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Bricks in hereditary length categories

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A length category $\mathscr{C}$ is an abelian category in which all objects have finite length. It is called hereditary, provided Ext^2 vanishes everywhere. Since a length category with only a set of isomorphism classes of objects can be considered as a category of modules of finite length over some suitable ring, we just will call the objects of $\mathscr{C}$ modules. And modules with endomorphism ring a division ring will be called *bricks*. We are going to study in which way a module in a hereditary length category can be built up from bricks, and derive a corresponding result for tilted algebras.

THEOREM. *Let $\mathscr{C}$ be a hereditary length category. Let M be an indecomposable module, not a brick. Then there exists a chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that for all $1 \le i \le r$, the module $Z_i = M_i/M_{i-1}$ is a brick, embeddable into M, and with $\mathrm{Ext}^1(Z_1, Z_1) \neq 0$.

Proof. In case we have constructed a chain of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$, we will say that M is an iterated extension of the modules M_i/M_{i-1}.

Now, first, we show that there exists a brick $Z_1 \subset M$, with $\mathrm{Ext}^1(Z_1, Z_1) \neq 0$ such that M/Z_1 is an iterated extension of submodules of M. In order to prove this, choose some $0 \neq \varphi \in \mathrm{End}\, M$ with image S of minimal length. We therefore have an epimorphism $\varepsilon : M \to S$ and a monomorphism $\mu : S \to M$ with $\varphi = \mu\varepsilon$. Clearly, $\varphi^2 = 0$, by the minimality of the length of S. Let $\bigoplus_{i=1}^r X_i$ be the kernel of φ, with all X_i indecomposable and with inclusion maps $\mu_i : X_i \to M$. Now S is contained in this kernel, let $\nu_i : S \to X_i$ be the corresponding projections, thus $\mu = \sum \mu_i\nu_i$. Note that $\mu_i\nu_i\varepsilon$ is an endomorphism of M, and its image is a factor module of S. Thus either $\mu_i\nu_i\varepsilon = 0$ or else $\mu_i\nu_i$ is a monomorphism, again due to the minimality of the length of S. Choose some j with $\mu_j\nu_j\varepsilon \neq 0$. Since $\mu_j\nu_j$ is a monomorphism, and $\mathscr{C}$ is hereditary, the induced map

$$\mathrm{Ext}^1(\mu_j\nu_j, X_j) : \mathrm{Ext}^1(X_j, X_j) \to \mathrm{Ext}^1(S, X_j)$$

has to be surjective. Note that $\mathrm{Ext}^1(S, X_j) \neq 0$, since the indecomposable module M is an extension of $\bigoplus X_i$ by S. Thus we have found a proper submodule $M'(=X_j)$ of M, which is indecomposable, satisfies $\mathrm{Ext}^1(M', M') \neq 0$, and such that M/M' is an extension of $\bigoplus_{i \neq j} X_i$ by S, both being submodules of M. Now either M' is a brick, then let $Z_1 = M'$. Otherwise, by induction, we can assume that there exists a brick $Z_1 \subset M'$, with $\mathrm{Ext}^1(Z_1, Z_1) \neq 0$, such that M'/Z_1 is an iterated extension of submodules of M'. Submodules of M' are also submodules of M, thus M/Z_1 being an extension of M'/Z_1 by M/M' is an iterated extension of submodules of M.

Now assume we have constructed a chain

$$0 = M_0 \subset M_1 \subset \cdots \subset M_s \subset M$$

of submodules such that for all $1 \leq i \leq s$, the module $Z_i = M_i/M_{i-1}$ is a brick, embeddable into M, with $\mathrm{Ext}^1(Z_1, Z_1) \neq 0$, and M/M_s being an iterated extension of submodules of M. Say, let

$$M_s = U_0 \subset U_i \subset \cdots \subset U_t = M$$

with U_i/U_{i-1} being a submodule of M, for all $1 \leq i \leq t$. We clearly may suppose that U_1/U_0 is indecomposable. Now either U_1/U_0 is a brick, then let $M_{s+1} = U_1$. Otherwise, we apply our first result to the module U_1/U_0. We obtain a brick $Z_{s+1} = M_{s+1}/U_0 \subset U_1/U_0$ such that $(U_1/U_0)/Z_{s+1} \approx U_1/M_{s+1}$ is an iterated extension of submodules of U_1/U_0, thus of submodules of M. Thus, we have in both cases obtained a chain

$$0 = M_0 \subset M_1 \subset \cdots \subset M_s \subset M_{s+1} \subseteq M$$

such that for all $1 \leq i \leq s+1$, the module $Z_i = M_i/M_{i-1}$ is a brick, embeddable into M, with $\mathrm{Ext}^1(Z_1, Z_1) \neq 0$, and M/M_{s+1} an iterated extension of submodules of M. After a finite number of steps, we must reach M. This finishes the proof.

By duality, we also have:

THEOREM*. *Let $\mathscr{C}$ be a hereditary length category. Let M be an indecomposable module, not a brick. Then there exists a chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that for all $1 \leq i \leq r$, the module $Z_i = M_i/M_{i-1}$ is a brick, and a factor module of M, such that, in addition $\mathrm{Ext}^1(Z_r, Z_r) \neq 0$.

COROLLARY. *Let $\mathscr{C}$ be a hereditary length category. Let M be an indecom-*

posable module, not a brick. Then there exists a chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_t \subseteq M$$

such that for all $1 \leq i \leq t$, *the module* $Z_i = M_i/M_{i-1}$ *is a brick with* $\mathrm{Ext}^1(Z_i, Z_i) \neq 0$, *and* M/M_t *is a direct sum of bricks.*

Proof. By the theorem, there exists $Z_1 = M_1 \subset M$, a brick with $\mathrm{Ext}^1(Z_1, Z_1) \neq 0$. Let $M/M_1 = \bigoplus_{i=1}^s Y_i$ with Y_i be indecomposable, and let $Y_1, \ldots, Y_r$ be bricks and $Y_{r+1}, \ldots, Y_s$ not bricks. For any i, $r+1 \leq i \leq s$, there exists $Y'_i = Y_i$ such that Y'_i is an iterated extension of bricks which have self-extensions and Y_i/Y'_i is a direct sum of bricks. Let $\bigoplus_{i=r+1}^s Y'_i = M'/M_1$, for some submodule M' with $M_1 \subseteq M' \subset M$. Then M' is an iterated extension of bricks with self-extensions, and $M/M' = (\bigoplus_{i=1}^r Y_i) \oplus (\bigoplus_{i=r+1}^s Y_i/Y'_i)$ is a direct sum of bricks.

Again, there is the dual result:

COROLLARY*. *Let* $\mathscr{C}$ *be a hereditary length category. Let* M *be an indecomposable module, not a brick. Then there exists a chain of submodules*

$$0 \subseteq M_1 \subset \cdots \subset M_t \subset M_{t+1} = M$$

such that for all $2 \leq i \leq t+1$ *the module* $Z_i = M_i/M_{i-1}$ *is a brick with* $\mathrm{Ext}^1(Z_i, Z_i) \neq 0$, *and* M_1 *is a direct sum of bricks.*

Remark. Note that the Jordan-Hölder-series usually will not satisfy any of these conditions. In fact, in case $\mathscr{C}$ is the category of modules of finite length over a hereditary artinian ring, then no simple module has self-extensions, and also any indecomposable module of length ≥ 2 has composition factors which are not embeddable into the module.

Application 1. Hereditary artinian rings of finite representation type

As first application, let us give rather short proofs of two well-known results in the representation theory of hereditary artinian rings:

(a) Let R be a hereditary artinian ring of finite representation type. Then any indecomposable module of finite length is a brick.

Proof. Assume M_R is indecomposable and not a brick. By the theorem, there exists a brick Z_R with $\mathrm{Ext}^1(Z, Z) \neq 0$. However, given a brick with self-extensions, one may easily construct arbitrarily long iterated extensions of copies of Z_R which are indecomposable (use the process of simplification, see [5]).

In particular, let A be a hereditary finite-dimensional algebra over an algebraically closed field k, and of finite representation type. Then the endomorphism ring of any indecomposable A-module is k. Recall that Gabriel asked in [2] for a direct proof of this result, since this immediately establishes the well-known bijection between the indecomposable representations of A and the positive roots of the corresponding quadratic form. Of course, in the meantime, there has been developed a completely different approach starting with the Coxeter functors of Bernstein-Gelfand-Ponomarev [1] and using the Auslander-Reiten-quiver of A (for a survey, see [3]).

Given any ring R, let $K_0(R)$ be its Grothendieck group of modules of finite length with respect to exact sequences, the element of $K_0(R)$ corresponding to the module M is denoted by $\mathbf{dim}\, M$, and elements of the form $\mathbf{dim}\, M$, with $M \neq 0$, are called positive. (Note that $K_0(R)$ is a free abelian group with basis the set of elements of the form $\mathbf{dim}\, S$, where S is simple, and the positive elements are defined with respect to this basis). In case A is a finite-dimensional algebra over some field k, and of finite global dimension, there is a (usually non-symmetric) bilinear form $\langle\ ,\ \rangle$ on $K_0(A)$ defined by

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle = \sum_{i \geq 0} \dim_k \operatorname{Ext}^i(M, N),$$

and the corresponding quadratic form will be denoted by q_A. Note that for A hereditary, we have

$$q_A(\mathbf{dim}\, M) = \dim_k \operatorname{End}(M) - \dim_k \operatorname{Ext}^1(M, M).$$

(b) Let A be a hereditary finite-dimensional algebra, and assume its quadratic form q_A is positive on positive elements of $K_0(A)$. Then any indecomposable module of finite length is a brick.

Proof. Assume again, there exists an indecomposable module of finite length which is not a brick. The theorem then gives a brick Z_A of finite length with $\operatorname{Ext}^1(Z_A, Z_A) \neq 0$. Since $\operatorname{Ext}^1(Z_A, Z_A)$ is an $\operatorname{End}(Z_A)$-vectorspace (both on the left and on the right), we have

$$\dim_k \operatorname{Ext}^1(Z_A, Z_A) \geq \dim_k \operatorname{End}(Z_A),$$

thus $q_A(\mathbf{dim}\, Z_A) \leq 0$, contrary to the assumption.

Application 2. Tilted algebras

Tilted algebras are the endomorphism rings of tilted modules over finite-dimensional hereditary algebras [4]. They seem to play a rather dominent role in representation theory: On the one hand, some basic questions concerning representations of finite-dimensional algebras can be reduced to questions about modules over tilted algebras. On the other hand, many properties of finite-dimensional hereditary algebras carry over to tilted algebras. For example, the existence of self-extending bricks can be transferred to tilted algebras as follows:

COROLLARY. *Let B be tilted algebra, and M an indecomposable B-module of finite length, not a brick. Then there exists a brick Z_B with* $\mathrm{Ext}^1(Z_B, Z_B) \neq 0$, *which is a submodule or a factor module of M.*

Proof. Let T_A be a tilting module with A finite-dimensional, hereditary, and $B = \mathrm{End}(T_A)$. We have to distinguish two cases: First, assume $\mathrm{Tor}_1^B(M_B, {}_BT) = 0$. Then there exists an A-module M'_A generated by T_A such that $M_B = F(M'_A)$, where $F = \mathrm{Hom}_A({}_BT_A, -)$. With M_B also M'_A is indecomposable and not a brick. Theorem* gives a chain of submodules

$$0 = M'_0 \subset M'_1 \subset \cdots \subset M'_r = M'$$

such that the modules $Z'_i = M'_i/M'_{i-1}$ are bricks and factor modules of M', and $\mathrm{Ext}^1(Z'_r, Z'_r) \neq 0$. Since all Z'_i are factor modules of M', they are also generated by T_A. Note that F is exact on exact sequences of modules generated by T_A. Thus, applying F, we obtain a chain of submodules

$$0 = F(M'_0) \subset F(M'_1) \subset \cdots \subset F(M'_r) = F(M') = M_B$$

with factors $F(Z'_i) = F(M'_1)/F(M'_{i-1})$. We have $\mathrm{End}(F(Z'_i)) \approx \mathrm{End}(Z'_i)$, thus all $F(Z'_i)$ are bricks. Also

$$\mathrm{Ext}^1_B(F(Z'_r), F(Z'_r)) \approx \mathrm{End}^1_A(Z'_r, Z'_r) \neq 0.$$

Thus, let $Z_B = F(Z'_r)$. This is a self-extending brick and a factor module of M.

Similarly, in the second case $M_B \otimes {}_BT = 0$, there exists an A-module M''_A satisfying $\mathrm{Hom}_A(T_A, M''_A) = 0$ and $F'(M''_A) = M_B$, where $F' = \mathrm{Ext}^1_A({}_BT_A, -)$. Here, M'' is indecomposable and not a brick. The theorem itself gives a chain

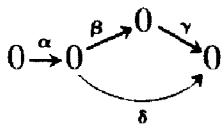
$$0 = M''_0 \subset M^{\circ}_1 \subset \cdots \subset M''_r = M''$$

such that all $Z''_i = M''_i/M''_{i-1}$ are bricks and embeddable into M'', with $\mathrm{Ext}^1(Z_1, Z_1) \neq 0$. Since the Z''_i are embeddable into M'', they also satisfy $\mathrm{Hom}_A(T_A, Z''_i) = 0$. Now F' is exact on exact sequences of modules X_A satisfying $\mathrm{Hom}_A(T_A, X_A) = 0$, and therefore we obtain a chain of submodules

$$0 = F'(M''_0) \subset F'(M''_1) \subset \cdots \subset F'(M''_r) = F'(M'') = M_B,$$

with factors $F'(Z''_i) = F'(M''_i)/F(M''_{i-1})$ all being bricks. In this case, let $Z_B = F'(Z''_1)$. then this is a brick, and a submodule of M_B, and also $\mathrm{Ext}^1_B(Z, Z) \neq 0$.

Remark. Note that we cannot improve this result as in the case of hereditary algebras. Consider, for example, the algebra B given by the quiver with relation



$\alpha\beta\gamma=0$

(it is easy to see that B is a tilted algebra), and the indecomposable representation M of dimenion type (1121), note that M is not a brick. Self-extending bricks are all of dimension type (0111), and H has such a brick Z as submodule, but no such brick as factor module. Also, if we factor out Z, we obtain a semi-simple module of length 2, and one of its summand cannot be embedded into M, namely that of dimension type (1000).

This result is sufficient in order to obtain the same consequences for tilted algebras as we have for hereditary algebras. In this way, we get several characterizations of tilted algebras of finite representation type:

THEOREM 2. *Let B be a tilted algebra. Then the following are equivalent*
(i) *B is of finite representation type.*
(ii) *Any indecomposable B-module of finite length is a brick.*
(iii) *For all bricks Z_B of finite length,* $\mathrm{Ext}^1_B(Z, Z)=0$.
(iv) *$q_B(x)>0$ for any positive x in* $K_0(B)$.

Proof. (i)$\Rightarrow$(iii): Assume there exists a brick Z_B of finite length, with $\mathrm{Ext}^1_B(Z, Z)\neq 0$. Note that $\mathrm{Ext}^2_B(Z, Z)=0$, since B is a tilted algebra and Z is indecomposable. Thus, we can construct arbitrarily long iterated extensions of copies of Z which are indecomposable (again, using the process of simplification [5]).

(iii)$\Rightarrow$(ii): This is a direct consequence of the existence theorem for self-extending bricks.

(iii)$\Rightarrow$(iv): Let x be a positive element in $K_0(B)$. Then $x=\mathbf{dim}\,M$ for some non-zero module M of finite length, and we may choose an M with endomorphism ring of smallest possible dimension. Let $M=\oplus M_i$ with M_i indecomposable. Then $\mathrm{Ext}^1(M_i, M_j)\neq 0$ for $i\neq j$, see [6]. We have seen that (iii) implies (ii), thus all the M_i are bricks, and using again (iii), we have $\mathrm{Ext}^1(M_i, M_i)=0$ for all i. Altogether we have $\mathrm{Ext}^1(M, M)=0$. Since the global dimension of a tilted algebra is ≤ 2, we see

$$q_B(x)=q_B(\mathbf{dim}\,M)=\dim\mathrm{End}(M)+\dim\mathrm{Ext}^2(M, M)>0.$$

(iv)$\Rightarrow$(ii): Assume there exists an indecomposable module of finite length which is not a brick. The existence theorem for self-extending bricks gives a brick Z of finite length satisfying $\mathrm{Ext}^1(Z, Z)\neq 0$. Note that $\mathrm{Ext}^2(Z, Z)=0$, since Z is indecomposable. Note that $\mathrm{Ext}^1(Z, Z)$ is an $\mathrm{End}(Z)$-vectorspace (both on the left and on the right), thus

$$q_B(\mathbf{dim}\, Z) = \dim \mathrm{End}(Z) - \dim \mathrm{Ext}^1(Z, Z) \leq 0,$$

using the fact that the global dimension of B is ≤ 2. This contradicts (iv).

(ii)$\Rightarrow$(i): This is true in general for finite-dimensional algebras. It follows directly from the representation theory of Schurian vectorspace categories (for a survey, see [7]). In our case, where B is a factor algebra of a finite-dimensional hereditary algebra, one considers the category of B-modules as the category of all representations of a bimodule of the form ${}_FM_C$, with F a division ring and C a proper factor algebra of B (see [8]), and uses induction.

Let us note that some of the implications have been known. As we have mentioned, the implication (ii)$\Rightarrow$(i) is true in general for finite-dimensional algebras which are not necessarily tilted algebras. If now B is a tilted algebra, and of finite representation type, then it was known that B has property (ii). In fact, one knows that the Auslander–Reiten quiver of B has no oriented cycles (see [4]), a much stronger assertion. Also, in case the base field k is algebraically closed, the implication (i)$\Rightarrow$(iv) has been shown in [4].

The main interest seems to lie in the implication (iii)$\Rightarrow$(i). Namely, we can reformulate this assertion as follows: If B is a tilted algebra and not of finite representation type, then there exists a self-extending brick. In fact, using elementary methods from algebraic geometry, we easily see that this implies the existence of even a family of self-extending bricks.

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