $GL_2(k)$, the normalizer of N modulo the centralizer of N is of order 2. Thus, always we have $H/H_1 = C_2$. On the other hand, the subgroup $G_1 = N \times H_1$ of G belongs to $\underline{\underline{G}}$, and since t divides the order of G_1/G_1', it follows that G_1 is commutative, thus by previous considerations, cyclic. This shows that G is either dihedral or binary dihedral. Also note that in case $\widetilde{\mathbb{CD}}_n$, the group G/G' has to be cyclic of order 4, since G/G' only has 3 irreducible representations. As a consequence, the order of G has to be $4t$, with t odd, and therefore n must be even.

In cases $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_{41}$ it follows from $|G| = 24$, $|G/G'| = 3$ that $|G'| = 8$, thus by the preceeding considerations, G' is either cyclic, dihedral or quaternion. The first two cases are impossible, since these groups do not allow an automorphism of order 3. Thus G is non-trivial semidirect product of $\mathcal{Q}_8$ by C_3, thus $G = \mathcal{T}$.

In the case $\widetilde{\mathbb{E}}_7$, we know from $|G| = 48$, $|G/G'| = 2$ that $|G'| = 24$. Also, it follows that $|G'/G''| = 3$, thus $G' = \mathcal{T}$ by the preceeding case. But there are just two non-trivial extensions of $\mathcal{T}$ by C_2, namely $GL_2(F_3)$ and $\mathcal{O}$. Since the Sylow groups of $GL_2(F_3)$ are semidihedral, this case cannot occur, thus $G = \mathcal{O}$.

Finally, in case $\widetilde{\mathbb{E}}_8$, the group G is perfect. Let H be a maximal normal subgroup of G, then G/H is a non-abelian simple group of order dividing 120, thus $G/H = A_5$ (see [5]). Also, H is of order 2, thus H is the center of G, and $G = SL_2(F_5) = \mathcal{J}$ (again, see [5]).

Also, note that the case $\mathbb{C}_{21}$ cannot occur. Namely, G/G' would be of order 4 with only two irreducible representations, impossible.

Finally, let us indicate the reason for the assertions concerning k. Note that k is a splitting field for C_n iff $\zeta_n \in k$. If T is a twodimensional faithful selfdual representation of C_n, then its trace is of the form ρ_n, thus we must have $\rho_n \in k$. Of course, conversely, $\begin{pmatrix} 0 & 1 \\ -1 & \rho_n \end{pmatrix}$ generates a cyclic subgroup of k of order n, and, in this way, we obtain a selfdual representation. For $G = \mathcal{T}$, there exists a faithful two-dimensional representation if and only if $\zeta_4 \in k$, and k is a splitting field for $\mathcal{T}$ iff in addition $\zeta_3 \in k$.

Remarks

1. On splitting fields. If T is defined over k, and the diagram of T is split, then k is a splitting field for G. (This is an immediate consequence of the considerations above).

2. On binary polyhedral groups. If we are only interested in the classical question of the finite subgroups of $SL_2(\mathbb{C})$, then one first notes that we may suppose that G is a subgroup of $SU_2(\mathbb{C})$, thus the canonical 2-dimensional representation is selfdual. (Namely, let (,) be the usual inner product on $\mathbb{C}^2$, and define a new inner product by $\langle x,y \rangle = \frac{1}{|G|} \sum_{g \in G} (gx,gy)$, then this is G-invariant). If $1 \neq g \in G$ is an involution, then g is diagonalizable, and since $G \subseteq SL_2(\mathbb{C})$, we see that g is given by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. This shows that g is central and is the only involution. In this way, we exclude the possibility of G being a dihedral group (or $C_2 x C_2$), thus (taking into account that k is a splitting field) only the cases $\tilde{\mathbb{A}}_n$, $\tilde{\mathbb{D}}_n$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$ $\tilde{\mathbb{E}}_8$ remain.

3. On infinite groups. If G is an infinite group with a two-dimensional selfdual representation T, and $\{M_i | i \in I\}$ is a set of irreducible representations with the property that for all $j \in I$, the tensor product $T \otimes M_j$ decomposes as a direct sum of some M_i, then similarly we obtain a generalized Cartan matrix whose components are either generalized Euclidean diagrams or else of the form

$$\mathbb{A}_\infty, \mathbb{A}_\infty^\infty, \mathbb{B}_\infty, \mathbb{C}_\infty, \mathbb{D}_\infty, \mathbb{L}_\infty.$$

(Again, the component containing the trivial representation cannot be of the form $\mathbb{C}_\infty$ or $\mathbb{L}_\infty$).

For example, consider the group $G = SL_2(\mathbb{C})$, and the irreducible rational representations. The canonical two-dimensional representation T is selfdual ([8],3.1.6), and we obtain the diagram $\mathbb{A}_\infty$. Namely, there is precisely one irreducible rational representation M_i of dimension i, and, for $i \geq 2$, we have $T \otimes M_i = M_{i-1} \oplus M_{i+1}$ (see[8], 3.2.1 and 3.2.4).

2. Generalized Cartan - matrices

Let I be an index set. A function $C: I \times I \to \mathbb{Z}$ will be called a generalized Cartan matrix in case

(1) $C_{ii} < 2$ for all $i \in I$,

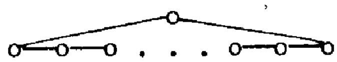
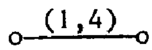
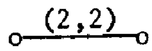
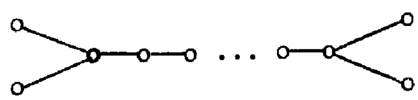
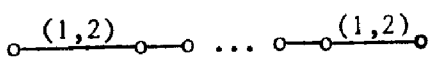
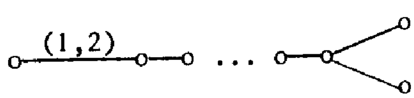
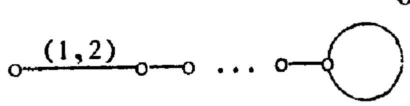
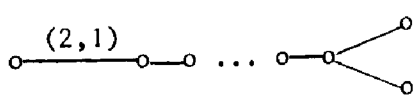
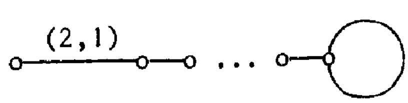
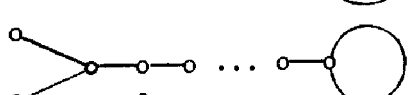
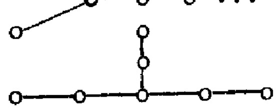
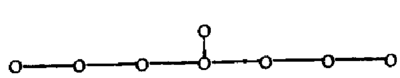
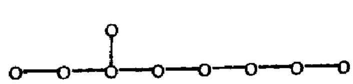
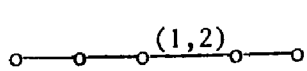
(2) $C_{ij} \leq 0$ for all $i \neq j$ in I, and

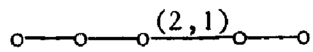
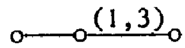
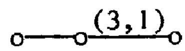
(3) $C_{ij} = 0$ if and only if $C_{ji} = 0$.

Note that we write C_{ij} instead of $C(i,j)$, and similarly $x_i = x(i)$ for any function $x: I \to \mathbb{Z}$. (This generalizes the notion of a Cartan matrix where one assumes instead of (1) the stronger property (1') $C_{ii} = 2$ for all $i \in I$, see[2]). The underlying graph of C has as vertices the elements of I, edges $\{i,j\}$ for all pairs $i \neq j$ with $C_{ij} \neq 0$, and there are $2-C_{ii}$ loops for every $i \in I$. In case for some $i \neq j$, we have $C_{ij}C_{ji} \neq 1$, then we add to the edge $\{i,j\}$ the pair of numbers $(|C_{ij}|,|C_{ji}|)$, and, in this way, we obtain a valued graph (possibly with loops). Clearly, C will be called connected in case the underlying graph is connected.

Let C be a generalized Cartan matrix. A symmetrization for C is a function $e: I \to \mathbb{N} = \{1,2,3,\dots\}$ satisfying $C_{ij}e_j = C_{ji}e_i$ for all i,j in I. An additive function for C is a function $d: I \to \mathbb{N}$ satisfying $\sum_{i \in I} d_i C_{ij} = 0$ for all $j \in I$. (The existence of an additive function for C of course immediately implies that for given $j \in I$ at most finitely many C_{ij} $(i \in I)$ are non-zero).

THEOREM 2. Let C be a connected generalized Cartan matrix on a finite index set with an additive function d. Then C is of one of the following form, d is a multiple of the listed additive function, and there exists a symmetrization (and all are multiples of the listed one).

Case	valued graph	additive function	symmetrization
$\tilde{\mathbb{A}}_n$		$111\ \overset{1}{\ldots}\ 111$	$111\ \overset{1}{\ldots}\ 111$
$\tilde{\mathbb{A}}_{11}$		21	14
$\tilde{\mathbb{A}}_{12}$		11	11
$\tilde{\mathbb{B}}_n$		111 ... 111	122...221
$\tilde{\mathbb{C}}_n$		122 ... 221	211...112
$\tilde{\mathbb{D}}_n$		${}^1_1 22 \ldots 22^1_1$	${}^1_1 11 \ldots 11^1_1$
$\tilde{\mathbb{L}}_n$		11 ... 11	11...11
$\tilde{\mathbb{BC}}_n$		222 ... 221	122...224
$\tilde{\mathbb{BD}}_n$		$222 \ldots 22^1_1$	$122 \ldots 22^2_2$
$\tilde{\mathbb{BL}}_n$		111 ... 11	122...22
$\tilde{\mathbb{CD}}_n$		$122 \ldots 22^1_1$	$211 \ldots 11^1_1$
$\tilde{\mathbb{CL}}_n$		122 ... 22	211...11
$\tilde{\mathbb{DL}}_n$		${}^1_1 22 \ldots 22$	${}^1_1 11 \ldots 11$
$\tilde{\mathbb{E}}_6$		$1\ 2\ \overset{\overset{1}{2}}{3}\ 2\ 1$	$1\ 1\ \overset{\overset{1}{1}}{1}\ 1\ 1$
$\tilde{\mathbb{E}}_7$		$1\ 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$	$111\overset{1}{1}111$
$\tilde{\mathbb{E}}_8$		$2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2\ 1$	$11\overset{1}{1}11111$
$\tilde{\mathbb{F}}_{41}$		1 2 3 2 1	1 1 1 2 2

$\tilde{\mathbb{F}}_{42}$		1 2 3 4 2	2 2 2 1 1
$\tilde{\mathbb{C}}_{21}$		1 2 1	1 1 3
$\tilde{\mathbb{C}}_{22}$		1 2 3	3 3 1

These generalized Cartan matrices will be called <u>generalized Euclidean</u> diagrams. Always, $n+1$ is the number of points, where n is the (first) index in $\tilde{A}_n$, $\tilde{A}_{11}$, etc.

<u>Proof</u>: One may apply Vinberg's theorem 3 of [10]. For the convenience of the reader, we give an outline of a short proof, similar to the proof given in [4].

A <u>subadditive function</u> $d: I \to \mathbb{N}$ for a generalized Cartan matrix is defined by the property $\sum_{i \in I} d_i C_{ij} \geq 0$, for all $j \in I$.

(1) Let C be a generalized Cartan matrix, and assume there exists an additive function ∂ for the transpose $C^{\perp}$. Then any subadditive function for C is additive.

Proof: From $C\partial^{\perp} = 0$ it follows $(dC)\partial^{\perp} = 0$. The components of dC are ≥ 0, those of ∂ are > 0, thus $dC = 0$.

(2) Every subadditive function for a generalized Euclidean diagram is additive.

Proof: One easily checks that the function given in the list above is additive, we can apply (1).

If $I' \subseteq I$, and C is a generalized Cartan matrix for I, and C' one for I', then we say that C' is a <u>restriction</u> of C if $C'_{ij} \geq C_{ij}$ for all $i,j \in I'$.

(3) Let C' be a proper restriction of a connected generalized Cartan matrix C, and d subadditive for C. Then $d|I'$ is subadditive and not additive.

Proof: Clearly $d|I'$ is subadditive, since for $j \in I'$

$$\sum_{i \in I'} d_i C'_{ij} \geq \sum_{i \in I'} d_i C_{ij} \geq \sum_{i \in I} d_i C_{ij} \geq 0.$$

In case $C'_{ij} > C_{ij}$ for some $i,j \in I'$, the first inequality sign will be proper, in case $j \in I'$, $i \in I \setminus I'$ are neighbors, the second inequality sign will be proper.

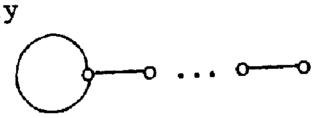
(4) Any connected generalized Cartan matrix with finite index set is either a restriction of a generalized Euclidean diagram, or has a generalized Euclidean diagram as its restriction.

Proof: This is a rather straightforward combinatorial verification.

The connected generalized Cartan matrices being proper restrictions of generalized Euclidean diagrams may be called <u>generalized Dynkin diagrams</u>; there is besides the usual Dynkin diagrams

$$\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$$

one additional case, namely

$\mathbb{L}_n$ 

(5) Let C be a generalized Dynkin diagram. Then this generalized Cartan matrix is regular, thus there is no non-zero function $f: I \to \mathbb{Q}$ satisfying $\sum_{i \in I} f_i C_{ij} = 0$ for all $j \in I$.

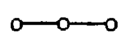
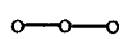
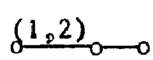
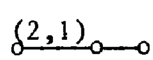
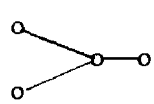
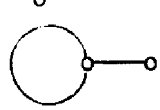
Proof: This is well-known (and easy to prove by case-by-case-inspection, using induction if necessery) for the usual Dynkin diagrams, and also easily established for $\mathbb{L}_n$.

(6) End of proof: If C has a proper restriction to a generalized Euclidean diagram C', then C cannot have any subadditive function, since its restriction to C' would be a subadditive function which is not additive by (3), a contradiction to (2). If C is a generalized Dynkin diagram, then there is no additive function by (5), or also by (2) and (3). - Thus, C has to be generalized Euclidean, by (4). If d', d'' are two additive functions for C, then some linear combination d of d', d'' vanishes for some $i \in I$. Assume $d \neq 0$. Let I be a connected component of its support. Then $C|I' \times I'$ is generalized Dynkin,

and the function $d|I'$ from I' to $\mathbb{Q}$ satisfies $\sum_{i\in I'} d_i C_{ij} = 0$ for all $j \in I'$. Thus $d|I'$ is zero by (5), a contraction. Thus, d', d'' are multiples of each other. - It is clear that for connected generalized Cartan matrices, all symmetrizations are multiples of each other.

Remarks

1. On infinite index sets. If C is a connected generalized Cartan matrix on an infinite index set, with an additive function d, then C is of one of the following form, d is a multiple of the listed additive function, and there exists a symmetrization (and all are multiples of the listed one)

case	valued graph	additive function	symmetrization
$\mathbb{A}_\infty$	 ...	1 2 3 ...	1 1 1 ...
$\mathbb{A}_\infty^\infty$	...  ...	... 1 1 1 ...	... 1 1 1 ...
$\mathbb{B}_\infty$	 ...	1 1 1 ...	1 2 2 ...
$\mathbb{C}_\infty$	 ...	1 2 2 ...	2 1 1 ...
$\mathbb{D}_\infty$	 ...	$\begin{matrix}1\\1\end{matrix}$ 2 2	$\begin{matrix}1\\1\end{matrix}$ 1 1 ...
$\mathbb{L}_\infty$	 ...	1 1	1 1 ...

(The proof is similar to the considerations above, and for Cartan matrices given in [4]).

2. On generalized Cartan matrices with subadditive functions. If C is a connected generalized Cartan matrix with a subadditive function which is not additive, then either C is a generalized Dynkin diagram, or else of type A_∞.

(The proof for finite index sets is contained in our proof above, the infinite case follows as in [4]).

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(Received February 11, 1980;
in revised form April 29, 1980)