Arch. Math., Vol. 50, 128–133 (1988)

0003-889X/88/5002-0128 $ 2.70/0

Modules of bounded length in Auslander-Reiten components

By

EUGENIA MARMOLEJO and CLAUS MICHAEL RINGEL

Let A be an artin algebra, and Γ a component of the Auslander-Reiten quiver of A. It has been asked [12] whether the number of isomorphism classes of indecomposable A-modules in Γ of fixed length is finite. This question is answered affirmatively for certain types of regular components. More generally, we consider regular components of full additive subcategories with sink maps in a length category.

Recall that a *length category* is an abelian category where every object X has a finite composition series; the length of such a composition series will be called the length of X and denoted by $|X|$. We denote by $[X]$ the isomorphism class of the object X, and we also write $|[X]|$ instead of $|X|$. Let $\mathscr{A}$ be a length category, and $\mathscr{K}$ a full additive subcategory. We consider $\mathscr{K}$ as a Krull-Schmidt category with short exact sequences, the short exact sequences being those in $\mathscr{A}$ which belong to $\mathscr{K}$. Thus, the Auslander-Reiten quiver $\Gamma(\mathscr{K})$ of $\mathscr{K}$ is defined (see [11]), but we consider $\Gamma(\mathscr{K})$ as a valued translation quiver, see [5], and we denote by $\tau_{\mathscr{K}}$ the Auslander-Reiten translation in $\mathscr{K}$. An (Auslander-Reiten) *component* $\mathscr{C}$ of $\mathscr{K}$ is, by definition, the full additive subcategory generated by the indecomposable objects X in $\mathscr{K}$ whose isomorphism classes belong to a fixed component Γ' of $\Gamma(\mathscr{K})$; of course, $\Gamma' = \Gamma(\mathscr{C})$. A component $\mathscr{C}$ of $\mathscr{K}$ is said to be *regular* provided $\Gamma(\mathscr{C})$ is a stable translation quiver. (Note that a component $\mathscr{C}$ of $\mathscr{K}$ is regular if and only if the following condition is satisfied: if X is indecomposable in $\mathscr{C}$, then there exists a source map f and a sink map g for X in $\mathscr{K}$, and there are short exact sequences (f, f') and (g', g) which lie inside $\mathscr{C}$.) *Let $\mathscr{C}$ be a regular component of $\mathscr{K}$. We assume in addition that there are indecomposable objects C_1, C_2 in $\mathscr{C}$ and X_1, X_2 in $\mathscr{K}\backslash\mathscr{C}$, with* $\operatorname{Hom}(X_1, C_1) \neq 0$ *and* $\operatorname{Hom}(C_2, X_2) \neq 0$. *Finally, assume there is a constant d such that for any indecomposable object X of $\mathscr{C}$, we have $|\tau_{\mathscr{K}} X| \leqq d\,|X|$.* These conditions are obviously satisfied in the case that $\mathscr{C}$ is a stable component of $\mathscr{A} = \mathscr{K} = A$-mod. (For d, we may take $(\dim_k A)^2$, see [10]). They are also satisfied in the following situation: Let ${}_AT$ be a tilting module, $\mathscr{K} = \mathscr{G}({}_AT)$ the full subcategory of all A-modules generated by ${}_AT$, and $\mathscr{A} = A$-mod. (Here, take for X_1 a suitable dirct summand of ${}_AT$, for X_2 a suitable indecomposable injective A-module; also if $\tau_{\mathscr{K}} X$ exists, it can be embedded into τX, see [7], thus, again let $d = (\dim_k A)^2$.)

A valued quiver $\Delta = (\Delta_0, \Delta_1, d, d')$ is given by a quiver (Δ_0, Δ_1) without multiple arrows, and two functions d, d' defined on Δ_1 with values in the set $\mathbb{N}_1$ of positive integers, if $\alpha: a \to b$ is an arrow, we write d_{ab} instead of $d(\alpha)$ and d'_{ab} instead of $d'(\alpha)$. A valued translation quiver $\Gamma = (\Gamma_0, \Gamma_1, d, d', \tau)$ is given by a valued quiver $(\Gamma_0, \Gamma_1, d, d')$ and a translation quiver $(\Gamma_0, \Gamma_1, \tau)$ such that for any non-projective vertex z and any arrow $y \to z$, we have $d_{\tau z, y} = d'_{yz}$ and $d'_{\tau z, y} = d_{yz}$. Given a valued quiver Δ, following Riedtmann [9, 5] we define a valued translation quiver $\mathbb{Z}\Delta$ as follows: $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$; the translation τ is given by $\tau((i, a)) = (i - 1, a)$, for $i \in \mathbb{Z}$, $a \in \Delta_0$; there are arrows $(i, a) \to (i, b)$ and $(i, b) \to (i + 1, a)$ for any arrow $a \to b$ in Δ and any $i \in \mathbb{Z}$, and the valuation for these arrow is

$$d_{(i,a)(i,b)} = d_{ab} = d'_{(i,b)(i+1,a)},$$

$$d'_{(i,a)(i,b)} = d'_{ab} = d'_{(i,b)(i+1,a)}.$$

Since our translation quivers are allowed to have loops (but not multiple arrows), the notion of a covering as defined by Riedtmann [9] has to be modified: let $(\Gamma_0, \Gamma_1, \tau)$ and $(\Gamma'_0, \Gamma'_1, \tau)$ be stable translation quivers (without multiple arrows); a map

$$\pi\colon (\Gamma_0, \Gamma_1, \tau) \to (\Gamma'_0, \Gamma'_1, \tau)$$

is of the form $\pi = (\pi_0, \pi_1)$, where $\pi_0\colon \Gamma_0 \to \Gamma'_0$, $\pi_1\colon \Gamma_1 \to \Gamma'_1$ are set-theoretical maps such that for $\alpha\colon x \to y$, we have $\pi_1(\alpha)\colon \pi_0(x) \to \pi_0(y)$, and $\pi_0(\tau(x)) = \tau(\pi_0(x))$, for all non-projective x. Such a map is called a *covering* provided π_0 and π_1 are surjective, and the restriction of π_0 to x^+ is a bijection from x^+ onto $\pi_0(x)^+$, and its restriction to x^- is a bijection from x^- onto $\pi_0(x)^-$. The covering π is said to be *finite*, provided $\pi_0^{-1}(x)$ is finite, for every vertex x of Γ'. If Γ, Γ' are valued translation quivers, a map $\pi\colon \Gamma \to \Gamma'$ is given by a map $\pi\colon (\Gamma_0, \Gamma_1, \tau) \to (\Gamma'_0, \Gamma'_1, \tau)$ such that $d_{\pi(x)\pi(y)} = d_{xy}$, $d'_{\pi(x)\pi(y)} = d'_{xy}$, for any arrow $x \to y$. A covering of valued translation quivers is a map of valued translation quivers which is a covering of translation quivers.

Proposition. *Assume there exists a covering $\pi\colon \mathbb{Z}\Delta \to \Gamma(\mathscr{C})$, where Δ is either a finite valued quiver without oriented cycles or else $\Delta = \mathbb{A}_\infty^\infty, \mathbb{B}_\infty, \mathbb{C}_\infty$, or $\mathbb{D}_\infty$. Then, for every $l \in \mathbb{N}$, there are only finitely many isomorphism classes of indecomposable modules in $\mathscr{C}$ which have length l (in $\mathscr{A}$). Also, π is a finite covering.*

Proof. On $\mathbb{Z}\Delta$, we define an additive function f by $f(x) = |\pi(x)|$. We show that for every $l \in \mathbb{N}$, the set $f^{-1}(l) = \{x \in (\mathbb{Z}\Delta)_0 \mid f(x) = l\}$ is finite. This obviously yields both assertions. Assume $f^{-1}(l)$ is an infinite set and let us derive a contradiction.

First, assume Δ is a finite valued quiver (without oriented cycles). Since $\mathbb{Z}\Delta$ has only finitely many τ-orbits, some τ-orbit contains infinitely many vertices from $f^{-1}(l)$. Thus, there is $a \in \Delta_0$ and an infinite subset $I \subseteqq \mathbb{Z}$, such that $f((i, a)) = l$, for all $i \in I$. We can assume that a is the unique sink of Δ. Let $\mathscr{C}(i)$ be the additive subcategory generated by all indecomposable objects X in $\mathscr{C}$, with $[X] = \pi((i, b))$ for some $b \in \Delta_0$. If Z is indecomposable in $\mathscr{C}$, then $|\tau_{\mathscr{K}} Z| \leqq d\,|Z|$; thus, if $Y \to Z$ is an irreducible map of $\mathscr{K}$, then $|Y| \leqq (d + 1)\,|Z|$. Since Δ has no oriented cycles, there is a bound n on the length of all paths in Δ. Given $b \in \Delta_0$, there is a path from b to a in Δ of length $\leqq n$. Thus, for $i \in \mathbb{Z}$, there is a path in $\Gamma(\mathscr{C})$ from $\pi(i, b))$ to $\pi((i, a))$ of length $\leqq n$. It follows that for $i \in I$,

$$|\pi((b, i))| \leqq (d + 1)^n\,|\pi((a, i))| = (d + 1)^n\, l.$$

Let $l' = (d + 1)^n\, l$. Then $|Z| \leqq l'$ for any indecomposable object Z in $\mathscr{C}(i)$ with $i \in I$.

Let X, X' be indecomposable objects in $\mathscr{K}\backslash\mathscr{C}$, and let C, C' be indecomposable objects in $\mathscr{C}$, such that $\operatorname{Hom}(X, C) \neq 0$, $\operatorname{Hom}(C', X') \neq 0$. Let $[C] = \pi((j, c))$, $[C'] = \pi((j', c'))$. First, suppose there are infinitely many $i \in I$ satisfying $i \leqq j$. Let $0 \neq \varphi: X \to C$. Using the sink map of the indecomposable objects in $\mathscr{C}(j)$, we can factor $\varphi = \psi_1 \varphi_1$, where $\psi_1 \in \operatorname{Hom}(X, C)$, $\varphi_1 \in \operatorname{rad}(C_1, C)$, with $C_1 \in \mathscr{C}(j-1)$. By induction, we obtain a factorization $\varphi = \psi_m \varphi_m \cdots \varphi_1$, where $\psi_m \in \operatorname{Hom}(X, C_m)$, $\varphi_i \in \operatorname{rad}((C_i, C_{i-1})$, with $C_i \in \mathscr{C}(j-i)$, and $C_0 = C$. However, as soon as the intervall $[j-m, j]$ in $\mathbb{Z}$ contains $2^{l'}$ elements from I, the Harada-Sai lemma [6, 13] asserts that $\psi_m \cdots \varphi_1 = 0$. This contradicts our assumption $\varphi \neq 0$, and shows that there are only finitely many $i \in I$ satisfying $i \leqq j$. Similarly, if we suppose that there are infinitely many $i \in I$ satisfying $i \geqq j'$, we use source maps in order to factorize any $0 \neq \varphi': C' \to X'$ in the form $\varphi' = \varphi'_1 \cdots \varphi'_m \psi'_m$, where $\varphi'_i \in \operatorname{rad}(C'_{i-1}, C'_i)$, $\psi'_m \in \operatorname{Hom}(C'_m, X')$, with $C'_0 = C'$, and $C'_i \in \mathscr{C}(j'+i)$, and again the Harada-Sai lemma gives a contradiction. This shows that there are only finitely many $i \in I$ satisfying $i \geqq j'$. Thus, I is finite. This finishes the proof in the case that Δ is finite.

Before we proceed to study the remaining cases, let us insert the following lemma on sectional paths due to Bongartz [2] and Igusa-Todorov [8].

Lemma. *Let $\mathscr{K}$ be a Krull-Schmidt category with short exact sequences, and having sink maps. Assume the sink map for any indecomposable object Z with $[Z]$ a projective vertex in $\Gamma(\mathscr{K})$, is a monomorphism in $\mathscr{K}$. Let $X_0, \ldots, X_n$ be indecomposable modules, and $f_i: X_{i-1} \to X_i$, $1 \leqq i \leqq n$, irreducible maps. Assume that for all $2 \leqq i \leqq n$, either X_i is projective, or $\tau_{\mathscr{K}} X_i \not\approx X_{i-2}$. Then $f_1 \cdots f_n \neq 0$.*

P r o o f. (For the convenience of the reader, we sketch a short proof). Assume we have $f_1 \cdots f_n = 0$. Since irreducible maps are non-zero, we have $n \geqq 2$. We can assume that the X_i, f_i are chosen in such a way that n is minimal. In particular, $[X_n]$ is not a projective vertex in $\Gamma(\mathscr{K})$, since otherwise $f_1 \ldots f_{n-1} = 0$. Since f_n is irreducible, there exists an Auslander-Reiten sequence of the form

$$0 \to \tau_{\mathscr{K}} X_n \xrightarrow{[g_n\ *]} X_{n-1} \oplus X'_{n-1} \xrightarrow{\left[\begin{smallmatrix} f_n \\ * \end{smallmatrix}\right]} X_n \to 0.$$

Assume for some $t \geqq 2$, the vertices $[x_i]$, $t \leqq i \leqq n$, are non-projective in $\Gamma(\mathscr{K})$. Then, for $n > i \geqq t$, we obtain inductively Auslander-Reiten sequences

$$0 \to \tau_{\mathscr{K}} X_i \xrightarrow{[g_i\ *\ *]} X_{i-1} \oplus \tau_{\mathscr{K}} X_{i+1} \oplus X'_{i-1} \xrightarrow{\left[\begin{smallmatrix} f_i \\ g_{i+1} \\ * \end{smallmatrix}\right]} X_i \to 0.$$

Here, we use that $\tau_{\mathscr{K}} X_{i+1} \not\approx X_{i-1}$. Since $f_1 \cdots f_n = 0$, there exists $h_n: X_0 \to \tau_{\mathscr{K}} X_n$ with $h_n g_n = f_1 \cdots f_{n-1}$. Inductively, we obtain for $n > i \geqq t$ maps $h_i: X_0 \to \tau_{\mathscr{K}} X_i$ with $h_i g_i = f_1 \cdots f_{i-1}$. If $t = 2$, then $h_t g_t = f_1$ is a factorization of the irreducible map f_1 with g_t not a split epimorphism, thus, h_t has to be a split monomorphism, thus an isomorphism, but this means $X_0 \approx \tau_{\mathscr{K}} X_2$, impossible. Thus, $t \geqq 3$, and we can assume that $[X_{t-1}]$ is a projective vertex in $\Gamma(\mathscr{K})$. The sink map for X_{t-1} is of the form

$$X_{t-2} \oplus \tau_{\mathscr{K}} X_t \oplus X'_{t-2} \xrightarrow{\left[\begin{smallmatrix} f_{t-1} \\ g_t \\ * \end{smallmatrix}\right]} X_{t-1},$$

and by assumption, this is a monomorphism in $\mathscr{K}$. Now

$$[f_1 \cdots f_{t-2}, -h_t, 0]\begin{bmatrix} f_{t-1} \\ g_t \\ * \end{bmatrix} = 0,$$

thus $f_1 \cdots f_{t-2} = 0$. But this contradicts the minimality of n. This finishes the proof.

Remark. This lemma gives a direct proof to a result due to Bautista and Smalø [4]: The Auslander-Reiten quiver of an artin algebra does not have sectional cyclic paths.

Next, consider the case $\Delta = \mathbb{A}_\infty^\infty$. We assume that $\mathbb{A}_\infty^\infty$ has a set of vertices the set $\mathbb{Z}$ of integers, and that there are arrows $a \to a+1$, for any $a \in \mathbb{Z}$. It follows that the set of vertices of $\mathbb{Z}\mathbb{A}_\infty^\infty$ is $\mathbb{Z} \times \mathbb{Z}$, with arrows $(i, a) \to (i, a+1)$ and $(i, a) \to (i+1, a-1)$, for all $i, a \in \mathbb{Z}$. Let g be an additive function on $\mathbb{Z}\mathbb{A}_\infty^\infty$. Then, for all $i, j, a, b \in \mathbb{Z}$,

$$g((i, a)) + g((j, b)) = g((i, b+j-i)) + g((j, a+i-j)).$$

In particular

$$g((i, a)) + g((0, 0)) = g((i, -i)) + g((0, a+i)),$$

for all $i, a \in \mathbb{Z}$. Assume that g takes values in $\mathbb{N}_0$, and let $l_0 = g((0, 0))$. If $g((i, a)) = l$, then both

$$g((i, -i)) \leqq l + l_0 \quad \text{and} \quad g((0, a+i)) \leqq l + l_0.$$

Assume there are infinitely many vertices (i, a) with $g((i, a)) = l$. If there exists some fixed i_0, and infinitely many different a_s, $s \in \mathbb{N}$, with $g((i_0, a_s)) = l$ for all $s \in \mathbb{N}$, then there are infinitely many b_s, $s \in \mathbb{N}$, with $g((0, b_s)) \leqq l + l_0$, namely all the $b_s = a_s + i_0$. Otherwise, there must exist infinitely many different i_s, $s \in \mathbb{N}$, and for any i_s some a_s' with $g((i_s, a_s')) = l$; in this case $g((i_s, -i_s)) \leqq l + l_0$. It follows that for some fixed value l' ($\leqq l + l_0$), there are infinitely many different vertices (i, a) with $g((i, a)) = l'$ and $i = 0$ or $i + a = 0$.

We apply this to the given additive function $f(x) = |\pi(x)|$. We see that there is a fixed value l' and infinitely many different $a \in \mathbb{Z}$ such that either $f((0, a)) = l'$ or $f((-a, a)) = l'$. First, assume there are infinitely many different $a \in \mathbb{Z}$ such that $f((0, a)) = l'$. Take integers u, v such that there are at least $2^{l'}$ different $a \in \mathbb{Z}$ with $u \leqq a \leqq v$ and $f((0, a)) = l'$. For every $b \in \mathbb{Z}$, choose a representative $M(b) \in \pi((0, b))$, note that there exists an irreducible map $\alpha_b \colon M(b) \to M(b+1)$. The Harada-Sai lemma [6, 10] asserts that $\alpha_u \alpha_{u+1} \cdots \alpha_{v-1} = 0$, whereas the lemma above yields that this composition is non-zero. In order to be able to apply our lemma, we have to observe that for all $i, a \in \mathbb{Z}$, we have $\pi((i-1, a)) \neq \pi((i, a-2))$, since π is assumed to be a covering. Similarly, we proceed in the case that there are infinitely many $a \in \mathbb{Z}$ such that $f((-a, a)) = l'$. This finishes the proof for $\Delta = \mathbb{A}_\infty^\infty$.

Consider now $\Delta = \mathbb{B}_\infty$, $\mathbb{C}_\infty$, *or* $\mathbb{D}_\infty$. The set of vertices of Δ is $\mathbb{N}_0$ for $\Delta = \mathbb{B}_\infty$, $\mathbb{C}_\infty$, and $\mathbb{N}_0 \cup \{0'\}$ for $\Delta = \mathbb{D}_\infty$; there are arrows $a \to a+1$, for $a \in \mathbb{N}_0$, and, for $\Delta = \mathbb{D}_\infty$, an additional arrow $0' \to 1$; the only nontrivial valuation is for the arrow $0 \to 1$ for $\Delta = \mathbb{B}_\infty$ and $\mathbb{C}_\infty$, namely $d_{01} = 1$, $d_{01}' = 2$ for $\Delta = \mathbb{B}_\infty$, and $d_{01} = 2$, $d_{01}' = 1$ for $\Delta = \mathbb{C}_\infty$.

The function f on $\mathbb{Z}\Delta$ defines a function g on $\mathbb{Z}\mathbb{A}_\infty^\infty$ as follows: let

$$g((i, a)) = \begin{cases} f((i, a)) & \text{for} \quad a \geqq 1 \\ f((i + a, -a)) & \phantom{\text{for}} \quad a \leqq -1 \end{cases},$$

and

$$g((i, 0)) = \begin{cases} 2 f((i, 0)) & \Delta = \mathbb{B} \\ f((i, 0)) & \text{provided} \quad \Delta = \mathbb{C}. \\ f((i, 0)) + f((i, 0')) & \Delta = \mathbb{D} \end{cases}$$

One easily checks that g is additive on $\mathbb{Z}\mathbb{A}_\infty^\infty$, since f is additive on $\mathbb{Z}\Delta$. It follows that for some fixed value l' there are infinitely many different vertices $(i, a) \in \mathbb{Z} \times \mathbb{Z}$ with $g((i, a)) = l'$ and $i = 0$ or $i + a = 0$. Consequently, there are infinitely many different $a \in \mathbb{N}_1$ such that either $f((0, a)) = l'$ or $f((-a, a)) = l'$. Again, we obtain a contradiction by using both the Harada-Sai lemma and our lemma above. This finishes the proof.

Remark 1. The special case of Δ being a star has been treated before in [1].

Remark 2. It is obvious that for $\Delta = \mathbb{A}_\infty^\infty, \mathbb{B}_\infty, \mathbb{C}_\infty, \mathbb{D}_\infty$, there are no finite coverings $\mathbb{Z}\Delta \to \Gamma$. This shows that under our assumptions, any component Γ with a covering $\mathbb{Z}\Delta \to \Gamma$, $\Delta = \mathbb{A}_\infty^\infty, \mathbb{B}_\infty, \mathbb{C}_\infty, \mathbb{D}_\infty$ is actually of the form $\mathbb{Z}\Delta$. For $\mathscr{A} = A$-mod, this has been observed in [5] in the cases $\Delta = \mathbb{B}_\infty, \mathbb{C}_\infty, \mathbb{D}_\infty$, and in [3] in the case $\Delta = \mathbb{A}_\infty^\infty$.

References

[1] M. AUSLANDER, R. BAUTISTA, M. I. PLATZECK, I. REITEN and S. O. SMALØ, Almost split sequences whose middle term has at most two indecomposable summands. Canad. J. Math. **31**, 942–960 (1979).

[2] K. BONGARTZ, On a result of Bautista and Smalø on cycles. Comm. Algebra **11**, 2123–2124 (1983).

[3] M. C. R. BUTLER and C. M. RINGEL, Auslander-Reiten sequences with few middle terms, with applications to string algebras. Comm. Algebra **15**, 145–179 (1987).

[4] R. BAUTISTA and S. O. SMALØ, Nonexistent cycles. Comm. Algebra **11**, 1755–1767 (1983).

[5] D. HAPPEL, U. PREISER and C. M. RINGEL, Vinberg's characterization of Dynkin diagrams using subadditive functions with application to DTr-periodic modules. LNM **832**, 280–294, Berlin-Heidelberg-New York 1980.

[6] M. HARADA and Y. SAI, On categories of indecomposable modules I. Osaka J. Math. **7**, 323–344 (1970).

[7] M. HOSHINO, Splitting torsion theories induced by tilting modules. Comm. Algebra **11**, 427–441 (1983).

[8] K. IGUSA and G. TODOROV, A characterization of finite Auslander-Reiten quivers. J. Algebra **89**, 148–177 (1984).

[9] CHR. RIEDTMANN, Algebren, Darstellungsköcher, Überlagerungen und zurück. Comment. Math. Helv. **55**, 199–224 (1980).

[10] C. M. RINGEL, Report on the Brauer-Thrall conjectures. LNM **831**, 104–136, Berlin-Heidelberg-New York 1980.

[11] C. M. RINGEL, Tame algebras and integral quadratic forms. LNM **1099**, Berlin-Heidelberg-New York 1984.

[12] C. M. RINGEL, Representation theory of finite dimensional algebras. Durham lectures. In: Representations of Algebras. London Math. Soc. Lecture Note Series. **116**, 7–79 (1986).
[13] C. M. RINGEL, The regular components of the Auslander-Reiten quiver of a tilted algebra. To appear in Chinese Ann. Math.

Eingegangen am 22. 11. 1986

Anschrift der Autoren:

Eugenia Marmolejo
Claus Michael Ringel
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 1
D-4800 Bielefeld 1