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AUSLANDER-REITEN SEQUENCES WITH FEW MIDDLE TERMS AND APPLICATIONS TO STRING ALGEBRAS

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dedicated to Maurice Auslander on his 60 th. birthday.

In the famous paper [AR-III] Auslander and Reiten introduced what now are called Auslander-Reiten sequences, and one consequence has been the definition of several numerical invariants both of individual modules and of artin algebras. Let A be such an algebra. Given an Auslander Reiten sequence

$$0 \longrightarrow X \longrightarrow \bigoplus_{i=1}^{r} Y_i \longrightarrow Z \longrightarrow 0 ,$$

with all Y_i indecomposable, the number $r = \alpha(Z)$ may be called the <u>number of middle terms</u>, and is defined for all indecomposable non-projective modules. Viewed as a function, α was considered by Auslander and Reiten in [AR-0] where they defined $\alpha(A)$ to be the supremum of $\alpha(Z)$ over all indecomposable non-projective

Z. In fact α is always bounded. For any A-module M, denote its length by $|M|$.

Lemma. Let $p = \max|P|$, where P runs through the indecomposable projective A-modules, let $q = \max|Q|$, where Q runs through the indecomposable injective A-modules. For any indecomposable non-projective A-module Z,

$$\alpha(Z) \leq (pq+1)^2.$$

Proof. Consider an Auslander-Reiten sequence as displayed above. We have $|X| \leq pq|Z|$, and $|Z| \leq (pq+1)|Y_i|$, for all i, see [Ri2]. Thus

$$r|Z| \leq (pq+1) \sum_{i=1}^{r} |Y_i| = (pq+1)(|X|+|Z|) \leq (pq+1)^2|Z|,$$

therefore $\alpha(Z) = r \leq (pq+1)^2$.

In [AR-IV], Auslander and Reiten asked whether every non-semisimple artin algebra has Auslander-Reiten sequences with just one middle term, and Martinez-Villa [M] gave an affirmative answer to this question. In section 1, we are going to exhibit explicitly a class of indecomposable non-projective modules V with $\alpha(V) = 1$.

Let Γ be a component of the Auslander-Reiten quiver Γ_A of A. We say that Γ is regular provided there is no projective and no injective module whose isomorphism class belongs to Γ. In section 2, we show that any regular component Γ with

$\alpha(Z) = 2$ for every module Z whose isomorphism class belongs to Γ, is of the form $\mathbb{Z}A_\infty^\infty$ or $\mathbb{Z}C_\infty$. This will be derived from a purely combinatorial characterization of $\mathbb{Z}A_\infty^\infty$.

Section 3 is devoted to a special class of algebras, the string algebras (in the terminology of [SW], these are the special biserial algebras whose indecomposable modules which are both projective and injective, are serial). It is well-known that a string algebra A satisfies $\alpha(A) \leq 2$. There are two methods known for obtaining a complete description of the Auslander-Reiten sequences of a string algebra. One method is based on the calculation of the indecomposable modules due to Gelfand-Ponomarev [GP]. It was developed in [BSh] for two special cases; and the general case has been treated in [WW]: first, one determines the Auslander-Reiten translate, and then the corresponding Auslander-Reiten sequences. The second method is based on covering theory, see [SW] and [DS]. Our aim is to demonstrate that the Gelfand-Ponomarev technique is well suited to showing that certain maps between indecomposable modules are irreducible, and that, in this way, one obtaines essentially all irreducible maps, and therefore also all Auslander-Reiten sequences.

1. Auslander-Reiten sequences with just one middle term.

Let $J = \text{rad } A$, and let e,f be primitive idempotents in A. Given a non-zero element $a \in fJe$, the A-module Ae/Aa is indecomposable and non-projective, and we consider the Auslander-Reiten sequence ending in Ae/Aa. Our main interest lies in the case

$a \in fJe \setminus fJ^2e$, and we show that under this assumption $\alpha(Ae/Aa) = 1$, see corollary 1 below. (Note that if A is given by a quiver Γ and relations, then any arrow of Γ gives, in this way, an indecomposable module V with $\alpha(V) = 1$, and different arrows give non-isomorphic modules; thus the number of Auslander-Reiten sequences with just one middle term is at least the number of arrows of Γ.) This result was inspired by the work of K. Erdmann [E], who noted, and used extensively, a special case. More generally, we consider non-zero elements $a \in fJe$ which are "non-supportive" in the sense of the following definition:

A non-zero element $a \in fJe$ is called non-supportive provided any indecomposable direct summand C of Je/Aa satisfies $C \cap (\mathrm{soc}(Ae/Ja))\pi \neq 0$, where $\pi : Ae/Ja \longrightarrow Ae/Aa$ is the canonical projection.

Theorem. Assume that $0 \neq a \in fJe$ is non-supportive. Then $\alpha(Ae/Aa) = 1$.

Proof. Let $V(a) = Ae/Aa$. By construction, there is the minimal projective presentation

$$Af \xrightarrow{.a} Ae \longrightarrow V(a) \longrightarrow 0$$

where $.a$ denotes the right multiplication by a. It follows that $U(a) := D\,\mathrm{Tr}\,V(a)$ is a submodule of the indecomposable injective module $D(fA)$; in particular, $U(a)$ has simple socle. Denote by

$$0 \longrightarrow U(a) \xrightarrow{\mu} N(a) \xrightarrow{\varepsilon} V(a) \longrightarrow 0$$

an Auslander-Reiten sequence ending in $V(a)$. If $V(a)$ is simple, then soc $N(a)$ lies in the kernel of ε (otherwise, ε would split), thus soc $N(a)$ = soc $U(a)$ is simple and $N(a)$ is indecomposable. Assume now that $V(a)$ is not simple. Let $N(a) = \bigoplus_{i=1}^{t} N_i$ with all N_i indecomposable, $\mu = [\mu_1,\ldots,\mu_t]$, $\varepsilon = [\varepsilon_1,\ldots,\varepsilon_t]^T$, where $\mu_i : U(a) \longrightarrow N_i$, $\varepsilon_i : N_i \longrightarrow V(a)$. Since $U(a)$ has simple socle, one of the μ_i has to be mono, say let μ_1 be mono. Let $C = \bigoplus_{i=2}^{t} N_i$, $p = [\mu_2,\ldots,\mu_t] : U(a) \longrightarrow C$, and $q = [\varepsilon_2,\ldots,\varepsilon_t]^T : C \longrightarrow V(a)$. With μ_1 also q is mono. Since $V(a)$ has a unique maximal submodule, it follows that ε_1 is epi, and therefore also p is epi. Since $U(a)$ has simple socle, we conclude that p vanishes on soc $U(a)$. We consider the canonical projection $\pi : Ae/Ja \longrightarrow Ae/Aa$. Since $\varepsilon = \begin{bmatrix}\varepsilon_1\\ q\end{bmatrix}$ is a sink map, there exists $[\pi_1,\pi_1'] : Ae/Ja \longrightarrow N_1 \oplus C$ with $\pi = \pi_1\varepsilon_1 + \pi_1'q$. Now, π is surjective, q maps into rad $V(a)$, thus $\pi_1\varepsilon_1$ is surjective. The kernel of π is Aa/Ja, let $\iota : Aa/Ja \longrightarrow Ae/Ja$ be the inclusion map. Note that there exists $\pi' : Aa/Ja \longrightarrow U(a)$ satisfying $\pi'[\mu_1,p] = \iota[\pi_1,\pi_1']$, since $0 = \iota\pi = \iota[\pi_1,\pi_1']\begin{bmatrix}\varepsilon_1\\ q\end{bmatrix}$. However, Aa/Ja is simple, and p vanishes on soc $U(a)$, thus $\iota\pi_1' = \pi'p = 0$. It follows that $\iota\pi_1\varepsilon_1 = \iota(\pi-\pi_1'q) = 0$. The length of Ae/Ja exceeds the length of $V(a)$ by 1, therefore

$$0 \longrightarrow Aa/Ja \xrightarrow{\iota} Ae/Ja \xrightarrow{\pi_1\varepsilon_1} V(a) \longrightarrow 0$$

is exact, thus there exists an automorphism η of $V(a)$ such that $\pi = \pi_1\varepsilon_1\eta$. Let $\varepsilon_1' = \varepsilon_1\eta$, and $q' = q\eta$. Since η is an automorphism, also the sequence

$$0 \longrightarrow U(a) \xrightarrow{[\mu_1\ p]} N_1 \oplus C \xrightarrow{\begin{bmatrix}\varepsilon_1' \\ q'\end{bmatrix}} V(a) \longrightarrow 0$$

is an Auslander-Reiten sequence. Note that ε_1' is epi, $\pi_1\varepsilon_1' = \pi$, and q' is mono. Given a map $f : X \longrightarrow Y$, we denote by soc f the induced map soc $X \longrightarrow$ soc Y. The functor soc is left exact, thus

$$0 \longrightarrow \operatorname{soc} U(a) \xrightarrow{[\operatorname{soc} \mu_1, 0]} \operatorname{soc} N_1 \oplus \operatorname{soc} C \xrightarrow{\begin{bmatrix}\operatorname{soc} \varepsilon_1' \\ \operatorname{soc} q'\end{bmatrix}} \operatorname{soc} V(a)$$

is exact. Actually, since we started with an Auslander-Reiten sequence, and soc $V(a)$ is a proper submodule of $V(a)$, it follows that $\begin{bmatrix}\operatorname{soc} \varepsilon_1' \\ \operatorname{soc} q'\end{bmatrix}$ is also epi. This shows that

$$\operatorname{soc} V(a) = (\operatorname{soc} N_1)\varepsilon_1' \oplus (\operatorname{soc} C)q'.$$

Since $\pi_1\varepsilon_1' = \pi$, we see that $(\operatorname{soc}(Ae/Ja))\pi \subseteq (\operatorname{soc} N_1)\varepsilon_1'$, thus $\operatorname{soc}(Cq') = (\operatorname{soc} C)q'$ intersects $(\operatorname{soc}(Ae/Ja))\pi$ trivially, consequently $C' \cap (\operatorname{soc}(Je/Ja))\pi = 0$ for any direct summand C' of Cq'. On the other hand, q' is an irreducible monomorphism with image contained in rad $V(a) = Je/Aa$, thus the image Cq' of q' is a direct summand of Je/Aa. Since we assume that a is non-supportive, it follows that Cq', and therefore C is zero. Thus $N(a)$ is indecomposable.

In order to apply the theorem, we have to exhibit non-supportive elements.

Lemma. Any element in $fJe \setminus fJ^2e$ is non-supportive.

Proof. Let $a \in fJe \setminus fJ^2e$, and consider the exact sequence

$$0 \longrightarrow Aa/Ja \longrightarrow Je/Ja \xrightarrow{\bar{\pi}} Je/Aa \longrightarrow 0$$

given by the inclusion $Aa \subseteq Je$, and the restriction $\bar{\pi}$ of π to Je/Ja. Since $a \notin J^2$, we see that the simple submodule $Aa/Ja \approx Af/Jf$ is not contained in $J(Je/Ja) = \text{rad}(Je/Ja)$, thus the sequence splits. It follows that $\text{soc}(Je/Aa)$ is the image of $\text{soc}(Je/Ja) = \text{soc}(Ae/Ja)$ under π, thus any indecomsposable direct summand of Je/Aa intersects $(\text{soc}(Ae/Ja))\pi = \text{soc}(Je/Aa)$ non-trivially.

Corollary 1. Let $a \in fJe \setminus fJ^2e$. Then $\alpha(Ae/Aa) = 1$.

Corollary 2. Let E be a simple non-projective A-module. Then there exists an Auslander-Reiten sequence $0 \to X \to Y \to Z \to 0$ with Y indecomposable and $\text{Hom}(Z,E) \neq 0$.

Proof. We have $E = Ae/Je$ for some primitive idempotent e. Since E is non-projective, $Je \neq 0$, thus J^2e is a proper submodule of Je. Choose some primitive idempotent f with $fJ^2e \neq fJe$. Let $a \in fJe \setminus fJ^2e$, and $Z = Ae/Aa$.

In case A is representation finite, this corollary was obtained by Auslander and Reiten in [AR-IV]. Of course, there is the following consequence.

Corollary 3. (Martinez-Villa [M]): Any artin algebra with non-zero radical has an Auslander-Reiten sequence with just one middle term.

The set of Auslander-Reiten sequences
$0 \longrightarrow U(a) \longrightarrow N(a) \longrightarrow V(a) \longrightarrow 0$ which we obtain using Corollary 1, with $V(a) = Ae/Aa$, is closed under duality. By definition, a module is of the form $V(a)$ if and only if it is the cokernel of a map φ between indecomposable projective A-modules such that φ is irreducible inside the category of projective A-modules. The dual description just furnishes a characterization of the module $U(a)$.

2. Regular components with α bounded by 2.

Let $\Gamma = (\Gamma_o, \Gamma_1, \tau, a)$ be a stable valued translation quiver (without multiple arrows or loops). [This means the following (see [HPR]): (Γ_o, Γ_1) is a locally finite quiver without multiple arrows or loops, $\tau : \Gamma_o \longrightarrow \Gamma_o$ is a bijective map such that $x^- = (\tau x)^+$ for any $x \in \Gamma_o$; here, x^- denotes the set of starting points of arrows with end point x, and x^+ the set of end points of arrows with starting point x. Finally, $a : \Gamma_1 \longrightarrow \mathbb{N}_1 \times \mathbb{N}_1$ is a function, the image of the arrow $x \longrightarrow y$ under a will be denoted by (a_{xy}, a'_{xy}), and this function satisfies $a_{\tau x,y} = a'_{yx}$.] Given a vertex $z \in \Gamma$, define $\alpha(z) = \sum_{y \in z^-} a'_{yz}$. Typical examples of stable valued translation quivers are the regular components C of the Auslander-Reiten quiver Γ_A of the artin algebra A. By abuse of language, we say that an indecomposable A-module M belongs to $\mathcal{C}$ provided the isomorphism class $[M]$ of M is a vertex of $\mathcal{C}$. The valuation $a : \mathcal{C}_1 \longrightarrow \mathbb{N}_1 \times \mathbb{N}_1$ is defined as follows: given two indecomposable A-modules M,N in $\mathcal{C}$, let $\mathrm{Irr}(M,N) = \mathrm{rad}(M,N)/\mathrm{rad}^2(M,N)$, and, if $\mathrm{Irr}(M,N) \neq 0$, denote by $a_{MN} = a_{[M][N]}$ its length as an $\mathrm{End}(N)$-module, by $a'_{MN} = a'_{[M][N]}$ its length as an $\mathrm{End}(M)$-module. In fact a'_{MN} is equal to the multiplicity of M occuring as middle term in the Auslander-Reiten sequence ending in N. We conclude that for any indecomposable A-module Z in $\mathcal{C}$, we have $\alpha(Z) = \alpha([Z])$.

Theorem. Let $\mathcal{C}$ be a regular component of Γ_A, and assume α is bounded by 2 on $\mathcal{C}$. Then $\mathcal{C}$ is of the form $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ (for some $n \in \mathbb{N}_1$), $\mathbb{Z}C_\infty$, or $\mathbb{Z}A_\infty^\infty$.

For the shape of the valued translation quivers $\mathbb{Z}A_\infty$, $\mathbb{Z}C_\infty$ and $\mathbb{Z}A_\infty^\infty$ we may refer to [HPR]. The proof of the theorem will be given in this section. We will need some additional terminology.

Let Γ be a stable valued translation quiver. An additive function f on Γ is a map $f : \Gamma_o \longrightarrow \mathbb{Z}$ satisfying

$$f(x) + f(\tau x) = \sum_{y \in x^-} f(y)a'_{yx}$$

for all vertices x of Γ. Such a function is said to be positive provided it takes values in $\mathbb{N}_1$, and unbounded provided it takes arbitrarily large values. Also, f is said to be of bounded growth provided there is $c \in \mathbb{N}_1$ with $\frac{1}{c}f(x) \leq f(y) \leq cf(x)$ for every arrow $x \longrightarrow y$. Obviously, if $\mathcal{C}$ is a regular component of Γ_A, then the length function ℓ (which sends $[M]$ to $\ell([M]) = |M|$)

is a positive additive function on C, it is of bounded growth (for c we can take $c = pq+1$ using the notation of the first lemma, see [Ri2]), and an important result of Auslander [A] asserts that ℓ is also unbounded.

According to Riedtmann [Rm], any stable translation quiver is of the form $\mathbb{Z}\Delta/G$, where Δ is an oriented tree and G an admissable automorphism group. In the same way, any stable valued translation quiver is of the form $\mathbb{Z}\Delta/G$, where Δ is a valued oriented tree and G is an admissable automorphism group, see [HPR]. Thus, consider a stable valued translation quiver $\Gamma = \mathbb{Z}\Delta/G$, where Δ is a valued oriented tree and G an admissable automorphism group. Suppose we have $\alpha(x) \leq 2$ for all vertices x of Γ. Then also $\alpha(x) \leq 2$ for the vertices x of $\mathbb{Z}\Delta$, therefore the Cartan class of Δ is one of A_n, C_n, $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{C}_n$, A_∞, C_∞, or A_∞^∞. There are no positive additive functions on $\mathbb{Z}A_n$ and $\mathbb{Z}C_n$, since A_n and C_n are Dynkin diagrams; also, any positive additive function on $\mathbb{Z}\tilde{A}_{11}$, $\mathbb{Z}\tilde{A}_{12}$ and $\mathbb{Z}\tilde{C}_n$ is bounded, since $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{C}_n$ are Euclidean diagrams, see [HPR]. Thus, if we suppose that there exists an unbounded positive additive function on Γ, then $\Delta = A_\infty$, C_∞, or A_∞^∞. In case $\Delta = A_\infty$ and C_∞, the automorphisms are just powers of τ. For $\Delta = A_\infty$, both $\mathbb{Z}A_\infty$ itself, as well as all proper quotiens $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ (with $n \in \mathbb{N}_1$) are known to be realisable as regular components of Γ_A, where A is a suitable artin algebra. For $\Delta = C_\infty$, only $\mathbb{Z}C_\infty$ itself can be realized in this way. For, it has been shown in [HPR] that $\mathbb{Z}C_\infty/\langle\tau^n\rangle$ (with $n \in \mathbb{N}_1$) does not admit an unbounded positive additive function.

It remains to consider $\Delta = A_\infty^\infty$. We recall that a path $x_o, x_1, \ldots, x_n$ in Γ is said to be <u>sectional</u> provided $x_{i-1} \neq \tau x_{i+1}$ for all $1 \leq i < n$ (and cyclic, provided $n \geq 1$, and $x_o = x_n$). According to Bautista and Smalø [BSm], there is no sectional cyclic path in Γ_A.

<u>Proposition</u>. Let G be an admissable group of automorphisms of $\mathbb{Z}A_\infty^\infty$, and let $\Gamma = \mathbb{Z}A_\infty^\infty/G$. Assume that there is no sectional cyclic path in Γ and that there exists a positive additive function on Γ which is unbounded and of bounded growth. Then $G = \{1\}$.

<u>Proof</u>. Assume $G \neq \{1\}$. Note that G contains no non-trivial element of finite order, since G is supposed to be admissable. We introduce coordinates in $\tilde{\Gamma} = \mathbb{Z}A_\infty^\infty$ as follows: as set of vertices of $\mathbb{Z}A_\infty^\infty$, we take the set of pairs $(x,y) \in \mathbb{Z}^2$ with $x \equiv y$ (mod 2), and with arrows $(x,y) \longrightarrow (x+1,y+1)$, $(x,y) \longrightarrow (x+1,y-1)$. Let g' be a non-trivial element of G, let $g'(0,0) = (x_o', y_o')$. Then $g'(1,1)$ is either $(x_o'+1, y_o'+1)$ or $(x_o'+1, y_o'-1)$. In the first case, let $g = g'$, in the second, let $g = (g')^2$. Denote $g(0,0) = (x_o, y_o)$ and note that $g(1,1) = (x_o+1, y_o+1)$, thus g is the translation of $\mathbb{Z}A_\infty^\infty$ by (x_o, y_o). Since g has infinite order, $(x_o, y_o) \neq (0,0)$, and we can assume $x_o + y_o \geq 0$ (otherwise, replace g by g^{-1}). We have $x_o+y_o \neq 0$, $x_o - y_o \neq 0$, since otherwise we obtain a sectional cyclic path in Γ. Consider the vertices $v = \frac{1}{2}(x_o+y_o, x_o+y_o)$, $w = \frac{1}{2}(x_o-y_o, -x_o+y_o)$ in $\mathbb{Z}A_\infty^\infty$. Note that v and w are linearly independent, and $v+w = (x_o, y_o) =$

$g(0,0)$. Consider the sublattice U of $\mathbb{Z}^2$ generated by v, w. We can consider U as a stable translation quiver: let two vertices u,u' be joined by an arrow $u \longrightarrow u'$ in U iff $u-u'$ is equal to v or w, and define $\tau_U u = u-v-w$. By assumption, there exists an additive function on Γ with various properties, and we may lift it to a function on $\mathbb{Z}A_\infty^\infty$. Thus, there is given a positive additive G-invariant function f on $\mathbb{Z}A_\infty^\infty$ which is unbounded and of bounded growth. We consider the restriction f_U of f to U. It is easy to see that f_U is additive on U. Since f is G-invariant, it follows that f_U is τ_U-invariant. But U itself is of the form $\mathbb{Z}A_\infty^\infty$, and, according to [HPR], any positive additive τ-invariant function on $\mathbb{Z}A_\infty^\infty$ is constant. Thus f_U is constant, say $f(u) = d$ for any $u \in U$. On the other hand, any vertex of $\mathbb{Z}A_\infty^\infty = \tilde{\Gamma}$ can be joined by a path of length $\leq x_o+y_o-1$ with a vertex in U. By assumption, f is of bounded growth, say $f(a) \leq cf(b)$ for any arrow $a \longrightarrow b$. Thus, if a is an arbitrary vertex of $\tilde{\Gamma}$, then $f(a) \leq c^t d$, with $t = x_o+y_o-1$. This contradicts the assumption that f is unbounded.

3. String algebras

If Q is a quiver, and β is an arrow of Q, let $s(\beta)$ be its starting point and $e(\beta)$ its end point. Recall that the category of representations of Q is the category of kQ^*-modules, where kQ^* is the path algebra of the opposite quiver of Q. A path in Q^* is of the form $\beta_1 \cdots \beta_n$, where β_i are arrows in Q with $s(\beta_i) = e(\beta_{i+1})$ for $1 \leq i < n$ and a zero relation on Q

is just given by a path of length ≥ 2. Let P be a set of zero relations in kQ^*, and $\langle P \rangle$ the ideal generated by P. Then $A = kQ^*/\langle P \rangle$ is called a <u>string algebra</u> provided the following conditions are satisfied:

(1) Any vertex of Q is starting point of at most two arrows.

(1*) Any vertex of Q is end point of at most two arrows.

(2) Given an arrow β, there is at most one arrow γ with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin P$.

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(3) Given an arrow β, there is some bound $n(\beta)$ such that any path $\beta_1 \ldots \beta_{n(\beta)}$ with $\beta_1 = \beta$ contains a subpath in P.

(3*) Given an arrow β, there is some bound $n'(\beta)$ such that any path $\beta_1 \ldots \beta_{n'(\beta)}$ with $\beta_{n'(\beta)} = \beta$ contains a subpath in P.

Note that we do not require Q to be finite, thus kQ^* may be an algebra without identity, but it always has sufficiently many primitive idempotents. Consequently, A has sufficiently many primitive idempotents e, and Ae and eA are always finite-dimensional.

In order to deal with the indecomposable modules for a string algebra, we need the following notation: Given an arrow β of Q, denote by β^{-1} a formal inverse for β, with $s(\beta^{-1}) := e(\beta)$ and $e(\beta^{-1}) := s(\beta)$, and write $(\beta^{-1})^{-1} = \beta$. We form "paths" $c_1 \ldots c_n$ of length $n \geq 1$, where the c_i are of the form β or β^{-1} and where $s(c_i) = e(c_{i+1})$ for $1 \leq i < n$, and we define

$(c_1 \ldots c_n)^{-1} = c_n^{-1} \ldots c_1^{-1}$, and $s(c_1 \ldots c_n) = s(c_n)$, $e(c_1 \ldots c_n) = e(c_1)$. A path $c_1 \ldots c_n$ of length $n \geq 1$ is called a <u>string</u> provided $c_{i+1} \neq c_i^{-1}$ for all $1 \leq i < n$, and no subpath $c_i c_{i+1} \ldots c_{i+t}$ nor its inverse belongs to P. In addition, we also want to have strings of length 0; by definition, for any vertex u of Q, there will be <u>two</u> strings of length 0, denoted by $1_{(u,1)}$ and $1_{(u,-1)}$, with both $s(1_{(u,t)}) = u$ and $e(1_{(u,t)}) = u$, for $t = -1, 1$, and we define $1_{(u,i)} = 1_{(u,-i)}$. In order to define the possible compositions, we choose quite arbitrarily two functions $\sigma, \varepsilon : Q_1 \longrightarrow \{-1, 1\}$ with the following properties:

(a) If $\beta_1 \neq \beta_2$ are arrows with $s(\beta_1) = s(\beta_2)$, then $\sigma(\beta_1) = -\sigma(\beta_2)$.

(b) If $\gamma_1 \neq \gamma_2$ are arrows with $e(\gamma_1) = e(\gamma_2)$, then $\varepsilon(\gamma_1) = -\varepsilon(\gamma_2)$.

(c) If β, γ are arrows with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin P$, then $\sigma(\beta) = -\varepsilon(\gamma)$.

[In practice, one may proceed as follows: Choose some vertex u. We are going to define $\sigma(\beta)$ for the arrows β with $s(\beta) = u$ and $\varepsilon(\gamma)$ for the arrows γ with $e(\gamma) = u$. In case there are arrows β_o, γ_o with $s(\beta_o) = u = e(\gamma_o)$ and $\beta_o\gamma_o \notin P$, choose such a pair and let $\sigma(\beta_o) = 1$ and $\varepsilon(\gamma_o) = -1$, then use (a) or (b) in order to define $\sigma(\beta)$ and $\varepsilon(\gamma)$ for the remaining arrows β and γ with $s(\beta) = u$, and $e(\gamma) = u$; note that the condition (c) will be satisfied automatically. In case there are no arrows β_o, γ_o with $s(\beta_o) = u = e(\gamma_o)$ and $\beta_o\gamma_o \notin P$, we only have to take care of the conditions (a) and (b), so there is no difficulty at all.] We extend the functions σ,ε to all strings as follows: if β is an arrow, let $\sigma(\beta^{-1}) = \varepsilon(\beta)$, $\varepsilon(\beta^{-1}) = \sigma(\beta)$; if $C = c_1\ldots c_n$ is a string of length $n \geq 1$, let $\sigma(C) = \sigma(c_n)$, and $\varepsilon(C) = \varepsilon(c_1)$; finally, define $\sigma(1_{(u,t)}) = -t$ and $\varepsilon(1_{(u,t)}) = t$. If $C = c_1\ldots c_n$ and $D = d_1\ldots d_m$ are strings of length ≥ 1, we say that the composition of C and D is defined provided $c_1\ldots c_n d_1\ldots d_m$ is a string, and write $CD = c_1\ldots c_n d_1\ldots d_m$; also, we say that the composition of $1_{(u,t)}$ and D is defined provided $e(D) = u$, $\varepsilon(D) = t$, and, in this case, let $1_{(u,t)}D = D$; we say that the composition of C and $1_{(u,t)}$ is defined provided $s(C) = u$, $\sigma(C) = -t$, and, in this case, let $C1_{(u,t)} = C$. Note that given arbitrary strings C and D such that CD is defined, then necessarily $\sigma(C) = -\varepsilon(D)$. We define $\mathcal{W}(u,t)$ to be the set of all strings C with $e(C) = u$, $\varepsilon(C) = t$. Thus, $\mathcal{W}(u,t)$ contains besides $1_{(u,t)}$ all strings $C = c_1\ldots c_n$ where either $c_1 = \beta$ is an arrow with $e(\beta) = u$, $\varepsilon(\beta) = t$, or $c_1 = \beta^{-1}$, where β is an arrow with $s(\beta) = u$ and $\sigma(\beta) = t$. Also, let $\mathcal{W}'(u,t)$ be the set of all strings C in $\mathcal{W}(u,t)$ such that, first of all, all powers C^n, $n \in \mathbb{N}_1$, are defined, and second, C itself is not the power of a string of smaller length. Let $\mathcal{W}$ be the set of all strings, and $\mathcal{W}'$ the subset of all strings C which belong to some $\mathcal{W}'(u,t)$. On $\mathcal{W}$, we consider the equivalence relation ρ which identifies every string C with its inverse C^{-1}. On $\mathcal{W}'$, we consider the equivalence relation ρ' which identifies every string $C = c_1\ldots c_n$ in $\mathcal{W}'$ with the cyclically permuted strings

$C_{(i)} = c_i c_{i+1} \dots c_n c_1 \dots c_{i-1}$ and their inverses $c_{(i)}^{-1}$, $1 \leq i \leq n$. We choose a complete set $\underset{\sim}{\mathcal{W}}$ of representatives of $\mathcal{W}$ relative to ρ, and a complete set $\underset{\sim}{\mathcal{W}}'$ of representatives of $\mathcal{W}'$ relative to ρ'.

For any string $C = c_1 \dots c_n$, or $C = 1_{(u,t)}$, we define a representation $M(C)$ of Q as follows: Let $u(i) = e(c_{i+1})$, $0 \leq i < n$, and $u(n) = s(C)$. Given a vertex v of Q, let $I_v = \{i \mid u(i) = v\} \subseteq \{0,1,\dots,n\}$. Then $M(C)_v$ will be a vectorspace of dimension the cardinality of I_v, say with base vectors z_i, $i \in I_v$. If $c_i = \beta$, an arrow, define $\beta(z_{i-1}) = z_i$, if $c_i = \beta^{-1}$, with β an arrow, define $\beta(z_i) = z_{i-1}$, for $1 \leq i \leq n$. If $\gamma : w \longrightarrow w'$ is an arrow and z_j is one of the base vectors of $M(C)_w$ and $\gamma(z_j)$ is not yet defined, let $\gamma(z_j) = o$. Obviously, $M(C)$ is a representation of Q which satisfies the relation in P, and $M(C)$ is called a string module. Note that $M(C)$ and $M(C^{-1})$ always are isomorphic (or equal, since $M(C)$ is defined only up to isomorphism), and $M(1_{(u,t)})$ is the simple representation corresponding to the vertex u.

Now, assume there is given a string $C = c_1 \dots c_n$ which belongs to $\mathcal{W}'$. Let Z be a k-vectorspace, and φ an automorphism of Z (thus, (Z,φ) may be viewed as an $k[T,T^{-1}]$-module where the action of the variable T on Z is given by applying φ). We are going to define a representation $M(C,\varphi)$ of Q. Given a vertex v of Q, let $I'_v = \{i \mid e(c_{i+1}) = v\} \subseteq \{0,1,\dots,n-1\}$, and let $M(C,\varphi)_v$ be the direct sum $\bigoplus_{i \in I'_v} Z_i$ of copies Z_i of Z.

If $c_1 = \beta$ is an arrow, define the action of β on Z_1 by sending $z \in Z_1$ to $\varphi(z) \in Z_o$; if $c_1 = \beta^{-1}$, with β an arrow, define the action of β on Z_o by sending $z \in Z_o$ to $\varphi^{-1}(z) \in Z_1$. For $2 \leq i \leq n$, if $c_i = \beta$ for an arrow β, define the action of β on Z_i to be the identity map from Z_i to Z_{i-1}; if $c_i = \beta^{-1}$ with β an arrow, define the action of β on Z_{i-1} to be the identity map from Z_{i-1} to Z_i. If $\gamma : w \longrightarrow w'$ is an arrow, and the action of γ is not yet defined on some $Z_j \subseteq M(C,\varphi)_w$, let $\gamma|Z_j$ be the zero map. In this way, we obtain a representation $M(C,\varphi)$ of Q which satisfies the relations in P, and $M(C,\varphi)$ is called a <u>band module</u>. Actually, we obtain in this way a faithful functor $M(C,-)$ from $k[T,T^{-1}]$-mod into A-mod. Let $\underset{\sim}{\Phi}$ be a complete set of representatives of indecomposable automorphisms of k-vectorspaces with respect to similarity.

<u>Theorem</u>. The modules $M(C)$, with $C \in \underset{\sim}{W}$, and the modules $M(C,\varphi)$ with $C \in \underset{\sim}{W}'$, $\varphi \in \underset{\sim}{\Phi}$, provide a complete list of indecomposable (and pairwise non-isomorphic) A-modules.

This theorem is essentially due to Gelfand-Ponomarev. In [GP], they have considered a special case, however their proof generalizes without serious difficulties to the general case. In fact, such generalizations were worked out in [Ri1] and then in [DF]. The essential features of the proof, the use of certain functors, will be repeated below and are explained in full detail in [Ri1]. We will show that these functors provide complete control of the irreducible maps in A-mod.

Given a vertex u of Q, let us construct certain subfunctors of the functor A-mod $\longrightarrow$ k-mod which sends the A-module M, or, equivalently, the representation M of Q, to the vectorspace M_u. We will denote the zero subspace of M_u by 0_u. Let $C \in \mathcal{W}(u,t)$ with $s(C) = v$. In case there exists an arrow $\beta : w \longrightarrow v$ with composition $C\beta$ defined, let $C^-(M) = C\beta M_w$; otherwise, let $C^-(M) = C0_v$. In case there exists an arrow $\gamma : v \longrightarrow w$ with $C\gamma^{-1}$ defined, let $C^+(M) = C\gamma^{-1}0_w$; otherwise, let $C^+(M) = CM_v$. It is obvious that $C^-(M) \subseteq C^+(M)$. We define

$$F_{1,c}(M) = (I^+_{(u,-t)} \cap C^+)(M)/[(I^+_{(u,-t)} \cap C^-)+(I^-_{(u,-t)} \cap C^+)](M).$$

In this way, we obtain a functor $F_{1,c}$: A-mod $\longrightarrow$ k-mod. Also, given a string $C \in \mathcal{W}'(u,t)$, let

$$C'(M) = \bigcup_n C^n 0_u, \quad C''(M) = \bigcap_n C^n M_u$$

Then $C'(M) \subseteq C''(M)$, and C induces on $C''(M)/C'(M)$ an automorphism, which we denote by $\varphi_{C,M}$. Let $F_C(M) = (C''(M)/C'(M), \varphi_{C,M})$ as a $k[T,T^{-1}]$-module. Note that we obtain in this way a functor F_C : A-mod $\longrightarrow$ $k[T,T^{-1}]$-mod.

On the other hand, we have embedding functors $S_{1,C}$: k-mod $\longrightarrow$ A-mod, sending the one-dimensional vectorspace to $M(C)$, where C is any string. For $C \in \mathcal{W}'$, we have the embedding functor S_C : $k[T,T^{-1}]$-mod $\longrightarrow$ A-mod which sends (Z,φ) to $M(C,\varphi)$. Denote by I the disjoint union of the set of pairs $(1,C)$ with $C \in \underset{\sim}{\mathcal{W}}$, and the set $\underset{\sim}{\mathcal{W}}'$, thus, for any $i \in I$, there are defined functors $S_i : \mathcal{A}_i \longrightarrow$ A-mod and F_i : A-mod $\longrightarrow \mathcal{A}_i$, where

A_i = k-mod, in case $i = (1,C)$, $C \in \underset{\sim}{W}$, and $A_i = k[T,T^{-1}]$-mod, in case $i \in \underset{\sim}{W}'$. The method of Gelfand and Ponomarev (as presented in [Ri1]) establishes the following result:

Proposition. The functors F_i, S_i $(i \in I)$ satisfy the following:

(i) $F_iS_i \simeq id_{A_i}$, $F_jS_i = 0$ for $i \neq j$.

(ii) The set $\{F_i \mid i \in I\}$ is locally finite and reflects isomorphisms.

(iii) For every M in A-mod, and every $i \in I$, there is a map $\gamma_{i,M} : S_iF_i(M) \longrightarrow M$ such that $F_i(\gamma_{i,M})$ is an isomorphism.

As an immediate consequence of these conditions, one obtains the following:

(iv) for any A-module M, the map $(\gamma_{i,M})_i : \bigoplus_I S_iF_i(M) \longrightarrow M$ is an isomorphism.

In particular, the indecomposable A-modules are of the form $S_i(X)$, with X indecomposable in A_i. Therefore:

(v) if M is an indecomposable A-module, and $i \in I$, then either $F_i(M) = 0$ or else M is isomorphic to $S_iF_i(M)$.

This shows that the theorem formulated above follows directly from the proposition. Also note the following consequence of (ii) and (v):

(vi) if X,Y are in A_i, and $f : S_i(X) \longrightarrow S_i(Y)$ is a map with $F_i(f)$ an isomorphism, then f is an isomorphism.

As a first observation which we are going to present here, we want to show that the functors S_i preserve irreducibility of maps.

Lemma. Let $\mathcal{A},\mathcal{B}$ be Krull-Schmidt categories, and $S : \mathcal{A} \to \mathcal{B}$, $F : \mathcal{B} \to \mathcal{A}$ be additive functors with the following properties:

(1) $FS \simeq id_{\mathcal{A}}$.

(2) If M is indecomposable in $\mathcal{B}$, then either $F(M) = 0$, or else M is isomorphic to $SF(M)$.

(3) If X,Y are in $\mathcal{A}$, and $f : S(X) \longrightarrow S(Y)$ is a map in $\mathcal{B}$ with $F(f)$ an isomorphism in $\mathcal{A}$, then f is an isomorphism.

Then the image of an irreducible map in $\mathcal{A}$ under S is irreducible in $\mathcal{B}$.

Proof. First we show that for M indecomposable in $\mathcal{B}$, either $F(M) = 0$ or else $F(M)$ is indecomposable. Let M be indecomposable in $\mathcal{B}$ and $F(M) \neq 0$. According to (2), we have $M \approx SF(M)$. Assume $F(M) = X_1 \oplus X_2$, with X_1,X_2 both non-zero. Then $SF(M) = S(X_1) \oplus S(X_2)$, and both $S(X_1)$, $S(X_2)$ are non-zero, according to (1). This contradicts the indecomposability of M.

Now, let $\gamma : X \longrightarrow Z$ be irreducible in $\mathcal{A}$. Using condition (1), one immediately observes that $S(\gamma)$ cannot be split mono or split epi. Let $f : S(X) \longrightarrow M$, $g : M \longrightarrow S(Z)$ be maps in $\mathcal{B}$ with $S(\gamma) = fg$. Apply F. Now, $FS(\gamma)$ is irreducible, by (1), and $FS(\gamma) = F(f)F(g)$, thus $F(f)$ is split mono or $F(g)$ is split epi. We consider the case of $F(f)$ being split mono. Decompose

$M = \bigoplus_{j\in J} M_j$, with M_j indecomposable, write $f = (f_j)_{j\in J}$ with $f_j : S(X) \to M_j$. Since $F(f) = (F(f_j))_{j\in J} : FS(X) \longrightarrow \bigoplus_{j\in J} F(M_j)$ is split mono, and all $F(M_j)$ are indecomposable or zero, there is a subset J' of J such that the map $(F(f_j))_{j\in J'} : FS(X) \longrightarrow \bigoplus_{j\in J'} F(M_j)$ is an isomorphism, and, moreover, all $F(M_j)$ are non-zero. According to (2), we can assume $M_j = S(Y_j)$ for some Y_j in $\mathcal{A}$, for all $j \in J'$. Let $Y = \bigoplus_{j\in J'} Y_j$, and apply (3) to the map

$f' = (f_j)_{j\in J'} : S(X) \longrightarrow S(Y) = \bigoplus_{j\in J'} S(Y_j) = \bigoplus_{j\in J'} M_j$: since $F(f')$

is an isomorphism in $\mathcal{A}$, it follows that f' is an isomorphism in $\mathcal{B}$. But this shows that f is split mono. A similar argument in the case that $F(g)$ is split epi shows that g is split epi.

We apply this to the functor $F_C : k[T,T^{-1}]\text{-mod} \to A\text{-mod}$, where $C \in \underset{\sim}{W}'$. Note that in $k[T,T^{-1}]$-mod, any indecomposable module X has an Auslander-Reiten sequence of the form $0 \to X \to Y \to X \to 0$, and that the corresponding Auslander-Reiten quiver is a family of homogeneous tubes. Any Auslander-Reiten sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} X \longrightarrow 0$ in $k[T,T^{-1}]$-mod gives rise to an exact sequence

$$0 \longrightarrow F_C(X) \xrightarrow{F_C(f)} F_C(Y) \xrightarrow{F_C(g)} F_C(X) \longrightarrow 0$$

where both $F_C(f)$ and $F_C(g)$ are irreducible (and where $F_C(X)$ is an arbitrary indecomposable band module). Thus we obtain in this way the Auslander-Reiten sequences for the indecomposable band modules, and we see that for any $C \in \underset{\sim}{W}'$, the set of modules of the form $M(C,\varphi)$ is closed under irreducible maps, and that the corresponding components of Γ_A are all homogeneous tubes.

It remains to consider the string modules. Of course, in this case our previous strategy does not give any information, since for $C \in \underset{\sim}{W}$, the category $A_{1,C}$ = k-mod has no irreducible maps. The key observation is that for any string C there is at most one arrow β with $C\beta$ a string, and at most one arrow γ with $C\gamma^{-1}$ a string (this is clear for strings of length ≥ 1, and the conventions concerning strings of length 0 have been made in order to preserve this property). We say that C starts <u>on a peak</u> provided there is no arrow β with $C\beta$ a string, and that C starts <u>in a deep</u> provided there is no arrow γ with $C\gamma^{-1}$ a string.

Let C,D be strings and β an arrow such that $C\beta D$ is a string. Then there is a canonical embedding of $M(C)$ into $M(C\beta D)$ as follows: let C be of length n, and D of length m, thus $C\beta D$ is of length $n+m+1$. By construction, $M(C\beta D)$ is given by $n+m+2$ base vectors $z_o,\ldots,z_{n+m+1}$ on which A operates according to the shape of $C\beta D$. It is obvious that the subspace with basis $z_o,\ldots,z_{n+1}$ is s submodule and of the form $M(C)$. Of course, the corresponding quotient $M(C\beta D)/M(C)$ is just $M(D)$, and we call the induced map $M(C\beta D) \longrightarrow M(D)$ the canonical projection.

<u>Lemma</u>. Let C be a string, not starting on a peak. Let $\beta_o,\beta_1,\ldots,\beta_r$ be arrows such that $C_h = C\beta_o\beta_1^{-1}\ldots\beta_r^{-1}$ is a string starting in a deep. Then the canonical embedding $M(C) \longrightarrow M(C_h)$ is irreducible.

Proof. Let C be of length n, belonging to $\mathcal{W}(u,t)$, thus C_h is of length $n+r+1$, and also in $\mathcal{W}(u,t)$. Denote by 1^+ and 1^- the functor $1^+ = 1^+_{(u,-t)}$ and $1^- = 1^-_{(u,-t)}$. In addition, we have to consider the functors C_h^-, C_h^+, C^-, C^+, all being subfunctors of the functor A-mod $\longrightarrow$ k-mod given by $N \longmapsto N_u$. Actually, we claim that $C_h^+ = C^-$. In order to see this, let $v = s(\beta_o)$, $w = e(\beta_r)$. Given an A-module N, we have
$C_h^+(N) = C\beta_o\beta_1^{-1}\ldots\beta_r^{-1}N_w = C\beta_oN_v = C^-(N)$.

By construction, $M(C_h)$ is given by base vectors $z_o,\ldots,z_{n+r+1}$, and $M(C)$ may be considered to be the subspace with base vectors $z_o,\ldots,z_n$. However, we prefer to denote the base vectors of $M(C_h)$ by $z'_o,\ldots,z'_{n+r+1}$, so that the canonical map $u : M(C) \longrightarrow M(C_h)$ is given by $z_iu = z'_i$, for $0 \le i \le n$. Note that z_o belongs to $(1^+ \cap C^+)(M(C))$ and not to $[(1^+ \cap C^-)+(1^- \cap C^+)](M(C))$, whereas z'_o belongs to $(1^+ \cap C_h^+)(M(C_h))$ and not to $[(1^+ \cap C_h^-)+(1^- \cap C_h^+)](M(C_h))$. Assume there is given a factorization $u = fg$, where $f : M(C) \longrightarrow N$, $g : N \longrightarrow M(C_h)$, and we consider z_of. Since $z_o \in (1^+ \cap C^+)(M(C))$, we see that $z_of \in (1^+ \cap C^+)(N)$. First, consider the case that z_of does not belong to $[(1^+ \cap C^-)+(1^- \cap C^+)](N)$. In this case, f induces a non-zero map $F_{1,C}(M(C)) \longrightarrow F_{1,C}(N)$, thus f is split mono. Second, assume that z_of belongs to $[(1^+ \cap C^-)+(1^- \cap C^+)](N)$, say $z_of = y_1+y_2$, where $y_1 \in (1^+ \cap C^-)(N)$ and $y_2 \in (1^- \cap C^+)(N)$. The images under g satisfy $y_1g \in (1^+ \cap C^-)(M(C_h))$, $y_2g \in (1^- \cap C^+)(M(C_h))$. Also $z'_o \in (1^+ \cap C_h^+)(M(C_h)) = (1^+ \cap C^-)(M(C_h))$. Since both $z'_o = z_ofg =$

y_1g+y_2g and y_1g belong to $(1^+ \cap C^-)(M(C_h))$, the same is true for y_2g, thus $y_2g \in (1^- \cap C^+)(M(C_h)) \cap (1^+ \cap C^-)(M(C_h)) = (1^- \cap C^-)(M(C_h)) = (1^- \cap C_h^+)(M(C_h))$. Since $z_o' = y_1g+y_2g$ does not belong to $[(1^+ \cap C_h^-)+(1^- \cap C_h^+)](M(C_h))$, it follows that y_1g does not belong to $[(1^+ \cap C_h^-)+(1^- \cap C_h^+)](M(C_h))$. This shows that the map $F_{1,C_h}(g) : F_{1,C_h}(N) \longrightarrow F_{1,C_h}(M(C_h))$ is non-zero, since it sends the residue class of the element $y_1 \in (1^+ \cap C^-)(N) = (1^+ \cap C_h^+)(N)$ to a non-zero residue class in $F_{1,C_h}(M(C_h))$. As a consequence, g is split epi. This finishes the proof.

We say that the string C <u>ends on a peak</u> provided there is no arrow β with $\beta^{-1}C$ a string, and that C <u>ends in a deep</u> provided there is no arrow γ with γC a string. Of course, C ends on a peak iff C^{-1} starts on a peak, and C ends in a deep iff C^{-1} starts in a deep. Also, if C and D are strings and β is an arrow such that $D\beta^{-1}C$ is a string, then there is a canonical embedding $M(C) \longrightarrow M(D\beta^{-1}C)$, and the corresponding quotient $M(D\beta^{-1}C)/M(C)$ is just $M(D)$; the induced map $M(D\beta^{-1}C) \longrightarrow M(D)$ is called the canonical projection. The previous lemma may be reformulated as follows:

<u>Lemma</u>. Let C be a string, not ending on a peak. Let $\beta_o, \beta_1, \ldots, \beta_r$ be arrows such that ${}_hC = \beta_r \ldots \beta_1 \beta_o^{-1} C$ is a string ending in a deep. Then the canonical embedding $M(C) \longrightarrow M({}_hC)$ is irreducible.

Also, using duality, we obtain the following dual versions:

Lemma. Let C be a string, not starting in a deep. Let $\gamma_o,\gamma_1,\ldots,\gamma_r$ be arrows such that ${}_cC = C\gamma_o^{-1}\gamma_1\ldots\gamma_r$ is a string starting on a peak. Then the canonical projection $M(C_c) \longrightarrow M(C)$ is irreducible.

Lemma. Let C be a string, not ending in a deep. Let $\gamma_o,\gamma_1,\ldots,\gamma_r$ be arrows such that ${}_cC = \gamma_r^{-1}\ldots\gamma_1^{-1}\gamma_oC$ is a string ending on a peak. Then the canonical projection $M({}_cC) \longrightarrow M(C)$ is irreducible.

We have now obtained a large number of irreducible maps between string modules, and we are going to show that we have obtained essentially all. We need some additional terminology: a string $C = c_1\ldots c_n$ is said to be direct provided all c_i are arrows, and inverse provided all c_i are inverses of arrows; by definition, the strings of length 0 are both direct and inverse. Note that $M(C)$ is serial (or "uniserial") provided C is direct or inverse.

First, let us consider a vertex u, and let $P(u)$ be the indecomposable projective module corresponding to u. Always, $P(u)$ is a string module, say $P(u) = M(C_1C_2)$ where C_1 is direct, C_2 is inverse, $s(C_1) = u$, and such that C_1C_2 both starts and ends in a deep. If the length of both C_1 and C_2 is zero, then $P(u)$ is simple, thus no irreducible map ends in $P(u)$. Now, suppose $C_1 = \beta_r\ldots\beta_1$ with $r \geq 1$. Then $M(\beta_r\ldots\beta_2)$ is a direct summand of rad $P(u)$ (for $r = 1$, $e(\beta_1) = v$, $\varepsilon(\beta_1) = t$, we mean by $\beta_r\beta_{r-1}\ldots\beta_2$ just $1_{(v,t)}$), and the inclusion map

$M(\beta_r\ldots\beta_2) \longrightarrow M(\beta_r\ldots\beta_2\beta_1\, C_2)$ is just the canonical embedding $M(\beta_r\ldots\beta_2) \longrightarrow M(\beta_r\ldots\beta_2)_h = M(\beta_r\ldots\beta_2\beta_1 C_2)$. Similarly, for $C_2 = \gamma_1^{-1}\ldots\gamma_s^{-1}$ with $s \geq 1$, the string module $M(\gamma_2^{-1}\ldots\gamma_s^{-2})$ is a direct summand of rad $P(u)$, and the corresponding inclusion map is the canonical embedding $M(\gamma_2^{-1}\ldots\gamma_s^{-1}) \longrightarrow {}_hM(\gamma_2^{-1}\ldots\gamma_s^{-1}) = M(C_1\gamma_1^{-1}\gamma_2^{-1}\ldots\gamma_s^{-1})$.

In the same way, the indecomposable injective module corresponding to the vertex u is of the form $M(D_1D_2)$, where D_1 is inverse, D_2 is direct, $s(D_1) = u$, and such that D_1D_2 both starts and ends on a peak. The dual consideration shows that we can write $M(D_1D_2)/\mathrm{soc}\, M(D_1D_2)$ as the direct sum of at most two serial modules and that the corresponding projections are just canonical projections.

Next, we are going to determine the Auslander-Reiten sequences containing string modules. First of all we consider those with just one middle term. Recall that we have constructed, in section 1, for every arrow β in Q an Auslander-Reiten sequence

$$0 \longrightarrow U(\beta) \longrightarrow N(\beta) \longrightarrow V(\beta) \longrightarrow 0,$$

with $N(\beta)$ indecomposable. Here, $N(\beta) = M(B)$, where $B = \gamma_r^{-1}\ldots\gamma_1^{-1}\beta\delta_1^{-1}\ldots\delta_s^{-1}$ starts in a deep and ends on a peak (with γ_i,δ_j being arrows), $U(\beta) = M(\gamma_r^{-1}\ldots\gamma_1^{-1})$, $V(\beta) = M(\delta_1^{-1}\ldots\delta_s^{-1})$, the map $U(\beta) \longrightarrow N(\beta)$ is the canonical embedding, and the map $N(\beta) \longrightarrow V(\beta)$ is the canonical projection. Note that $B = (\gamma_r^{-1}\ldots\gamma_1^{-1})_h = {}_c(\delta_1^{-1}\ldots\delta_s^{-1})$, and that $C = \gamma_r^{-1}\ldots\gamma_1^{-1}$ is a string which ends on a peak, but does not start on a peak (whereas $\delta_1^{-1}\cdots\delta_s^{-1}$ is a string which starts in a deep but does not end in a deep). We call this Auslander-Reiten sequence a canonical exact sequence.

Consider now a string C with neither $M(C)$ injective, nor isomorphic to any $U(\beta)$. First, assume C neither starts nor ends on a peak, thus both ${}_hC$ and C_h, and also ${}_hC_h (= {}_h(C_h) = ({}_hC)_h)$ are defined. The exact sequence

$$0 \longrightarrow M(C) \xrightarrow{[u\ u]} M({}_hC) \oplus M(C_h) \xrightarrow{\begin{bmatrix} u \\ -u \end{bmatrix}} M({}_hC_h) \longrightarrow 0$$

(here, and in the following, we denote a canonical embedding by u, a canonical projection by p) will be called a canonical exact sequence. Next, assume C does not start on a peak but ends on a peak, thus C_h is defined. Since C is not inverse (otherwise $M(C)$ would be of the form $U(\beta)$), we can write C in the form $C = \gamma_r^{-1}\cdots\gamma_1^{-1}\gamma_0 D$ with $r \geq 0$, thus $C = {}_cD$. Of course, D itself does not start on a peak, thus also D_h is defined, and there is the following exact sequence

$$0 \longrightarrow M(C) \xrightarrow{[p\ u]} M(D) \oplus M(C_h) \xrightarrow{\begin{bmatrix} u \\ -p \end{bmatrix}} M(D_h) \longrightarrow 0,$$

called a canonical exact sequence. In case C starts on a peak, but does not end on a peak, we similarly write $C = D_c$ for some D, and have the following exact sequence which is called canonical:

$$0 \longrightarrow M(C) \xrightarrow{[u\ p]} M({}_hC) \oplus M(D) \xrightarrow{\begin{bmatrix} p \\ -u \end{bmatrix}} M({}_hD) \longrightarrow 0.$$

Finally, suppose C both starts and ends on a peak. Since $M(C)$ is not injective, C is not of the form $C = C_1C_2$ with C_1 inverse, C_2 direct, thus C is of the form
$C = \gamma_r^{-1}\cdots\gamma_1^{-1}\gamma_o D\beta_o^{-1}\beta_1\cdots\beta_s$ with $r,s \geq 0$, thus $C = {}_cD_c$, and there is the following exact sequence

$$0 \longrightarrow M(C) \xrightarrow{[p\ p]} M(D_c) \oplus M({}_cD) \xrightarrow{\begin{bmatrix} p \\ -p \end{bmatrix}} M(D) \longrightarrow 0,$$

again called canonical.

Proposition ([WW,SW,DS]). The canonical exact sequences are the Auslander-Reiten sequences containing string modules.

Proof. We only have to verify that the two maps occuring in the exact sequence are irreducible. This is clear in case the middle term is indecomposable (of course, in this case, we also may use the general result of section 1, but we do not need this), and it is also clear in case there are two middle terms which are not isomorphic (for if $f_i : X \longrightarrow Y_i$ are irreducible maps, $i = 1,2$, with X, Y_1, Y_2 indecomposable, and Y_1, Y_2 not isomorphic, then $[f_1,f_2] : X \longrightarrow Y_1 \oplus Y_2$ is irreducible, and there is a dual assertion). Thus, we only have to consider the case where C neither starts nor ends on a peak and $M({}_hC) \approx M(C_h)$, and the dual case where D neither starts nor ends in a deep and $M(D_c) \approx M({}_cD)$. We treat the first case in detail , the second follows by duality:

Lemma. Let β_o, β_1 be arrows with

(*) $\beta_o \neq \beta_1$, $e(\beta_o) = e(\beta_1)$, $s(\beta_o) = s(\beta_1)$, and $\gamma\beta_i \in P$ for all arrows γ, and $i = 1,2$.

Let $C = (\beta_o\beta_1^{-1})^s$ for some $s \geq o$ (where $(\beta_o\beta_1^{-1})^o = 1_{(u,t)}$ provided $\beta_o \in W(u,t)$). Then ${}_hC = (\beta_o\beta_1^{-1})^{s+1} = C_h$, the space $Irr(M(C),M(C_h))$ is two-dimensional, and a basis of $Irr(M(C),M(C_h))$ is given by the (residue classes of the) canonical embeddings $M(C) \longrightarrow M(C_h)$ and $M(C) \longrightarrow M({}_hC)$.

Conversely, assume C is a string which neither starts nor ends on a peak, and assume $M({}_hC) \approx M(C_h)$. Then there are arrows β_o,β_1 satisfying (*) such that $C = (\beta_o\beta_1^{-1})^s$ for some $s \geq o$.

<u>Proof</u>. First, let β_o, β_1 be arrows satisfying (*), and $C = (\beta_o\beta_1^{-1})^s$, for some $s \geq o$. In order to show that the residue classes of the two canonical embeddings $M(C) \longrightarrow M(C_h)$ and $M(C) \longrightarrow M({}_hC)$ are linearly independent in $Irr(M(C),M(C_h))$, we may restrict to the subquiver given by the arrows β_o, β_1 (and the vertices $s(\beta_o)$, $e(\beta_o)$), thus we deal either with the Kronecker quiver (in case $s(\beta_o) \neq e(\beta_o)$), or with the algebra $k[X,Y]/(X^2,Y^2,XY)$ (in case $s(\beta_o) = e(\beta_o)$). However, both algebras and their representations are well-known.

For the converse, let C be a string which neither starts nor ends on a peak, and assume $M({}_hC) \approx M(C_h)$. We claim that we must have ${}_hC = C_h$. [Otherwise, $C_h = ({}_hC)^{-1} = (C^{-1})_h$, thus $C = C^{-1}$. However, if the length of C is odd, the middle letters of C and of C^{-1} have different exponent, whereas for C of even length

$\neq o$, the two middle letters would be of the form $\beta\beta^{-1}$ or $\beta^{-1}\beta$ with β an arrow, and this is excluded. Finally, our conventions for strings of length zero imply also in this case $C \neq C^{-1}$.]

Let ${}_hC = \gamma_q\dots\gamma_1\gamma_o^{-1}C$, and $C_h = C\beta_o\beta_1^{-1}\dots\beta_r^{-1}$. Since ${}_hC = C_h$, we have $r = q$. Also, $r \geq 1$. [Otherwise, C starts with γ_o^{-1}, since C_h does; but then C starts with $\gamma_o^{-1}\gamma_o^{-1}$, and so on, this giving a contradiction.] First, consider the case of C of length zero. Then we must have $r = s = 1$, and the arrows β_o, β_1 satisfy (*). Second, assume the length of C is non-zero. Then $C = \gamma_r C'$, and also $C = C''\beta_r^{-1}$, thus C contains as letters both arrows and inverses of arrows. Therefore $C = \gamma_r\dots\gamma_1\gamma_o^{-1}D'$ and also $C = D''\beta_o\beta_1^{-1}\dots\beta_r^{-1}$. It follows from $C_h = {}_hC$ that $D' = D''$, so we denote D' by D. Now D is a string which neither starts nor ends on a peak, ${}_hD = D_h$, and the length of D is properly smaller than the length of C. By induction, we may assume that D is of the form $(\beta_o\beta_1^{-1})^s$ where β_o,β_1 satisfy (*) and $s \geq o$, and therefore $C = D_h = (\beta_o\beta_1^{-1})^{s+1}$. This finishes the proof.

We note the following consequence of the proposition.

Corollary. The only Auslander-Reiten sequences containing string modules and having a unique middle term are those of the form $0 \longrightarrow U(\beta) \longrightarrow N(\beta) \longrightarrow V(\beta) \longrightarrow 0$, with β an arrow of Q.

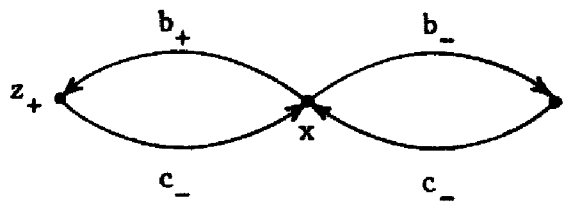
It seems to be of interest that in this way, we obtain a complete description of the boundary of the components of Γ_A containing string modules. The boundary is given by the arrows

$[X] \to [Y]$ with X,Y indecomposable and either X injective or Y projective, and by the 1-simplices corresponding to Auslander-Reiten sequences with a unique middle term. Let a be the number of arrows of Q. Then, there are precisely a arrows $[X] \to [Y]$ in Γ_A with X injective, namely the arrows given by the maps $Q(\beta) : Q(x) \longrightarrow Q(y)$, where β is an arrow with $e(\beta) = x$, $s(\beta) = y$. There are precisely a arrows $[X] \to [Y]$ in Γ_A with Y projective, namely the arrows given by the maps $P(\beta)$: $P(x) \longrightarrow P(y)$, where again β is an arrow with $e(\beta) = x$, $s(\beta) = y$. and, as we have seen above, there are precisely a Auslander-Reiten sequences with a unique middle term and containing string modules; again, these Auslander-Reiten sequences are indexed, in a natural way, by the arrows of Q.

Theorem. Let A be a string algebra. Then $\alpha(A) \leq 2$, and all but finitely many components of Γ_A are of the from $\mathbb{Z}A_\infty^\infty$ and $\mathbb{Z}A_\infty/\langle\tau\rangle$.

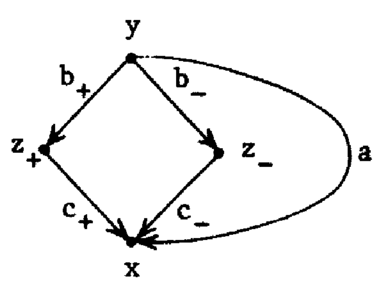
Proof. All components of Γ_A which contain band modules are homogeneous tubes, the components of Γ_A which contain string modules and which are without boundary, are regular, and therefore of the form $\mathbb{Z}A_\infty^\infty$, according to section 2.

Our interest in elucidating the irreducible maps for string algebras arose from a study of the representation theory of the quiver with relations



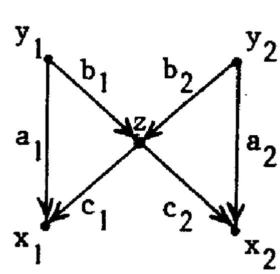
$: c_+b_+ = c_-b_-$

proposed by I.M. Gelfand [G]. It is clear that its finite-dimensional representations form the full subcategory C of those representations M of the finite-dimensional algebra A given by the quiver with relations



$: c_+b_+ = c_-b_-$

for which $M(a)$ is an isomorphism. The algebra A admits an automorphism group G of order 2 (its generator interchanging z_+, z_-, b_+, b_-, c_+, c_-, and fixing x, y, and a) and the twisted group algebra $B = A[G]$, in the sense of [RR], is given, <u>in characteristic not equal to</u> 2, by the quiver with relations



$: c_2b_1 = o, \ c_1b_2 = o.$

Thus B is a string algebra and the results above may be used to obtain its indecomposables and its Auslander-Reiten quiver. In characteristics other than 2, the methods of [RR] then yield the indecomposables for A and its Auslander-Reiten quiver, and these in turn give corresponding information about the category C of representations of the Gelfand quiver. In this way, we see that the regular components of Γ_A are of the form $\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}D_\infty$, $\mathbb{Z}A_\infty/\langle\tau\rangle$ and $\mathbb{Z}A_\infty/\langle\tau^2\rangle$, and there is one non-regular component. The indecomposable A-modules M with [M] in a component of the from $\mathbb{Z}A_\infty/\langle\tau\rangle$ or $\mathbb{Z}A_\infty/\langle\tau^2\rangle$ all belong to C. Each of the remaining regular components contains precisely nine indecomposable modules belonging to C. Finally, the non-regular component contains precisely one indecomposable module from C. In addition one can show that the irreducible maps in C remain irreducible as maps in A-mod. We should mention that the Gelfand problem first has been treated, in arbitrary characteristic, by Nazarova and Rojter in [NR]. In [K], Khoroshkin gave a solution in characteristic not equal to 2, using an approach similar to the one outlined above.

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