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THE REGULAR COMPONENTS OF THE AUSLANDER-REITEN QUIVER OF A TILTED ALGEBRA

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Abstract

Let B be a connected finite-dimensional hereditary algebra of infinite representation type. It is shown that there exists a regular tilting B-module if and only if B is wild and has at least three simple modules. In this way, the author determines the possible form of regular components which arise as a connecting component of the Auslander-Reiten quiver $\Gamma(A)$ of a tilted algebra A. The second result asserts that for a tilted algebra A, any regular component of $\Gamma(A)$ which is not a connecting component, is quasi -serial.

Let k be a field. A finite dimensional k-algebra A is said to be a tilted algebra provided A is the endomorphism ring $\mathrm{End}({}_BT)$ of a tilting module ${}_BT$ over a finite dimensional hereditary k-algebra B (see [10]). A component of the Auslander-Reiten quiver $\Gamma(A)$ of A which does not contain indecomposable modules which are projective or injective, is said to be regular. In this paper, we are going to determine the structure of the regular components of $\Gamma(A)$, when A is a tilted algebra.

So suppose B is a finite dimensional connected hereditary k-algebra of infinite representation type, ${}_BT$ a tilting module, and $A=\mathrm{End}({}_BT)$. If ${}_BT$ is preprojective or preinjective (so that A is a concealed algebra), the regular components of A correspond to the regular components of B, thus all are quasi-serial [13, 2]. So we may assume that A is not a concealed algebra. In this case, A has precisely one connecting component[16], and the connecting component is regular if and only if ${}_BT$ is a regular B-module. If the connecting component is regular, it is of the form $Z\Delta(B^*)$ where $\Delta(B^*)$ is the valued quiver of the opposite algebra B^*. Our first Theorem will give the precise conditions on B for the existence of a regular tilting B-module; in this way we determine the possible form of regular components which arise as connecting components for tilted algebras. This theorem has been announced, under the additional assumption on k to be algebraically closed, at the Puebla conference 1980. Our second result asserts that for a tilted algebra all other regular components are quasi-serial. The proof follows rather closely that of the corresponding result for hereditary

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algebras. The combinatorial part of the proof is separated in Section 2. Note that according to [15], an algebra A is a tilted algebra if and only if the category A-mod of all A-modules contains a slice. In an appendix, we present a modification of the definition of a slice which shows directly that the notion of a slice is self-dual. The terminology used in this paper follows rather closely [15].

§ 1. Existence of Regular Tilting Modules

Theorem. *Let B be a connected finite-dimensional hereditary algebra of infinite representation type. There exists a regular tilting module if and only if B is wild and has at least three simple modules.*

Proof If B has precisely two simple modules, any indecomposable regular module X satisfies $\mathrm{Ext}^1(X, X)\neq 0$ (see [12]), thus there cannot exist a regular tilting module. If B is tame, and X is a regular tilting module, then $\dim X$ lies in the proper subspace of $K_0(B)$ given by all vectors with zero defect, whereas the dimension vectors of the indecomposable summands of a tilting module will generate $K_0(B)$.

For the proof of the converse, we will need some preparations. The following lemma is well-known. (Let us remark that one may strengthen the conclusion considerably (see [6, 14].) For the convenience of the reader, we sketch the proof.

Lemma 1. *Let C be a connected finite-dimensional hereditary algebra of infinite representation type. Let P, P' be indecomposable projective C-modules. Then* $\dim_k \mathrm{Hom}$ $(P, \tau^{-n}P')$ *is unbounded, for* $n\in \mathbf{N}$.

Proof Let $P_1, \cdots, P_m$ be the indecomposable projective B-modules, $Q_1, \cdots, Q_m$ the indecomposable injective B-modules, with top $P_i=\mathrm{soc}\, Q_i$, for $1\leqslant i\leqslant m$. Let $z_{ij}=\overline{\lim\limits_{n}}\dim_k \mathrm{Hom}(P_i, \tau^{-n}P_j)$, an element in $\mathbf{N}\cup\{\infty\}$. Since C is of infinite representation type, the length of the indecomposable preprojective C-modules is unbounded[1], thus not all z_{ij} can be finite. Let $\mathrm{rad}^2(P_s, P_t)\neq \mathrm{rad}(P_s, P_t)$. Then, the Auslander-Reiten sequence starting in $\tau^{-n}P_s$ has $\tau^{-n}P_t$ as a direct summand of its middle term, thus $z_{it}\leqslant 2z_{is}$ for all i. Also, the Auslander-Reiten sequence starting in $\tau^{-n}P_t$ has $\tau^{-n+1}P_s$ as a direct summand of its middle term, thus $z_{is}\leqslant 2z_{it}$ for all i. Note that $\dim_k \mathrm{Hom}(P_i, \tau^{-n}P_j)=\dim_k \mathrm{Hom}(\tau^n Q_i, Q_j)$, for all i, j, and n. Since the Auslander-Reiten sequence ending in $\tau^n Q_t$ has $\tau^n Q_s$ as a direct summand of its middle term, we see that $z_{sj}\leqslant 2z_{tj}$ for all j. Finally, the Auslander-Reiten sequence ending in $\tau^n Q_s$ has $\tau^{n+1}Q_t$ as a direct summand of its middle term, thus $z_{tj}\leqslant 2z_{sj}$ for all j. It follows that $z_{ij}=\infty$ for all i, j.

If C, D are rings and M is a C-D-bimodule, we may form the ring $B=\begin{bmatrix} C & M \\ O & D\end{bmatrix}$.

The B-modules may be written in the form $X=[X_0, X_\omega, \gamma_X)$, where X_0 is a C-

modules, X_ω a D-module, and γ_X: $M \underset{D}{\otimes} X_\omega \to X_0$ a C-linear map. There are two full embeddings of the category of all C-modules into the category of all B-modules. First of all, we may identify the C-module Y with the B-module (Y, O, o); second, we may send the C-module Y to the B-module $\overline{Y} = (Y, \mathrm{Hom}_C(M, Y), e_Y)$, where e_Y: $M \underset{D}{\otimes} \mathrm{Hom}(M, Y) \to Y$ is the evaluation map. In case D is a division ring, the B-module $P(\omega) = (M, D, \mu)$ with μ: $M \underset{D}{\otimes} D \to M$ the multiplication map, is indecomposable and projective, its radical is M, and we denote its top by $E(\omega) = P(\omega)/M$.

As a consequence, we obtain the following lemma.

Lemma 2. *Let C be a connected finite-dimensional hereditary k-algebra of infinite representation type, and M a non-zero projective C-module. Let D be a k-subalgebra of $\mathrm{End}(_CM)$ which is a division ring. Let $B = \begin{bmatrix} C & M \\ O & D \end{bmatrix}$. Then almost all indecomposable preprojective C-modules are regular when considered as B-modules.*

Proof Let $M = M_1 \oplus M_2$ with M_1 indecomposable. The exact sequence

$$0 \to M_1 \oplus M_2 \to P(\omega) \to E(\omega) \to 0$$

shows that $\dim_k \mathrm{Ext}^1_B(E(\omega), N) \geqslant \dim_k \mathrm{Hom}_B(M_1, N)$ for any C-module N. According to the lemma, $\dim_k \mathrm{Hom}_B(M_1, -) = \dim_k \mathrm{Hom}_C(M_1, -)$ is unbounded on the set of indecomposable preprojective C-modules. Thus, also $\dim_k \mathrm{Ext}^1_B(E(\omega), -)$ is unbounded on the set of indecomposable preprojective C-modules. Choose some indecomposable preprojective C-module N with

$$\dim_{\mathrm{End}(E(\omega))} \mathrm{Ext}^1(E(\omega), N) \cdot \dim \mathrm{Ext}^1(E(\omega), N)_{\mathrm{End}(N)} \geqslant 4.$$

Of course, $\mathrm{End}(E(\omega)) = D$; and we denote by E the endomorphism ring of N.

Let N be the full subcategory of all B-modules $(X_0, X_\omega, \gamma_X)$ with X_0 a direct sum of copies of N. The category N is equivalent to the category of representations of the E-D-bimodule $\mathrm{Hom}_k(\mathrm{Ext}^1(\mathrm{E}(\omega), N), k)$. By our choice of N, this bimodule is of infinite representation type, thus there are infinitely many isomorphism classes of indecomposable B-modules belonging to N. The B-module $\overline{N}$ is injective as an object, in N. Thus, for any indecomposable object X in N different from $E(\omega)$, there is a non-trivial map $X \to \overline{N}$. It follows that $\overline{N}$ cannot be a preprojective B-module. On the other hand, $\tau_B^- \overline{N} = \tau_C^- N$ (see [15], 2.5.6). Thus, the preprojective C-module $\tau_C^- N$ is not a preprojective B-module. Since the set of preprojective modules is closed under predecessors, it follows that there are at most finitely many indecomposable C-modules which are preprojective as B-modules. Of course, an indecomposable preprojective C-module cannot be preinjective as B-module. This finishes the proof.

The following result is a variant of an argument due to Bongartz[4].

Lemma 3. *Let C be a finite-dimensional k-algebra, let M be a non-zero* *projective C-module, and let D be a k-subalgebra of* $\mathrm{End}({}_CM)$, *which is a division ring. Let* $B=\begin{bmatrix} C & M \\ O & D \end{bmatrix}$. *Let S be a tilting C-module. Then, there exists an indecomposable B-module Y with the following properties:*

(a) *$S\oplus Y$ is a tilting B-module,*

(b) *there is an exact sequence $O\to S'\to Y\to E(\omega)\to o$ with S' a non-zero direct sum of direct summands of S.*

Proof Let $E_1, \cdots, E_n$ be a k-basis of $\mathrm{Ext}^1_B(E(\omega), S)$. Consider the corresponding exact sequence

$$(E_i)_i: O\to \bigoplus_n S\to Y'\to E(\omega)\to o.$$

One easily checks that $S\oplus Y'$ has no self-extensions (first, apply $\mathrm{Hom}(-, S)$ in order to see that $\mathrm{Ext}^1(Y', S)=0$; apply $\mathrm{Hom}(S, -)$ in order to see that $\mathrm{Ext}^1(S, Y')=0$; finally, apply $\mathrm{Hom}(Y', -)$ in order to see that $\mathrm{Ext}^1(Y', Y')=0$). Fix some decomposition of Y' into indecomposables. All but one direct summand will be isomorphic to direct summands of S, the remaining one, say Y, will map onto $E(\omega)$ with kernel S' a direct sum of direct summands of S. In order to see that $S'\neq 0$, we only have to show $\mathrm{Ext}^1_B(E(\omega), S)\neq 0$. We apply $\mathrm{Hom}(-, S)$ to the exact sequence

$$(*)\quad O\to M\to P(\omega)\to E(\omega)\to o$$

and obtain $\mathrm{Ext}^1_B(E(\omega), S)\approx \mathrm{Hom}_B(M, S)$, the latter being non-zero, since M is a non-zero projective C-module and S is a tilting C-module. Also note that $(*)$ shows that $\mathrm{proj{\cdot}dim{\cdot}}E(\omega)=1$, thus $\mathrm{proj{\cdot}dim{\cdot}}Y\leqslant 1$. This finishes the proof.

The lemma will be applied in two different situations.

Corollary 1. *Let C be a connected finite-dimensional hereditary k-algebra of infinite representation type, let M be a non-zero projective C-module, and let D be a k-subalgebra of* End $({}_CM)$ *which is a division ring. Let* $B=\begin{bmatrix} C & M \\ O & D \end{bmatrix}$. *Let S be a preprojective tilting C-module which is regular as a B-module. Then there exists an indecomposable B-module Y such that $S\oplus Y$ is a regular tilting B-module.*

Proof Choose Y as in Lemma 2. Since Y has the non-zero regular submodule S', we see that Y cannot be preprojective. Also note that Y can be embedded into $\bar{S}'$. Since S' is a preprojective C-module, there are infinitely many indecomposable C-modules N_i with $\mathrm{Hom}_C(S', N_i)\neq 0$, and therefore $\mathrm{Hom}_B(\bar{S}', \bar{N}_i)\neq 0$. This shows that $\bar{S}'$ cannot be preinjective, thus Y is not preinjective.

Corollary 2. *Let C be a connected finite-dimensional hereditary algebra, let M be a non-zero C-module which is projective but not injective, and let D be a k-subalgebra of* $\mathrm{End}({}_CM)$ *which is a division ring. Let* $B=\begin{bmatrix} C & M \\ O & D \end{bmatrix}$. *Let Q be the minimal injective* *cogenerator in C-mod. Then there exists an indecomposable B-module Y such that $Q\oplus Y$ is a tilting B-module, and* $\mathrm{Hom}_B(Q, Y)\neq 0$, $\mathrm{Hom}_B(Y, Q)\neq 0$. *In particular, if Q is a regular B-module, then $Q\oplus Y$ is a regular tilting B-module.*

Proof Since Q is a tilting C-module, we can apply the lemma. We obtain Y with $Q\oplus Y$ a tilting B-module and with an exact sequence

$$0\to Q'\to Y\to E(\omega)\to 0,$$

where Q' is non-zero and injective. In particular, $\mathrm{Hom}(Q, Y)\neq 0$. We claim that top Y is not simple. Otherwise, Y is of the form $Y=P(\omega)/U$ for some proper submodule U of $P(\omega)$. First, assume $U=0$. Applying $\mathrm{Hom}(Q, -)$ to the exact sequence $0\to M\to P(\omega)\to E(\omega)\to 0$, we obtain

$$0=\mathrm{Hom}_B(Q, E(\omega))\to \mathrm{Ext}^1_B(Q, M)\to \mathrm{Ext}^1_B(Q, P(\omega)).$$

Since $Q\oplus P(\omega)$ is a tilting B-module, $\mathrm{Ext}^1_B(Q, P(\omega))=0$, thus $\mathrm{Ext}^1_B(Q, M)=0$. But this implies that M is injective (consider a minimal injective resolution of M, it is a short exact sequence which has to split). This contradiction shows that $U=0$ is impossible. Thus, assume $U\neq 0$. We apply $\mathrm{Hom}(-, Q)$ to the exact sequence $0\to U\to P(\omega)\to Y\to 0$, and obtain

$$\mathrm{Hom}_B(U, Q)\approx \mathrm{Ext}^1_B(Y, Q).$$

Since $U\neq 0$, and Q is a cogenerator, $\mathrm{Hom}_B(U, Q)\neq 0$. Since $Q\oplus Y$ is a tilting module, $\mathrm{Ext}^1_B(Y,Q)=0$. This contradiction shows that also $U\neq 0$ is impossible. Altogether, we see that top Y is not simple. Thus Y maps onto a simple factor module E of Q'. Since E is injective, E is a direct summand of Q, thus $\mathrm{Hom}(Y, Q)\neq 0$.

Let B be a connected finite-dimensional hereditary k-algebra. We assume that B is basic and not simple, and we denote by $e_1, \cdots, e_n$ a complete set of orthogonal primitive idempotents. We denote by $P(i)=Be_i$ the indecomposable projective B-module corresponding to e_i. Let J be the radical of B. The species $(F_i, {}_iM_j)_{1\le i,j\le n}$ of B is obtained as follows: let $F_i=e_iBe_i$, this is a division ring, and let ${}_iM_j=e_i(J/J^2)e_j$, this is an F_i-F_j-bimodule. Denote $d_{ij}=\dim({}_iM_j)_{F_j}$, $d'_{ij}=\dim_{F_i}({}_iM_j)$. The valued quiver $\Delta(B)$ of B has n vertices, say indexed by the number $1, \cdots, n$, there is an arrow $i\leftarrow j$ provided ${}_iM_j\neq 0$, and this arrow is endowed with the pair of numbers (d_{ij}, d'_{ij}). Note that the species of B, or even the valued quiver of B determines the representation type of B, but B-mod is not necessarily equivalent to the category of representations of its species (see [6]). In case ω is a source of $\Delta(B)$ let $e=\sum_{i\neq\omega} e_i$, let $C=eBe$, $M=eBe_\omega$, $D=F_\omega$. Then B is isomorphic to $\begin{bmatrix} C & M \\ O & D\end{bmatrix}$.

With B also its opposite algebra B^* is a finite-dimensional hereditary k-algebra; we obtain $\Delta(B^*)$ from $\Delta(B)$ by changing the orientation of any arrow, replacing at the same time the pair (d_{ij}, d'_{ij}) by the pair (d'_{ij}, d_{ij}). The category of all B^*-modules is dual to the category of all B-modules, a duality is given by forming the k-dual.

Note that the k-dual T^* of a regular tilting B-module T is a regular tilting B^*-module. There is a second possibility of changing the algebra but keeping regular tilting modules: Let a be a sink in the valued quiver $\Delta(B)$. Then $U(a)=\tau^- P(a) \oplus \bigoplus_{i\neq a} P(i)$ is a preprojective tilting B-module, and $B(a)=\mathrm{End}(U(a))$ is a finite-dimensional hereditary k-algebra. This process is a special case of a socalled APR-tilt (see [3]), it generalizes the well-known reflection functors of Bernstein-Gelfand-Ponomarev; we will call it the reflection at the vertex a. The valued quiver of $B(a)$ is obtained from $\Delta(B)$ by changing the orientation of all arrows of $\Delta(B)$ ending in a, and by replacing for these arrows the numbers (d_{aj}, d'_{aj}) by (d'_{aj}, d_{aj}). Given any vertex b of $\Delta(B)$, there exists a finite sequence of reflections such that b becomes a sink, and also a finite sequence of reflections such that b becomes a source (we even may require that b becomes the unique sink, or the unique source, respectively). Note that for T a regular tilting B-module, and a a sink of $\Delta(B)$, the $B(a)$-module $\mathrm{Hom}_B(U(a), T)$ is a regular tilting $B(a)$-module.

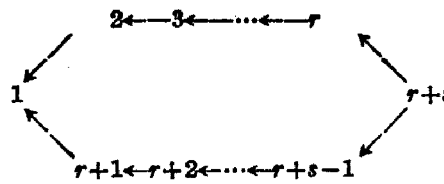
With these preparations, we are going to show the existence of regular tilting modules. Let B be a connected finite-dimensional hereditary k-algebra which is wild and has at least three simple modules. We may assume that B is basic. We choose, as above, a complete set $e_1, \cdots, e_n$ of orthogonal primitive idempotents, and use the notation introduced there.

Let Δ' be a connected valued subquiver of $\Delta(B)$ which is not a Dynkin diagram, and suppose there are vertices of $\Delta(B)$ which do not belong to Δ'. It is easy to see that there exists a vertex ω of $\Delta(B)$ such that the quiver Δ'' obtained from $\Delta(B)$ by deleting ω (and all arrows starting or ending in ω) is connected. We may assume that ω is a source, replacing otherwise B by an algebra obtained from B by a finite sequence of reflections. We write B in the form $\begin{bmatrix} C & M \\ O & D\end{bmatrix}$, where $\Delta(C)=\Delta''$ and $D=F_\omega$. Since Δ'' is connected, and not a Dynkin diagram. C is of infinite representation type. Lemma 2 shows that there are preprojective tilting C-modules S which are regular when considered a B-modules. Now Corollary 1 shows the existence of a regular tilting B-module.

Let us derive several consequences. Since $n\geqslant 3$, we see that the existence of an arrow with valuation (d_{ij}, d'_{ij}) satisfying $d_{ij}d'_{ij}\geqslant 4$ implies that B has a regular tilting module. Thus, we can assume $d_{ij}d'_{ij}\leqslant 3$ for any arrow. Also, if $d_{ij}d'_{ij}=3$ for some i, j, and $n\geqslant 4$, then B has a regular tilting module. Thus, if $d_{ij}d'_{ij}=3$ for some i, j, then we will assume $n=3$. Also, if $\Delta(B)$ contains a cycle, then $\Delta(B)$ is just a primitive cycle (with some valuation).

First, we consider the case of $\Delta(B)$ being a primitive cycle with some valuation.

Up to duality and reflections, we may assume that the underlying quiver of $\Delta(B)$ is of the form



with $r\geq 2$, $s\geq 1$, and $d_{12}=1$, $d'_{12}\geq 2$.

For, since B is wild, at least one pair i, j satisfies $d_{ij}d'_{ij}\geq 2$. We can assume that $d_{ij}=1$, otherwise replace B by B^*. After a sequence of reflections at suitable vertices different from i and j, we can assume that i is tne only sink. Now, if j is not a source, then B is as stated, with $i=1$, $j=2$. So, assume j is a source. In this case, reflection at the vertex i and afterwards replacing the algebra by its opposite gives the desired form, again with $i=1$, $j=2$.

We write $B=\begin{bmatrix} C & M \\ O & D \end{bmatrix}$, where $\Delta(C)$ has the vertices $1, \cdots, r+s-1$, whereas $D=F_{r+s}$. We denote by $Q_C(i)$, $1\leq i\leq r+s-1$ the indecomposable injective C-module with socle $E(i)$, thus $Q=\bigoplus_{i=1}^{r+s-1} Q_C(i)$ is the minimal injective cogenerator in C-mod. We will show that Q is a regular B-module. As a consequence, Corollary 2 asserts the existence of a regular tilting B-module.

Consider the indecomposable projective B-module $P(r+s)$, it has a sub-module U which is a direct sum of $d'_{r,r+s}$ copies of $P(r)$. Let $X=P(r+s)/U$. Note that $\mathrm{Hom}_B(P(i), X)=0$, for $2\leq i<r$. The modules $E(2), \cdots, E(r)$, and X are pairwise orthogonal bricks, and $\mathrm{Ext}^1_B(E(i+1), E(i))\neq 0$ for $2\leq i<r$, $\mathrm{Ext}^1_B(E(r), X)\neq 0$, $\mathrm{Ext}^1_B(X, E(2))\neq 0$. Thus $E(2), \cdots, E(r)$, and X belong to a cycle in B-mod. This shows that all these modules are regular B-modules; in particular $Q_C(r)=E(r)$ is a regular B-module. Similarly, for $s\geq 2$, the module $Q_C(r+s-1)=E(r+s-1)$ is a regular B-modules. For $1\leq i\leq r+s-1$, the module $Q_C(i)$ maps non-trivially to $Q_C(r)$ or to $Q_C(r+s-1)$, thus all $Q_C(i)$ are preprojective or regular, On the other hand, $Q_C(1)$ maps non-trivially to any $Q_C(i)$, $1\leq i\leq r+s-1$. We will show that $Q_C(1)$ is not preprojective, this then implies that none of the modules $Q_C(i)$, $1\leq i\leq r+s-1$ can be preprojective. Take non-zero maps $h: E(1)\to P(r)$ and $h': E(1)\to X$, and denote by Y the pushout of h and h', thus the cokernel of $\begin{bmatrix} h \\ h' \end{bmatrix}: E(1)\to P(r)\oplus X$. It is easy to see that Y is indecomposable. Also, using a projective resolution of Y, one checks without difficulty that $\mathrm{Ext}^1_B(Y, Y)\neq 0$. The socle of $P(r)$ is a direct sum of copies of $E(1)$, and, since $d'_{12}\geq 2$, the socle of $P(r)$ is not simple. Note that Y has a (unique) submodule X' isomorphic to X, and a submodule P' isomorphic to $P(r)$ with $P'+X'$

$=Y$, and $P'\cap X'$ isomorphic to $E(1)$. It follows that $Y/X'\simeq P'/P'\cap X'$ has a submodule of the form $E(1)$. Since $P'/P'\cap X'$ is a C-module, $\mathrm{Hom}_C(Y/X', Q_C(1))\neq 0$, thus $\mathrm{Hom}_B(Y, Q_C(1))\neq 0$. Since Y is an indecomposable B-module with $\mathrm{Ext}^1_B(Y, Y)\neq 0$, we see that $Q_C(1)$ cannot be preprojective.

It remains to consider the case where $\Delta(B)$ does not contain a cycle. Thus, let $\Delta(B)$ be a tree with some valuation. First, let $d_{ij}d'_{ij}<1$ for all i, j, thus we may neglect the valuation. It is well-known (and easy to see) that in this case $\Delta(B)$ contains a subquiver which is of the form $\tilde{\mathbb{D}}_n$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$, or $\tilde{\mathbb{E}}_8$, thus the existence of a regular tilting B-module has been established above. Similarly, assume next that $d_{ij}d'_{ij}\leqslant 2$ for all i, j, and $d_{ij}d'_{ij}=2$ for at least some pair i, j. Now, either there is a second pair s, t with $d_{st}d'_{st}=2$, thus $\Delta(B)$ contains a subquiver of the form $\widetilde{BB}_n$, $\widetilde{BC}_n$, or $\widetilde{CC}_n$; or there is a branching vertex in $\Delta(B)$, and $\Delta(B)$ contains a subquiver $\widetilde{BD}_n$ or $\widetilde{CD}_n$; or, finally, $\Delta(B)$ contains a subquiver of the form $\tilde{F}_{41}$ or $\tilde{F}_{42}$. Always, there is a connected subquiver which is not Dynkin, thus there exists a regular tilting B-module. Thus, assume now that $d_{ij}d'_{ij}=3$ for some i, j. By previous considerations, we know that $n=3$, thus, up to reflections, $\Delta(B)$ is of the form

$$\underset{1}{\circ}\longleftarrow\underset{2}{\circ}\longleftarrow\underset{3}{\circ}$$

with valuation $(d_{12}, d'_{12})=(a, b)$, $(d_{23}, d'_{23})=(c, d)$, where $ab=3$, and $2<cd<3$ (the case $cd=1$ would be tame). The valuation may be read off from the dimension vectors of the indecomposable projective B-modules $P(i)$ and the indecomposable injective B-modules $Q(i)$, namely

$$\dim P(1)=[1, 0, 0],\ \dim P(2)=[b, 1, 0],\ \dim P(3)=[bd, d, 1],$$
$$\dim Q(1)=[1, a, ac],\ \dim Q(2)=[0, 1, c],\ \dim Q(3)=[0, 0, 1].$$

The Coxeter transformation Φ_B (it is determined by the equalities $(\dim P(i))\Phi_B=-\dim Q(i))$ is given as follows:

$$\Phi_B=\begin{bmatrix}-1 & -a & -ac\\ b & 2 & 2c\\ 0 & d & cd-1\end{bmatrix},$$

where we have used $ab=3$. Consider now the module $Q_C(1)$; its dimension vector is $\dim Q_C(1)=[1, a, 0]$, and

$$[1, a, 0]\Phi_B=[2, a, ac],$$
$$[1, a, 0]\Phi_B^2=[1, acd, ac(cd-1)].$$

Since $Q_C(1)$ is not projective, $\dim \tau_B Q_C(1)=(\dim Q_C(1))\Phi_B=[2, a, ac]$, thus $\tau_B Q_C(1)$ is not projective. Therefore $\dim \tau_B^2 Q_C(1)=[1, acd, ac(cd-1)]$. We see that there is a non-trivial map $\tau_B^2 Q_C(1)\to Q(1)$, its kernel has dimension vector $[0, u, v]$ with $u\geqslant acd-a=a(cd-1)\geqslant 1$. Thus, $\tau_B^2 Q_C(1)$ has a submodule of the form $E(2)=Q_C(2)$. This shows that the B-modules $Q_C(2)$ and $Q_C(1)$ belong to a cycle in B-mod, thus they are regular B-modules. As a consequence, there exists a regular tilting B-module.

This finishes the proof of the theorem.

In general, a valued quiver $\Delta=(\Delta_0, \Delta_1, d, d')$ is given by a quiver (Δ_0, Δ_1) without multiple arrows, and two functions d, d' defined on Δ_1 with values in the set $\mathbb{N}_1$ of positive integers; if α: $a\to b$ is an arrow in Δ_1, we usually write d_{ab} instead of $d(\alpha)$, and d'_{ab} instead of $d'(\alpha)$. Recall that for any finite-dimensional algebra B, there is defined its valued quiver $\Delta(B)$. A connected valued quiver Δ will be said to be wild, provided $\Delta=\Delta(B)$ for some wild finite-dimensional hereditary algebra B. Note that Δ is wild if and only if it is neither a Dynkin diagram nor a Euclidean diagram. A valued translation quiver $\Gamma=(\Gamma_0, \Gamma_1, d, d', \tau)$ is given by a valued quiver $(\Gamma_0, \Gamma_1, d, d')$ and a translation quiver $(\Gamma_0, \Gamma_1, \tau)$ such that for any arrow $y\to z$, with z non-projective, we have $d_{\tau z, y}=d'_{yz}$ and $d'_{\tau z, y}=d_{yz}$. Given a valued quiver $\Delta=(\Delta_0, \Delta_1, d, d')$, there is defined, in the usual way, a valued translation quiver $\mathbb{Z}\Delta$ as follows: $(\mathbb{Z}\Delta)_0=\mathbb{Z}\times\Delta_0$, thus the vertices of $\mathbb{Z}\Delta$ are pairs (i, a) with $i\in\mathbb{Z}$, $a\in\Delta_0$; the translation τ is given by $\tau((i, a))=(i-1, a)$; there are arrows $(i, a)\to(i, b)$ and $(i, b)\to(i+1, a)$, for any arrow $a\to b$ in Δ, and any $i\in\mathbb{Z}$ and the valuation for these arrows is given by

$$d_{(i,a)(i,b)}=d_{ab}=d'_{(i,b)(i+1,a)},\ d'_{(i,a)(i,b)}=d'_{ab}=d_{(i,b)(i+1,a)}.$$

Let B be a connected, finite-dimensional hereditary algebra, and ${}_BT$ a tilting module. The component of the Auslander–Reiten quiver $\Gamma(A)$ of $A=\mathrm{End}({}_BT)$ containing the isomorphism classes $[\mathrm{Hom}_B(T, Q)]$, with Q an indecomposable injective B-module, is called a connecting component. In case ${}_BT$, is not preprojective or preinjective, $\Gamma(A)$ has a unique connecting component; in case ${}_BT$ is regular, the connecting component is of the form $\mathbb{Z}\Delta(B^*)$. Thus, there is the following consequence:

Corollary. *Let Δ be a wild connected valued quiver with at least three vertices. Then there exists a finite-dimensional algebra A such that the Auslander–Reiten quiver $\Gamma(A)$ has a component of the form $\mathbb{Z}\Delta$.*

Proof Let $\Delta=\Delta(B^*)$ for some finite-dimensional hereditary algebra B, let ${}_BT$ be a regular tilting B-module, and $A=\mathrm{End}({}_BT)$. Then, the connecting component of A has the form $\mathbb{Z}\Delta$.

§ 2. The Stable Valued Translation Quivers with a Monotone, Strict Additive Function

Let $\Gamma=(\Gamma_0, \Gamma_1, a, d', \tau)$ be a valued translation quiver. A function f: $\Gamma_0\to\mathbb{Z}$ is called an additive function for Γ provided for any non-projective vertex $z\in\Gamma_0$,

$$f(\tau z)+f(z)=\sum_{y\in z^-} f(y)d'_{yz}.$$

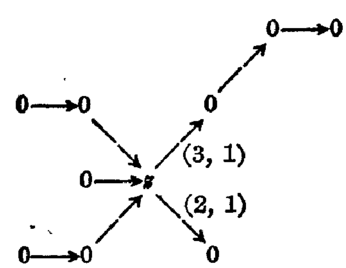
A function f: $\Gamma_0\to\mathbb{Z}$ will be called strict provided $f(x)\neq f(y)$ for every arrow $x\to y$. Also, f will be called monotone provided for every arrow $x\to y$, with both vertices x, y non-projective, $f(x)\leqslant f(y)$ implies $f(\tau x)\leqslant f(\tau y)$. Of course, a strict function is monotone if and only if for every arrow $x\to y$ with both x, y non-projective, $f(x)< f(y)$ implies $f(\tau x)<f(\tau y)$, and this happens if and only if for every arrow $x\to y$ with both x, y non-injective, $f(x)>f(y)$ implies $f(\tau^- x)>f(\tau^- y)$. We are going to analyse the stable valued translation quivers which admit a monotone, strict, additive function with values in the set $\mathbf{N}_0$ of non-negative integers.

We denote by $\mathbf{A}_\infty$ the valued quiver with $\mathbf{N}_1$ as set of vertices, with arrows $a\to a+1$, for all $a\in\mathbf{N}_1$, and with valuation $d_{a,a+1}=1=d'_{a,a+1}$, for all $a\in\mathbf{N}_1$. Note that $\mathbf{A}_\infty$ with the prescribed orientation has a unique source and no sink.

A finite valued quiver $\Delta=(\Delta_0, \Delta_1, d, d')$ will be called a star with center c, provided the underlying graph of (Δ_0, Δ_1) is a star with center c, and such that, in addition,

$$d_{ab}=1 \text{ for } a\neq c, \text{ and}$$
$$d'_{ab}=1 \text{ for } b\neq c;$$

the number $\sum_a d'_{ac}+\sum_b d_{cb}$ will be called the number of branches of Δ. A typical example of a star is the following valued quiver:



for any arrow $a\to b$, the numbers d_{ab}, d'_{ab} are indicated in the form $a\xrightarrow{(d_{ab},\, d'_{ab})} b$, provided at least one of the numbers is different from 1; the number of branches of this star is 8.

Proposition. *Let Δ be a connected valued quiver. Then the following conditions are equivalent*:

(i) *$\mathbb{Z}\Delta$ admits a monotone, strict, additive function with values in $\mathbf{N}_0$,*

(i′) *$\mathbb{Z}\Delta$ admits a monotone, strict, additive function with values in $\mathbf{N}_1$,*

(ii) *Δ is either $\mathbf{A}_\infty$ or a star, but not a Dynkin diagram.*

The proof will be done in several steps. Some of the partial results require less restrictive assumptions. We recall that a function f: $\Gamma_0\to\mathbb{Z}$ is said to be a subadditive function for the valued translation quiver $\Gamma=(\Gamma_0, \Gamma_1, d, d', \tau)$ provided for every non-projective vertex $z\in\Gamma_0$,

$$f(\tau z)+f(z)\geqslant\sum_{y\in z^-}f(y)d'_{yz}.$$

If f is a subadditive function for Γ, and z is a non-projetive vertex, $z^-=\{y_1, \cdots, y_t\}$ with pairwise different vertices $y_1, \cdots, y_t$, then

$$f(\tau z)<\sum_{i=1}^{s}f(y_i)d'_{y_iz} \text{ implies } \sum_{i=s+1}^{t}f(y_i)d'_{y_iz}<f(z).$$

If f is a strict function for Γ, then f is monotone if and only if for every arrow $x\to y$ with both x, y non-projective,

$$f(x)<f(y) \quad \text{implies} \quad f(\tau x)<f(\tau y),$$

and this is equivalent to the condition that for every arrow $x\to y$ with both x, y non-injective,

$$f(x)>f(y) \quad \text{implies} \quad f(\tau^- x)>f(\tau^- y).$$

Given a valued quiver Δ, and a a vertex of Δ, we call the number $\beta(a)=\sum_i d'_{ia}+\sum_j d_{aj}$ the branching number at the vertex a.

Lemma 1. *Let Δ be a valued quiver, and f a monotone, strict, subadditive function for $\mathbb{Z}\Delta$ with values in $\mathbb{N}_0$. Let a be a vertex of Δ, and suppose that $\beta(a)\geqslant 3$. Let $v=(i, a)$ for some $i\in\mathbb{Z}$.*

(1) *If $f(v)\leqslant f(\tau v)$, then $f(u)<f(\tau v)$ for all $u\in v^-$.*

(1*) *If $f(v)\leqslant f(\tau^- v)$, then $f(w)<f(\tau^- v)$ for all $w\in v^+$.*

(2) *$f(u)<f(v)$ for all $u\in v^-$, or $f(u)<f(\tau v)$ for all $u\in v^-$.*

(3) *There exists $n\in\mathbb{N}$, such that for all $w\in v^+$, both $f(\tau^{n+1}w)<f(\tau^n v)$, and $f(\tau^n w)<f(\tau^n v)$.*

Proof (1) Assume $f(v)\leqslant f(\tau v)$. We want to show that $f(u)<f(\tau v)$ for all $u\in v^-$. Assume for the contrary, $f(u)\geqslant f(\tau v)$ for some $u\in v^-$, say $u=u_1$, where $u_1, \cdots, u_t$ are pairwise different vertices and $v^-=\{u_1, \cdots, u_t\}$. Since f is strict, $f(\tau u)<f(u_1)$, thus

$$f(u_1)(d'_{u_1v}-1)+\sum_{i=2}^{t}f(u_i)d'_{u_iv}<f(v),$$

since f is subadditive. Since f is monotone, $f(\tau v)<f(u_1)$ implies $f(\tau^2 v)<f(\tau u_1)$, thus also

$$f(\tau u_1)(d'_{u_1v}-1)+\sum_{i=2}^{t}f(\tau u_i)d'_{u_iv}<f(\tau v),$$

where we use that $d'_{xy}=d'_{\tau x,\tau y}$, for all $x\to y$. The subadditivity for f gives

$$f(\tau u_i)+f(u_i)\geqslant f(\tau v).$$

Since $\tau v\in u_i^-$, for all $1\leqslant i\leqslant t$. Let $\beta=\sum_i d'_{u_iv}$, note that $\beta=\beta(a)$, thus $\beta\geqslant 3$. Then

$$f(\tau v)+f(v)>(f(\tau u_1)+f(u_1))(d'_{u_1v}-1)+\sum_{i=2}^{t}(f(\tau u_i)+f(u_i))d'_{u_iv}$$

$$\geqslant f(\tau v)(d'_{u_1v}-1)+\sum_{i=2}^{t}f(\tau v)d'_{u_iv}=f(\tau v)(\beta-1),$$

therefore

$$f(v)>f(\tau v)(\beta-2)\geqslant f(\tau v);$$

since $\beta\geqslant 3$. This contradiction shows that $f(u)<f(\tau v)$ for all $u\in v^-$.

(1^*) follows by duality.

(2, 3) Choose $j\in\mathbb{Z}$ with $f((j, a))$ being minimal under all $f((i, a))$, $i\in\mathbb{Z}$. Let $z=(j, a)$. Then $f(z)\leqslant f(\tau z)$, thus $f(\tau z)>f(y)$ for all $y\in z^-$, by (1). Since f is monotone, $f(\tau v)>f(u)$ for all $u\in v^-$, provided $v=(i, a)$ with $i\geqslant j$. Also, $f(z)\leqslant f(\tau^- z)$, thus $f(w)<f(\tau z)$ for all $w\in z^+=(\tau^- z)^-$, by (1^*). Using again the fact that f is monotone, $f(u)<f(v)$ for all $u\in v^-$, provided $v=(i, a)$, with $i\leqslant j+1$. Thus, for $v=(i, a)$, and $u\in v^-$, the following holds: if $i=j$ or $i=j+1$, then both $f(u)<f(v)$ and $f(u)<f(\tau v)$; if $i<j$, then $f(u)<f(v)$; if $i>j+1$, then $f(u)<f(\tau v)$. This finishes the proof of (2). Also, we see that for $i=j-1$, j, or $j+1$, and $v=(i, a)$, $w\in v^+$, we have $f(\tau w)<f(v)$ and $f(w)<f(v)$. For general i, let $n=i-j$. Then $f(\tau^n w)<f(v)$ and $f(w)<f(v)$. This proves (3).

Lemma 2. *Let Δ be a valued quiver, and f a monotone, strict, additive function for $\mathbb{Z}\Delta$ with values in $\mathbb{N}_0$. Let a be a vertex of Δ, and suppose $\beta(a)=2$. Then a has precisely two neighbors; if b is a neighbor of a, then $d_{ab}=1$ provided $a\to b$, and $d'_{ba}=1$ provided $b\to a$. Let $z=(i, a)$ for some $i\in\mathbb{Z}$, let $z^-=\{y_1, y_2\}$, and suppose $f(y_1)\leqslant f(y_2)$. Then, for all $n\in Z$, $f(\tau^n y_1)<f(\tau^n z)<f(\tau^n y_2)$, $f(\tau^n y_1)<f(\tau^{n+1} z)<f(\tau^n y_2)$.*

Proof Since $\beta(a)=2$, given any $i\in\mathbb{Z}$ and $z=(i, a)$, either $z^-=\{y_1, y_2\}$ with $y_1\neq y_2$, and $d'_{y_1z}=1=d'_{y_2z}$, or else $z^-=\{y\}$ with $d'_{yz}=2$; in the second case, let $y_1=y_2=y$. Since f is additive,

$$f(\tau z)+f(z)=f(y_1)+f(y_2).$$

We can assume $f(y_1)\leqslant f(y_2)$. We claim $f(y_1)<f(\tau z)$. Assume for the contrary $f(\tau z)<f(y_1)$, thus also $f(\tau z)<f(y_2)$. The additivity shows that $f(y_1)<f(z)$ and $f(y_2)<f(z)$. Since f is monotone, we conclude that for all $n\geqslant 0$,

$$f(\tau^{n+1}z)<f(\tau^n y_1)<f(\tau^n z).$$

In this way, we obtain an infinite decreasing sequence

$$f(z)>f(\tau z)>f(\tau^2 z)>\cdots$$

of non-negative integers, impossible. This shows that $f(y_1)<f(\tau z)$. By the additivity of f, this implies $f(z)<f(y_2)$. By duality, we also have $f(y_1)<f(z)$, and therefore also $f(\tau z)<f(y_2)$. This gives the case $n=0$ of the stated inequalities. At the same time, we have shown that $f(y_1)<f(y_2)$, thus $y_1\neq y_2$. This excludes the case $z^-=\{y\}$, thus a has precisely two neighbors. If $b\to a$, then (i, b) is one of y_1, y_2, and therefore $d'_{ab}=1$. If $a\to b$, then $(i-1, b)$ is one of y_1, y_2, and therefore $d_{ab}=1$.

Finally, consider $\tau z=(i-1, a)$. Now, $(\tau z)^-=\{\tau y_1, \tau y_2\}$, and we claim that $f(\tau y_1)\leqslant f(\tau y_2)$. Otherwise, the previous considerations, applied to τz instead of z, yield $f(\tau z)<f(\tau y_1)$. But, since f is monotone, $f(y_1)<f(z)$ yields $f(\tau y_1)<f(\tau z)$.

This shows that $f(y_1) \leqslant f(y_2)$ yields $f(\tau^n y_1) \leqslant f(\tau^n y_2)$ for all $n>0$. By duality, $f(y_1) \leqslant f(y_2)$ also yields $f(\tau^n y_1) \leqslant f(\tau^n y_2)$ for all $n<0$. Thus, the inequalities for $n=0$ imply those for arbitrary $n \in \mathbb{Z}$. This finishes the proof.

Given a function f on $(\mathbb{Z}\Delta)$ with values in $\mathbb{N}_0$, where Δ is a valued quiver, and given $a \in \Delta_0$, denote by $\mu_f(a) = \mu(a)$ the minimum of all values $f((i, a))$, with $i \in \mathbb{Z}$.

Lemma 3. *Let Δ be a valued quiver, and f a monotone, strict, additive function for $\mathbb{Z}\Delta$ with values in $\mathbb{N}_0$. Let a be a vertex of Δ.*

(1) *If $\beta(a) \geqslant 3$, then for all neighbors b of a, we have $\beta(b) \leqslant 2$.*

(2) *If $\beta(a) = 2$, then there exists a neighbor b of a with $\beta(b) \leqslant 2$, and $\mu(b) < \mu(a)$.*

Proof (1) Assume $\beta(a) \geqslant 3$. Part (3) of Lemma 1 asserts that for suitable $j \in \mathbb{Z}$, and $z = (j, a)$,

$$f(\tau w) < f(z),\ f(w) < f(z)$$

for all $w \in z^+$. Let b be a neighbor of a. If $a \to b$ in Δ, let $w = (j, b)$; if $b \to a$ in Δ, let $w = (j+1, b)$. Then $w \in z^+$, and $z \in w^-$. It follows that $\beta(b) \leqslant 2$. Otherwise, apply part (2) of Lemma 1 to $v = w$ and $u = z$: we obtain $f(z) < f(w)$, or $f(z) < f(\tau w)$, a contradiction.

(2) Assume $\beta(a) = 2$. Choose $i \in \mathbb{Z}$, such that for $z = (i, a)$, we have $f(z) = \mu(a)$. Lemma 2 asserts that we can write $z^- = \{y_1, y_2\}$ with

$$f(y_1) < f(z) \quad \text{and} \quad f(\tau^- y_1) < f(z).$$

Since $y_1 \in z^-$, it follows that $y_1 = (i, b)$ or $= (i+1, b)$ for a neighbor b of a. As previously, it follows that $\beta(b) \leqslant 2$. Otherwise, apply part (2) of Lemma 1 to $v = \tau^- y_1$ and $u = z$, and obtain a contradiction. Also, we see that

$$\mu(b) \leqslant f(y_1) < f(z) = \mu(a).$$

Proof of the implication (i)$\Rightarrow$(ii): Let Δ be a connected valued quiver and f a monotone, strict, additive function for $\mathbb{Z}\Delta$ with values in $\mathbb{N}_0$. Assume a_1 is a vertex of Δ with $\beta(a_1) = 2$. According to Lemma 3, there exists a neighbor a_2 of a_1 with $\beta(a_2) \leqslant 2$ and $\mu(a_2) < \mu(a_1)$. We use induction in order to obtain a sequence $a_1, a_2, \cdots, a_r$ with a_{i+1} being a neighbor of a_i, and $\beta(a_i) = 2$, for $1 \leqslant i \leqslant r-1$, such that

$$\mu(a_1) > \mu(a_2) > \cdots > \mu(a_r).$$

Since the numbers $\mu(a_i)$ are non-negative integers, such a sequence must stop eventually; therefore, we can assume $\beta(a_r) = 1$. Thus, a_{r-1} is the only neighbor of a_r. Any vertex a_i, $2 \leqslant i \leqslant r-1$, has as neighbors just the vertices a_{i-1}, and a_{i+1}. By Lemma 2, the vertex a_1 has besides a_2 an additional neighbor, say a_0. The following is already known about the valuation: if $a \to b$, then $d_{ab} = 1$ provided $a \in \{a_1, \cdots, a_r\}$, and $d'_{ab} = 1$ provided $b \in \{a_1, \cdots, a_r\}$. In case all vertices a of Δ satisfy $\beta(a) \leqslant 2$, we easily see that $\Delta = \mathsf{A}_\infty$ or else $\Delta = \mathsf{A}_n$ for some n, thus Δ is a tree (the latter case $\Delta = \mathsf{A}_n$ actually cannot occur, as we will see below). Now assume there exists a vertex c of Δ

with $\beta(c)\geqslant 3$. Lemma 8 asserts that all neighbors a of c satisfy $\beta(a)<2$. But then our previous considerations show that Δ is a star with center c. If Δ is a Dynkin diagram, then $\mathbf{Z}\Delta$ does not admit any non-trivial additive function with values in $\mathbf{N}_0$ (see [9]), thus Δ cannot be a Dynkin diagram. This finishes the proof of the implication (i)$\to$(ii).

Lemma. *Let Δ be either $\mathbf{A}_\infty$, or a star whose center is the only sink, and let f be an additive function for $\mathbf{Z}\Delta$ with values in $\mathbf{N}_1$. Then, for every arrow α: $a\to b$ in Δ, and $i\in\mathbf{Z}$,*

$$f((i, a))<f((i, b)),\ f((i+1, a))<f((i, b)).$$

Proof By our choice of orientation, there is a path $a_0\to a_1\to\cdots\to a_{t-1}\to a_t$ in Δ such that a_0 is a source and $a_{t-1}=a$, $a_t=b$; moreover, $\beta(a_0)=1$, $\beta(a_j)=2$ for $1\leqslant j<t$. The additivity of f yields by induction on $j\geqslant 1$

$$f((i, a_j))=f((i, a_{j-1}))+f((i+j, a_0)),$$

thus we conclude

$$\begin{aligned}f((i, b))-f((i, a))&=f((i, a_t))-f((i, a_{t-1}))\\&=f((i+t, a_0))>0.\end{aligned}$$

Similarly,

$$f((i, b))-f((i+1, a))=f((i, a_0))>0.$$

Corollary. *Let Δ be either $\mathbf{A}_\infty$ or a star. Then any additive function on $\mathbf{Z}$ with values in $\mathbf{N}_1$ is strict and monotone.*

Proof of the implication (ii)$\to$(i'): First, consider the case $\Delta=\mathbf{A}_\infty$, recall that the set of vertices of $\Delta=\mathbf{A}_\infty$ is just $\mathbf{N}_1$. The function f on $(\mathbf{Z}\Delta)_0$ defined by $f((i, a))=a$ obviously is additive.

Assume now that Δ is a star, but not a Dynkin diagram. If Δ is a Euclidean diagram, theh $\mathbf{Z}\Delta$ admits an additive function f with values in $\mathbf{N}_1$ which even is τ-invariant (as in the case $\Delta=\mathbf{A}_\infty$). Thus, we can assume that Δ is wild. If Δ_0 has at least three vertices, consider $\mathbf{Z}\Delta$ as a connecting component for some tilted algebra, as constructed in section 1, and let f be the length function. Then f is additive, and takes values in $\mathbf{N}_1$. Finally, it remains to consider a wild star with precisely two vertices. It is sufficient to deal with

$$\underset{a}{0}\xrightarrow{(d,\ 1)}\underset{b}{0},$$

where $d\geqslant 5$. In this case, the additive function f defined by $f((0, a))=2$ and $f((0, b))=1$ takes values in $\mathbf{N}_1$ (see [13]). This finishes the proof.

Every stable valued translation quiver Γ is of the form $\Gamma=\mathbf{Z}\Delta/G$, where Δ is a valued quiver whose underlying graph is a tree, and G is a group of automorphisms of $\mathbb{Z}\Delta$, and let π: $\mathbb{Z}\Delta \to \mathbb{Z}\Delta/G = \Gamma$ be the corresponding projection (it is a covering). Any function f: $\Gamma_0 \to \mathbb{Z}$ gives rise to the function f: $(\mathbb{Z}\Delta)_0 \to \mathbb{Z}$ defined by $f(x) = f(\pi(x))$. If f is additive, or subadditive, or strict, or monotone, the function f will have the same property. As a consequence, there is the following corollary:

Corollary. *Let Γ be a stable valued translation quiver which admits a monotone, strict, additive function with values in $\mathbb{N}_0$. Then $\Gamma = \mathbb{Z}\Delta/G$, where Δ is either $\mathbf{A}_\infty$, or a star, but not a Dynkin diagram, and G is a group of automorphisms of $\mathbb{Z}\Delta$.*

§ 3. The Quasi-Serial Components

Components of the form $\mathbb{Z}\mathbf{A}_\infty/G$, with G a group of automorphisms of $\mathbb{Z}\mathbf{A}_\infty$, are called quasi-serial (see [13]).

Theorem. *Let A be a tilted algebra. Then, any regular component of $\Gamma(A)$ which is not a connecting component, is quasi-serial.*

Proof Let B be a finite-dimensional, hereditary k-algebra, and ${}_BT$ a tilting module with $A = \mathrm{End}({}_BT)$. The regular components of $\Gamma(A)$ different from the connecting component correspond to the regular components of the relative Auslander-Reiten quivers of $F({}_BT)$ and $G({}_BT)$. Here $F({}_BT)$ is the full subcategory of B-mod given by all B-modules X with $\mathrm{Hom}_B(T, X) = 0$, and $G({}_BT)$ is the full subcategory of B-mod given by all B-modules which are generated by ${}_BT$. Up to duality, we may assume that we deal with a regular component Γ of the relative Auslander-Reiten quiver of $G({}_BT)$, and we want to show that Γ is quasi-serial. We need the following result due to Hoshino; recall that the subcategories $F({}_BT)$ and $G({}_BT)$ form a torsion pair in B-mod, the modules in $G({}_BT)$ being the torsion modules, those in $F({}_BT)$ the torsionfree modules.

Lemma[8]. (Hoshino) *Let $0 \to X \to Y \to Z \to 0$ be a relative Auslander-Reiten sequence in $G({}_BT)$. Then X is the $G({}_BT)$-torsion submodule of τZ.*

Here $\tau Z = DTrZ$ denotes the usual Auslander-Reiten translate of Z in B-mod. Since B is hereditary, τ actually is a functor (from B-mod into itself), and is left exact. We write $\tau_G Z$ for the $G({}_BT)$-torsion submodule of τZ, for any B-module Z. The Hoshino lemma asserts that the restriction of τ_G to $G({}_BT)$ just yields the relative Auslander-Reiten translation. Of course, with τ also τ_G is a functor, since the assignement of the torsion submodule with respect to a fixed torsion pair is functorial. There is the following consequence:

Corollary. *The endo-functor τ_G of B-mod preserves monomorphisms.*

Proof Let α: $X \to Y$ be a monomorphism in B-mod. Since τ is left exact, $\tau\alpha$: $\tau X \to \tau Y$ is a monomorphism. But $\tau_G\alpha$ is the restriction of $\tau\alpha$ to the $G({}_BT)$-torsion submodule $\tau_G X$ of τX, thus also $\tau_G \alpha$ is a monomorphism.

Let Γ be a regular component of the relative Auslander–Reiten quiver of $G({}_BT)$. Let D be the additive subcategory generated by the indecomposable B–modules X with $[X]$ belonging to Γ. For X indecomposable in D, let $f([X]) = |X|$, the length of X as a B–module. Of course, f is an additive function on Γ with values in $\mathbb{N}_1$. Note that any relative irreducible map in $G({}_BT)$ is a monomorphism or an epimorphism in B–mod (for, its image is again generated by ${}_BT$, thus in $G({}_BT)$. As a consequence, f is strict. We claim that f is also monotone. For, assume X, Y are indecomposables in $G({}_BT)$, there exists a relative irreducible map α: $X \to Y$ and $|X| < |Y|$. Then α cannot be an epimorphism in B–mod, thus α is a monomorphism in B–mod. Since τ_G preserves monomorphisms, we conclude $|\tau_G X| < |\tau_G Y|$. Thus, we can apply the proposition in section 2, and conclude that Γ is either quasi–serial or of the form $\mathbb{Z}\Delta/G$, where Δ is a star but not a Dynkin–diagram, and G a group of automorphisms of $\mathbb{Z}\Delta$. Thus, assume Γ is of the form $\mathbb{Z}\Delta/G$, with Δ a star but not a Dynkin diagram, let $\pi:\mathbb{Z}\Delta \to \mathbb{Z}\Delta/G = \Gamma$ be the canonical projection. Let c be the center of Δ, and for any $i \in \mathbb{Z}$ let $\pi((i, c)) = [M_i]$, where M_i is an indecomposable module in D. Note that $\tau_G M_i = M_{i-1}$, for all $i \in \mathbb{Z}$. We claim that M_i is cogenerated by any M_j, with $i < j$. It is sufficient to show that M_i is cogenerated by M_{i+1}. Since Δ is not a Dynkin diagram, the number β of branches of Δ is at least three, thus the relative Auslander–Reiten sequence ending in M_{i+1} is of the form

$$0 \to M_i \to \bigoplus_{j=1}^{\beta} Y_{ij} \xrightarrow{(\alpha_j)_j} M_{i+1} \to 0,$$

with $\beta \geq 3$. Note that since Δ is a tree, all the given maps α_j: $Y_{ij} \to M_{i+1}$ are monomorphisms in B–mod, thus we obtain a monomorphism in B–mod

$$M_i \to \bigoplus_{j=1}^{\beta} Y_{ij} \xrightarrow{\oplus \alpha_j} \bigoplus_{j=1}^{\beta} M_{i+1}.$$

Let $m = |M_0|$. Fix some $i \geq 0$. Since M_0 is cogenerated by M_i, there actually is a monomorphism $M_0 \to \bigoplus_m M_i$ in B–mod. Applying τ_G^i, we obtain a monomorphism

$$M_{-i} = \tau_G M_0 \to \bigoplus_m M_0.$$

Now, $|\bigoplus_m M_0| = m^2$, thus we see that $|M_{-i}| \leq m^2$, for all $i \geq 0$. But it has been shown in [11] that π is a finite covering and that the number of isomorphism classes in D containing modules of a fixed length is finite. This contradiction shows that Γ has to be quasi–serial.

Appendix: The notion of a slice.

Let A be a finite dimensional k–algebra. Let $\mathscr{S}$ be a module class in A–mod which satisfies the following two conditions:

(α) $\mathscr{S}$ is sincere.

(β) $\mathscr{S}$ is path closed.

Then the following two conditions are equivalent:

(δ) Let M be indecomposable and not projective, and let S be indecomposable and in $\mathscr{S}$. If there exists an irreducible map $S \to M$, then M or τM belongs to $\mathscr{S}$.

(δ') Let N, S be indecomposable and suppose S belongs to $\mathscr{S}$. If there exists an irreducible map $N \to S$, then either N belongs to $\mathscr{S}$, or N is not injective and $\tau^- N$ belongs to $\mathscr{S}$.

Proof Clearly (δ') implies (δ): Let M, S be indecomposable, M not projective, S in $\mathscr{S}$, and assume there exists an irreducible map $S \to M$. Let $N = \tau M$. Then there is an irreducible map $N \to S$, and by (δ'), we conclude one of N, $\tau^- N$ belongs to $\mathscr{S}$.

Conversely, assume (δ), and let N, S be indecomposable, S in $\mathscr{S}$, with an irreducible map $N \to S$. If N is not injective, then (δ) applied to $M = \tau^- N$ yields that one of N, $\tau^- N$ is in $\mathscr{S}$. Thus, assume N is injective. Since $\mathscr{S}$ is supposed to be sincere, there is an indecomposable S' in $\mathscr{S}$ with $\mathrm{Hom}(S', N) \neq 0$. Thus $S' \leqslant N \leqslant S$, and therefore N belongs to $\mathscr{S}$, according to (β).

In addition, the following condition is of interest:

(γ) If M is indecomposable and not projective, then at most one of M, and τM belongs to $\mathscr{S}$.

We call $\mathscr{S}$ a slice (in A-mod) provided the conditions (α), (β), (γ) and (δ) are satisfied. Note that all these conditions are self-dual, thus $\mathscr{S}$ is a slice in A-mod if and only if $\mathscr{S}^* = \{\mathrm{Hom}_k(S, k) \mid S \in \mathscr{S}\}$ is a slice in A^{op}-mod. According to the consideration above, this notion of a slice coincides with that of [15].

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