JOURNAL OF ALGEBRA 87, 150–201 (1984)

Coherent Tubes

GABRIELLA D'ESTE

Seminario Matematico, Università di Padova, Via Belzoni 7, I-35100 Padova, Italy

AND

CLAUS MICHAEL RINGEL

Fakultät für Mathematik, Universität, D-4800 Bielefeld, West Germany

Communicated by W. Feit

Received September 23, 1982

A translation quiver containing a closed oriented path, with underlying topological space being homeomorphic to $S^1 \times \mathbb{R}_+$, will be called a *tube*. Tubes arise rather frequently as components of the Auslander–Reiten quiver of (tame) artin algebras [5, 11] and partially ordered sets [3]. Here, we try to give a rather systematic investigation into their properties. In this way, we present rigorous proofs of some results announced in [5, 11]. However, the main objective of the present paper is our interest in separating tubular series as considered in Section 4.

In the first section, we consider some fundamental properties of tubes. In particular, we show that there are two kinds of arrows in a tube, those pointing to the mouth and those pointing to infinity (Proposition 1 in Section 1.2). Also, we investigate in greater detail some special classes of tubes, the reduced ones and the smooth ones. Note that the regular components of the Auslander–Reiten quiver of a tame hereditary artin algebra, or more general, of a tame concealed artin algebra, are smooth tubes.

We work with a process of enlarging a given tube T, namely, of inserting several rays in order to obtain a new tube $T[v, n]$. It will be seen in Section 3 that, starting from a smooth tube, and using several times this and the dual construction, we obtain precisely the coherent tubes which admit length functions, or, equivalently, the coherent tubes of rank $\geqslant 1$ (Theorem 3.3). Any coherent tube of rank $\geqslant 1$ actually appears as a component of the Auslander–Reiten quiver of a suitable artin algebra (Theorem 4.2).

Given a ring A, an A-module V, and $n \in \mathbb{N}_1$, we consider the ring $A[V, n]$ (see Section 2.3; note that $A[V, 1]$ is just the one-point extension of A by V).

0021-8693/84 $3.00

In case A is an artin algebra with a component T of its Auslander–Reiten quiver being a tube, and V is a rather special indecomposable A-module in T, a socalled "ray module" (see Section 2.2), we calculate completely the component of $A[V, n]$ containing the module V: it is of the form $T[v, n]$ (Theorem 2.3). In this case, we will call $A[V, n]$ a simple tubular extension; the dual construction will be called a cosimple tubular extension.

We introduce the general notion of a tubular extension of a tame concealed algebra A, for an algebra being obtained from A by a sequence of successive simple or cosimple tubular extensions. Let B be a tubular extension of a tame concealed algebra A. Then there is a class $\mathscr{T}(B, A)$ of indecomposable B-modules formed by the modules belonging to a family of tubes. The remaining indecomposable B-modules fall into two separate classes $\mathscr{P}(B, A)$ and $\mathscr{Q}(B, A)$, such that $\operatorname{Hom}(X, Y) = 0$ for $X \in \mathscr{Q}(B, A)$, $Y \in \mathscr{P}(B, A)$, also for $X \in \mathscr{Q}(B, A)$, $Y \in \mathscr{T}(B, A)$, and for $X \in \mathscr{T}(B, A)$, $Y \in \mathscr{P}(B, A)$ and that any homomorphism $X \to Y$ with $X \in \mathscr{P}(B, A)$, $Y \in \mathscr{Q}(B, A)$ factors through a direct sum of modules in $\mathscr{T}(B, A)$. We therefore say that $\mathscr{T}(B, A)$ is a separating tubular series. There are factor algebras B^+ and B^- of B with A being the push-out of B^+ and B^- such that the modules in $\mathscr{P}(B, A)$ are B^+-modules, and those in $\mathscr{Q}(B, A)$ are B^--modules (Theorem 4.3). Note that both B^+ and B^- may be of wild representation type, whereas the only indecomposable sincere representations will belong to $\mathscr{T}(B, A)$.

In the final section, we consider a component of the Auslander–Reiten quiver of an artin algebra which is a coherent tube T and determine the modules in T which are preprojective or preinjective in the sense of Auslander–Smalø (Theorem 5.3).

Throughout the paper, we will denote by $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ the set of natural numbers, and $\mathbb{N}_1 = \mathbb{N}_0 \setminus \{0\}$.

1. TUBES

1.0. A *quiver* $Q = (Q_0, Q_1)$ (without multiple arrows) is given by a set Q_0 of "vertices" and a subset $Q_1 \subseteq Q_0 \times Q_0$ of "arrows." If $\alpha = (x, y)$ belongs to Q_1, we will write more suggestively $\alpha: x \to y$, and call x the starting point and y the endpoint of α. The set of all starting points of arrows with endpoint y is denoted by y^-, the set of all endpoints of arrows with starting point x is denoted by x^+. An arrow $\alpha: x \to x$ is called a loop. A (finite) path of length n is given by an n-tuple $\pi = (\alpha_1, \ldots, \alpha_n)$ of arrows $\alpha_i: x_{i-1} \to x_i$, both the arrows α_i $(1 \leqslant i \leqslant n)$ as well as the vertices x_i $(0 \leqslant i \leqslant n)$ will be said to belong to π, and x_0 is called the starting point; x_n is the endpoint of π. A *circuit* in Q is given by a set $\omega = \{\alpha_1, \ldots, \alpha_n\}$ of arrows $\alpha_i: x_{i-1} \to x_i$ with pairwise different vertices $x_0, x_1, \ldots, x_{n-1}$, and $x_n = x_0$.

A *translation quiver* $Q = (Q_0, Q_1, \tau)$ is given by a quiver (Q_0, Q_1) without multiple arrows (but note that we allow loops, in contrast to [2]), together with an injective map $\tau: Q_0' \to Q_0$, where Q_0' is a subset of Q_0, such that $z^{-} = (\tau z)^{+}$ for all $z \in Q_0'$. The pairs $(\tau z, z)$ with $z \in Q_0'$ will be called extensions, and more suggestively denoted by $\tau z \llcorner z$, and Q_2 will denote the set of all extensions. The vertices in $Q_0 \backslash Q_0'$ are said to be projective, those in $Q_0 \backslash \tau(Q_0')$ are said to be injective. The set of arrows $\alpha: y \to z$ with $z \in Q_0'$ will be denoted by Q_1'. For any $\alpha \in Q_1'$, there is a unique arrow $\tau z \to y$, and it will be denoted by $\sigma\alpha$. If both $y, z \in Q_0'$, and $\alpha: y \to z$, then there is an arrow $\tau y \to \tau z$, and it will be denoted by $\tau\alpha$ $(=\sigma^2\alpha)$. A generalized path from x_0 to x_n is given by a sequence $(\varepsilon_1, ..., \varepsilon_n)$ such that for any $1 \leqslant i \leqslant n$, ε_i is either an arrow $x_{i-1} \to x_i$ or an extension $x_{i-1} \llcorner x_i$. A generalized circuit is given by a set $\{\varepsilon_1, ..., \varepsilon_n\}$ such that ε_i is either an arrow $x_{i-1} \to x_i$ or an extension $x_{i-1} \llcorner x_i$ with pairwise different vertices $x_0, x_1, ..., x_{n-1}$, and $x_n = x_0$. Usually we will denote a generalized path or a generalized circuit just by the sequence $(x_0, x_1, ..., x_n)$ of vertices. Given a translation quiver $Q = (Q_0, Q_1, \tau)$, there exists the dual translation quiver $Q^* = (Q_0^*, Q_1^*, \tau^*)$ with $Q_0^* = Q_0$, with $(x, y) \in Q_1^*$ iff $(y, x) \in Q_1$, and $\tau^* = \tau^{-1}: \tau(Q_0') \to Q_0$. Using this duality, any assertion concerning translation quivers leads to a dual assertion. Following Gabriel and Riedtmann, we may consider any translation quiver (Q_0, Q_1, τ) as a 2-dimensional simplicial complex as follows: The 0-simplices are the elements of Q_0; there are two kinds of 1-simplices, namely, the elements $\alpha: x \to y$ of Q_1, with boundary being given by x, y, and the elements $x \llcorner z$ of Q_2 with boundary x, z. Finally, for every $\alpha: y \to z$ in Q_1', there is a 2-simplex (or "triangle") with boundary the 1-simplices α, $\sigma\alpha$ and $\tau z \llcorner z$; we denote it by $(\tau z, y, z)$. The geometric realization of this complex will be called the *underlying topological space* $|Q|$ of Q.

Given a translation quiver $Q = (Q_0, Q_1, \tau)$, a *translation subquiver* is a translation quiver of the form $P = (P_0, P_1, \tau')$ with $P_0 \subseteq Q_0, P_1 \subseteq Q_1$ and $P_0' \subseteq Q_0'$, with τ' being the restriction of τ to P_0'. P is said to be a *full* translation quiver provided we have, in addition, $P_1 = Q_1 \cap (P_0 \times P_0)$, and such that for $x, z \in P_0$, there exists the extension $x \llcorner z$ in P if and only if two conditions are satisfied: this extension exists in Q, and any arrow $y \to z$ in Q_1 actually belongs to P_1. Note that a full translation subquiver P of Q is uniquely determined by P_0 and Q. If x is a vertex of the translation quiver $Q = (Q_0, Q_1, \tau)$, we say that P is obtained from Q by *deleting* x, provided P is the full translation subquiver of Q defined by $Q_0 \backslash \{x\}$.

1.1. DEFINITION. A translation quiver T is called a *tube* provided it contains a circuit and the underlying topological space is homeomorphic to $S^1 \times \mathbb{R}_+$ (where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle, and $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geqslant 0\}$ is the space of non-negative real numbers). If T is a tube, the 1-simplices belonging to the boundary of $|T|$ are said to form the *mouth* of T.

If Q is any translation quiver, an arrow $\alpha: x \to y$ of Q lies on at most two triangles, namely, on $(\tau y, x, y)$ in case y is not projective, and on $(x, y, \tau^{-1}x)$, in case x is not injective. The number of triangles an extension $x \urcorner z$ lies on is given by the cardinality of x^+. Thus let T be a tube. Then, given an arrow $\alpha: x \to y$ of T, it is impossible that both x is injective and y is projective; and *$\alpha: x \to y$ belongs to the mouth of T if either x is injective or y is projective.* For an extension $x \urcorner z$ of T, the set x^+ can never be empty, and *$x \urcorner z$ belongs to the mouth if and only if x^+ contains a single vertex.* In general, *for every vertex x of the tube T, both sets x^+ and x^- can contain at most two vertices.* Namely, for x injective, one shows that x^+ has at most two elements by considering the mouth of T which has to be a 1-sphere, whereas for x not injective, the same follows from the fact that $|T|$ is a manifold with boundary. The assertion that x^- contains at most two elements follows by duality.

1.2. We are going to state the main results of Section 1. Let T be a tube. A function d: $T_1 \to \{\pm 1\}$ is called a *direction function* for T if and only if the following properties are satisfied

(1) If α, β are two arrows with the same starting point or the same endpoint, then $d(\alpha) \neq d(\beta)$.

(2) If $\alpha \in T_1'$, then $d(\sigma\alpha) \neq d(\alpha)$.

(3) For any infinite path

$$x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} x_2 \xrightarrow{\alpha_3} x_3 \longrightarrow \cdots$$

with pairwise different arrows, there exists some i with $d(\alpha_i) = 1$.

(3*) For any infinite path

$$\cdots \longrightarrow y_3 \xrightarrow{\beta_3} y_2 \xrightarrow{\beta_2} y_1 \xrightarrow{\beta_1} y_0,$$

with pairwise different arrows, there exists some i with $d(\beta_i) = -1$.

PROPOSITION 1. *Any tube has a unique direction function.*

If d is the direction function of the tube T, an arrow α of T with $d(\alpha) = 1$ will be said *to point to infinity*; an arrow β with $d(\beta) = -1$ will be said *to point to the mouth.*

An infinite path

$$x = x(0) \to x(1) \to x(2) \to \cdots$$

of arrows all pointing to infinity and being pairwise different will be called a *ray* and denoted by $[x, \infty)$; note that any ray is uniquely determined by its starting point. In dealing with rays, we always will stick to the notation $x(i)$ for the endpoint of the ith arrow. Two rays $[x, \infty)$ and $[y, \infty)$ will be said to belong to the same *ray class* provided they have a common arrow, or equivalently, if $y = x(i)$ or $x = y(i)$ for some i. Dually, an infinite path $\cdots \to x(-2) \to x(-1) \to x(0) = x$ ending in x and consisting of pairwise different arrows all pointing to the mouth will be called a *coray* and denoted by $(\infty, x]$. Two corays with a common arrow will be said to belong to the same *coray class*. A ray which is not a proper subset of another ray is called a *maximal* ray; any ray class contains a unique maximal ray. Similarly, any coray class contains a unique maximal coray (which is not properly contained in another coray).

Given a circuit $\omega = \{\alpha_1, ..., \alpha_n\}$ of length n, the number of arrows α_i pointing to the mouth will be denoted by $q(\omega)$, the number of arrows α_i pointing to infinity will be denoted by $p(\omega)$, and $(p(\omega), q(\omega))$ is called the *type* of ω.

PROPOSITION 2. *Let T be a tube. All circuits of T have the same type (p, q), and q is the number of ray classes and p is the number of coray classes of T.*

The common type (p, q) of all circuits of T will also be called the *type of T*.

The proof of these two propositions will be given in Section 1.6. Now let T be a tube.

1.3. We may identify the underlying topological space $|T| = S^1 \times \mathbb{R}_+$ of T with the subset $\{(r, s) \in \mathbb{R}^2 \mid r^2 + s^2 \geqslant 1\}$ of the real plane, thus the underlying space $|\omega|$ of a circuit ω is a Jordan curve, and therefore divides $|T|$ into an interior and an exterior part. The translation subquiver of T with underlying space being the closure of the interior part will be denoted by $\bar{\omega}$, the translation subquiver of T with underlying space being the closure of the exterior part will be denoted by $T(\omega)$. Note that $\bar{\omega}$ is a finite quiver, but not necessarily connected, whereas $T(\omega)$ is infinite and connected.

PROPOSITION 3. *Let T be a tube. If ω is a circuit, then $T(\omega)$ is again a tube. Also, there exists a unique circuit ω_0 such that any other circuit is contained in $T(\omega_0)$.*

The unique circuit ω_0 given by the proposition will be called the *circuit next to the mouth.*

Proof of Proposition 3. A circuit ω of the tube T cannot be contractible since otherwise $\bar{\omega}$ would be a simply connected translation quiver with a circuit, contrary to Proposition 1.6 of [2]. Thus, using the identification of $|T|$ with $\{(r,s) \in \mathbb{R}^2 \mid r^2 + s^2 \geqslant 1\}$, we see that $|\omega|$ runs around the hole $\{(r,s) \in \mathbb{R}^2 \mid r^2 + s^2 < 1\}$, and therefore $|T(\omega)|$ again is homeomorphic to $S^1 \times \mathbb{R}_+$.

Now, for a circuit ω, let $t(\omega)$ denote the number of triangles inside $\bar{\omega}$, and choose a circuit ω_0 with $t(\omega_0)$ being minimal. We show that any other circuit ω is contained in $T(\omega_0)$. Let $\omega \neq \omega_0$ be a circuit. Since $t(\omega_0)$ is minimal, ω cannot be inside $\bar{\omega}_0$. Now assume ω is not contained in $T(\omega_0)$. Then ω runs both through vertices not in $\bar{\omega}_0$ and through vertices not in $T(\omega_0)$, and therefore, ω and ω_0 intersect. Choose x_0 lying both on ω and ω_0 and being the starting point of an arrow α_0: $x_0 \to x_1$ in ω with x_1 not in $T(\omega_0)$. The circuit ω now has the form $\omega = \{\alpha_1, ..., \alpha_n\}$ with $\alpha_i : x_{i-1} \to x_i$, and we choose $j \geqslant 2$ minimal with x_j again belonging both to ω and ω_0. Now, there is a path from x_j to x_0 in ω_0, say

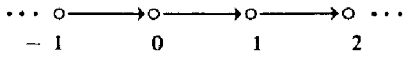
$$x_j = y_0 \xrightarrow{\beta_1} y_1 \longrightarrow \cdots \longrightarrow y_{r-1} \xrightarrow{\beta_r} y_r = x_0$$

with pairwise different y_i. Since also no x_i with $0 \leq i < j$ belongs to ω_0, we see that $\omega' = \{\alpha_1, ..., \alpha_j, \beta_1, ..., \beta_r\}$ is a circuit, and ω' is properly contained in $\bar{\omega}_0$, contrary to the choice of ω_0. This shows that ω is contained in $T(\omega_0)$. On the other hand, using the fact that ω is not contractible, we see that ω_0 is contained in $\bar{\omega}$, thus not in $T(\omega)$. This shows the unicity of ω_0.

1.4. Let us construct a special class of tubes which are both of interest in representation theory and also will be used in order to deal with general tubes. To begin with, recall from [8] the definition of the translation quiver $\mathbb{Z}Q$ for a given quiver Q. The set $(\mathbb{Z}Q)_0$ of vertices is given by $\mathbb{Z} \times Q_0$; for any arrow $\alpha: x \to y$, $i \in \mathbb{Z}$, there are two arrows

$$(i,x) \xrightarrow{(i,\alpha)} (i,y) \qquad \text{and} \qquad (i,y) \xrightarrow{(i,\alpha^*)} (i+1,x);$$

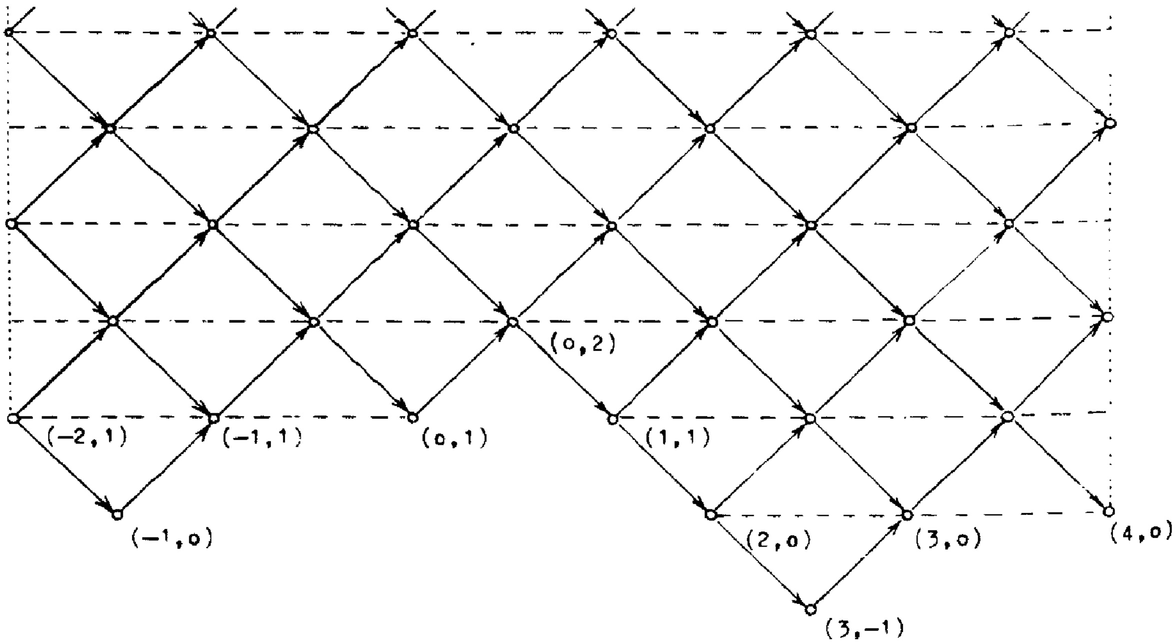
and there are the extensions $(i,x) \urcorner (i+1,x)$ for all $i \in \mathbb{Z}, x \in Q_0$. In particular, we will consider $\mathbb{Z}\mathbb{A}_\infty^\infty$, where $\mathbb{A}_\infty^\infty$ is the quiver



with $\mathbb{Z}$ being the set of vertices, and with arrows $i \to i+1$ for all $i \in \mathbb{Z}$. Then the set of vertices $\mathbb{Z}\mathbb{A}_\infty^\infty$ is given by $\mathbb{Z}^2$, and the underlying topological space of $\mathbb{Z}\mathbb{A}_\infty^\infty$ is of the form $\mathbb{R}^2$. Note that any pair (a,b) of integers induces a translation of $\mathbb{Z}\mathbb{A}_\infty^\infty$, defined on the vertices by $(i,x) \mapsto (i+a, x+b)$.

Let π be a (finite) generalized path in $\mathbb{Z}\mathbb{A}_\infty^\infty$, starting from (i, x) and ending in $(j, y) = (i + a, y + b)$, and we assume that π is not constant, thus $(a, b) \neq (0, 0)$. Let g be the translation of $\mathbb{Z}\mathbb{A}_\infty^\infty$ given by (a, b). We denote by $\tilde{\pi}$ the infinite generalized path given by the arrows and extensions which are images under the translations g^z $(z \in \mathbb{Z})$ of the arrows and extensions in π. Now $\tilde{\pi}$ divides $\mathbb{R}^2 = |\mathbb{Z}\mathbb{A}_\infty^\infty|$ into two connected components, both being homeomorphic to open half-planes, and we denote by $\tilde{T}(\pi)$ the translation subquiver of $\mathbb{Z}\mathbb{A}_\infty^\infty$ with underlying topological space being the closure of the open half-plane containing the vertex $(i - 1, x + 2)$. In particular, the boundary of $|\tilde{T}(\pi)|$ is given by $|\tilde{\pi}|$. Note that g defines an automorphism of $\tilde{T}(\pi)$, and the quotient of $\tilde{T}(\pi)$ with respect to the action of g will be denoted by $T(\pi) = \tilde{T}(\pi)/g$. Obviously, $T(\pi)$ is a tube and the mouth of $T(\pi)$ is given by $\tilde{\pi}/g$. A closed fundamental domain in $|\tilde{T}(\pi)|$ is given by the elements $(r, s) \in \mathbb{R}^2$ which belong to $|\tilde{T}|$ and satisfy $2i + x \leqslant 2r + s \leqslant 2j + y$.

As an example, consider the generalized path $\pi = ((-2, 1), (-1, 0), (-1, 1), (0, 1), (0, 2), (1, 1), (2, 0), (3, -1), (3, 0), (4, 0))$. Here, $(a, b) = (6, -1)$, the fundamental domain considered above looks as follows:



We will characterize the tubes of the form $T(\pi)$ in several ways. We will say that the mouth of a tube is *oriented*, in case it is given by a generalized circuit.

Proposition 4. *The following conditions are equivalent for a tube T.*

(i) *T is of the form $T(\pi)$, for some generalized path π in $\mathbb{Z}\mathbb{A}_\infty^\infty$.*

(ii) *The mouth of T is oriented.*

(iii) *If x is injective, then $|x^+| = 1$; if y is projective, then $|y^-| = 1$.*

(iv) *Any vertex belongs to a circuit.*

(v) *There is no sink and no source.*

Proof of Proposition 4. (i) $\Rightarrow$ (ii): This follows from the construction of $T(\pi)$.

(ii) $\Rightarrow$ (i): Let $x_0, x_1, ..., x_{n-1}$ be pairwise different vertices, and $x_n = x_0$, such that for any $0 \leqslant i < n$, there is either an arrow $x_i \to x_{i+1}$ in the mouth or an extension $x_i \llcorner x_{i+1}$ in the mouth. Choose $\tilde{x}_0 = (0, 0) \in \mathbb{Z}\mathbb{A}_\infty^\infty$, and define $\tilde{x}_i$ inductively. Namely, let $\tilde{x}_i = (s, t)$ being defined. If $x_i \llcorner x_{i+1}$ belongs to the mouth, let $\tilde{x}_{i+1} = (s+1, t)$. If there is an arrow $x_i \to x_{i+1}$, and x_{i+1} is not projective, let $\tilde{x}_{i+1} = (s, t+1)$. Otherwise, if there is an arrow $x_i \to x_{i+1}$ with x_{i+1} being projective, let $\tilde{x}_{i+1} = (s+1, t-1)$. In this way, we obtain a generalized path $\pi = (\tilde{x}_0, \tilde{x}_1, ..., \tilde{x}_n)$ in $\mathbb{Z}\mathbb{A}_\infty^\infty$. We want to construct a covering map from $\tilde{T}(\pi)$ to T. By construction, the number of triangles a vertex x_i of the mouth of T belongs to coincides with the number of triangles in $\tilde{T}(\pi)$ to which the vertex $\tilde{x}_i$ belongs. (Suppose first $n > 2$; then this number is 1 for x_i projective-injective. It is 2, in case $x_{i-1} \llcorner x_i$ is an extension in the mouth and x_i is injective, and also in the dual case. It is 3, in case $x_{i-1} \llcorner x_i$ and $x_i \llcorner x_{i+1}$ both belong to the mouth, in case x_{i-1} and x_i, both are injective, and in case x_i, x_{i+1}, both are projective. It is 4, in case $x_{i-1} \llcorner x_i$ belongs to the mouth, x_{i+1} is projective and in the dual case. Finally, it is 5, in case x_{i-1} is injective and x_{i+1} projective. If $n \leqslant 2$, then a similar result follows more easily.) Of course, a vertex in the interior both of T as well as of $\tilde{T}(\pi)$ belongs to precisely 6 triangles if $n > 2$, and to 6 or 4 triangles otherwise. This shows that the map $\tilde{x}_i \mapsto x_i$ extends in a unique way to a simplicial map from $\tilde{T}(\pi)$ to T, and obviously this gives a covering map of translation quivers. In this way, we define an isomorphism from $T(\pi) = \tilde{T}(\pi)/g$ onto T.

(ii) $\Rightarrow$ (iii): If x is injective, and $y_1 \neq y_2$ belong to x^+, then both arrows $x \to y_1$, $x \to y_2$ belong to the mouth, thus the mouth is not oriented.

(iii) $\Rightarrow$ (ii): If the mouth contains an arrow, say $\alpha: x \to y$, let $x_0 = x$, $x_1 = y$. Otherwise, there exists an extension on the mouth, and we label it $x_0 \llcorner x_1$. Now assume, we have constructed a sequence $x_0, x_1, ..., x_j$ of vertices on the mouth such that for any $0 \leqslant i < j$, there is either an arrow $x_i \to x_{i+1}$ or an extension $x_i \llcorner x_{i+1}$ in the mouth. If x_j is injective, let $x_j \to x_{j+1}$ be the unique arrow starting at x_j. If x_j is not injective, and $|x_j^+| = 1$, let $x_{j+1} = \tau^- x_j$. Finally, assume x_j is not injective, and there are arrows $x_j \to y$, $x_j \to y'$ with $y \neq y'$. If neither y nor y' would be projective, then x_j^- contains the two different vertices τy and $\tau y'$, thus x_j cannot be projective; however, this contradicts the fact that x_j lies on the mouth. Thus one of y, y', say y, is projective. In this case, let $x_{j+1} = y$. In this way, we obtain an infinite generalized path $(x_0, x_1, x_2, ...)$ consisting of arrows and extensions on the mouth, and since the mouth is homeomorphic to S^1, there is some n with $x_n = x_0$.

(i) $\Rightarrow$ (iv): We use the notation used in the construction of $T(\pi)$. Any path in $\tilde{T}(\pi)$ starting from a vertex of the form (s, t) with $s \leqslant i$, $t = x + 2i - 2s$, and ending in $(s + a, t + b)$ gives rise to a circuit in $T(\pi)$, using the projection $\tilde{T}(\pi) \to T(\pi)$. Let $\pi = (u_0, u_1, ..., u_m)$. In general, this is a generalized path, and not a path. In order to replace it by a path π', we insert, for every extension $u_i \llcorner u_{i+1}$ a new vertex $u_i' = u_i + (0, 1)$ between u_i and u_{i+1}. Let $\pi' = (v_0, v_1, ..., v_n)$. Then any vertex of $T(\pi)$ is the image of a vertex of the form $v_i + (-n, 2n)$, for some $n \geqslant 0$, and therefore belongs to the circuit obtained in the following way: we consider the path obtained from π' by using the translation by $(-n, 2n)$, and then use the projection $\tilde{T}(\pi) \to T(\pi)$. This finishes the proof of the implication (i) $\Rightarrow$ (iv).

Since (iv) $\Rightarrow$ (v) is trivial, it remains to show the implication (v) $\Rightarrow$ (ii). Let ω_0 be the circuit next to the mouth, say $\omega_0 = \{\alpha_1, ..., \alpha_n\}$ with $\alpha_i : x_{i-1} \to x_i$. Assume there exists some arrow $\beta : y \to x_i$ inside $\bar{\omega}_0$ with $y \neq x_{i-1}$. We claim that y does not belong to ω_0 and that there is no path from any x_j to y. For otherwise take a path γ from x_j to y of smallest possible length (with γ being of length zero in case $y \in \omega_0$). Since y is in $\bar{\omega}_0$, the whole path is inside $\bar{\omega}_0$. Let α be the path from x_i to x_j inside ω_0. Thus, we obtain a circuit ω by using first γ, then β, then α, and ω is contained inside $\bar{\omega}_0$, and $\omega \neq \omega_0$. This contradicts Proposition 3. As a consequence, any path ending in y runs only through the finitely many vertices which belong to $\bar{\omega}_0$ and not to ω_0. Also, the vertices of such a path must be pairwise different since otherwise we would obtain a circuit which does not belong to $T(\omega_0)$. Thus there are only finitely many paths ending in y, and therefore, there exists a source, contrary to assumption (v). Similarly, the existence of an arrow $\beta' : x_i \to y'$ inside $\bar{\omega}_0$ with $y' \neq x_{i+1}$ implies the existence of a sink, again in contrast to (v). As a consequence, the only triangles contained in $\bar{\omega}_0$ are of the form (x_{i-1}, x_i, x_{i+1}), and in case such a triangle exist, the extension $x_{i-1} \llcorner x_{i+1}$ belongs to the mouth. Thus, deleting from ω_0 those vertices x_i, for which the triangle (x_{i-1}, x_i, x_{i+1}) exists and belongs to $\bar{\omega}_0$, we obtain a generalized circuit, and this generalized circuit is the mouth of T. Thus the mouth of T is oriented. This finishes the proof of Proposition 4.

1.5. Tubes which satisfy the equivalent conditions of Proposition 4 are said to be *reduced*. Note that there is a reduction procedure for obtaining from a tube a reduced tube in a finite number of steps, namely, in each step one deletes a sink or a source. In fact, there is the following corollary to Proposition 4.

COROLLARY. *Let T be a tube with ω_0 being the circuit next to the mouth. Deleting from T successively sinks and sources, one obtains after a finite* *number of steps the full translation subquiver* T^r *of T defined by the vertices of* $T(\omega_0)$.

Note that $T(\omega_0)$ may be a proper translation subquiver of T^r, namely, $T(\omega_0)$ is obtained from T^r by deleting all extensions belonging to its mouth.

Proof. Deleting a sink or a source of the tube T, the full translation subquiver T^r of T remains untouched, thus the procedure of deleting sinks and sources has to stop after a finite number of steps since at most those vertices which belong to $\bar{\omega}_0$ and not to ω_0 itself may be deleted. Thus, after a finite number of steps we reach a full translation subquiver T'' of T which contains T^r and which does not have a sink or a source. But then T'' is a tube and the mouth of T'' is a generalized circuit, according to Proposition 4, thus there exists a circuit ω in T'' which contains all vertices of its mouth. It follows that $\omega = \omega_0$ and therefore $T'' = T^r$.

1.6. *Proof of Propositions* 1 *and* 2. First, consider a reduced tube $T = T(\pi)$ with π being a generalized path in $\mathbb{Z}\mathbb{A}_\infty^\infty$, and denote by $e\colon \tilde{T}(\pi) \to T(\pi)$ the canonical projection and by d a direction function for $T(\pi)$. Then $d \circ e\colon \tilde{T}(\pi)_1 \to \{\pm 1\}$ is a function satisfying conditions (1) and (2) of 1.2, thus $d \circ e$ takes a fixed value on the arrows of the form $(i, \alpha)\colon (i, x) \to (i, x+1)$, and the other value on the arrows of the form $(i, \alpha^*)\colon (i, x) \to (i+1, x-1)$. Let (i, x) be the starting point of π. Consider first the case where the endpoint of π is of the form $(i, x+b)$ for some $b \geqslant 1$. In this case the vertices $y_j = e(i-j, x+j)$ for $j \geqslant 0$ are pairwise different, there are arrows

$$\cdots \longrightarrow y_3 \longrightarrow y_2 \longrightarrow y_1 \longrightarrow y_0$$

and all have the same direction. It follows from Condition (3*) that d takes the value -1 on these arrows, thus $de(i, \alpha) = 1$, $de(i, \alpha^*) = -1$ for all i, α. In case the endpoint of π is not of the form $(i, x+b)$ for any b, the vertices $x_j = e(i, x+j)$ for $j \geqslant 0$ are pairwise different and there are arrows

$$x_0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots$$

all of which have the same direction. Condition (3) implies that d takes the value 1 on these arrows, thus again $de(i, \alpha) = 1$, $de(i, \alpha^*) = -1$ for all i, α. This shows that there is at most one direction function on T. Conversely, it is obvious that the function $d'\colon \tilde{T}(\pi)_1 \to \{\pm 1\}$ given by $d'(i, \alpha) = 1$, $d'(i, \alpha^*) = -1$ induces a direction function on $T(\pi)$.

If T is a general tube, consider the circuit ω_0 next to the mouth and use induction on the number n of vertices belonging to $\bar{\omega}_0$ and not to ω_0. If $n = 0$, then T is reduced. Otherwise, there exists a vertex x which is a sink or a source. First, assume there exists a source x with x^+ containing a single element y, say with arrow $\alpha : x \to y$. Then α and $x \llcorner \tau^- x$ belong to the mouth of T, thus $\sigma^- \alpha$ cannot belong to the mouth, and therefore the full translation quiver Q of T obtained by deleting the vertex x is a tube, with n decreased by 1. By induction, there is a unique direction function d on Q, and d has a unique extension to T, namely, $d(\alpha) = -d(\sigma^- \alpha)$. Next, assume there exists a source x with $\tau^- x$ not belonging to the mouth. In this case, there exist two different arrows $\alpha : x \to y$, $\alpha' : x \to y'$ starting at x. Again, the full translation quiver Q of T obtained by deleting x is a tube, with n decreased by 1, and we extend the direction function d of Q by setting $d(\alpha) = -d(\sigma^- \alpha)$, $d(\alpha') = -d(\sigma^- \alpha')$. In case there exists a sink z with either z^- containing a single element or with τz not belonging to the mouth, we proceed dually. Thus assume, for every source x, the set x^+ contains two elements and $\tau^- x$ belongs to the mouth, and for every sink z, the set z^- contains two elements and τz belongs to the mouth. Given any extension $x \llcorner z$ not belonging to the mouth, with x, z both belonging to the mouth, $x \llcorner z$ divides $|T|$ into two parts, and we denote by $\overline{x \llcorner z}$ the closure of the bounded part. Now choose an extension $x \llcorner z$ with x a source or z a sink such that $\overline{x \llcorner z}$ contains a minimal number of triangles. We consider the case of x a source (the other case being dual). By assumption, x^+ contains two different elements y, y', say with arrows $\alpha : x \to y$ and $\alpha' : x \to y'$. Now one of y, y' belongs to $\overline{x \llcorner z}$, say y, and assume y is not injective. Considering a maximal path starting with $\tau^- \alpha : \tau^- x \to \tau^- y$, we obtain a sink z' inside $\overline{x \llcorner z}$. By assumption, $\tau z'$ belongs to the mouth and $(z')^-$ contains two elements, thus we may consider again $\overline{x' \llcorner z'}$. However, $\overline{x' \llcorner z'}$ is a proper subset of $\overline{x \llcorner z}$, thus contradicting the minimality of $\overline{x \llcorner z}$. This shows that y is injective. Now consider the full translation subquiver Q of T obtained by deleting x and y. Since α, α', $\sigma^- \alpha$ belong to the mouth, $\sigma^- \alpha'$ does not belong to the mouth, thus Q is a tube, and n is now reduced by 2. The unique direction function d of Q is extended to T by setting $d(\alpha) = -d(\alpha') = -d(\sigma^- \alpha) = d(\sigma^- \alpha')$. This finishes the proof of Proposition 1.

For the proof of Proposition 2, we may assume T to be reduced, thus $T = T(\pi)$ for some generalized path π in $\mathbb{Z}\mathbb{A}_\infty^\infty$. We may assume that $(0, 0)$ is the starting point, and (a, b) the endpoint of π. Now, a is the number of ray classes, whereas $a + b$ is the number of coray classes. Also, any circuit of $T(\pi)$ lifts to a path in $\tilde{T}(\pi)$ starting from a vertex of the form (s, t) with $s \leqslant 0$ and $t = -2s$ and ending in $(s + a, t + b)$. Thus, it involves a arrows of the form (j, α^*), and $a + b$ arrows of the form (j, α). This finishes the proof of Proposition 2.

1.7. We want to characterize a rather small class of tubes which occur frequently in the representation theory of tame algebras. Recall that $\mathbb{A}_\infty$ denotes the quiver

$$\circ \longrightarrow \circ \longrightarrow \circ \;\cdots\; \circ \longrightarrow \circ \;\cdots\;.$$
$$0 \qquad 1 \qquad 2 \qquad\quad i \qquad i+1$$

For $r \geqslant 1$, the shift map $(i, x) \mapsto (i + r, x)$ of $\mathbb{Z}\mathbb{A}_\infty$ will be denoted by r. Also recall that a vertex x of a translation quiver is called periodic provided $\tau^t x = x$ for some $t \geqslant 1$.

Lemma. *The following conditions are equivalent for a tube T:*

(i) *T is of the form $\mathbb{Z}\mathbb{A}_\infty/r$ for some $r \geqslant 1$.*

(ii) *The mouth of T contains only extensions.*

(iii) *There are no projective or injective vertices.*

(iv) *All vertices are periodic.*

A tube T satisfying these properties will be called *smooth*, and $r(T)$ will denote the (uniquely determined) number r in condition (i).

Proof. Clearly (i) implies both (ii) and (iv). Also, there are the obvious implications (ii) ⇒ (iii) and (iv) ⇒ (iii). If T satisfies (iii), then Proposition 4 shows that $T = T(\pi)$ for some generalized path π in $\mathbb{Z}\mathbb{A}_\infty^\infty$. Without loss of generality, we may assume that π has the starting point $(0, 0)$. Let (r, s) be its endpoint. Since $T(\pi)$ has no projective or injective vertex, it follows that $s = 0$ and that all the extensions $(i, 0) \urcorner\!\llcorner (i + 1, 0)$ with $0 \leqslant i \leqslant r - 1$ belong to π. Thus T is of the form $\mathbb{Z}\mathbb{A}_\infty/r$. This finishes the proof.

Note that in fact it has been shown in [5] that any translation quiver with a length function (in the sense of 3.1) and with all vertices being periodic is a smooth tube.

1.8. Let T be an arbitrary tube. A ray $[z, \infty)$ will be called *regular* provided z is not projective, $\tau z \urcorner\!\llcorner z$ belongs to the mouth, and both z and τz belong to the reduced tube T^r. Dually, the coray $(\infty, x]$ is said to be *regular* provided x is not injective, $x \urcorner\!\llcorner \tau^- x$ belongs to the mouth, and both x, $\tau^- x$ belong to T^r. (Note that we may weaken the conditions: if $\tau z \urcorner\!\llcorner z$ belongs to the mouth, τz is in T^r, and $[z, \infty)$ exists, then also z belongs to T^r. Namely, no $z(i)$, $i \in \mathbb{N}_0$, can be projective since otherwise we would obtain a hole between the rays $[z, \infty)$, $[\tau z(1), \infty)$, and the arrow $\tau z(1) \to z$, which is impossible, since, by assumption, $\tau z \urcorner\!\llcorner z$ belongs to the mouth. This shows that $\tau(z(i)) = (\tau z)(i)$, for all $i \in \mathbb{N}_0$. Now, given a circuit going through τz, it will go along $[\tau z, \infty)$ up to some $\tau z(i)$, then to $z(i - 1)$ and further. We obtain a different circuit by going $\tau z \to \tau z(1) \to z$ and then along $[z, \infty)$ to $z(i - 1)$ and further.)

Thus, regular rays and regular corays always come in pairs: Given an extension $x \urcorner\!\llcorner z$ in the mouth, with both x, z belonging to T^r, there are attached the regular ray $[z, \infty)$ and the regular coray $(\infty, x]$. Therefore, the number of regular rays coincides with the number of regular corays (and also with the number of extensions $x \urcorner\!\llcorner z$ belonging to the mouth, with x, z both in T^r), and this number will be called the *rank* $r(T)$ of T. Let ω be the circuit next to the mouth. Then $r(T)$ is the number of paths of the form $x \to^{\alpha} y \to^{\beta} z$ contained in ω, with $x \llcorner z$ belonging to the mouth. Obviously, for such a path $x \to^{\alpha} y \to^{\beta} z$, the arrow α points to infinity, whereas the arrow β points to the mouth. This shows that always $r(T) \leqslant p(T)$, $r(T) \leqslant q(T)$, and that $r(T) = p(T) = q(T)$ if and only if T is a smooth tube.

The vertices which belong both to a regular ray as well as to a regular coray are called *regular vertices*. Clearly, a tube is smooth if and only if all its vertices are regular. Given a regular vertex x, belonging to the regular ray $[v, \infty)$ and to the regular coray $(\infty, w]$, say $x = v(n) = w(-m)$, for some n, $m \in \mathbb{N}_0$, then the following two properties are equivalent:

(i) no $v(i)$, $0 \leqslant i < n$, belongs to a regular coray.

(ii) no $w(-j)$, $0 \leqslant j < m$, belongs to a regular ray.

If these equivalent conditions are satisfied, then x is called a *simple regular* vertex.

1.9. We are interested in tubes since some components of the Auslander–Reiten species of artin algebras are tubes. Recall that for an artin algebra A, its Auslander–Reiten species $\Gamma(A)$ is given as follows: the vertices of $\Gamma(A)$ are given by the set Γ_0 of isomorphism classes $|X|$ of indecomposable modules X, and we denote by $F(X)$ the factor ring of $\operatorname{End}(X)$ modulo its radical. For X, Y indecomposable modules, let $\operatorname{Irr}(X, Y)$ be the bimodule of irreducible maps (see [10]), it is an $F(X) - F(Y)$-bimodule (note that we always consider left modules and homomorphisms are assumed to operate on the opposite side as the scalars, thus the composition of maps $f\colon X \to Y$, $g\colon Y \to Z$ is given by fg). The underlying quiver of $\Gamma(A)$ is given by (Γ_0, Γ_1) with arrows $|X| \to |Y|$ provided $\operatorname{Irr}(X, Y) \neq 0$. It becomes a translation quiver by defining $\tau(|X|) = |D\operatorname{Tr} X|$, for X indecomposable and not projective, with $D\operatorname{Tr}$ denoting the Auslander–Reiten translation "dual of transpose." A connected component of the Auslander–Reiten species of A will be called a *component of* A.

DEFINITION. A component of an artin algebra is called a tube provided its underlying translation quiver is a tube and for any arrow $[X] \to [Y]$ in the component, the $F(X) - F(Y)$-bimodule $\operatorname{Irr}(X, Y)$ is of length one both as an $\operatorname{End}(X)$-module as well as an $\operatorname{End}(Y)$-module.

In case the underlying translation quiver of a component is a tube T, it will be convenient to denote this component just by T, and to denote a fixed representative of a vertex x of T by M_x and a fixed representative of an arrow α of T by f_α. (In this way, we obtain a representation $\mathscr{M} = (M_x, f_\alpha)_{x \in T_0, \alpha \in T_1}$ of the quiver (T_0, T_1). Note that (M_x, f_α) may not satisfy any commutativity relation!)

A representation $\mathcal{M} = (M_x, f_\alpha)$ of a translation quiver will be called *exact*, provided the following sequence

$$\begin{tikzcd} 0 \arrow[r] & M_x \arrow[r, "(f_{xy})_y"] & \bigoplus_{y \in x^+} M_y \arrow[r, "(f_{yz})_y"] & M_z \arrow[r] & 0 \end{tikzcd}$$

is exact for every extension $x \urcorner z$. (For $\alpha: x \to y$, we also denote f_α by f_{xy}.) Note that in case $\mathcal{M}$ is an exact representation of a tube T, for x^+ consisting of a single vertex y and $z = \tau^- x$, there is the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M_x \arrow[r, "f_{xy}"] & M_y \arrow[r, "f_{yz}"] & M_z \arrow[r] & 0, \end{tikzcd} \qquad (*)$$

whereas, for x^+ consisting of the two vertices y, y' and $z = \tau^- x$, there is the commuting diagram

$$\begin{tikzcd} M_x \arrow[r, "f_{xy}"] \arrow[d, "-f_{xy'}"'] & M_y \arrow[d, "f_{yz}"] \\ M_{y'} \arrow[r, "f_{y'z}"] & M_z \end{tikzcd} \qquad T[v, 3]$$

and this diagram is cartesian (it is both a pushout and a pullback diagram). Sometimes, it will be more convenient to consider *commutative* representations instead of exact representations: here we require that, for x^+ consisting of a single vertex y and $z = \tau^- x$, the sequence $(*)$ is exact, and that, for x^+ consisting of the two vertices y, y' and $z = \tau^- x$, the diagram

$$\begin{tikzcd} M_x \arrow[r, "f_{xy}"] \arrow[d, "f_{xy'}"'] & M_y \arrow[d, "f_{yz}"] \\ M_{y'} \arrow[r, "f_{y'z}"] & M_z \end{tikzcd}$$

is cartesian.

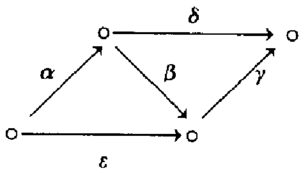
For any tame hereditary algebra A, the regular A-modules form an abelian category $\mathcal{R}$ which is the product of serial hereditary categories $\mathcal{R}_t$, all with only finitely many simple objects [4, 9]. The indecomposable A-modules belonging to a fixed $\mathcal{R}_t$ form a single component which is a smooth tube T_t, and $r(T_t)$ is the number of simple regular objects in $\mathcal{R}_t$. More generally, let B be a tame concealed algebra. By definition, $B = \operatorname{End}({}_AM)$ for a preprojective tilting module ${}_AM$ with A being a tame hereditary algebra [6, 7]. Then the category of B-modules contains a full subcategory which is equivalent to the category $\mathcal{R}$ of regular A-modules and which is closed under irreducible maps; the B-modules contained in this subcategory again will be called regular, and it follows that the indecomposable regular B-modules form components which are smooth tubes.

Also, it has been observed by Bünermann [3] that for any one-parameter partially ordered set S, all components of the Auslander–Reiten species of S but two are tubes.

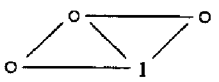
1.10. Finally, let us mention some examples of components which have some features in common with tubes, but without being tubes.

(a) Let A_1 be the path algebra of a quiver of type $\tilde{A}_n$, with $n > 1$, and with at least one sink. The underlying topological space of the preprojective component of A_1 is homeomorphic to $S^1 \times \mathbb{R}_+$, but it does not contain an oriented cycle. Thus it is not a tube.

(b) Let A_2 be the path algebra of the quiver



with $\beta\alpha = 0$, $\gamma\varepsilon = 0$. The component containing the representation with dimension vector



is depicted in [11, p. 271]. Again, the underlying topological space is homeomorphic to $S^1 \times \mathbb{R}_+$, and it does not contain an oriented cycle.

(c) Of course, many examples of components containing oriented cycles, but without being of the form $S^1 \times \mathbb{R}_+$, are known; for example, the Auslander–Reiten quiver of $k[X]/(X^n)$ with $n \geqslant 2$.

2. SIMPLE TUBULAR EXTENSIONS

2.1. Let T be a tube. A vertex v in T will be called a *ray vertex* provided it satisfies the following two properties:

(1) There exists the ray $[v, \infty)$.

(2) If $v(i) \to w$ is an arrow in T pointing to the mouth, then $i \geqslant 1$, and the extension $v(i-1) \urcorner w$ exists.

Let v be a ray vertex in the tube T and $n \in \mathbb{N}_1$. We are going to define a tube $T[v, n]$ obtained from T by *inserting rays*. The set $T[v, n]_0$ of vertices is obtained from T_0 by adding additional vertices of the form (t, i), with $1 \leqslant t \leqslant n$ and $i \geqslant 1 - t$. The set $T[v, n]_1$ of arrows is obtained from T_1 by replacing any arrow of the form $v(i) \rightarrow w$ pointing to the mouth by an arrow $(1, i) \rightarrow w$, and adding additional arrows

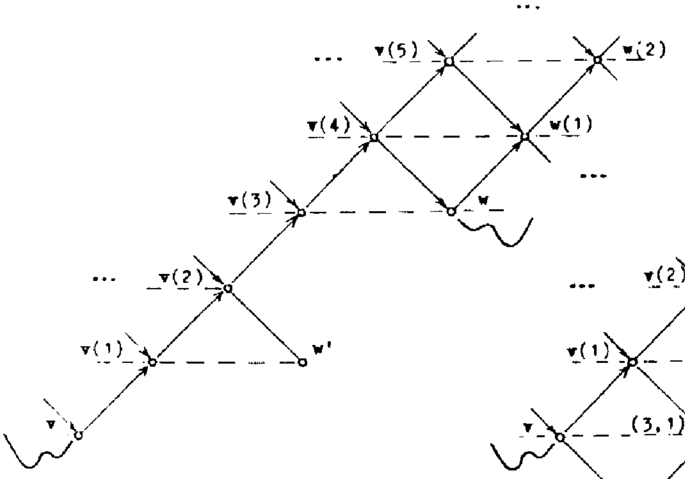
$$\begin{array}{lll} v(i) \rightarrow (n, i) & \text{for} \quad i \geqslant 0, & \\ (t, i) \rightarrow (t-1, i) & \text{for} \quad 2 \leqslant t \leqslant n & \text{and} \quad i > 1 - t, \\ (t, i) \rightarrow (t, i+1) & \text{for} \quad 1 \leqslant t \leqslant n & \text{and} \quad i \geqslant 1 - t. \end{array}$$

Finally, the set $T[v, n]_2$ of extensions is obtained from T_2 by replacing any extension of the form $v(i) \daleth z$ by an extension $(1, i) \daleth z$ and adding additional extensions:

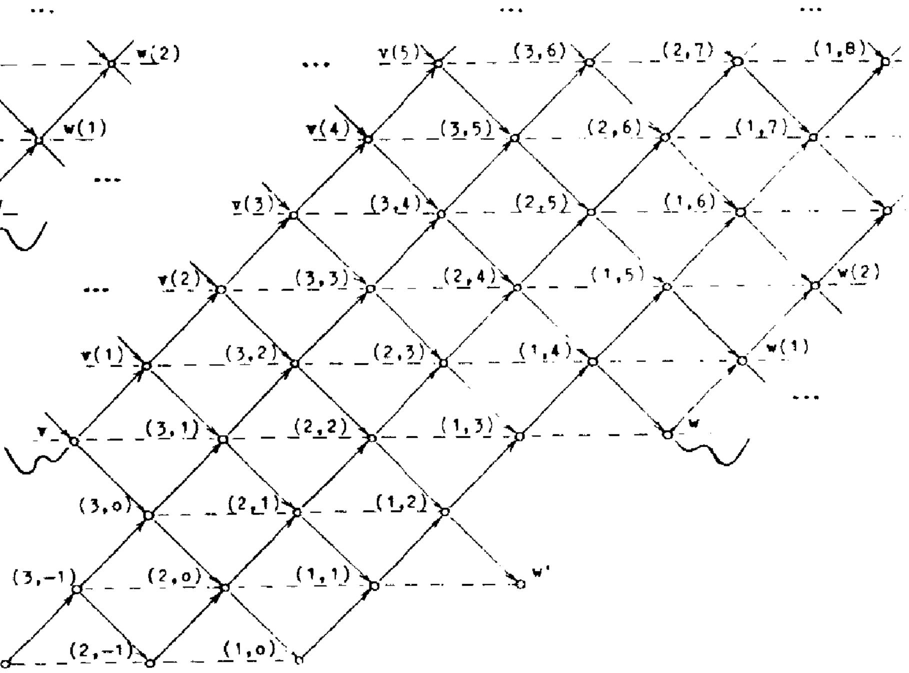
$$v(i) \daleth (n, i+1) \qquad \text{for } i \geqslant 0,$$

$$(t, i) \daleth (t-1, i+1) \qquad \text{for } 2 \leqslant t \leqslant n \text{ and } i \geqslant 1 - t.$$

We sketch an example with $n = 3$:



T



The dual process will be called *insertion of corays*. Given a tube T, a *coray vertex* in T is by definition a vertex v of T such that v^* is a ray vertex in T^*, and the n-fold *coray insertion* determined by the coray vertex v is given by $(T^*[v^*, n])^*$.

2.2. Let A be a finite dimensional algebra. If X, Y are indecomposable A-modules with $|X|$, $|Y|$ belonging to a component of A which is a tube, and $f\colon X \to Y$ is an irreducible map, then we will say that f points to the mouth or to infinity, provided the arrow $|X| \to |Y|$ in T has the corresponding property. Let V be a (finite-dimensional) A-module with endomorphism ring D being a division ring. Assume the component of A containing $v = |V|$ is a tube T, and that the ray $[v, \infty)$ exists. Choose indecomposable modules $V(i)$ with $|V(i)| = v(i)$. Then V is called a *ray module* provided the following conditions are satisfied:

(1) If $V = X_0 \to X_1 \to \cdots \to X_n$ is a chain of irreducible maps and indecomposable modules and one of the maps points to the mouth, then the composition is zero.

(2) The vectorspaces ${}_D\mathrm{Hom}(V, V(i))$ are one-dimensional for $i \in \mathbb{N}_0$.

We also will introduce the dual notion of a coray module. Again assume that W is an A-module with $\mathrm{End}(W) = D$, a division ring, such that $w = |W|$ belongs to a tube T, and such that the coray $(\infty, w]$ exists. We choose indecomposable modules $W(-i)$ with $|W(-i)| = w(-i)$, $i \in \mathbb{N}_0$. Then W is a *coray module*, provided:

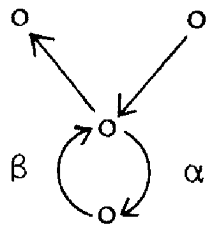
1*) If $X_{-n} \to X_{-n+1} \to \cdots \to X_{-1} \to X_0 = W$ is a chain of irreducible maps and indecomposable modules and one of the maps points to infinity, then the composition is zero.

(2*) The vectorspaces $\mathrm{Hom}(W(-i), W)_D$ are one-dimensional for $i \in \mathbb{N}_0$.

Note that for all the results concerning ray modules we will prove, there also is a dual version concerning coray modules, but we will not state the dual results explicitly.

EXAMPLES. If A is a tame concealed algebra, and V is an A-module, then V is a ray module if and only if V is simple regular, if and only if V is a coray module.

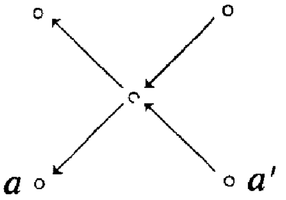
We obtain a component which is a smooth tube T with two vertices $v = |V|$, and $w = |W|$ on the mouth such that V is both a ray module and a coray module, and W is neither a ray module nor a coray module by considering the following algebra A: Let A be given by the quiver



with the relation $\alpha\beta = 0$, and let T be the tube containing the indecomposable modules V, W of dimension type

$$\begin{array}{c}0\ 0\\1\\0\end{array} \quad \text{and} \quad \begin{array}{c}1\ 1\\1\\2\end{array}.$$

(Namely, $\alpha\beta = 0$ is a splitting zero relation [11], thus we consider besides A also the path algebra B of the quiver



of type $\tilde{D}_4$. The Auslander–Reiten quiver of A is obtained from that of B by identifying the vertices corresponding to the two simple modules $S(a)$ and $S(a')$.)

Given a ray module V, we always will choose irreducible maps $\alpha_i: V(i-1) \to V(i)$ for $i \in \mathbb{N}_1$, and define $\mu_i = \alpha_1 \cdots \alpha_i: V \to V(i), \mu_0 = 1$.

PROPOSITION. *Let ${}_AV$ be a ray module belonging to the tube T. Then $\mu_i: V \to V(i)$ is a nonzero map, thus a generator of the one-dimensional vector space ${}_{\mathrm{End}(V)}\mathrm{Hom}(V, V(i))$, for all $i \in \mathbb{N}_0$. If W is an indecomposable A-module also belonging to T, but not isomorphic to any $V(i)$, with $i \in \mathbb{N}_0$, then* $\mathrm{Hom}(V, W) = 0$.

The proof of this proposition will be given in Section 2.5.

COROLLARY. *If V is a ray module belonging to the tube T, then $[V]$ is a ray vertex in T.*

Proof. Let W be indecomposable, and $\beta: V(i) \to W$ an irreducible map pointing to the mouth. Then $i \geqslant 1$, according to property (1) of a ray module. Again using this property, we see that $\mu_i\beta = 0$. Now W cannot be projective since otherwise β is a monomorphism; thus $\mu_i = 0$ contrary to the proposition. Denote by $[\beta]$ the arrow $[V(i)] \to [W]$ in T. Since W is not projective, there exists the vertex $\tau[W]$, with an arrow $\sigma[\beta]: \tau[W] \to [V(i)]$ pointing to infinity. Thus $\tau[W] = [V(i-1)]$. This finishes the proof.

2.3. Given a ring A and ${}_AV$ an A-module with endomorphism ring

D and $n \in \mathbb{N}_1$, let $A[V, n]$ be the ring of $(n+1) \times (n+1)$ matrices of the following form:

$$A[V,n] = \left(\begin{array}{c|c|ccc} A & V & & 0 & \\ \hline 0 & D & & 0 & \\ \hline & D & D & & 0 \\ 0 & \vdots & \vdots & \ddots & \\ & D & \underbrace{D \quad \cdots}_{n-1} & & D \end{array}\right)\left.\vphantom{\begin{array}{c} D \\ \vdots \\ D \end{array}}\right\} n-1$$

with the usual addition and multiplication (using the bimodule structure ${}_AV_D$). Note that any A-module also can be considered as an $A[V, n]$-module. We will use a rather convenient way for describing the $A[V, n]$-modules. Namely, any $A[V, n]$-module is given by the following data $(M, U_t, \varphi_t)_{1 \leqslant t \leqslant n}$, with M being a left A-module, U_t a left D-module for $1 \leqslant t \leqslant n$, $\varphi_1 : {}_AV_D \otimes_D U_1 \to {}_AM$ an A-linear map, and the remaining $\varphi_t : U_{t-1} \to U_t$ being D-linear; conversely, these data always determine an $A[V, n]$-module which may be written in a suggestive way as a set of column vectors, namely,

$$\begin{pmatrix} M \\ U_1 \\ \vdots \\ U_n \end{pmatrix}.$$

Of course, the $A[V, n]$-modules which are in fact A-modules are those of the form $({}_AM, 0, 0)$, and we will just write ${}_AM$ instead of $({}_AM, 0, 0)$. Given two $A[V, n]$-modules (M, U_t, φ_t) and (M', U'_t, φ'_t), an $A[V, n]$-linear map $(M, U_t, \varphi_t) \to (M', U'_t, \varphi'_t)$ is of the form $(f, u_t)_{1 \leqslant t \leqslant n}$, with $f: {}_AM \to {}_AM'$ being A-linear, and all $u_t : U_t \to U'_t$ being D-linear, such that $\varphi_1 f = (1_V \otimes u_1)\, \varphi'_1$ and $\varphi_t u_t = u_{t-1} \varphi'_t$ for all $2 \leqslant t \leqslant n$.

THEOREM. *Let A be a finite-dimensional algebra with a component T which is a tube, and let ${}_AV$ be a ray module with $v = [V]$ belonging to T. Then the component of $A[V, n]$ containing $[V]$ is of the form $T[v, n]$.*

In fact, let us exhibit representatives for the vertices of $T[v, n]$. For any vertex x which actually belongs to T_0, we have to take the corresponding A-module M_x. For $i \geqslant 0$, let $M_{(n,i)} = \overline{V(i)} = (V(i), D, ..., D; \mu_i, 1, ..., 1)$, where as before $\alpha_i : V(i-1) \to V(i)$ is a fixed irreducible map and $\mu_i = \alpha_1 \cdots \alpha_i : V \to V(i)$, $\mu_0 = 1$. Let $J = (0, D, ..., D, \varphi_t)$, with $\varphi_1 = 0$ and $\varphi_t = 1$ for $t \geqslant 2$; and let $M_{(n,i)}$ be the unique submodule of J of length $i + n$, with

$1-n\leqslant i\leqslant -1$. Then $M_{(t,i)}=M_{(n,i)}/M_{(n,-t)}$ for $1\leqslant t\leqslant n-1$ and $i\geqslant 1-t$. Note that for $i\geqslant 0$, $M_{(1,i)}=(V(i), D, 0,\ldots, 0;\, \mu_i, 0, \cdots, 0)$.

The proof of the theorem will be given in Section 2.8.

If ${}_AV$ is a ray module, then $A[V,n]$ will be called a *simple tubular extension* of A. The dual construction will be called a *cosimple tubular extension*: here, we start with a coray module ${}_AW$, with $\operatorname{End}({}_AW)=D$, and form the ring

$$A[W,n]^*=\left(\begin{array}{c|c|ccc} A & 0 & & 0 & \\ \hline W^* & D & D & \cdots & D \\ \hline & & D & \cdots & D \\ 0 & 0 & & \ddots & \vdots \\ & & 0 & & D \end{array}\right)\left.\vphantom{\begin{array}{c}D\\ \vdots\\ D\end{array}}\right\} n-1, \qquad \underbrace{\hphantom{D\cdots D}}_{n-1}$$

with W^* being the $D-A$-bimodule dual to W.

2.4. Let us start with the proof of Proposition 2.2 by showing the following lemma.

LEMMA. *Let V be a ray module with $D=\operatorname{End} V$. Then $\mu_i\neq 0$, thus μ_i is a generator for the one-dimensional D-space ${}_D\operatorname{Hom}(V, V(i))$, for all $i\in\mathbb{N}_0$. Also, if $\gamma\colon W\to V(i)$ is an irreducible map pointing to the mouth and $\delta\colon V\to W$ an arbitrary map, then $\delta\gamma=0$.*

Proof. By induction on i. Let $i=0$. By definition $\mu_0\neq 0$, and if $\gamma\colon W\to V$ is irreducible and $\delta\colon V\to W$ arbitrary, then $\delta\gamma=0$ since otherwise $\delta\gamma\in\operatorname{End}(V)=D$ would be invertible and therefore γ would be a split epimorphism.

Now assume $i>0$. First, assume that $\mu_i=0$. Then $V(i)$ cannot be projective since otherwise α_i is a monomorphism and thus $\mu_{i-1}=0$, contrary to the induction hypotheses. Let

$$0\longrightarrow Y\xrightarrow{(\varphi,\psi)} V(i-1)\oplus Z\xrightarrow{\binom{\alpha_i}{\xi}} V(i)\longrightarrow 0$$

be the Auslander–Reiten sequence ending in $V(i)$ with Z indecomposable or zero. Note that φ is an irreducible map pointing to the mouth. Factorizing the map $(\mu_{i-1}, 0)\colon V\to V(i-1)\oplus Z$ through the kernel of $\binom{\alpha_i}{\xi}$, we obtain $\delta\colon V\to Y$ such that $\delta\varphi=\mu_{i-1}$ (and $\delta\psi=0$). However, by induction hypothesis, we have on the one hand $\delta\varphi=0$ since $\varphi\colon Y\to V(i-1)$ is irreducible and points to the mouth, and $\mu_{i-1}\neq 0$ on the other hand.

Next, assume $\gamma: W \to V(i)$ is an irreducible map pointing to the mouth, and $\delta: V \to W$ is arbitrary. By induction, μ_i generates the one-dimensional D-vectorspace ${}_D\mathrm{Hom}(V, V(i))$; thus there is $\varepsilon \in D = \mathrm{End}(V)$ with $\varepsilon\mu_i = \delta\gamma$. If $V(i)$ is projective, both γ and α_i are inclusions of direct summands with zero intersection; thus $(\varepsilon\mu_{i-1})\alpha_i = \delta\gamma$ implies $\delta = 0$. We can then assume that $V(i)$ is not projective. Thus there is an Auslander–Reiten sequence of the form

$$0 \longrightarrow Y \xrightarrow{(\varphi,\,\psi)} V(i-1) \oplus W \xrightarrow{\binom{\alpha_i}{\gamma}} V(i) \longrightarrow 0.$$

Factorizing $(\varepsilon\mu_{i-1}, -\delta)$ through the kernel of $\binom{\alpha_i}{\gamma}$, we obtain $\eta: V \to Y$ such that $\eta\varphi = \varepsilon\mu_{i-1}$ (and $\eta\psi = -\delta$). Now φ is an irreducible map pointing to the mouth; then by induction, $\eta\varphi = 0$. Since by induction μ_{i-1} is a generator of the one-dimensional D-space ${}_D\mathrm{Hom}(V, V(i-1))$, we conclude that $\varepsilon = 0$; thus $\delta\gamma = 0$.

This finishes the proof.

2.5. Before we continue to consider the second assertion of Proposition 2.2, let us introduce a simply connected translation quiver $\bar{T}$ (depending on T and v), and a certain exact representation $\mathcal{M}$ of $\bar{T}$. Both $\bar{T}$ and $\mathcal{M}$ will also be used in the proof of Theorem 2.3. Recall that a representation (M_x, f_α) of a translation quiver Δ is called *exact* provided for every extension $x \llcorner z$; the sequence

$$0 \longrightarrow M_x \xrightarrow{(f_{xy})_y} \bigoplus_{y \in x^+} M_y \xrightarrow{(f_{yz})_y} M_z \longrightarrow 0 \qquad (*)$$

is exact (for $\alpha: x \to y$, we also denote the map f_α by f_{xy}). It is clear that an exact sequence $(*)$ with all maps f_{xy}, f_{yz} $(y \in x^+)$ being irreducible is an Auslander–Reiten sequence. Also, given a translation quiver Δ, let $\mathcal{S}_i(\Delta)$ denote the set of all vertices z of Δ, with i being a bound for the length of all paths ending in x. Thus, $\mathcal{S}_0(\Delta)$ is the set of sources of Δ, and z belongs to $\mathcal{S}_i(\Delta)$ iff z^- is contained in $\mathcal{S}_{i-1}(\Delta)$.

Given a vertex v of a tube T, which belongs to the mouth and such that the ray $[v, \infty)$ exists, let us define $\bar{T}$ (it is something like a fundamental domain in the universal covering [2] of T). Namely, $\bar{T}$ is obtained from T by first deleting from T all vertices of the form $v(i)$ with $i \in \mathbb{N}_0$ and all arrows and extensions involving such vertices; we then insert new vertices $\underline{v}(i)$ and $\bar{v}(i)$, and arrows $\underline{v}(i) \to \underline{v}(i+1)$, $\bar{v}(i) \to \bar{v}(i+1)$, for $i \in \mathbb{N}_0$. For every old arrow $v(i) \to x$ pointing to the mouth, we insert a new arrow $\underline{v}(i) \to x$; for every old arrow $z \to v(i)$ pointing to the mouth, we insert an arrow $z \to \bar{v}(i)$. If there is an old arrow $w \to v(0)$ pointing to infinity, we insert a new arrow $w \to \bar{v}(0)$. Also, an old extension $v(i) \llcorner y$ gives rise to an extension $\underline{v}(i) \llcorner y$, and old extension $y \llcorner v(i)$ to an extension $y \llcorner \bar{v}(i)$. Thus, $\bar{T}$ is obtained from

T by cutting T along the ray $[v, \infty)$. Obviously, for any vertex $x \in \bar{T}_0$, the length of all paths ending in x is bounded; thus $\bar{T}_0 = \bigcup_{i \in \mathbb{N}_0} \mathscr{S}_i(\bar{T})$. Also note that $\underline{v}(i) \in \mathscr{S}_i(\bar{T}) \backslash \mathbf{S}_{i-1}(\bar{T})$.

We are going to define an exact representation $\mathscr{M} = (M_x, f_\alpha)$ of $\bar{T}$ in the category of A-modules which furnishes a complete picture of the component of A given by T. This will be done by induction on i.

For $x \in \mathscr{S}_0(\bar{T})$, with $x \neq \underline{v}, \bar{v}$ choose any M_x with $[M_x] = x$, and note that there are no arrows ending in x; for $x = \underline{v}$, we choose $M_x = V$. Also, if $x = \bar{v} \in \mathscr{S}_0(\bar{T})$, choose $M_x = V$. Now let $i \geqslant 1$, and assume that for any $x \in \mathscr{S}_{i-1}(\bar{T})$ and any arrow $\alpha: y \rightarrow x$, an indecomposable module M_x with $[M_x] = x$ and an irreducible map $f_\alpha = f_{yx}$ are defined and let $z \in \mathscr{S}_i(\bar{T}) \backslash \mathscr{S}_{i-1}(\bar{T})$. If $z = \underline{v}(i)$, let $M_{\underline{v}(i)} = V(i)$, and $f_{\underline{v}(i-1)\underline{v}(i)} = \alpha_i$. Next assume z is a projective vertex different from $\underline{v}(i)$, thus z is not of the form $\underline{v}(j)$ for any j. If $z \neq \bar{v}(j)$ for any j, then z also is a projective vertex when considered as vertex of T; thus an indecomposable module M_z with $[M_z] = z$ is projective. If $z = \bar{v}(j)$, then we choose $M_z = V(j)$, and so again M_z is indecomposable projective. Always, for $y \in z^-$, there is given an indecomposable module with $[M_y] = y$ and rad $M_z \approx \oplus_{y \in z^-} M_y$, thus let $f_{yz}: M_y \rightarrow M_z$ be the corresponding map. Finally, in case z is not projective, let $x = \tau z$. There are given indecomposable modules M_x with $[M_x] = x$ and M_y with $[M_y] = y$ and irreducible maps $f_{xy}: M_x \rightarrow M_y$, where $y \in z^- = x^+$. (In case $x = \underline{v}(j)$, then the condition should mean that $[M_x] = v(j)$, and similarly for y.) Thus, let M_z and all f_{yz} be defined by the exact sequence $(*)$. Note that $x \neq \bar{v}(j)$ for all j. Since x is not an injective vertex of $\bar{T}$, it cannot be injective when considered as vertex of T (or, if $x = \underline{v}(j)$, then $v(j)$ will not be an injective vertex); thus M_x will not be injective. Also the number of arrows starting in x is the same both in T or in $\bar{T}$ (or, if $x = \underline{v}(j)$, the number of arrows starting in $\underline{v}(j)$ is the same as the number of arrows starting in $v(j)$.) Thus, the map $(f_{xy})_y: M_x \rightarrow \oplus_{y \in x^+} M_y$ is minimal left almost split, and thus the left-hand side of an Auslander–Reiten sequence. Therefore, M_z is indecomposable, the maps f_{yz} are irreducible, and $[M_z] = z$ (or $= v(i)$ in case $z = \bar{v}(i)$). This finishes the induction step. Altogether, we have constructed an exact representation $\mathscr{M} = (M_x, f_\alpha)$ of $\bar{T}$ with $[M_{\underline{v}(j)}] = [M_{\bar{v}(j)}] = v(j)$, for all j, $[M_x] = x$ for the remaining x, and all f_α being irreducible.

End of the proof of Proposition 2.2. Let W be an indecomposable A-module belonging to T, and not of the form $V(j)$ for any $j \in \mathbb{N}_0$. We have to show that $\operatorname{Hom}(V, W) = 0$. Let $w = [W]$; thus, by construction of $\mathscr{M}$, $W \approx M_w$, where we now consider w as an element of $\bar{T}_0$. Thus, we may assume $W = M_w$. The assertion will be shown by induction on i, where $w \in \mathscr{S}_i(\bar{T}) \backslash \mathscr{S}_{i-1}(\bar{T})$. If $i = 0$, then either $w = \underline{v}(0)$, but then $M_w \approx V$, contrary to the assumption, or else M_w is a simple projective module, and not isomorphic to V, thus $\operatorname{Hom}(V, M_w) = 0$. Now assume $i \geqslant 1$ and

$\mathrm{Hom}(V, M_x) = 0$ for any $x \in \mathscr{S}_{i-1}(\bar{T})$ with M_x not isomorphic to any $V(j)$. Let $w \in \mathscr{S}_i(\bar{T})$, with $w \neq \underline{v}(j)$, $\bar{v}(j)$ for all j and consider any map $g: V \to M_w$. By construction of $\mathscr{M}$, the map $(f_{yw})_y : \bigoplus_{y \in w^-} M_y \to M_w$ is right almost split, thus $g = \sum_{y \in w^-} g_y f_{yw}$ for some maps $g_y : V \to M_y$. If M_y is not of the form $V(j)$ for any j, then $g_y = 0$ since $y \in \mathscr{S}_{i-1}(\bar{T})$. Thus assume that M_y is of the form $V(j)$ for some j. Then $y = \underline{v}(j)$ (since $y = \bar{v}(j)$ is impossible because otherwise $w = \bar{v}(j+1)$, contrary to our assumption). Also, the arrow $y \to w$ points to the mouth since otherwise we would have $w = \underline{v}(j+1)$. However, any $V \to V(j)$ is a composition of irreducible maps, by the first part of the proposition; thus $g_y f_{yw}$ is a composition of irreducible maps with one of them pointing to the mouth, and $g_y f_{yw} = 0$ by definition of a ray module. This finishes the proof.

2.6. For the proof of Theorem 2.3 we will need some preliminary results which we are going to derive now. Up to the end of the section, we will assume that ${}_AV$ is a ray module belonging to a tube T. First, let us determine some indecomposable injective $A[V, n]$ – modules. (In fact, in this way, we obtain all indecomposable injective $A[V, n]$-modules belonging to $T[V, n]$).

LEMMA. *Let I be an indecomposable injective A-module belonging to T. If $I = V(i)$ for some i, then $(V(i), D, 0,\ldots, 0; \mu_i, 0,\ldots, 0)$ is an injective $A[V, n]$-module. If I is not of the form $V(i)$ for any i, then I itself is also injective as an $A[V, n]$-module.*

Proof. First, assume $I = V(i)$, and consider any inclusion (f, u_t): $(V(i), D, 0,\ldots, 0; \mu_i, 0,\ldots, 0) \to ({}_AM, U_t, \varphi_t)$. Now f has to be a split monomorphism, since $V(i)$ is injective, thus ${}_AM = V(i) \oplus C$ with f being given by $(1, 0)$ and $\varphi_1 : {}_AV \otimes U_1 \to V(i) \oplus C$ by (σ, ρ). Now $\sigma: {}_AV \otimes U_1 \to V(i)$ factors through μ_i, say $\sigma = (1 \otimes \delta)\mu_i$ for some $\delta: U_1 \to D$, and thus $(\sigma, \rho)\binom{1}{0} = \sigma = (1 \otimes \delta)\mu_i$ shows that $(\binom{1}{0}, \delta, 0,\ldots, 0)$ is a homomorphism from $({}_AM, U_t, \varphi_t)$ to $(V(i), D, 0,\ldots 0; \mu_i, 0,\ldots, 0)$. Also, $(1 \otimes u_1\delta)\mu_i = (1 \otimes u_1)(\sigma, \rho)\binom{1}{0} = \mu_i(1, 0)\binom{1}{0} = \mu_i$, together with the fact that $\mathrm{Hom}(V, V(i))$ is a one-dimensional $\mathrm{End}(V)$-vectorspace generated by μ_i, shows that $u_1\delta = 1$. Thus we have obtained a retraction for (f, u_t).

Now consider the case of I not being of the form $V(i)$ and let (f, u_t): $(I, 0,\ldots, 0;\ 0,\ldots, 0) \to ({}_AM, U_t, \varphi_t)$ be an inclusion. Similar to the previous case ${}_AM$ is now of the form $I \oplus C$ with f being given by $(1, 0)$ and φ_1 by (σ, ρ). However, Proposition 2.2 states that $\mathrm{Hom}(V, I) = 0$, thus $\sigma = 0$, and therefore $(\binom{1}{0}, 0,\ldots, 0)$ is a retraction for (f, u_t).

2.7. Our next aim is to show that certain $A[V, n]$-module homomorphisms are irreducible. We need the following preliminary result on A-modules:

Lemma 1. *Let C be an A-module, and assume that no direct summand of C is of the form $V(j)$ with $j < i$. Then any homomorphism φ: $V \to C$ factors through μ_i.*

Proof. By induction on j, with $0 \leqslant j \leqslant i$, we show that φ: $V \to C$ factors through μ_j. This is clear for $j = 0$. Assume we know $\varphi = \mu_j \varphi_j$ with φ_j: $V(j) \to C$ and $0 \leqslant j < i$. Let (α_{j+1}, γ): $V(j) \to V(j+1) \oplus Y$ be a minimal left almost split map starting with $V(j)$, and with Y either zero or indecomposable. Since φ_j is not split mono, there exist φ_{j+1}: $V(j+1) \to C$ and δ: $Y \to C$ such that $\varphi_j = \alpha_{j+1}\varphi_{j+1} + \gamma\delta$. Now $\mu_j \gamma = 0$ since V is a ray module and either $\gamma = 0$ or γ is an irreducible map pointing to the mouth. Thus

$$\varphi = \mu_j \varphi_j = \mu_j \alpha_{j+1} \varphi_{j+1} + \mu_j \gamma \delta = \mu_{j+1} \varphi_{j+1}.$$

This finishes the proof.

By $\overline{V(i)}$ we denote the $A[V, n]$-module $\overline{V(i)} = (V(i), D, ..., D;\ \mu_i, 1..., 1)$, and by $\overline{\alpha_i}$: $\overline{V(i-1)} \to \overline{V(i)}$ the $A[V, n]$-module homomorphism $(\alpha_i, 1, ..., 1)$.

Lemma 2. *For all $i \in \mathbb{N}_1$, the $A[V, n]$-homomorphisms α_i: $V(i-1) \to V(i)$ and $\overline{\alpha_i}$: $\overline{V(i-1)} \to \overline{V(i)}$ are irreducible.*

Proof. First consider the case of α_i considered as an $A[V, n]$-homomorphism. Let $({}_AM, U_t, \varphi_t)$ be an $A[V, n]$-module, with homomorphisms (f, u_t): $V(i-1) \to ({}_AM, U_t, \varphi_t)$ and (f', u_t'): $({}_AM, U_t, \varphi_t) \to V(i)$ such that $(f, u_t)(f', u_t') = (\alpha_i, 0)$. Note that all the maps u_t and u_t' are zero maps. Now $ff' = \alpha_i$; thus f is split mono or f' is split epi since α_i is an irreducible A-homomorphism. If f' is split epi, say $gf' = 1_{V(i)}$ for some g, then also (f', u_t') is split epi with coretraction $(g, 0)$. Thus, consider the case where f' is not epi, therefore f is split mono. We can replace ${}_AM$ by $V(i-1) \oplus C$, and f by $(1, 0)$; thus f' is of the form $\binom{\alpha_i}{\beta}$ for some $\beta: C \to V(i)$. Also φ_1 is of the form (σ, ρ) for some σ: $V \otimes U_1 \to V(i-1)$, ρ: $V \otimes U_1 \to C$. Since $(f', 0)$ is a homomorphism of $A[V, n]$-modules, $\varphi_1 f' = 0$; thus $\sigma\alpha_i + \rho\beta = 0$. Consider the minimal right almost split A-map $\binom{\alpha_i}{\gamma}$: $V(i-1) \oplus Y \to V(i)$ ending in $V(i)$ with either $Y = 0$ or else γ: $Y \to V(i)$ being an irreducible map pointing to the mouth. Now β cannot be split epi since otherwise f' would be split epi. Thus β can be lifted to $V(i-1) \oplus Y$; there are maps $\beta': C \to V(i-1)$, $\beta'': C \to Y$ such that $\beta = (\beta', \beta'')\binom{\alpha_i}{\gamma} = \beta'\alpha_i + \beta''\gamma$. Note that $\rho\beta''\gamma = 0$. Namely, either $\gamma = 0$ or else we use Proposition 2.2: we deal with γ, an irreducible map pointing to the mouth and $\rho\beta''$ being defined on ${}_AV \otimes U_1$ which is a direct sum of copies of V. Then

$$0 = \sigma\alpha_i + \rho\beta = \sigma\alpha_i + \rho\beta'\alpha_i + \rho\beta''\gamma = \sigma\alpha_i + \rho\beta'\alpha_i = (\sigma, \rho)\begin{pmatrix} 1 \\ \beta' \end{pmatrix} \alpha_i,$$

thus also $0 = (\sigma, \rho)\binom{1}{\beta'}$. It follows that $(\binom{1}{\beta'}, 0)$: $(V(i-1) \oplus C, U_t, \varphi_t) \to$

$V(i-1)$ is a homomorphism of $A[V,n]$-modules, and clearly a retraction for $(f,u_t)=((1,0),0)$.

Next consider the case of $\overline{\alpha_i}$. Assume, there are given an $A[V,n]$-module $({}_AM, U_t, \varphi_t)$ and maps $(f,u_t)\colon \overline{V(i-1)} \to ({}_AM, U_t, \varphi_t)$, $(f',u_t')\colon ({}_AM, U_t, \varphi_t) \to \overline{V(i)}$ such that $(f,u_t)(f',u_t') = \overline{\alpha_i}$. Since $ff'=\alpha_i$ and α_i is irreducible, either f is split mono of f' is split epi. Consider first the case f being split mono, say $M = V(i-1)\oplus C$, $f=(1,0)$, $\varphi_1=(\sigma,\rho)$ with $\sigma\colon V\otimes U_1 \to V(i-1)$, $\rho\colon V\otimes U_1 \to C$, and $f' = \left(\begin{smallmatrix}\alpha_i\\ \beta\end{smallmatrix}\right)$ with $\beta\colon C\to V(i)$. If β is not split epi, we consider a minimal right almost split A-map $\left(\begin{smallmatrix}\alpha_i\\ \gamma\end{smallmatrix}\right)\colon V(i-1)\oplus Y\to V(i)$, ending in $V(i)$, with either $Y=0$ or else $\gamma\colon Y\to V(i)$ an irreducible map pointing to the mouth, and lift β to $V(i-1)\oplus Y$. We obtain maps $\beta'\colon C\to V(i-1)$, $\beta''\colon C\to Y$ such that $\beta=\beta'\alpha_i+\beta''\gamma$, and note that $\rho\beta''\gamma=0$. Then

$$(1\otimes u_1')\mu_{i-1}\alpha_i = (1\otimes u_1')\mu_i = \sigma\alpha_i+\rho\beta = \sigma\alpha_i+\rho\beta'\alpha_i = (\sigma,\rho)\begin{pmatrix}1\\ \beta'\end{pmatrix}\alpha_i,$$

thus $(1\otimes u_1')\mu_{i-1} = (\sigma,\rho)\left(\begin{smallmatrix}1\\ \beta'\end{smallmatrix}\right)$, and consequently $(\left(\begin{smallmatrix}1\\ \beta'\end{smallmatrix}\right), u_t')$ is a homomorphism of $A[V,n]$-modules, and obviously it is a retraction for (f,u_t). However, if β is split epi, then we can write $C=V(i)\oplus C'$, with $\beta=\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$; thus $f\colon V(i-1)\to V(i-1)\oplus V(i)\oplus C'$ is given by $(1\ 0\ 0)$, and $f'\colon V(i-1)\oplus V(i)\oplus C'\to V(i)$ is given by

$$\begin{pmatrix}\alpha_i\\ 1\\ 0\end{pmatrix}.$$

Also, $u_1u_1'=1$, thus we decompose $V\otimes U_1 = V\oplus W$ with $1_V\otimes u_1$ given by $(1,0)$ and $1_V\otimes u_1'$ by $\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$. Write

$$\varphi_1 = \begin{pmatrix}\varphi_{11} & \varphi_{12} & \varphi_{13}\\ \varphi_{21} & \varphi_{22} & \varphi_{23}\end{pmatrix}\colon V\oplus W\to V(i-1)\oplus V(i)\oplus C'.$$

Then clearly $\varphi_{11}=\mu_{i-1}, \varphi_{12}=0$, $\varphi_{13}=0$, and also $\varphi_{21}=\tau\mu_{i-1}, \varphi_{22}=-\tau\mu_i$ for some $\tau\colon W\to V$. [Namely, $\operatorname{Hom}(V, V(i-1))$ is generated by μ_{i-1}, as an $\operatorname{End}(V)$-vectorspace, and W is a direct sum of copies of V; thus $\varphi_{21}=\tau\mu_{i-1}$ for some τ, and similarly $\varphi_{22}=\tau'\mu_i$ for some $\tau'\colon W\to V$. It follows from

$$\begin{pmatrix}\mu_{i-1} & 0 & 0\\ \tau\mu_{i-1} & \tau'\mu_i & \varphi_{23}\end{pmatrix}\begin{pmatrix}\alpha_i\\ 1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}\mu_i$$

that $\tau'=-\tau$.] Decompose $C'=X\oplus Y$, with $X=\bigoplus_{s=1}^{m} V(j_s)$ being a direct sum of modules of the form $V(j_s)$, where $j_s < i$, and Y having no direct summand of this form. Let $X' = \oplus_{s=1}^{m} V$, and $\mu = \oplus_{s=1}^{m} \mu_{j_s} : X' = \oplus_{s=1}^{m} V \to \oplus_{s=1}^{m} V(j_s) = X$, and denote by $\pi : X \oplus Y \to X$ the projection onto X. Then $\varphi_{23}\pi$: $W \to X$ factors through μ since W is a direct sum of copies of V and using Proposition 2.2. Say $\varphi_{23}\pi = \bar{\varphi}_{23}\mu$. Let W' be the kernel of $\bar{\varphi}_{23}$, and W'' the image of $\bar{\varphi}_{23}$; thus $\bar{\varphi}_{23} = \varphi'\varphi''$ with $\varphi' : W \to W''$ and φ'': $W'' \to X'$. Since W and X' are direct sums of copies of V, the same is true for W' and W''. We have to distinguish two cases.

First assume $\tau(W') = 0$. Thus τ factors through $\varphi' : W \to W''$ and since $\varphi'' : W'' \to X'$ is injective, there exists $\bar{\tau}$ with $\bar{\varphi}_{23}\bar{\tau} = \tau$. Let $\mu' = \oplus_{s=1}^{m} \alpha_{j_s+1} \cdots \alpha_{i-1} : X = \oplus_{s=1}^{m} V(j_s) \to \oplus_{s=1}^{m} V(i-1)$, then $\mu\mu' = \oplus_{s=1}^{m} \mu_{i-1} : \oplus_{s=1}^{m} V \to \oplus_{s=1}^{m} V(i-1)$. Since $\mathrm{End}(V) \cdot \mu_{i-1} = \mu_{i-1} \cdot \mathrm{End}(V(i-1))$, there exists ζ': $\oplus_{s=1}^{m} V(i-1) \to V(i-1)$ with $\mu\mu'\zeta' = (\oplus_{s=1}^{m} \mu_{i-1})\,\zeta' = \bar{\tau}\mu_{i-1}$. Thus, $\zeta = \pi\mu'\zeta' : C' \to V(i-1)$ satisfies $\varphi_{23}\zeta = \tau\mu_{i-1}$. |Namely, $\varphi_{23}\zeta = \varphi_{23}\pi\mu'\zeta' = \bar{\varphi}_{23}\mu\mu'\zeta' = \bar{\varphi}_{23}\bar{\tau}\mu_{i-1} = \tau\mu_{i-1}$.| It follows that (f, u_t) is a coretraction with retraction

$$\left(\begin{pmatrix} 1 \\ 0 \\ -\zeta \end{pmatrix}, u_t'\right).$$

|Namely, we only have to check the following relation

$$\begin{pmatrix} \mu_{i-1} & 0 & 0 \\ \tau\mu_{i-1} & -\tau\mu_i & \varphi_{23} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -\zeta \end{pmatrix} = \begin{pmatrix} \mu_{i-1} \\ \tau\mu_{i-1} - \varphi_{23}\zeta \end{pmatrix} = \begin{pmatrix} \mu_{i-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_{i-1};$$

everything else is obvious.|

Next assume $\tau(W') \neq 0$, and denote by v: $W' \to W$ the inclusion. Since W' is a direct sum of copies of V, there exists λ: $V \to W'$ such that $\lambda v\tau = 1_V$. Also, W' is mapped under φ_{23} into Y since $v\varphi_{23}\pi = v\varphi'\varphi''\mu = 0$. Denote the restriction of φ_{23} to W' by $\underline{\varphi}$: $W' \to Y$. Using the definition of Y and Lemma 1, we see that we can factor $\lambda\underline{\varphi}$ through μ_i, thus there exists $\lambda' : V(i) \to Y$ such that $\lambda\underline{\varphi} = \mu_i\lambda'$. Let $\eta = \lambda v$ and $\eta' = \lambda' v'$, with v': $Y \to X \oplus Y$ denoting the inclusion. Then we have shown that $\eta\tau = \lambda v\tau = 1_V$, and $\eta\varphi_{23} = \lambda v\varphi_{23} = \lambda\underline{\varphi}v' = \mu_i\lambda'v' = \mu_i\eta'$. As a consequence,

$$(1, -\eta) \begin{pmatrix} \mu_{i-1} & 0 & 0 \\ \tau\mu_{i-1} & -\tau\mu_i & \varphi_{23} \end{pmatrix} = (0 \;\; \mu_i \;\; -\eta\varphi_{23}) = \mu_i(0 \;\; 1 \;\; -\eta').$$

Thus, let $f'' = (0 \;\; 1 \;\; -\eta')$: $V(i) \to V(i-1) \oplus V(i) \oplus C' = M$, $1_V \otimes u_1'' =$

$(1, -\eta)$: $V = V \otimes D \rightarrow V \otimes U_1 = V \oplus W$, and $u_t'' = u_{t-1}'' \varphi_t$ for $2 \leqslant t \leqslant n$. Then (f'', u_t'') is a coretraction for (f', u_t').

It remains to consider the case of f not being split mono. Thus f' is split epi, say $M = V(i) \oplus C$, $f' = \binom{1}{0}$, $\varphi_1 = (\sigma, \rho)$ with $\sigma: V \otimes U_1 \rightarrow V(i)$ and $\rho: V \otimes U_1 \rightarrow C$. Then $f = (\alpha_i, \delta)$ for some $\delta: V(i-1) \rightarrow C$. Now δ cannot be split mono since otherwise f is split mono. Let (α_i, ε): $V(i-1) \rightarrow V(i) \oplus W$ be a minimal left almost split homomorphism of A-modules starting with $V(i-1)$, where W either is zero or indecomposable. Since δ is not split mono, there exist $\delta': V(i) \rightarrow C$, $\delta'': W \rightarrow C$ such that $\delta = \alpha_i \delta' + \varepsilon \delta''$. We claim that $((1, \delta'), u_t)$ is a coretraction for (f', u_t'). We only have to show that it is a map of $A[V, n]$-modules. We have

$$\begin{aligned}(1 \otimes u_1)(\sigma, \rho) &= \mu_{i-1}(\alpha_i, \delta)\\ &= (\mu_{i-1}\alpha_i, \mu_{i-1}\delta) = (\mu_{i-1}\alpha_i, \mu_{i-1}\alpha_i\delta' + \mu_{i-1}\varepsilon\delta'')\\ &= (\mu_{i-1}\alpha_i, \mu_{i-1}\alpha_i\delta') = \mu_i(1, \delta'),\end{aligned}$$

where we have used the fact that $\mu_{i-1}\varepsilon = 0$ according to the definition of a ray module. Namely, ε is either zero or an irreducible map pointing to the mouth.

This finishes the proof.

2.8. *Proof of Theorem* 2.3. Recall that in 2.5 we have defined a simply connected translation quiver $\bar{T}$ (depending on T and v) and an exact representation $\mathscr{M}$ of $\bar{T}$. Similar to the definition of $\bar{T}$, we also define $\bar{T}[v, n]$ by cutting $T[v, n]$ along to the ray $[v, \infty)$.

Now we want to construct an exact representation of $\bar{T}[v, n]$. Note that $\bar{T}_0$ is a subset of $\bar{T}[v, n]_0$, and all arrows of $\bar{T}$ but those involving $\underline{v}(i)$ and pointing to the mouth are also arrows of $\bar{T}[v, n]$, and we will endow the vertices of $\bar{T}$ with the same modules as before, and the arrows of $\bar{T}[v, n]$ which also belong to $\bar{T}$ with the same maps as before. For $i \geqslant 0$, let $M_{(t,i)} = (V(i), U_j, \varphi_j)$, with $U_j = D$ for $1 \leqslant j \leqslant t$, $U_j = 0$ otherwise, and $\varphi_1 = \mu_i, \varphi_j = \mathrm{id}$ for $2 \leqslant j \leqslant t$, and $\varphi_j = 0$, otherwise. Thus $M_{(n,i)} = \overline{V(i)}$. For $i < 0$, let $M_{(t,i)} = (0, U_j, \varphi_j)$, with $U_j = D$ for $-i+1 \leqslant j \leqslant t$, $U_j = 0$, otherwise, and $\varphi_j = \mathrm{id}$ for $-i+2 \leqslant j \leqslant t$, and $\varphi_j = 0$, otherwise. The arrows of the form $(t, i-1) \rightarrow (t, i)$ will be endowed with the maps $(\alpha_i, 1, \ldots, 1)$ for $i \geqslant 1$, and with the obvious canonical inclusions for $i \leqslant 0$. Any arrow of the form $\underline{v}(i) \rightarrow (n, i)$ will be endowed with the map $(-1)^i(1, 0, \ldots, 0)$, an arrow of the form $(t, i) \rightarrow (t-1, i)$, with $2 \leqslant t \leqslant n$ and $i + t > 1$, will be endowed with the map $(-1)^i(1, u_j)$ with $u_j = 1$ for $1 \leqslant j \leqslant t-1$, and $u_j = 0$, otherwise. Finally, consider the arrows of the form $(1, i) \rightarrow y$ pointing to the mouth. If such an arrow exists in $\bar{T}[v, n]$, there is an arrow $\alpha: \underline{v}(i) \rightarrow y$ in $\bar{T}$, and therefore there is given an irreducible map $f_\alpha : V(i) \to M_y$. We endow the arrow $(1, i) \to y$ with the map

$$(f_\alpha, 0, \dots, 0) \colon M_{(1,i)} = (V(i), D, 0, \dots, 0; \mu_i, 0, \dots, 0) \to (M_y, 0, \dots, 0; 0, \dots, 0).$$

In this way, we obtain a representation of $\bar{T}[v, n]$, which again will be denoted by (M_x, f_α), or also by $\mathscr{M}[v, n]$. Instead of f_α we usually will just write f.

Let us list the main properties of the representation $\mathscr{M}[v, n] = (M_x, f_\alpha)$ of $\bar{T}[V, n]$.

(i) $\mathscr{M}[v, n]$ is exact, and all modules M_x are indecomposable.

(ii) If z is a projective vertex of $\bar{T}[v, n]$ and not of the form $\underline{v}(j)$ for any j, then M_z is a projective $A[V, n]$-module, $(f)_y \colon \oplus_{y \in z^-} M_y \to M_z$ is minimal right almost split, and the corresponding bimodules $\operatorname{Irr}(M_y, M_z)$ of irreducible maps are of the form ${}_D D_D$.

(iii) If M_y is isomorphic to a direct summand of the radical of an indecomposable projective module P, then $P \approx M_z$ for some $z \in \bar{T}[v, n]_0$.

(iv) If x is an injective vertex of $\bar{T}[v, n]$ and not of the form $\bar{v}(i)$, then M_x is an injective $A[V, n]$-module.

(v) The maps $f \colon M_{\underline{v}(i-1)} \to M_{\underline{v}(i)}$ and $f \colon M_{(n, i-1)} \to M_{(n,i)}$ with $i \geqslant 1$ are irreducible; the bimodule of irreducible maps $\operatorname{Irr}(M_{\underline{v}(i-1)}, M_{\underline{v}(i)})$ is of the form ${}_D D_D$.

(vi) The modules $M_{\underline{v}(i)}$ and $M_{\bar{v}(i)}$ are isomorphic, for all $i \in \mathbb{N}_0$, and there are no other isomorphisms between modules of the form M_x, $x \in \bar{T}[v, n]_0$.

Proof. (i) The modules M_x, with $x \in \bar{T}_0$, are indecomposable by construction of $\mathscr{M}$. The remaining ones are obviously also indecomposable. Also, it is easy to check that $\mathscr{M}[v, n]$ is an exact representation.

(ii) Let z be a projective vertex of $\bar{T}[x, n]$, and not of the form $\underline{v}(j)$. If $z \in \bar{T}_0$, then also $z^- \subseteq \bar{T}_0$. (Namely, the only arrows $y \to z$ with $z \in \bar{T}_0$ and $y \notin \bar{T}_0$ are arrows pointing to the mouth with $y = (1, i)$, $i \geqslant 1$. However, then there exists the extension $(1, i-1) \urcorner z$, thus z is not projective.) Thus the assertion follows from the corresponding assertion for $\bar{T}$. If $z \notin \bar{T}_0$, then $z = (n, i)$ with $i \leqslant 0$, and the assertion is obviously true by the construction of $\mathscr{M}[v, n]$.

(iii) Let P be indecomposable projective, and assume M_y is a direct summand of $\operatorname{rad} P$. If P is an A-module, then also M_y is an A-module, thus with $[M_y]$ also $z = [P]$ belongs to T. Thus $P \approx M_z$ (or $P \approx M_{\bar{v}(j)}$ in case $z = v(j)$). If P is not an A-module, then P is isomorphic to one of $M_{(n,i)}$, $i \leqslant 0$. Assertion (iv) has been shown in 2.6 and assertion (v) has been shown in 2.7; the assertion concerning the bimodule of irreducible maps carries over from T. Finally, property (vi) is satisfied by the construction of $\mathcal{M}|v, n|$.

We are going to derive some consequences. First, we consider the case $n \geqslant 2$ and we want to show by induction on i:

(a) Let $n \geqslant 2$. The map $(f)_y : \bigoplus_{y \in z} M_y \to M_z$ is minimal right almost split for $z = (n, i)$ and $i \geqslant 1 - n$. For $i \geqslant 2 - n$, one has $D \operatorname{Tr} M_{(n-1,i)} \approx M_{(n,i-1)}$, and for $i \geqslant 0$, one has $D \operatorname{Tr} M_{(n,i+1)} \approx M_{\underline{v}(i)}$.

Proof. For $i = 1 - n$, the module $M_{(n,1-n)}$ is simple projective, thus $0 \to M_{(n,1-n)}$ is minimal right almost split. For $2 - n \leqslant i < 0$, the map $f : M_{(n,i-1)} \to M_{(n,i)}$ is the inclusion of the radical of the projective module $M_{(n,i)}$, thus minimal right almost split, and the corresponding bimodule of irreducible maps is of the form ${}_DD_D$. Since $M_{(n,1-n)}$ is simple projective, the middle term of the Auslander–Reiten sequence starting with $M_{(n,1-n)}$ is a direct sum of copies of all indecomposable projective modules P with $M_{(n,1-n)}$ being a direct summand of $\operatorname{rad} P$, thus according to (iii) and (ii), the module $\operatorname{Tr} D\, M_{(n,1-n)}$ is given as the cokernel of $f : M_{(n,1-n)} \to M_{(n,2-n)}$; thus according to ($i$) by $M_{(n-1,2-n)}$. Assume we know by induction that $\operatorname{Tr} D\, M_{(n,i)} \approx M_{(n-1,i+1)}$, for some $1 - n \leqslant i < -1$. Then $M_{(n-1,i+1)}$ is the only nonprojective indecomposable module X with an irreducible map $M_{(n,i+1)} \to X$, and according to (iii) and (ii), the minimal left almost split map starting with $M_{(n,i+1)}$ is of the form (f,f): $M_{(n,i+1)} \to M_{(n-1,i+1)} \oplus M_{(n,i+2)}$ and its cokernel $M_{(n-1,i+2)}$ is $\operatorname{Tr} D\ M_{(n,i+1)}$. Now consider $i = 0$. The map $(f,f) : M_{(n,-1)} \oplus M_{\underline{v}(0)} \to M_{(n,0)}$ is the inclusion of the radical of the indecomposable projective module $M_{(n,0)}$, thus minimal right almost split, and both bimodules of irreducible maps are of the form ${}_DD_D$. Since $M_{(n,0)}$ is not a direct summand of the radical of a projective module, the middle term of the almost split sequence starting with $M_{(n,0)}$ has two summands. Since there are given the two irreducible maps $M_{(n,0)} \to M_{(n-1,0)}$ and $M_{(n,0)} \to M_{(n,1)}$, we see that $\operatorname{Tr} D\, M_{(n,0)}$ is the cokernel of (f,f): $M_{(n,0)} \to M_{(n-1,0)} \oplus M_{(n,1)}$, thus $M_{(n-1,1)}$. Also there must be an irreducible map from $M_{(n,0)}$ to $\operatorname{Tr} D\, M_{\underline{v}(0)}$, thus $\operatorname{Tr} D\, M_{\underline{v}(0)} \approx M_{(n,1)}$ [since $D \operatorname{Tr} M_{(n-1,0)} \approx M_{(n,-1)}$]. Now assume the assertions are shown for some $i \geqslant 0$. For the exact sequence

$$0 \longrightarrow M_{\underline{v}(i)} \xrightarrow{(f,f)} M_{(n,i)} \oplus M_{\underline{v}(i+1)} \xrightarrow{\binom{f}{f}} M_{(n,i+1)} \longrightarrow 0,$$

the map (f,f) is irreducible and $D \operatorname{Tr} M_{(n,i+1)} \approx M_{\underline{v}(i)}$, so this must be an Auslander–Reiten sequence, and $\binom{f}{f}$ is minimal right almost split. Since $\binom{f}{f}$: $M_{(n,i-1)} \oplus M_{\underline{v}(i)} \to M_{(n,i)}$ is minimal right almost split, and $M_{(n,i)}$ is not a summand of the radical of an indecomposable projective module, the Auslander–Reiten sequence starting with $M_{(n,i)}$ has two middle terms. Since both $f: M_{(n,i)} \to M_{(n-1,i)}$ and $f: M_{(n,i)} \to M_{(n,i+1)}$ are irreducible, it follows that $\operatorname{Tr} D\, M_{(n,i)}$ is the cokernel of $(f,f): M_{(n,i)} \to M_{(n-1,i)} \oplus M_{(n,i+1)}$. Finally, there must be an irreducible map from $M_{(n,i+1)}$ to $\operatorname{Tr} D\, M_{\underline{v}(i+1)}$, thus $\operatorname{Tr} D\, M_{\underline{v}(i+1)} \approx M_{(n,i+2)}$ [since $D \operatorname{Tr} M_{(n-1,i+1)} \approx M_{(n,i)}$].

Similarly one proves by induction on i:

(a′) Let $n = 1$. For $i \geqslant 0$, the map $(f)_y: \oplus_{y \in z^-} M_y \to M_z$ is minimal right almost split where $z = (n, i)$, and $D \operatorname{Tr} M_{(n,i+1)} \approx M_{\underline{v}(i)}$. Also if there is an arrow $(n, i) \to y$ pointing to the mouth, then $D \operatorname{Tr} M_y \approx M_{(n,i-1)}$.

In general, we therefore have:

(b) Let y be in $\bar{T}(v, n)$. If y is not of the form $\underline{v}(j)$ for any j, then the map $(f)_x: \oplus_{x \in y^-} M_x \to M_y$ is minimal right almost split. If y is not of the form $\bar{v}(j)$ for any j, then the map $(f)_z: M_y \to \oplus_{z \in y^+} M_z$ is minimal left almost split, and all bimodules of irreducible maps $\operatorname{Irr}(M_x, M_y)$, $x \in y^-$ and $\operatorname{Irr}(M_y, M_z)$, $z \in y^+$, are of the form ${}_D D_D$.

Proof. By induction on i, where $y \in \mathcal{S}_i(\bar{T}[v, n])\backslash\mathcal{S}_{i-1}(\bar{T}[v, n])$. For $i = 0$, and $y \neq \underline{v}(0)$ the vertex y is a source, thus projective, and $0 \to M_y$ is minimal right almost split by (ii). Suppose now $y \neq \bar{v}(0)$. If $y = \underline{v}(0)$, then $(f)_z: M_y \to \oplus_{z \in y^+} M_z$ clearly is minimal amost left split. If $y \neq \underline{v}(0)$, then M_y is simple projective, and if $M_y \to Z$ is irreducible with Z indecomposable, then Z is projective, and by (iii) of the form M_z for some z. According to (ii), there is an arrow $y \to z$. Always, if $y \to z$ is an arrow, then z is a projective vertex (since y is a source), M_z is projective, $f: M_y \to M_z$ is irreducible, and $\operatorname{Irr}(M_y, M_z) \approx {}_D D_D$, again by (ii). Thus $(f)_z$: $M_y \to \oplus_{z \in y^+} M_z$ is minimal left almost split. Now assume $y \in \mathcal{S}_{i+1}(\bar{T}[v, n])\backslash\mathcal{S}_i(\bar{T}[v, n])$. Suppose first that y is not of the form $\underline{v}(j)$. If y is projective, $(f)_x: \oplus_{x \in y^-} M_x \to M_y$ is minimal right almost split by (ii). If y is not projective, consider τy. If $\tau y = \underline{v}(j)$ for some j, then $y = (n, j+1)$, thus $(f)_x$: $\oplus_{x \in y^-} M_x \to M_y$ is minimal right almost split by (a) and (a′). Assume $w = \tau y$ is not of the form $\underline{v}(j)$. Then by induction, we have the minimal left almost split map $(f)_x: M_w \to \oplus_{x \in w^+} M_x$. Its cokernel is the map $(f)_x: \oplus_{x \in y^-} M_x \to M_y$ since $\mathcal{M}[v, n]$ is exact. Since $M_y \neq 0$, we see that M_w cannot be injective, and thus the map $(f)_x: \oplus_{x \in y^-} M_x \to M_y$ is minimal right almost split. In all cases, it follows that $\operatorname{Irr}(M_x, M_y) \approx {}_D D_D$ for all $x \in y^-$. Assume now that y is not of the form $\bar{v}(j)$. Let Z be indecomposable, with an irreducible map $M_y \to Z$. If Z is projective, then $Z \approx M_z$ for some z, according to (iii), and there is an arrow $y \to z$ according to (ii) and $\operatorname{Irr}(M_y, M_z) \approx {}_D D_D$. If Z is not projective, then there is an irreducible map $D \operatorname{Tr} Z \to M_y$, thus $D \operatorname{Tr} Z \approx M_x$ for some $x \in y^-$. Now M_x is not injective, thus x cannot be an injective vertex by (iv), thus there is z with $x \urcorner z$. Note that $x \in \mathcal{S}_i(\bar{T}[v, n])$, and again by induction there is the minimal left almost split map $(f)_{y_j}: M_x \to \oplus_{y_j \in x^+} M_{y_j}$. Its cokernel is on the one hand

$\operatorname{Tr} D\, M_x = Z$; on the other hand M_z since $\mathcal{M}[v, n]$ is exact. Also, since there is an irreducible map $M_x \to M_y$, we have $y = y_j$ for some j. Thus we have $Z \approx M_z$, an arrow $y \to z$, and $f: M_y \to M_z$ being irreducible. Also, since $\operatorname{Irr}(M_x, M_y) \approx {}_D D_D$, $\operatorname{Irr}(M_y, M_z) \approx {}_D D_D$. Conversely, if there is an arrow $y \to z$, then either z is a projective vertex and $f: M_y \to M_z$ is irreducible by (ii) or there is the extension $x \daleth z$, and as above we see that $f: M_y \to M_z$ is irreducible. This shows that the maps $f: M_y \to M_z$ with $z \in y^+$ furnish a complete set of irreducible maps with starting point y; thus we conclude that $(f)_z: M_y \to \bigoplus_{z \in y^+} M_z$ is minimal left almost split.

This finishes the proof of Theorem 2.3.

2.9. We assume that A is a finite-dimensional algebra with a component T which is a tube, and such that ${}_AV$ is a ray module with $v = [V]$ belonging to T. We want to determine the ray modules in $T[v, n]$.

PROPOSITION. (a) *Let W be a ray A-module in T, with $w = [W]$. If the rays $[v, \infty)$ and $[w, \infty)$ do not belong to the same ray class, then W is also a ray $A[V, n]$-module.*

(b) *If $V(i)$ is a ray A-module, for some $i \in \mathbb{N}_0$, then $M_{(1,i)}$ is a ray $A[V, n]$-module.*

(c) *The modules $M_{(t,1-t)}$, with $1 \leqslant t \leqslant n$, are ray $A[V, n]$-modules.*

Proof. (a) By construction of $T[v, n]$, we know that w stays a ray vertex in $T[v, n]$. If

$$W = W(0) \to W(1) \to \cdots \to W(i) \to W(i+1) \to \cdots$$

is a chain of indecomposable $A[V, n]$-modules and irreducible maps, all pointing to infinity, then all these modules are in fact A-modules, and the maps are irreducible A-maps pointing to infinity. Also, since the rays $[v, \infty)$ and $[w, \infty)$ do not belong to the same ray class, any indecomposable $A[V, n]$-module X with an irreducible map $W(i) \to X$ pointing to the mouth is in fact an A-module and the map is an irreducible A-map pointing to the mouth. This shows that properties (1) and (2) of the definition of a ray module are satisfied for the $A[V, n]$-module W.

(b) Assume $V(i)$ is a ray A-module, for some $i \in \mathbb{N}_0$. Now $M_{(1,j)} = (V(j), D, 0\ldots, 0; \mu_j, 0,\ldots, 0)$, with $\mu_j = \alpha_1 \cdots \alpha_j$ for $j \geqslant 1$, and $\mu_0 = 1$. Any map $f: V(i) \to V(j)$ can be uniquely lifted to a map $(f, \varphi_t)_t: M_{(1,i)} \to M_{(1,j)}$ since we must have $\mu_i f = \varphi_1 \mu_j$, and ${}_D\operatorname{Hom}(V, V(j))$ is one-dimensional. Since $\operatorname{Irr}(V(j), V(j+1))$ is of length one both as an $\operatorname{End}(V(j))$-module as well as an $\operatorname{End}(V(j+1))$-module, we can identify the residue division rings of the endomorphism rings $\operatorname{End}(V(j))$ with D choosing fixed irreducible maps $V(j-1) \to V(j)$. Since we assume that $V(i)$ is a ray A-module, $\operatorname{End}(V(i))$ is itself a division ring, thus $\operatorname{End}(V(i)) = D$. It follows that for $j \geqslant i$, the End($V(i)$)-vectorspace ${}_D\text{Hom}(V(i), V(j))$ is one-dimensional. For the proof of property (1) of a ray module, let X be an indecomposable $A[V, n]$-module with an irreducible map $(g, \psi_t)\colon M_{(1,j)} \to X$ pointing to the mouth. By construction, X is an A-module. Now consider a chain of irreducible maps of $A[V, n]$-modules

$$M_{(1,i)} \xrightarrow{(f^i, \varphi_t^i)} M_{(1,i+1)} \xrightarrow{(f^{i+1}, \varphi_t^{i+1})} \cdots \xrightarrow{(f^{j-1}, \varphi_t^{j-1})} M_{(1,j)} \xrightarrow{(g, \psi_t)} X.$$

It is easy to see that all the maps f^s, $i \leqslant s \leqslant j-1$, and g are irreducible A-homomorphisms, with g being an A-homomorphism pointing to the mouth; thus the composition $f^i \cdots f^{j-1} \cdot g$ is 0, and therefore also $(f^i, \varphi_t^i) \cdots (f^{j-1}, \varphi_t^{j-1})(g, \psi_t)$.

(c) The module $M_{(1,0)}$ is a ray $A[V, n]$-module according to (b) since $V(0)$ is a ray A-module. Thus consider $M_{(t,1-t)}$ with $2 \leqslant t \leqslant n$, and denote it by S_t. It is a simple $A[V, n]$-module with endomorphism ring D, and the only indecomposable $A[V, n]$-modules X belonging to $T[v, n]$ with $\text{Hom}(S_t, X) \neq 0$ are the modules of the form $M_{(t,i)}$; the dimension of ${}_D\text{Hom}(S_t, M_{(t,i)})$ is equal to 1. This immediately gives the proof of all the properties (1) and (2) required for a ray module. For example, if

$$S_t = M_{(t,1-t)} \to M_{(t,2-t)} \to \cdots \to M_{(t,i)} \to M_{(t-1,i)}$$

is a chain of arbitrary maps, then the composition is zero since $\text{Hom}(S_t, M_{(t-1,i)}) = 0$.

This finishes the proof of the proposition.

2.10. Again, we assume that V is a ray A-module with $v = [V]$ belonging to a tube T. We are going to determine the coray modules in $T[V, n]$.

PROPOSITION. *Any coray A-module W in T is also a coray $A[V, n]$-module. If V itself is a coray module, then also $M_{(1,0)}$ is a coray $A[V, n]$-module.*

Proof. Let W be a coray module in T, thus $w = [W]$ is a coray vertex both in T and $T[v, n]$. Recall that the coray ending in $w = [W]$ is denoted by $(\infty, w]$ and is given by the following vertices:

$$\cdots \to w(-j-1) \to w(-j) \cdots \to w(-1) \to w(0) = w.$$

We will use this notation for the coray in $T[v, n]$. (The corresponding coray in T ending in w is obtained from $(\infty, w]$ by deleting several vertices and arrows!) Also, we denote the module $M_{w(-j)}$ by $W(-j)$, where $j \in \mathbb{N}_0$.

First assume that $W(-j)$ is an A-module, thus $\text{Hom}(W(-j), W)_{\text{End}(W)}$ is one-dimensional, say with generator ν_j. Also if $\beta\colon M_x \to W(-j)$ is an irreducible $A[V, n]$-homomorphism pointing to infinity, then M_x is in fact an A-module, and β, as an A-module homomorphism, is also irreducible and points to infinity. Thus, given in addition a chain of irreducible $A[V, n]$-maps from $W(-j)$ to W, say with composition $v_j\varepsilon$, where $\varepsilon \in \mathrm{End}(W)$, then $\beta v_j \varepsilon = 0$, since v_j also is the composition of irreducible A-homomorphisms. Second, let $W(-j)$ be of the form $M_{(t,i)}$ for some $1 \leqslant t \leqslant n, i \geqslant 1$ (the cases $i \leqslant 0$ being impossible). Thus either $t = n$ or $W(-j) = M_{(n,i)}/M_{(n,-t)}$, with $M_{(n,i)} = V(i)$. Consider a homomorphism $v: M_{(n,i)} \to W$, and note that v has the form $v = (f, 0)$ with $f: V(i) \to W$ being an arbitrary map. Namely, for any map $f: V(i) \to W$, $\mu_i f = 0$, since μ_i is a proper composition of irreducible A-maps all pointing to infinity (according to $i \geqslant 1$), f is a composition of irreducible A-maps pointing to the mouth, and V is a ray module. Since $\mathrm{Hom}(V(i), W)_{\mathrm{End}(W)}$ is one-dimensional, the same is true for $\mathrm{Hom}(M_{(n,i)}, W)_{\mathrm{End}(W)}$ and then also for $\mathrm{Hom}(M_{(t,i)}, W)_{\mathrm{End}\,W}$ where $t < n$. If $\gamma: M_{(t,i-1)} \to M_{(t,i)}$ is an irreducible map, then $\gamma = (g, u_t)$ with $g: V(i-1) \to V(i)$ being an irreducible A-map pointing to infinity; thus $gf = 0$ and therefore $\gamma v = 0$.

Now assume that V itself is a coray A-module and consider $M = M_{(1,0)}$. By the previous considerations, we know that V is a coray $A[V, n]$-module. Note that given an indecomposable $A[V, n]$-module X which is not of the form $M_{(t,i)}$ for some $i \leqslant 0$, then any homomorphism $\delta: X \to M$ factors through V (namely, lift δ along the minimal right almost split maps ending in the modules $M_{(t,j)}$ with $j \leqslant 0$). Also, note that $\mathrm{Hom}(V, M)$ is one-dimensional as well as an $\mathrm{End}(M)$-vectorspace and an $\mathrm{End}(V)$-vectorspace, generated by the obvious inclusion $V \to M$. Thus, for $j \geqslant n$, we have $\dim \mathrm{Hom}(M(-j), M)_{\mathrm{End}(M)} = \dim \mathrm{Hom}(M(-j), V)_{\mathrm{End}(V)} = 1$, and given an irreducible map $X \to M(-j)$ pointing to infinity, its composition with an arbitrary map $M(-j) \to M$ is zero. Also, it is easy to see that $\mathrm{Hom}(M_{(t,i)}, M)$ is one-dimensional as an $\mathrm{End}(M)$-space for $i = 0$, and zero for $i < 0$.

3. COHERENT TUBES WITH LENGTH FUNCTIONS

3.1. Our aim in this section is a combinatorial description of those tubes which are obtained from smooth tubes by suitable ray insertions and coray insertions. First, we recall the notion of a length function on a translation quiver Q. Note that for Q an Auslander–Reiten component, the function $f: Q_0 \to \mathbb{N}_1$ given by $f([X])$, the length of the module X, will satisfy the axioms for a length function; it will be called the canonical length function.

DEFINITION. Let Q be a translation quiver. A function $f: Q_0 \to \mathbb{N}_0$ is said to be a *length function* for Q, provided f satisfies the following properties:

(1) (Additivity): For any extension $x \mathrel{\rceil\!\!\lfloor} z$, one has $f(x)+f(z)=\sum_{y\in x^+} f(y)$.

(2) If z is a projective vertex, $f(z)=1+\sum_{y\in z^-} f(y)$.

(2*) If x is an injective vertex, $f(x)=1+\sum_{y\in x^+} f(y)$.

The set of length functions may be considered as a subset of the rational vector space of all functions $Q_0 \rightarrow \mathbb{Q}$. The dimension of the affine subspace generated by all length functions will be called the *rank* of the set of length functions.

For example, for the smooth tube $T=\mathbb{Z}\mathbb{A}_\infty/r$, any length function f is uniquely determined by its values on the vertices belonging to the mouth, and these values may be arbitrary numbers in $\mathbb{N}_0$. Thus, the rank of the set of length functions on $\mathbb{Z}\mathbb{A}_\infty/r$ is just r.

3.2. LEMMA. *Let T be a tube, and v a ray vertex in T. Any length function f for T has a unique extension to a length function f' for $T[v, n]$, and any length function for $T[v, n]$ occurs in this way. The extension f' of a length function f for T does not take the value zero outside T_0. If f, g are length functions with $f \leqslant g$, then $f' \leqslant g'$.*

Proof. Let f be a length function for T. Define $f'\colon T[v, n]_0 \rightarrow \mathbb{N}_0$ as follows: $f'(w)=f(w)$ for $w\in T_0$, $f'(t, i)=t+f(v(i))$, for $i\geqslant 0$, and $f'(t, i)=t+i$, for $i<0$.

First, we show the additivity of f'. Assume there is an extension $v(i) \mathrel{\rceil\!\!\lfloor} z$ in T; thus there is the extension $(1, i) \mathrel{\rceil\!\!\lfloor} z$ in $T[v, n]$. If $v(i+1)$ is the only element of z^- in T, then

$$f'(z)+f'(1, i)=f(z)+1+f(v(i))=1+f(v(i+1))=f'(1, i+1).$$

If z^- in T contains two elements, say $v(i+1)$ and w, then

$$\begin{aligned} f'(z)+f'(1, i) &= f(z)+1+f(v(i))=1+f(v(i+1))+f(w)\\ &= f'(1, i+1)+f'(w). \end{aligned}$$

For the extensions $v(i) \mathrel{\rceil\!\!\lfloor} (n, i+1)$, we have

$$\begin{aligned} f'(v(i))+f'(n, i+1) &= f(v(i))+n+f(v(i+1))\\ &= f'(n, i)+f'(v(i+1)). \end{aligned}$$

Finally, consider the extensions $(t, i) \mathrel{\rceil\!\!\lfloor} (t-1, i+1)$ in $T[v, n]$ with $2\leqslant t\leqslant n$. If $i\geqslant 0$, then

$$\begin{aligned} f'(t, i)+f'(t-1, i+1) &= 2t-1+f(v(i))+f(v(i+1))\\ &= f'(t-1, i)+f'(t, i+1). \end{aligned}$$

If $i=-1$, and $t+i>1$, then

$$f'(t,-1)+f'(t-1,0)=2t-2+f(v(0))=f'(t-1,-1)+f'(t,0).$$

If $i\leqslant -2$, and $t+i>1$, then

$$f'(t,i)+f'(t-1,i+1)=2t+2i=f'(t-1,i)+f'(t,i+1).$$

For $(t,i)=(2,-1)$, we have

$$f'(2,-1)+f'(1,0)=2+f(v(0))=f'(2,0),$$

and for $i\leqslant -2$, and $t+i=1$, we have

$$f'(t,i)+f'(t-1,i+1)=2t+2i=2=t+i+1=f'(t,i+1).$$

The vertices (n,i) with $1-n\leqslant i\leqslant 0$, are projective, and one has $f'(n,1-n)=1$, $f'(n,i)=1+f'(n,i-1)$ for $1-n<i<0$, and $f'(n,0)=n+f(v(0))=1+f'(n,-1)+f'(v(0))$. Finally, assume that $v(i)$ is injective in T for some i, thus $(1,i)$ is injective in $T[v,n]$. If $v(i+1)$ is the only element in $v(i)^+$, then $(1,i)^+$ consists of $(1,i+1)$, and

$$f'(1,i)=1+f(v(i))=2+f(v(i+1))=1+f'(1,i+1).$$

If $v(i)^+$ contains two elements, say $v(i+1)$ and w, then

$$\begin{aligned}f'(1,i)&=1+f(v(i))=2+f(v(i+1))+f(w)\\&=1+f'(1,i+1)+f'(w).\end{aligned}$$

Conversely, assume f' is a length function for $T[v,n]$. By induction first on $n-t$, then on i, one shows that

$$f'(t,i)=\begin{cases}t+i & \text{for } \ i<0\\ t+f'(v(i)) & \text{for } \ i\geqslant 0;\end{cases}$$

thus f' is uniquely determined by its restriction to T_0. Also, given an extension $v(i)\urcorner\!\llcorner z$ in T, then $(1,i)\urcorner\!\llcorner z$ is an extension in $T[v,n]$, and

$$\begin{aligned}f'(v(i))+f'(z)&=-1+f'(1,i)+f'(z)=-1+f'(1,i+1)\\&=f'(v(i+1)),\end{aligned}$$

in case $v(i+1)$ is the only vertex belonging to z^- in T, whereas

$$\begin{aligned}f'(v(i))+f'(z)&=-1+f'(1,i)+f'(z)=-1+f'(1,i+1)+f'(w)\\&=f'(v(i+1))+f'(w)\end{aligned}$$

in case z^- in T consists of the two vertices $v(i+1)$ and w. Altogether we see that the restriction of f' to T_0 is a length function for T, and that this restriction uniquely determines f'.

The last two assertions are obvious. This finishes the proof of the lemma.

PROPOSITION. *Let T be obtained from the smooth tube $\mathbb{Z}\mathbb{A}_\infty/r$ by a sequence of ray insertions and coray insertions. Then T has a unique minimal length function f_0 and $f_0^{-1}(0) = (\mathbb{Z}\mathbb{A}_\infty/r)_0$. Also, the set $\{f - f_0 | f$ a length function on $T\}$ is closed under addition and scalar multiplication by elements of $\mathbb{N}_0$, and it generates an r-dimensional subspace of the space of functions $T_0 \to \mathbb{Q}$. In particular the rank of the set of length functions on T is r, and coincides with the rank $r(T)$ of T.*

Proof. By induction, starting with a smooth tube. For a smooth tube, the constant zero function obviously is a length function, and therefore the unique minimal length function. The case of a ray insertion has been treated in 3.2, the case of a coray insertion is dual. Recall that the rank $r(T)$ of T is given by the number of paths $x \to y \to z$ contained in the circuit next to the mouth such that $x \urcorner\!\!\llcorner z$ exists and belongs to the mouth. Obviously, $r(T)$ is not changed by a ray insertion or a coray insertion, and $r(\mathbb{Z}\mathbb{A}_\infty/r) = r$. This finishes the proof.

3.3. Let us introduce the notion of a coherent tube. It is easy to see that a tube obtained from a coherent tube by inserting rays or corays is coherent again. Since smooth tubes are always coherent, we obtain a large class of coherent tubes by starting with smooth tubes and forming successively ray insertions and coray insertions. The main result of this section will show that in this way we obtain all coherent tubes which admit a length function.

DEFINITION. A tube is said to be *coherent*, provided it satisfies the following conditions:

(1) The τ-orbit of any projective vertex contains a vertex which belongs to a circuit.

(1*) The τ-orbit of any injective vertex contains a vertex which belongs to a circuit.

(2) For any projective vertex x, the ray $[x, \infty)$ exists.

(2*) For any injective vertex x, the coray $(\infty, x]$ exists.

THEOREM. *The following properties are equivalent for a tube T:*

(i) *T is coherent and has a length function.*

(ii) *T is coherent and $r(T) > 0$.*

(iii) *T is obtained from a smooth tube by a sequence of ray insertions followed by a sequence of coray insertions.*

(iii*) *T is obtained from a smooth tube by a sequence of coray insertions followed by a sequence of ray insertions.*

(iv) *T is obtained from a smooth tube by a sequence of ray insertions and coray insertions.*

Note that for a tube T satisfying these properties, $r(T)$ is equal to the rank of the set of its length functions; see 3.2 (whereas for $r(T)=0$, there is no length function for T, thus the rank of the set of length functions is -1).

3.4. The proof of Theorem 3.3 will be given in 3.5. Here, we start with some preliminary results.

We define two kinds of cancellations in a tube T. First, let a be a projective ray vertex of T such that none of the vertices $a(i)$, $i \geqslant 1$, is projective. Let $z=\tau a(1)$. There exists the ray $|z, \infty)$, and $z(i)=\tau a(i+1)$ for all $i \geqslant 0$. Also, z is the only vertex in a^-. Now $\partial_a T$ is obtained from T as follows: we delete all vertices of the form $a(i)$, $i \geqslant 0$, and all arrows and extensions involving any of the $a(i)$, and add a new arrow $z(i) \rightarrow y$ for any arrow $a(i) \rightarrow y$ pointing to the mouth, and a new extension $z(i) \urcorner\!\llcorner y$ for any extension $a(i) \urcorner\!\llcorner y$ in T. In this case, we will say that a allows a *cancellation of the first kind.*

Next, let a be a projective ray vertex such that also $a(1)$ is projective, and none of the vertices $a(i)$, $i \geqslant 0$, is injective. Let $b=\tau^- a$. There exists the ray $|b, \infty)$, and $b(i)=\tau^- a(i)$ for all $i \geqslant 0$. Also, there is no arrow ending in a. In this case $\partial_a T$ is obtained from T in the following way: we delete all vertices of the form $a(i)$, $i \geqslant 0$, and all arrows and extensions involving any of the $a(i)$, and we add a new arrow $x \rightarrow b(i-1)$ for all arrows $x \rightarrow a(i)$ pointing to the mouth, and the extension $x \urcorner\!\llcorner b(i-1)$ for all extensions $x \urcorner\!\llcorner a(i)$ in T. In this case, we say that a allows a *cancellation of the second kind.*

(a) Assume that T is a coherent tube and that a is a vertex of T which allows cancellation of the first or the second kind. Then $\partial_a T$ is again a tube, and $p(\partial_a T)=p(T)$, $q(\partial_a T)=q(T)-1$. Also, $\partial_a T$ is coherent and $r(\partial_a T)=r(T)$. If there exists a length function f for T, then $f|(\partial_a T)_0$ is a length function for $\partial_a T$.

Proof. Any circuit of T must contain a sequence of arrows of the following form:

$$z' \rightarrow a(i) \rightarrow a(i+1) \rightarrow \cdots \rightarrow a(i+s) \rightarrow b'$$

with $z' \rightarrow a(i)$ and $a(i+s) \rightarrow b'$ both pointing to the mouth, and $s \geqslant 0$. In case of a cancellation of the first kind, $z' = z(i)$, and we replace this path of T by the path in $\partial_a T$

$$z(i) \to z(i+1) \to \cdots \to z(i+s) \to b'.$$

In case of a cancellation of the second kind, one has $i \geqslant 1$ and $b' = b(i+s-1)$, so we replace this path of T by the path in $\partial_a T$

$$z' \to b(i-1) \to b(i) \to \cdots \to b(i+s-1).$$

This shows that $\partial_a T$ again has a circuit. Also, it is rather easy to check that with T also $\partial_a T$ is a coherent tube, and that $p(\partial_a T) = p(T)$, $q(\partial_a T) = q(T) - 1$, and $r(T) = r(\partial_a T)$.

The proof that $f|(\partial_a T)_0$ is a length function is similar to the proof of Lemma 3.1. We only note that in the case of a cancellation of the second type, and an arrow $x \to a(i)$ pointing to the mouth, one has $f(x) < f(a(i))$, by induction on i; thus x cannot be an injective vertex.

(b) Let T be a coherent tube. If T is neither smooth nor the mouth of T is a circuit, then there exists a vertex which allows cancellation or cocancellation of the first or second kind.

Proof. Assume no vertex of T allows cancellation. Let ω be the circuit next to the mouth, and

$$x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} x_3$$

arrows belonging to ω and pointing to the mouth. We claim that x_3 has to be projective. Otherwise, we may assume, without loss of generality, that the next arrow $x_3 \to x_4$ belonging to ω either points to infinity or that x_4 is projective. [Namely, if all arrows in ω point to the mouth, then one of the vertices involved has to be projective since otherwise ω is not next to the mouth.] Choose $m \geqslant 1$ such that $\tau^{m-1}x_3$ is not projective, whereas $a = \tau^m x_3$ is projective. [Note, that such an m must exist since otherwise we obtain a circuit involving the arrow $\tau x_3 \to x_2$ pointing to infinity, whereas by assumption the arrow $x_1 \to x_2$ points to the mouth and belongs to the circuit next to the mouth.] There does not exist an arrow $\tau^m x_3 \to y$ pointing to the mouth. [Again, some $\tau^s y$, $s \geqslant 0$, has to be projective, and therefore the τ-orbit of y contains some vertex in a circuit. Since the vertices $\tau^{-i}y$, with $i \leqslant m-1$, do not belong to circuits, we obtain an arrow $x_3 \to \tau^{-m}y$ pointing to the mouth. Since x_3, and some $\tau^{-i}y$, $i \geqslant m$, belong to circuits, we see that also $\tau^{-m}y$ belongs to a circuit, and therefore to ω. But this contradicts the choice of x_3.] Since $a = \tau^m x_3$ is projective, there exists the ray $[a, \infty)$. No vertex of the form $a(i)$ can be injective. [Otherwise, we would obtain an additional hole inside the closed curve formed by the extensions $\tau^i x_3 \mathrel{\urcorner\!\llcorner} \tau^{i-1} x_3$, $1 \leqslant i \leqslant m$, the arrows $a(i-1) \to a(i)$, $1 \leqslant i \leqslant s$ where $a(s)$ is first vertex of the ray $[a, \infty)$ belonging to ω, and the arrows from $a(s)$ to x_3 belonging to ω.] As a consequence, a is a projective ray vertex with none of the vertices $a(i)$, $i \geqslant 0$, being injective. In case $a(1)$ is projective, a allows cancellation of the second kind. Thus assume $a(1)$ is not projective. Let $z = \tau a(1)$. We claim that the ray $[z, \infty)$ exists, that none of the vertices $z(i)$ is injective, and that $\tau a(i+1) = z(i)$ for all $i \geqslant 0$. Namely, there exists $s \geqslant 0$ such that $y = \tau^s z$ is projective [otherwise we would obtain a circuit involving a], and there cannot exist a hole inside the closed curve given by the extensions $\tau^i z \mathrel{\urcorner\!\llcorner} \tau^{i-1} z$, $1 \leqslant i \leqslant s$, the arrow $z \to a$, the extensions $\tau^i x_3 \mathrel{\urcorner\!\llcorner} \tau^{i-1} x_3$, $1 \leqslant i \leqslant m$, the arrows $y(i-1) \to y(i)$, $1 \leqslant i \leqslant s$, where $y(s)$ is the first vertex of the ray $[y, \infty)$ belonging to ω, and the arrows from $y(s)$ to x_3 belonging to ω. This shows that a allows a cancellation of the first kind.

Since we assume from the beginning that T does not allow any cancellation, it follows that any vertex of T belongs to a circuit, using the property shown above and its dual. Now, assume the mouth contains both extensions and arrows. Let $x \mathrel{\urcorner\!\llcorner} z$ be an extension, and $w \to x$ an arrow, both contained in the mouth. First assume that $w \to x$ points to the mouth, then it is clear that x allows a cancellation of the first kind. In case $w \to x$ points to infinity, let

$$a = w(-s) \to \cdots \to w(-1) \to w \to x$$

be arrows belonging to the mouth, all pointing to infinity, with s being maximal. If the extension $\tau a \mathrel{\urcorner\!\llcorner} a$ belongs to the mouth, then a allows cocancellation of the first type. Otherwise, a is projective, and there is an arrow $b \to a$ pointing to the mouth and belonging to the mouth. In this case again, a allows cancellation of the first kind.

This shows that either the mouth contains only arrows, and then the mouth is a circuit, or else the mouth contains only extensions, and then T is smooth. This finishes the proof of (b).

We say that a vertex a of the tube T arrows *n-fold joint cancellation* provided a has the following properties: the vertices $a_i = \tau^{i-1} a$, $1 \leqslant i \leqslant n$, exist and are ray vertices, the vertices $a_n(i)$, $0 \leqslant i < n$ are projective, and none of the vertices $a_n(i)$, $i \geqslant n$, is projective. In this case, the sequence of cancellations $\partial_{a_1} \cdots \partial_{a_n} T$ is defined, the first $n-1$ cancellations being of the second kind, the last cancellation being of the first kind. For $n = 1$, the n-fold joint cancellation just reduces to a cancellation of the first kind. Note that the joint cancellation is the reverse process to the ray insertion, namely, we have

$$\partial_{(1,0)} \cdots \partial_{(n,1-n)} T(x, n) = T$$

for any ray vertex x in T, and $n \geqslant 1$.

(c) Let T be a coherent tube with $r(T) > 0$. If $q(T) > r(T)$, then there exists a vertex which allows joint cancellation. If $p(T) > r(T)$, then there exists a vertex which allows joint cocancellations.

Proof. Assume $q(T) > r(T)$. Let ω be the circuit next to the mouth, and $x_1 = \tau z \to y \to z$ a path in ω with $y \to z$ pointing to the mouth. Let s be maximal with a path in ω

$$a = x_s \to x_{s-1} \to \cdots \to x_1$$

of arrows pointing to infinity. Thus $\tau x_{s-1} \to a$ belongs to ω. Since $q(T) > r(T)$, without loss of generality we can assume that either a is projective or else $\tau a \to \tau x_{s-1}$ does not belong to ω. If a is projective, then a allows cancellation of the first kind, thus a 1-fold joint cancellation. If a is not projective, then one of the vertices $\tau a, \tau^2 a, \ldots$, say $b = \tau^t a$, is projective. If b is not a source, then b allows a cancellation of the first kind, otherwise there exists some n with $2 \leqslant n \leqslant t+1$ such that all the vertices $b(i)$, $0 \leqslant i < n$, are projective, whereas $b(n)$ (and therefore all $b(i)$, $i \geqslant n$) are not projective. In this case, the element $\tau^{t-n+1}a$ allows n-fold joint cancellation. Thus, in case there does not exist a vertex which allows joint cancellation, $q(T) = r(T)$. Dually, if no joint cocancellation is possible, $p(T) = r(T)$.

3.5. For the proof of the theorem, we will need an additional result:

LEMMA. *Let T be a tube with the mouth being a circuit. Then T does not admit a length function.*

Proof. First, consider the case where all arrows on the mouth point to the mouth, thus all vertices on the mouth are projective, and let

$$a_0 \to a_1 \to a_2 \to \cdots \to a_q = a_0$$

be the circuit which forms the mouth of T. Assume there exists a length function f, then $f(a_{i+1}) = f(a_i) + 1$, and $f(a_0) = f(a_q) = f(a_0) + q$, a contradiction. In the same way, we obtain a contradiction in case all arrows of the mouth point to infinity, thus, all vertices on the mouth are injective. Therefore, assume now there are as well arrows on the mouth which point to the mouth and arrows which point to infinity, and let f be a length function for T. If $x \to y$ is an arrow on the mouth which points to infinity, then x is injective and $f(y) = f(x) - 1$. It follows that for any arrow $x \to y$ which points to infinity, $f(y) = f(x) - 1$. Since the mouth is a circuit containing arrows pointing to the mouth, the ray $[x, \infty)$ exists for any vertex x. Now $f(x(n)) = f(x) - n$ shows that f takes negative values, a contradiction.

Proof of the theorem. $(i) \Rightarrow (ii)$. Let T be a coherent tube with a length function f. Since any cancellation reduces $q(T)$, any cocancellation reduces

$p(T)$; the process of cancellations and cocancellations must stop after a finite number of steps with a tube T^s. By (b), T^s is either smooth or the mouth of T^s is a circuit. But the mouth of T^s cannot be a circuit since with T also T^s has a length function. Thus T^s is smooth, and therefore $r(T) = r(T^s) > 0$.

(ii) $\Rightarrow$ (iii). Let T be a coherent tube with $r(T) > 0$. By (c), we can use a sequence of first joint cocancellations and then joint cancellations in order to obtain a tube T^s with $p(T^s) = r(T^s) = q(T^s)$, thus T^s is smooth. Since joint cancellation is the reverse process to ray insertion, joint cocancellation the reverse process to coray insertion, we obtain T from T^s by using first ray insertions and then coray insertions. Also, $r(T^s) = r(T)$, thus $T^s = \mathbb{Z}A_\infty/r(T)$. By duality, one also has (ii) $\Rightarrow$ (iii*). The proof of (iv) $\Rightarrow$ (i) follows from 3.2.

3.6. We have seen above that a coherent tube T with length function has rank $r(T) > 0$. Thus, if a component of the Auslander–Reiten quiver of a finite-dimensional algebra is a coherent tube T, we always must have $r(T) > 0$. However, there are coherent tubes T with $r(T) = 0$ which are of interest in the representation theory of orders.

EXAMPLE. Let R be a discrete rank one valuation ring, and consider the R-lattices over the 2×2 lower triangular matrix ring $S = \left(\begin{smallmatrix} R & 0 \\ R & R \end{smallmatrix}\right)$ over R. Then there is a component T of type (1, 1) with $r(T) = 0$.

Namely, let I be the maximal ideal of R, say with generator π. We consider the following lattice of column vectors $L(i) = \left(\begin{smallmatrix} I^i \\ R \end{smallmatrix}\right)$ with $i \in \mathbb{N}_0$ (where $I^0 = R$). There are inclusion maps $L(i) \to L(i-1)$, all of them will be denoted by ι, and the maps $L(i-1) \to L(i)$ given by multiplication with π, all of them will be denoted just by π. Then T is given by the lattices $L(i)$, $i \in \mathbb{N}_0$, the various maps ι and π, and the extensions

$$0 \longrightarrow L(i) \xrightarrow{(\iota, \pi)} L(i-1) \oplus L(i+1) \xrightarrow{\left(\begin{smallmatrix} \pi \\ -\iota \end{smallmatrix}\right)} L(i) \longrightarrow 0,$$

where $i \in \mathbb{N}_1$.

4. TUBULAR EXTENSIONS OF TAME CONCEALED ALGEBRAS

4.1. Let $A = A_0$ be a tame concealed algebra, with $T_0(\lambda)$, $\lambda \in \Lambda$, being the set of all tubes of A. The algebra B is said to be a *tubular* extension of A, provided there exists a finite sequence of algebras $A = A_0, A_1, \ldots, A_m = B$ with A_{i+1} being obtained from A_i by a simple (or cosimple) tubular extension using some ray (or coray) module $V_i \in T_i(\lambda_i)$, for some $\lambda_i \in \Lambda$, and where $T_{i+1}(\lambda)$, $\lambda \in \Lambda$, is the corresponding set of tubes of A_{i+1} with $T_{i+1}(\lambda)_0 \supseteq T_i(\lambda)_0$. The *type* of this extension is, by definition, the function $(p,q): \Lambda \to \mathbb{N}_1^2$ where $p(\lambda)=p(T_m(\lambda))$, $q(\lambda)=q(T_m(\lambda))$, and the class of modules belonging to the various $T(\lambda)=T_m(\lambda)$, $\lambda \in \Lambda$, will be denoted by $\mathscr{T}(B,\Lambda)$.

Note that for a tame concealed algebra A, all simple regular A-modules are ray modules, and, in general, there is the following result:

PROPOSITION. *Let B be a tubular extension of a tame concealed algebra A. Let M be an indecomposable B-module belonging to $\mathscr{T}(B,\Lambda)$. Then M is a ray module if and only if $|M|$ is a ray vertex of $T(\lambda)$, and M is a coray module, if and only if $|M|$ is a coray vertex of $T(\lambda)$, for a suitable $\lambda \in \Lambda$.*

Proof. Let $A=A_0, A_1,..., A_m=B$ be a sequence of algebras as given in the definition above. In case $m=0$, the assertions are obvious. The general case now follows by induction, using Propositions 2.9 and 2.10 in case A_m is a simple tubular extension of A_{m-1}, and the duals of 2.9 and 2.10 in case A_m is a cosimple tubular extension.

4.2 THEOREM. *Let A be a tame concealed algebra with a simple regular module M of period r. Let T be a coherent tube of rank r. Then there exists a tubular extension B of A such that the component of B containing $|M|$ is of the form T.*

Proof. We use Theorem 3.3. By assumption, $r>0$; thus T is obtained from the smooth tube $\mathbb{Z}\mathbb{A}_\infty/r$ by a sequence of ray insertions and coray insertions. Now the component of A containing M is of the form $\mathbb{Z}\mathbb{A}_\infty/r$; thus the theorem follows from Proposition 4.1.

COROLLARY. *Let T be a coherent tube with a length function f such that $f(x) \geqslant 2$ for some simple regular vertex x. Then there exists a tubular extension B of some tame concealed algebra A with a component of the form T such that f coincides with the canonical length function of this component.*

Proof. By Theorem 3.3, T is obtained from a smooth tube of the form $\mathbb{Z}\mathbb{A}_\infty/r$, with $r \geqslant 1$, by a sequence of ray insertions and coray insertions. The simple regular vertices of $\mathbb{Z}\mathbb{A}_\infty/r$ are the orbits of the pairs $(i,0) \in \mathbb{Z} \times \{0\}$ under the translation $i \to i+r$, and we denote the orbit containing $(i,0)$ by $\bar{i}$. Note that $(\mathbb{Z}\mathbb{A}_\infty/r)_0$ is a subset of T_0, and the vertices $\bar{1},..., \bar{r}$ are just the simple regular vertices of T. Let $s=\sum_{i=1}^{r} f(\bar{i})$.

Let us construct a quiver Q of type $\tilde{\mathbb{A}}_{r+s}$ as follows: the vertices of Q are given by the pairs (u,v), where $1 \leqslant u \leqslant r$ and $1 \leqslant v \leqslant f(\bar{u})$, and there are arrows $(u,v) \to (u,v-1)$, for $1 < v \leqslant f(\bar{u})$ and all $\bar{u}$, and also arrows $(u, f(\bar{u})) \to (u+1, 1)$, for all $1 \leqslant u < r$, and $(r, f(\bar{r})) \to (1,1)$. Since $f(\bar{i}) \geqslant 2$ for at least one i, the path algebra $A=kQ$ is finite-dimensional. We define indecomposable representations $E(i)$, $1 \leqslant i \leqslant r$, as follows: let $E(i)_{(u,v)} = k$ for $i = u$, and 0, otherwise, with maps $E(i)_{(i,v)} \to E(i)_{(i,v-1)}$ being the identity for $1 < v \leqslant f(\bar{i})$, and all other maps being zero. Note that the representations $E(i)$ are regular, and belong to a fixed component T'' which is a tube. In fact, they are all the simple regular representations of T'', and, by construction, the length of $E(i)$ is equal to $f(\bar{i})$.

According to the theorem, there exists a tubular extension B of A such that the component of B containing the representations $E(i)$ is of the form T. The canonical length function on T coincides with f on the simple regular vertices of T, thus on all of T.

4.3. Given an artin algebra B, let us recall the definition of the quiver $Q(B)$ of B. Its vertices are the isomorphism classes of simple B-modules, and given two simple B-modules S, T, there is an arrow $[S] \to [T]$ if and only if $\operatorname{Ext}^1(S, T) \neq 0$. Also recall that a Serre subcategory $\mathscr{S}$ of the module category ${}_B\mathscr{M}$ is a full subcategory closed under submodules, factor modules and extensions; note that $\mathscr{S}$ is uniquely determined by the set of simple modules contained in $\mathscr{S}$. A Serre subcategory uniquely determines a two-sided ideal I of B such that $\mathscr{S}$ is the category of all B/I-modules. Actually, the ideal I is generated by some idempotent e. We will say that a subcategory $\mathscr{S}$ is *convex*, provided it is a Serre subcategory and the set of simple modules contained in $\mathscr{S}$ is path closed (given a path $S_0 \to S_1 \to \cdots \to S_n$ in the quiver of B, with S_0, S_n both in $\mathscr{S}$, then all S_i belong to $\mathscr{S}$). If $\mathscr{S}$ is a convex subcategory of ${}_B\mathscr{M}$, and M is an arbitrary B-module, then we denote by $M|_{\mathscr{S}}$ the restriction of M to $\mathscr{S}$; it is a subquotient M'/M'' of M belonging to $\mathscr{S}$, with $M'' \subseteq M' \subseteq M$, such that neither M/M' nor M'' has any composition factor in $\mathscr{S}$. Note that $M|_{\mathscr{S}}$ is (up to isomorphism) uniquely determined by $\mathscr{S}$. Finally, it is easy to see that for $\mathscr{S}$ being convex, and I and e as before, the algebra $A = (1-e)B(1-e)$ satisfies $B = I \oplus A$; thus we may consider $\mathscr{S}$ as the category of all A-modules.

Also, given a class $\mathscr{S}$ of B-modules, let $\mathfrak{l}(\mathscr{S})$ denote the class of indecomposable B-modules X satisfying $\operatorname{Hom}(X, S) = 0$, for all $S \in \mathscr{S}$, and $\mathfrak{r}(\mathscr{S})$ the class of all indecomposable B-modules Y satisfying $\operatorname{Hom}(S, Y) = 0$ for all $S \in \mathscr{S}$.

Now, assume that B is a tubular extension of a tame concealed algebra A. By construction, A is a factor algebra of B, and the full subcategory ${}_A\mathscr{M}$ of all A-modules in the category ${}_B\mathscr{M}$ is a convex subcategory. Given a B-module M, we denote its restriction to ${}_A\mathscr{M}$ just by $M|_A$. We also consider the factor algebras B^+ and B^- of B, where ${}_{B^+}\mathscr{M}$ is the Serre subcategory of ${}_B\mathscr{M}$ generated by the A-modules and those simple B-modules for which the injective envelope belongs to $\mathscr{C}(B, A)$, and similarly, ${}_{B^-}\mathscr{M}$ is the Serre subcategory of ${}_B\mathscr{M}$ generated by the A-modules and those simple B-modules with projective cover belonging to $\mathscr{T}(B,A)$. Note that also ${}_{B+}\mathscr{M}$ and ${}_{B-}\mathscr{M}$ are convex subcategories of ${}_B\mathscr{M}$.

THEOREM. *Let B be a tubular extension of a tame concealed algebra A. Then the indecomposable B-modules fall into three pairwise disjoint classes: namely, $\mathscr{P}(B,A) := \mathbf{r}(\mathscr{T}(B,A))$, $\mathscr{T}(B,A)$, and $\mathscr{Q}(B,A) := \mathbf{l}(\mathscr{T}(B,A))$. If $X \in \mathscr{Q}(B,A)$, $Y \in \mathscr{P}(B,A)$, then $\mathrm{Hom}(X,Y)=0$, and, for $X \in \mathscr{P}(B,A)$, $Y \in \mathscr{Q}(B,A)$, any homomorphism $X \to Y$ factors through a direct sum of modules belonging to $\mathscr{T}(B,A)$. Any $M \in \mathscr{P}(B,A)$ is a B^+-module, and $M|_A$ is a preprojective A-module. Any $M \in \mathscr{Q}(B,A)$ is a B^--module, and $M|_A$ is a preinjective A-module. Finally, for $M \in \mathscr{T}(B,A)$, the restriction $M|_A$ is either an indecomposable regular A-module or zero.*

Proof. The proof is by induction. If $B=A$, then $\mathscr{T}(A,A)$ is the class of indecomposable regular modules, $\mathbf{r}(\mathscr{T}(A,A))$ is the class of indecomposable preprojective A-modules, and $\mathbf{l}(\mathscr{T}(A,A))$ is the class of indecomposable preinjective A-modules.

Now assume the theorem is true for a tubular extension B of A, and assume C is obtained from B by a simple tubular extension using the ray module ${}_BV$ of $\mathscr{T}(B,A)$, say $C = B[V,n]$. Let $D = \mathrm{End}(V)$, and consider the D-vectorspace category ${}_D\mathscr{F} = {}_D\mathrm{Hom}_B({}_BV, {}_B\mathscr{M})$. We denote the functor ${}_D\mathrm{Hom}_B({}_BV, -)\colon {}_B\mathscr{M} \to {}_D\mathscr{F}$ by F. The indecomposable object ${}_DF(V(i))$ is one-dimensional both as a left D-vectorspace, as well as a right $\mathrm{End}(F(V(i)))$-vectorspace, with μ_i being a generator. Note that for any indecomposable B-module ${}_BW$, not isomorphic to any $V(j)$, with $j<i$, the object $F(W)$ in ${}_D\mathscr{F}$ is generated by $F(V(i))$. (Namely, any homomorphism $\varphi \in F(W)$ factors through μ_i, according to Lemma 1 in 2.7.). Let B' be the path algebra over D of the quiver

$$\underset{2}{\circ}\longrightarrow\underset{3}{\circ}\longrightarrow\underset{4}{\circ}\ \cdots\ \underset{n-1}{\circ}\longrightarrow\underset{n}{\circ}\ ,$$

and V' the indecomposable projective-injective B'-module. The D-vectorspace category $\mathrm{Hom}_{B\times B'}(V \times V', {}_{B\times B'}\mathscr{M})$ is the direct sum category of ${}_D\mathscr{F}$ and ${}_D\mathscr{F}' = \mathrm{Hom}_{B'}(V', {}_{B'}\mathscr{M})$, and ${}_D\mathscr{F}'$ is the additive category generated over D by a linearly ordered set of $n-1$ elements. Now C is the one-point extension of $B \times B'$ using the module $V \times V'$; see [11]; thus the indecomposable C-modules ${}_CM$ not belonging to ${}_{B\times B'}\mathscr{M}$ correspond bijectively to the indecomposable objects $(U, \mathrm{Hom}_{B\times B'}(V \times V', X \times X'), \varphi)$ in $\mathscr{U}({}_D\mathscr{F} \amalg {}_D\mathscr{F}')$ with $U \neq 0$, and such that no indecomposable direct summand of $X \times X'$ belongs to $\mathbf{r}(V \times V')$. Given such an object in $\mathscr{U}({}_D\mathscr{F} \amalg {}_D\mathscr{F}')$, either $X = V(i)$ for some i, and then $U = {}_DD$, X' is indecomposable and ${}_CM$ belongs to the component of C containing V or else all indecomposable summands of X belong to $\mathscr{Q}(B, A)$. Let $\mathscr{T}(C, A)$ denote the class of indecomposable C-modules which are either in $\mathscr{T}(B, A)$ or correspond to an object in $\mathscr{U}({}_D\mathscr{T} \amalg {}_D\mathscr{T}')$ of the form $({}_DD, \operatorname{Hom}_{B \times B'}(V \times V', V(i) \times X'), \varphi)$. Similarly, let $\mathscr{Q}(C, A)$ denote the class of indecomposable C-modules which are either in $\mathscr{Q}(B, A)$ or correspond to some $(U, \operatorname{Hom}_{B \times B'}(V \times V', X \times X'), \varphi)$ with $U \neq 0$ and X a direct sum of modules in $\mathscr{Q}(B, A)$. Finally, let $\mathscr{P}(C, A) = \mathscr{P}(B, A)$. It follows that $\operatorname{Hom}({}_CM, {}_CN) = 0$ for $M \in \mathscr{T}(C, A)$, $N \in \mathscr{P}(C, A)$, and also for $M \in \mathscr{Q}(C, A)$ and $N \in \mathscr{P}(C, A) \cup \mathscr{T}(C, A)$.

On the other hand, if $\operatorname{Hom}({}_CM, {}_CN) = 0$ for all $M \in \mathscr{T}(C, A)$, then N cannot belong to $\mathscr{T}(C, A)$ nor to $\mathscr{Q}(B, A)$. Now assume ${}_CN$ does not belong to $\mathscr{P}(C, A)$, thus it corresponds to some indecomposable $(U, \operatorname{Hom}_{B \times B'}(V \times V', X \times X'), \varphi)$ with $U \neq 0$ and X a direct sum of modules in $\mathscr{Q}(B, A)$. Then $X = 0$ since otherwise there is some $Y \in \mathscr{T}(B, A) \subseteq \mathscr{T}(C, A)$ with $\operatorname{Hom}_B(Y, X) \neq 0$. But then there is a nonzero homomorphism from $({}_DD, \operatorname{Hom}_{B \times B'}(V \times V', V \times V'), \delta)$ with $\delta\colon D \to D \times D$ being the diagonal map, to $(U, \operatorname{Hom}_{B \times B'}(V \times V', 0 \times X'), \varphi)$, and $({}_DD, \operatorname{Hom}_{B \times B'}(V \times V', V \times V'), \delta)$ corresponds to the C-module $(V, D, \ldots, D; 1, \ldots, 1)$ in $\mathscr{T}(C, A)$. This shows that $\mathscr{P}(C, A) = \mathbf{r}(\mathscr{T}(C, A))$. For the proof that $\mathscr{Q}(C, A) = \mathbf{l}(\mathscr{T}(C, A))$, we only have to note that a C-module ${}_CM$ with $\operatorname{Hom}({}_CM, {}_CN) = 0$ for all ${}_CN \in \mathscr{T}(C, A)$ cannot belong to $\mathscr{T}(C, A)$, and also not to $\mathscr{P}(C, A) = \mathscr{P}(B, A)$, since $\mathscr{T}(B, A) \subseteq \mathscr{T}(C, A)$.

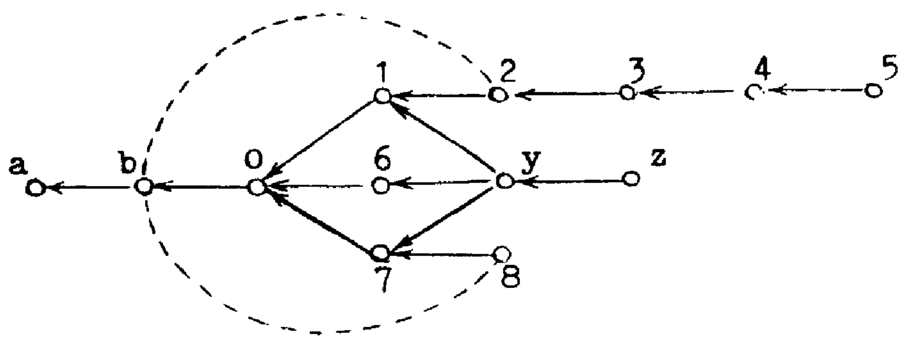
Now assume, ${}_CN \in \mathscr{P}(C, A)$, ${}_CM \in \mathscr{Q}(C, A)$, and $f\colon {}_CN \to {}_CM$ is a homomorphism. We want to show that f factors through a direct sum of modules in $\mathscr{T}(C, A)$. Let M' be the largest submodule of M which is a B-module. Note that M just corresponds to some object $(U, \operatorname{Hom}_{B \times B'}(V \times V', M' \times X'), \varphi)$ in $\mathscr{U}({}_D\mathscr{T} \amalg {}_D\mathscr{T}')$, thus, by definition, M' is a direct sum of modules in $\mathscr{Q}(B, A)$. Since ${}_CN$ is in fact a B-module, f factors through M', and, by induction, the induced map $N \to M'$ factors through a direct sum of modules in $\mathscr{T}(C, A)$.

Finally, we note that $C^+ = B^+$ and $C^- = B^-[V|B^-, n]$, thus all modules in $\mathscr{P}(C, A)$ are C^+-modules, and all modules in $\mathscr{Q}(C, A)$ are C^--modules. This finishes the proof in case C is a simple tubular extension of B by a ray module in $\mathscr{T}(B, A)$. The case of C being a cosimple tubular extension of B by a coray module in $\mathscr{T}(B, A)$ is dual.

4.4. In view of the preceding theorem, let us introduce the following definition: a set $\mathscr{T}$ of indecomposable modules is said to be *separating*, provided the indecomposable modules not belonging to $\mathscr{T}$ fall into two disjoint classes $\mathscr{P}$ and $\mathscr{Q}$ such that $\operatorname{Hom}(X, Y) = 0$ for $X \in \mathscr{Q}$, $Y \in \mathscr{P}$, also for $X \in \mathscr{Q}$, $Y \in \mathscr{T}$, and for $X \in \mathscr{T}$, $Y \in \mathscr{P}$, and such that, moreover, any map $X \to Y$ with $X \in \mathscr{P}$, $Y \in \mathscr{Q}$ factors through a direct sum of modules in $\mathscr{T}$. (Usually, these classes $\mathscr{P}$ and $\mathscr{Q}$ are uniquely determined by $\mathscr{T}$.) In case $\mathscr{T}$ are the modules belonging to a family of tubes, and $\mathscr{T}$ is separating, then

$\mathscr{T}$ will be said to be a *separating tubular series*. The above theorem asserts, in particular, that given a tubular extension B of a tame concealed algebra A, the set $\mathscr{T}(B, A)$ is a separating tubular series.

Also, we have seen that for a tubular extension B of a tame concealed algebra A, any indecomposable B-module not belonging to the separating tubular series $\mathscr{T}(B, A)$ is either a B^+-module or a B^--module. Let us consider as an example the path algebra B of the following quiver with all



commutativity relations and the two indicated zero relations. It is a tubular extension of the path algebra A of the full subquiver given by the vertices 0, 1,..., 8. Here, B^+ is given by the vertices 0, 1,..., 8, a, b, and B^- is given by the vertices 0, 1,..., 8, y, z. Note that both B^+ and B^- are wild (B^+ contains the wild subquiver given by a, b, 0, 1, 6, 7, whereas B^- contains the wild subquiver given by 1, 6, 7, 8, y, z). The tubular series $\mathscr{T}(B, A)$ contains several indecomposable sincere representations, namely, the following: let V be the simple regular A-module with $V_i = k$ for $i = 0, 1, 6, 7$, and zero otherwise; now given an indecomposable regular A-module $W \not\simeq V$ with regular socle and regular top both of the form V, there is precisely one indecomposable sincere B-module $\hat{W}$ with restriction to A being W; it satisfies $\hat{W}_i = k$ for $i = a$, b, y, z, and these are all the indecomposable sincere B-modules.

We also note the following: suppose B is a tubular extension of a tame concealed algebra A. If we consider a one-point extension by a module X in $\mathscr{Q}(B, A)$, or a one-point coextension by a module Y in $\mathscr{P}(B, A)$, and also if we repeat these processes, the family $\mathscr{T}(B, A)$ will remain the union of certain components which are tubes and will remain to have the separation property of a separating tubular series.

5. COHERENT TUBES AS COMPONENTS OF ALGEBRAS

5.1. Let $T = (T_0, T_1, \tau)$ be a coherent tube which is the component of an algebra A. Any vertex x of T is in fact an isomorphism class of indecomposable A-modules, and we will denote a representative of x by M_x, and similarly, for $\alpha: x \to y$ an arrow of T, let $f_\alpha: M_x \to M_y$ be some irreducible map. Thus, we deal with a certain representation $\mathscr{M} = (M_x, f_\alpha)$ of the quiver (T_0, T_1) in the category of A-modules. Note that we have shown in Section 3 that the rank of T satisfies $r(T) > 0$ since there is given the canonical length function. In particular, we also have $p > 0$, $q > 0$, where (p, q) is the type of T.

PROPOSITION. *Let T be a coherent tube which is the component of an algebra A. Let $(\alpha_1, \ldots, \alpha_n)$ be a path with all arrows $\alpha_i: x_{i-1} \to x_i$, $1 \leqslant i \leqslant n$, pointing to the mouth.*

(a) *If x_n belongs to a regular ray, or to no ray, then the composition $f_{\alpha_1} \cdots f_{\alpha_n}$ is an epimorphism.*

(b) *If x_n belongs to a ray, and x_0 belongs to a regular ray, whereas no x_i, $1 \leqslant i \leqslant n$, belongs to a regular ray, then the composition $f_{\alpha_1} \cdots f_{\alpha_n}$ is a monomorphism.*

The proof will be given in the next section.

5.2. Given a vertex v on the mouth of a tube T such that the ray $[v, \infty)$ exists, we have introduced in 2.5 the translation quiver $\bar{T}$ which is obtained from T by cutting along $[v, \infty)$. Given a vertex w in $\bar{T}$, let $\Delta = \Delta(v, w)$ be the full translation subquiver of $\bar{T}$ with Δ_0 being the set of all vertices $y \in \bar{T}_0$ with a path from y to w. Obviously, Δ_0 is a finite set, and the underlying space $|\Delta|$ of Δ is contractible. We usually will deal with the following situation: there is a chain of arrows (pointing to the mouth)

$$w(-n) \xrightarrow{\beta_n} w(-n+1) \xrightarrow{\beta_{n-1}} \cdots \xrightarrow{\quad} w(-1) \xrightarrow{\beta_1} w(0) = w$$

such that $w(-n) = \underline{v}(m)$ for some $n, m \in \mathbb{N}_0$.

We may lift the representation $\mathscr{M} = (M_x, f_\alpha)$ of T to a representation of $\bar{T}$ (with $M_{\underline{v}(i)} = M_{\bar{v}(i)} = M_{v(i)}$, and so on), and then we may consider the restriction to Δ. However, we would like to deal with a commutative representation $\mathscr{M}^c_\Delta = (M^c_x, f^c_\alpha)$ which coincides on all β_i with the given representation (such that $|M^c_x| = |M_x|$ and with all f^c_α being irreducible). This is easily established by keeping, in addition to the f_{β_i}, also certain f_α, where α comes from an arrow in T pointing to infinity, and forming pullbacks, in order to obtain the remaining M^c_x and f^c_α. We say that $\mathscr{M}^c_\Delta$ is derived from $\mathscr{M}$.

Proof of Proposition 5.1(a). Let $(\alpha_1, \ldots, \alpha_n)$ be a proper path of arrows $\alpha_i: x_{i-1} \to x_i$, $1 \leqslant i \leqslant n$ all pointing to the mouth, and such that x_n either belongs to a regular ray or to no ray. In both cases, we note that the coray $(\infty, x_n|$ exists, and it will contain vertices lying on regular rays. Without loss of generality we may assume that x_0 lies on a regular ray (otherwise, extend the path to the left and renumber the arrows). First, assume that x_n belongs to a regular ray $[z, \infty)$, say $x_n = z(m-1)$ for some $m \in \mathbb{N}_1$. We may suppose that no x_i with $1 \leqslant i < n$ is regular (otherwise, we divide the path into two paths $(\alpha_1, \dots, \alpha_i)$, $(\alpha_{i+1}, \dots, \alpha_n)$, and use induction). Since τz and z both belong to the reduced tube T^r, there exists the following diagram

$$\begin{tikzcd}
x_{00} \arrow[r] \arrow[d] & x_{01} \arrow[r] \arrow[d] & \cdots \arrow[r] & x_{0,n-1} \arrow[d] & \\
x_{10} \arrow[r] \arrow[d] & x_{11} \arrow[r] \arrow[d] & \cdots \arrow[r] & x_{1,n-1} \arrow[r] \arrow[d] & x_{1n} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
x_{m0} \arrow[r] & x_{m1} \arrow[r] & \cdots \arrow[r] & x_{m,n-1} \arrow[r] & x_{m.n}
\end{tikzcd} \tag{$*$}$$

of vertices and arrows from T^r, where all horizontal arrows point to the mouth, all vertical arrows point to infinity, and $x_{0,n-1} = \tau z$, $x_{1n} = z$ and finally $x_{mi} = x_i$ for all $0 \leqslant i \leqslant n$. Similarly, in case x_n belongs to no ray, choose a path of maximal length of arrows pointing to infinity $x_{1n} \to \cdots \to x_{mn} = x_n$. Then x_{1n} lies on the mouth, and it cannot be projective since x_{1n} does not lie on a ray, thus the extension $\tau x_{1n} \lrcorner x_{1n}$ lies on the mouth. Let $x_{0,n-1} = \tau x_{1n}$, and note that all the corays $(\infty, x_{0,n-1}]$ and $(\infty, x_{jn}]$, $1 \leqslant j \leqslant m$ exist. Thus, in both cases we have the diagram $(*)$ with $x_{0,n-1} \lrcorner x_{1n}$ lying on the mouth. By assumption, there is a regular ray $[v, \infty)$ containing x_0 (and therefore all x_{j0}, $0 \leqslant j \leqslant m$). We consider now the corresponding diagram in $\bar{T}$. (This is possible, since by assumption, those rays $[x_{0i}, \infty)$, $0 \leqslant i \leqslant n-1$ which exist are pairwise nonequivalent. Note that in case $r(T) = 1$, the rays $[x_0, \infty)$ and $[x_n, \infty)$ are equivalent; in this case, some of the vertices x_{in} may be of the form $\bar{v}(j)$; however, we will denote the diagram in $\bar{T}$ using the same letters as before.) Let $\Delta = \Delta(v, x_{mn})$, and consider a commutative representation $\mathcal{M}^c_\Delta$ derived from $\mathcal{M}$. Since $\mathcal{M}^c_\Delta$ is commutative, all the small squares of $(*)$ are cartesian squares, and we add an additional vertex x_{0n}, with $M^c_{x_{0n}} = 0$, and with zero maps $M^c_{x_{0,n-1}} \to M^c_{x_{0n}}$, $M^c_{x_{0n}} \to M^c_{x_{1n}}$, in order to obtain an additional cartesian square. In this way, $(*)$ becomes a large square, which is cartesian. Since the upper morphism $M^c_{x_{00}} \to M^c_{x_{0n}} = 0$ is surjective, the same is true for the lower morphism $M^c_{x_{m0}} \to M^c_{x_{mn}}$, and this map is just the composition $f_{\alpha_1} \cdots f_{\alpha_n}$.

Proof of Proposition 5.1(b). Now, let $(\alpha_1, \dots, \alpha_n)$ be a path, with all arrows $\alpha_i : x_{i-1} \to x_i$, $1 \leqslant i \leqslant n$, pointing to the mouth, such that x_0 belongs to a regular ray $[v, \infty)$, say $x_0 = v(m)$, and such that no other vertex x_i, with $1 \leqslant i \leqslant n$, belongs to a regular ray. We use induction on n. Choose $i \in \mathbb{N}_0$

minimal such that there exists a path $(\gamma_1,\dots,\gamma_n)$ of length n of arrows $\gamma_j\colon x_{i,j-1}\to x_{ij}$ pointing to the mouth, with $x_{i0}=v(i)$, as well as a path of length $m-i$ with arrows pointing to infinity from x_{in} to x_n. We claim that one of the vertices x_{ij}, $1\leqslant j\leqslant n$ has to be projective. Namely, assume no x_{ij}, $1\leqslant j\leqslant n$ is projective. Since x_{i1} is not projective, the arrow $\tau x_{i1}\to x_{i0}$ is an arrow pointing to infinity and ending in $x_{i0}=v(i)$. Thus $i>0$ since $v(0)=v$ generates a regular ray. Also, the extension $\tau x_{in}\rceil\!\lfloor x_{in}$ belongs to the mouth, since otherwise there is an arrow $\delta\colon x_{i-1,n}\to x_{in}$ pointing to infinity, and thus we obtain both a path $(\tau\gamma_2,\dots,\tau\gamma_n,\sigma\delta)$ from $v(i-1)$ to $x_{i-1,n}$ with arrows pointing to the mouth as well as a path of length $m-i+1$ with arrows pointing to infinity, from $x_{i-1,n}$ to x_n, thus contradicting the minimality of i. Since v belongs to a circuit, the coray $(\infty, v|$ exists. By assumption, the ray $|x_n,\infty)$ exists, thus also $|x_{in},\infty)$ exists. Now the coray $(\infty, v|$ and the ray $|x_{in},\infty)$ intersect, and, in this way, we obtain a closed path containing τx_{in} and x_{in}, as follows: we go from v to $v(i-1)$ along the ray $|v,\infty)$, then from $v(i-1)$ to τx_{in} using $(\tau\gamma_2,\dots,\tau\gamma_n)$, then along $\sigma\gamma_n$ and γ_n to x_{in}, finally along the ray $|x_{in},\infty)$ up to an intersection point with the coray $(\infty, v|$ and back to v along this coray. This shows that τx_{in} and x_{in} belong to the reduced tube T^r, and since $\tau x_{in}\rceil\!\lfloor x_{in}$ lies on the mouth, $|x_{in},\infty)$ is a regular ray. Since this contradicts the assumption that x_n does not lie on a regular ray, it follows that one of the vertices x_{ij}, $1\leqslant j\leqslant n$, is projective.

We fix j maximal such that x_{ij} is projective. First, we consider the case $j<n$. Let $x_{i-1,r}=\tau x_{i,r+1}$ for $j\leqslant r<n$. There also may be an arrow ending in x_{in} and pointing to infinity. In case it exists, its starting point will be denoted by $x_{i-1,n}$. Thus, we obtain the following diagram of vertices and arrows from T (the two dotted arrows and the vertex $x_{i-1,n}$ may or may not exist):

$$\begin{tikzcd}
 & x_{i-1,j} \arrow[r] \arrow[d, "\delta_{ij}"] & \cdots \arrow[r] & x_{i-1,n-1} \arrow[r, dotted] \arrow[d] & x_{i-1,n} \\
x_{i,j-1} \arrow[r, "\alpha_{ij}"] \arrow[d] & x_{i,j} \arrow[r] \arrow[d] & \cdots \arrow[r] & x_{i,n-1} \arrow[r, "\alpha_{in}"] \arrow[d] & x_{i,n} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
x_{m,j-1} \arrow[r, "\alpha_{mj}"] & x_{m,j} \arrow[r] & \cdots \arrow[r] & x_{m,n-1} \arrow[r, "\alpha_{mn}"] & x_{m,n}
\end{tikzcd} \tag{**}$$

Here, all horizontal arrows point to the mouth, all vertical arrows point to infinity, and $x_{mr}=x_r$, $\alpha_{mr}=\alpha_r$ and $\alpha_{ir}=\gamma_r$. The horizontal map ending in x_{ij} is denoted by α_{ij}, the vertical by δ_{ij}. Consider now the corresponding diagram in $\bar{T}$ (this is possible since the rays $[x_i, \infty)$ are pairwise inequivalent), and the finite translation subquiver $\Delta = \Delta(v, x_{mn})$. Let $\mathscr{M}^c_\Delta$ be a commutative representation derived from $\mathscr{M}$. In case there does not exist an arrow ending in x_{in} and pointing to infinity, let $M^c_{x_{i-1,n}} = 0$ and add zero maps $M^c_{x_{i-1,n-1}} \to M^c_{x_{i-1,n}}$, and $M^c_{x_{i-1,n}} \to M^c_{x_{in}}$. In this way, the whole diagram $(**)$, including the dotted arrows, is composed from small squares all of which are, up to sign, cartesian. We want to show that the composition $f_{\alpha_j} \cdots f_{\alpha_n} = f^c_{\alpha_{mj}} \cdots f^c_{\alpha_{mn}}$ is a monomorphism. Using part of the diagram $(**)$, we see that it is enough to show that $f^c_{\alpha_{ij}} \cdots f^c_{\alpha_{in}}$ is a monomorphism. Now $M^c_{x_{ij}}$ is a projective module, with radical being given by the direct sum of the images of $f^c_{\alpha_{ij}}$ and $f^c_{\delta_{ij}}$, both being monomorphisms. Thus, it is enough to show that the image of $\mu \colon U \to M^c_{x_{ij}}$ (the canonical inclusion of the kernel of $f^c_{\alpha_{i,j+1}} \cdots f^c_{\alpha_{in}}$) is contained in the image of $f^c_{\delta_{ij}}$. However, again using part of the diagram $(**)$, we have the following cartesian square

$$\begin{tikzcd}
M^c_{x_{i-1,j}} \arrow[r, "f^c_{\alpha_{i-1,j+1}} \cdots f^c_{\alpha_{i-1,n}}"] \arrow[d, "f^c_{\delta_{ij}}"] & M^c_{x_{i-1,n}} \arrow[d, "f^c_{\delta_{in}}"] \\
M^c_{x_{ij}} \arrow[r, "f^c_{\alpha_{i,j+1}} \cdots f^c_{\alpha_{i,n}}"] & M^c_{x_{in}}.
\end{tikzcd}$$

Using the pullback property, we factor the pair $(\mu, 0) \colon U \to M^c_{x_{ij}} \oplus M^c_{x_{i-1,n}}$ through $M^c_{x_{i-1,j}}$, thus we obtain $\mu' \colon U \to M^c_{x_{i-1,j}}$ such that $\mu = \mu' f^c_{\delta_{ij}}$. This shows that in case $j < n$, the map $f_{\alpha_j} \cdots f_{\alpha_n}$ is a monomorphism. In case $j = n$, the same result follows more easily: $f_{\alpha_{ij}}$ is a monomorphism since $M^c_{x_{ij}}$ is projective, thus also $f^c_{\alpha_{mj}}$ is a monomorphism, using the corresponding cartesian square. Thus, always $f_{\alpha_j} \cdots f_{\alpha_n}$ is a monomorphism. By induction, we also have that $f_{\alpha_1} \cdots f_{\alpha_{j-1}}$ is a monomorphism. This finishes the proof.

5.3. Recall that Auslander and Smalø [1] have introduced the notion of a preprojective module over a general artin algebra: An indecomposable module M is said to be *preprojective* if and only if there exists a proper submodule M' of M such that there are only finitely many pairwise nonisomorphic indecomposable modules N_i having a homomorphism $N_i \to M$ with image not contained in M'.

THEOREM. *Let T be a component of A which is a coherent tube, and let M be an indecomposable module in T. Then M is preprojective if and only if M belongs to a ray, but not to a regular ray, and M is preinjective if and only if M belongs to a coray, but not to a regular coray.*

Proof. Let M be an indecomposable module with $w = |M|$ belonging to T_0. First, assume w belongs to a regular ray, or to no ray at all. In both cases there exists the coray $(\infty, w|$

$$\cdots \to w(-i) \xrightarrow{\beta_i} w(-i+1) \xrightarrow{\beta_{i-1}} \cdots \xrightarrow{\beta_2} w(-1) \xrightarrow{\beta_1} w(0) = w.$$

Choose indecomposable modules $M(-i)$ with $[M(-i)] = w(-i)$, and irreducible maps $f_i : M(-i) \to M(-i+1)$. By the proposition, all the compositions $f_i \cdots f_1$, for all $i \in \mathbb{N}_1$, are surjective. This shows that M cannot be preprojective.

Now, assume that w does not belong to a regular ray, but to a ray. If w does not belong to a coray, then there are only finitely many paths ending in w; thus there are only finitely many indecomposable modules having a path of irreducible maps to M, and M is preprojective. Thus, we can assume that the coray $(\infty, w|$ exists. Let n be the smallest number in $\mathbb{N}_1$ such that $w(-n)$ belongs to a regular ray, say to $|v, \infty)$, thus $w(-n) = v(m)$ for some $m \in \mathbb{N}_0$. Choose indecomposable modules $M(-i)$ with $|M(-i)| = w(-i)$, $1 \leqslant i \leqslant n$, and irreducible maps $f_i : M(-i) \to M(-i+1)$. Let $\mathcal{N}$ be the set of vertices x in T which have a path $x = x_0 \to x_1 \to \cdots \to x_t = w$ from x to w such that no x_i belongs to $[v, \infty)$. We claim that for any indecomposable module N with $[N] \notin \mathcal{N}$, any homomorphism $N \to M$ factors through $f_n \cdots f_1$. Since $\mathcal{N}$ is finite, and $f_n \cdots f_1$ is a monomorphism according to Proposition 5.1, this implies that M is preprojective.

Let $\Delta = \Delta(v, w)$, and choose a commutative representation $\mathscr{M}^c_\Delta = (M^c_x, f^c_\alpha)$ derived from $\mathscr{M}$. Note that for the vertices z of Δ which are not of the form $\underline{v}(i)$, the maps

$$\bigoplus_{y \in z^-} M^c_y \xrightarrow{(f^c_{yz})} M^c_z$$

are minimal right almost split. Now assume there is given an indecomposable module N, with $[N] \notin \mathcal{N}$, and a homomorphism $g : N \to M$. Factorising g through the various minimal right almost split maps inside Δ, we can write g as a sum of maps $g'g''$ with $g' : N \to M^c_{\underline{v}(i)}$, where $0 \leqslant i \leqslant n$, and $g'' : M^c_{\underline{v}(i)} \to M$ being a composition of irreducible maps of the form f^c_α, with $\alpha \in \Delta_1$. However, since $\mathscr{M}^c_\Delta$ is commutative, we see that g'' factors through $f_n \cdots f_1$, thus g factors through $f_n \cdots f_1$. This finishes the proof of the first assertion of the theorem. The second assertion follows by duality.

5.4. Given a coherent tube T, in view of the preceding theorem, it seems to be natural to call a maximal nonregular ray a *preprojective* ray, a maximal nonregular coray a *preinjective* coray. The number of preprojective rays is $q(T) - r(T)$, the number of preinjective corays is $p(T) - r(T)$. We may reformulate the preceding theorem as follows: in case T is the component of an artin algebra A, then an indecomposable A-module M in T is preprojective if and only if $|M|$ belongs to a preprojective ray, and preinjective if and only if $|M|$ belongs to a preinjective coray. We also mention two obvious consequences:

COROLLARY 1. *Let T be a component of A which is a coherent tube, and M an indecomposable A-module in T. Then M is neither preprojective nor preinjective if and only if $|M|$ is a regular vertex of T.*

COROLLARY 2. *Let T be a component of A which is a coherent tube. There are infinitely many isomorphism classes of indecomposable modules belonging to T which are both preprojective and preinjective if and only if both $p(T) > r(T)$ and $q(T) > r(T)$.*

REFERENCES

1. M. AUSLANDER AND S. O. SMALØ, Preprojective modules over artin algebras, *J. Algebra* **66** (1980), 61–122.
2. K. BONGARTZ AND P. GABRIEL, Covering spaces in representation theory, *Invent. Math.* **65** (1982), 331–378.
3. D. BÜNERMANN, Auslander–Reiten–Köcher von einparametrigen teilweise geordneten Mengen, Dissertation, Düsseldorf, 1981.
4. V. DLAB AND C. M. RINGEL, Indecomposable representation of graphs and algebras, *Mem. Amer. Math. Soc.* **173** (1976).
5. D. HAPPEL, U. PREISER, AND C. M. RINGEL, Vinberg's Characterization of Dynkin Diagrams Using Subadditive Functions with an Application to D Tr-Periodic Modules, Lecture Notes in Mathematics No. 832, pp. 280–294, Springer-Verlag, Berlin/Heidelberg/New York, 1980.
6. D. HAPPEL AND C. M. RINGEL, "Tilted Algebras," *Trans. Amer. Math. Soc.* **274** (1982), 399–443.
7. D. HAPPEL AND C. M. RINGEL, "Construction of Tilted Algebras," Proceedings ICRA III, Lecture Notes in Mathematics, No. 903, Springer-Verlag, Berlin/Heidelberg/New York, 1982.
8. CHR. RIEDTMANN, Algebren, Darstellungsköcher, Überlagerungen und zurück, *Comm. Math. Helv.* **55** (1980), 199–224.
9. C. M. RINGEL, Representations of K-species and bimodules, *J. Algebra* **41** (1976), 269–302.
10. C. M. RINGEL, "Report on the Brauer–Thrall Conjectures: Rojter's Theorem and the Theorem of Nazarova–Rojter," Lecture Notes in Mathematics No. 831, pp. 104–136, Springer-Verlag, Berlin/Heidelberg/New York, 1980.
11. C. M. RINGEL, "Tame Algebras," Lecture Notes in Mathematics No. 831, pp. 137–287, Springer-Verlag, Berlin/Heidelberg/New York, 1980.