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SYMMETRIES AND VARIATION OF SPECTRA

To Sujit Kumar Mitra on his Sixtieth birthday

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ABSTRACT An interesting class of matrices is shown to have the property that the spectrum of each of its elements is invariant under multiplication by p-th roots of unity. For this class and for a class of Hamiltonian matrices improved spectral variation bounds are obtained.

1. **Introduction.** One of the basic facts of perturbation theory is that a perturbation of order ϵ in the entries of an $n \times n$ matrix leads, in general, to a perturbation of order $\epsilon^{1/n}$ in its eigenvalues [1], [10], [13], [14]. Quantitative expressions of this phenomenon are found in various *spectral perturbation bounds* of the kind described below.

Let $\mathbb{C}^n$ be the complex Euclidean space with its Euclidean norm $\| \; \|$. Let A be an $n \times n$ matrix, identified, as usual, with a linear operator on $\mathbb{C}^n$. The *operator norm* of A, also called the *bound norm* or the *spectral norm* is defined as $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$. Let A, B be two matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ respectively, each eigenvalue being counted as often as its multiplicity. A distance between these two n-tuples can be defined as

$$v(A,B) = \min_{\sigma} \max_{1 \le i \le n} |\alpha_i - \beta_{\sigma(i)}|, \tag{1}$$

where σ runs over all permutations on n symbols. One wants to find bounds for this *eigenvalue variation* in terms of the variation $\|A - B\|$ between the matrices.

It can be proved that

$$v(A,B) \le c(2M)^{1-1/n} \|A - B\|^{1/n}, \tag{2}$$

where, $M = \max(\|A\|, \|B\|)$ and c is a constant. See [1, Chapter 5] and references therein; the quantity $v(A,B)$ is denoted by $d(\operatorname{Eig} A, \operatorname{Eig} B)$ in [1].

As the discussion in [1, p. 98] shows, the exponent $1/n$ occurring in (2) is an essential feature of the general spectral variation problem. Until recently the best inequality of this type involved a constant c that depended on and grew with the dimension n. Only recently

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it has been established that a bound like (2) with the constant c bounded by 4 holds for all values of n. See [5], [12].

In other words,

$$v(A,B) \leq 4(2M)^{1-1/n}\|A-B\|^{1/n}, \tag{3}$$

for all matrices A, B of order n. Somewhat stronger results are known [5], [11]; however, we will use (3) in our discussion.

In this paper we are concerned not with improving the constant c but with the question: when can the exponent $1/n$ be improved? It is well-known that when the matrices A and B are from some special classes, then one can do better. For example, when A and B are normal then one can prove that

$$v(A,B) \leq 3\|A-B\|. \tag{4}$$

See [3], [4]. The factor 3 in the inequality (4) can be dropped when A, B are both Hermitian or both unitary and in some other cases. However, Holbrook has given examples showing that in (4) the factor 3 cannot be replaced by 1 in general. See [1] and references therein.

In [2] it was shown that when the eigenvalues of A and B have a certain symmetry, viz., the eigenvalues occur in pairs, $\pm\lambda$ then the exponent $1/n$ occurring in (2) can be improved. Matrices belonging to the Lie algebras of the complex, orthogonal group and of the complex symplectic group have this property.

Here we will show that similar improvements can be obtained for some other classes of matrices whose eigenvalues are symmetrically distributed. In Section 2 we identify an interesting class of matrices whose eigenvalue have the following property: λ is an eigenvalue iff so are $\omega\lambda, \omega^2\lambda, \ldots, \omega^{p-1}\lambda$ for some primitive p-th root of unity ω. In Section 4 we study the class of Hamiltonian matrices, whose eigenvalues are symmetric about the imaginary axis. In both cases improved spectral variation bounds are obtained.

While the motivation for our work came from our interest in eigenvalue variation, our analysis leads to some other matters of general interest; these are also discussed below.

2. Matrices with Carrollian spectra.

Let n, p, r be positive integers, $n = pr$. We will say that an n-tuple of complex numbers is *p-Carrollian* if its elements can be enumerated as

$$(\lambda_1, \ldots, \lambda_r, \omega\lambda_1, \ldots, \omega\lambda_r, \ldots, \omega^{p-1}\lambda_1, \ldots, \omega^{p-1}\lambda_r), \tag{5}$$

where ω is a primitive p-th root of unity. We will say that an $n \times n$-matrix has a *p-Carrollian spectrum* if its eigenvalues counted according to their algebraic multiplicities can be enumerated as above. In [2] and [1, p. 92] the special case $p = 2$ was examined; what was called Carrollian there becomes 2-Carrollian in our present terminology.

Let X be a matrix in a block-partitioned form having the following pattern

$$X = \begin{bmatrix} 0 & A_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & 0 & \cdots & A_{p-1} \\ A_p & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{6}$$

Here $A_1, \dots, A_p$ are square matrices of order r and X is a matrix of order $n = pr$. Let $Y = \operatorname{diag}(I_r, \omega I_r, \dots, \omega^{p-1} I_r)$, then obviously $Y^{-1}XY = \omega X$, X is similar to ωX.

Thus the matrix (6) has a p-Carrollian spectrum. (A little more generally, the same is true if the partitioning (6) is such that all the diagonal zero blocks are square but the other blocks are rectangular).

In [7] Choi proved the following interesting proposition. Let

$$Z = \begin{bmatrix} R & A_1 \\ A_2 & -R \end{bmatrix}, \tag{7}$$

where A_1, A_2 and R are $r \times r$ matrices such that R commutes with A_1. Then Z has a 2-Carrollian spectrum. Notice that we can write

$$Z = X + Y, \tag{8}$$

where

$$Y = \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}. \tag{9}$$

Choi's Proposition then says that when $p = 2$, the property of having a p-Carrollian spectrum is preserved when we perturb a matrix of the form (6) by adding to it a matrix of the form (9), provided R commutes with A_1. We shall prove that this phenomenon takes place for all values of p.

But let us remark beforehand that, though Z in (7) has a 2-Carrollian spectrum, Z is not similar to $-Z$ in general, see the example

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

THEOREM 1. *Let p be any positive integer and let $R, A_1, A_2, \dots, A_p$ be $r \times r$ matrices such that R commutes with all A_i except one. Let*

$$Z = X + Y, \tag{10}$$

where, X is as in (6) and Y is the block-diagonal matrix

$$Y = \begin{bmatrix} \alpha_1 R & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 R & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & \alpha_p R \end{bmatrix}, \tag{11}$$

$\{\alpha_1, \ldots, \alpha_p\} = \{1, \omega, \ldots, \omega^{p-1}\}$ and where ω is the primitive p-th root of unity. Then the spectrum of Z is p-Carrollian.

The proof of Theorem 1 is delayed until the end of this section.

We will call a $p \times p$ block matrix *cyclic* if it has the form displayed in (6), *i.e.*, if all the blocks except those on the first super-diagonal and the one in the southwest corner are zero. We will denote by

$$D = \operatorname{diag}(D_1, D_2, \ldots, D_p), \tag{12}$$

a $p \times p$ block-diagonal matrix. Let

$$T = D + X, \tag{13}$$

where D and X are as in (12) and (6), respectively. We call such a matrix *diagonal + cyclic block matrix*.

Every 2×2 block matrix has the above form. If D_1 is invertible such a matrix can be factorised as

$$\begin{bmatrix} D_1 & A_1 \\ A_2 & D_2 \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ A_2 & D_2 - A_2 D_1^{-1} A_1 \end{bmatrix} \begin{bmatrix} I & D_1^{-1} A_1 \\ 0 & I \end{bmatrix}. \tag{14}$$

Our first observation is that such a factorization can also be done for diagonal + cyclic block matrices of higher order. To assist the reader let us first write the 3×3 case: if D_1 and D_2 are invertible we can write

$$\begin{bmatrix} D_1 & A_1 & 0 \\ 0 & D_2 & A_2 \\ A_3 & 0 & D_3 \end{bmatrix} = LU, \tag{15}$$

where,

$$L = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ A_3 & -A_3 D_1^{-1} A_1 & D_3 + A_3 D_1^{-1} A_1 D_2^{-1} A_2 \end{bmatrix}, \tag{16}$$

$$U = \begin{bmatrix} I & D_1^{-1} A_1 & 0 \\ 0 & I & D_2^{-1} A_2 \\ 0 & 0 & I \end{bmatrix}. \tag{17}$$

In general, we have:

THEOREM 2. *Let T be a $p \times p$ diagonal + cyclic block matrix defined by (6), (12) and (13). Let $D_1, D_2, \ldots, D_{p-1}$ be invertible. Then we can factorize T as $T = LU$ where L and U are block lower and upper triangular matrices respectively, such that*

(i) $L = D + S$,

(ii) S is a $p \times p$ block matrix with all its blocks except those on the last row zero,

(iii) the last row of S has entries

$$S_{p1} = A_p,$$
$$S_{p2} = -A_p D_1^{-1} A_1,$$
$$S_{p3} = A_p D_1^{-1} A_1 D_2^{-1} A_2,$$
$$\vdots$$
$$S_{pp} = (-1)^{p-1} A_p D_1^{-1} A_1 D_2^{-1} A_2 D_3^{-1} A_3 \cdots D_{p-1}^{-1} A_{p-1},$$

(iv) $U = I + W$, *where* W *is a* $p \times p$ *block matrix all whose blocks are zero except those on the first superdiagonal and these are given by*

$$W_{12} = D_1^{-1} A_1, W_{23} = D_2^{-1} A_2, \ldots, W_{p-1,p} = D_{p-1}^{-1} A_{p-1}.$$

In particular

$$\det T = \det(D_1 \cdots D_{p-1}) \det\left(D_p + (-1)^{p-1} A_p D_1^{-1} A_1 D_2^{-1} A_2 \cdots D_{p-1}^{-1} A_{p-1}\right) \tag{18}$$

We remark that the factorization in Theorem 2 is just the block LU-decomposition used in many contexts in numerical linear algebra.

PROOF OF THEOREM 1. Apply the above Theorem 2 with $D = Y - \lambda I$, where Y is given by (11). For any λ which is not an eigenvalue of R we get from (18)

$$\det(Z - \lambda I) = \det\left(\prod_{j=0}^{p-1} (\omega^j R - \lambda) + (-1)^{p-1} A_p A_1 \cdots A_{p-1}\right) = \det(Z - \omega\lambda I),$$

as the product is unchanged when λ is replaced by $\omega\lambda$.

By continuity, this holds for all λ. Hence the spectrum of Z is p-Carrollian. ∎

The proof given above was inspired by the one given by Choi [7] for the case $p = 2$.

3. **Eigenvalue variation under Carrollian symmetry.** Following the ideas in [2] we now obtain improved spectral variation bounds for matrices with Carrollian spectra. Let

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n, \tag{19}$$
$$g(z) = z^n + b_1 z^{n-1} + \cdots + b_n, \tag{20}$$

be two monic polynomials of degree n with roots $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ respectively. Let

$$\gamma = \max\{\alpha : \exists 0 \le t \le 1, \text{ s.t. } tf(\alpha) + (1-t)g(\alpha) = 0\}. \tag{21}$$

Then the roots α_i and β_i can be enumerated in such a way that

$$\max_{1 \le i \le n} |\alpha_i - \beta_i| \le 4 \left\{ \sum_{k=1}^{n} |a_k - b_k| \gamma^{n-k} \right\}^{1/n}. \tag{22}$$

If the eigenvalues of A and B are outside a ρ-neighbourhood of 0 then

$$\nu(A,B) \leq \frac{4}{\rho^{p-1}} c_{r,p} M^{p-1/r} \|A-B\|^{1/r}. \tag{34}$$

REMARK. If $n = rp$ and ω is the primitive p-th root of 1 we can write

$$\sum_{k=0}^{r} kp\binom{n}{kp} = \sum_{k=0}^{n} k\binom{n}{k}\left\{\frac{1}{p}\sum_{j=0}^{p-1}\omega^{kj}\right\} = \frac{1}{p}\sum_{j=0}^{p-1}\sum_{k=0}^{n} k\binom{n}{k}\omega^{kj}$$
$$= \frac{n}{p}\sum_{j=0}^{p-1}\omega^{j}(1+\omega^{j})^{n-1} = r\sum_{j=0}^{p-1}\omega^{j}(1+\omega^{j})^{n-1}.$$

Here, in the first step we used the fact that $\sum_{j=0}^{p-1}\omega^{kj} = p$ if k is an integral multiple of p, and is 0 otherwise. In the third step we have used the identity $\sum_{k=0}^{n} k\binom{n}{k}x^k = nx(1+x)^{n-1}$.

For $p = 2$ the above identity reduces to the better known identity

$$\sum_{k=0}^{r} 2k\binom{n}{2k} = n2^{n-2},$$

while the case $p = 3$, $n = 3r$ yields

$$\sum_{k=0}^{r} 3k\binom{n}{3k} = r\left(2^{n-1} + (-1)^r\right)$$

(as pointed out by the referee). While for small p one can certainly get a closed form, a general simple formula seems not to be known.

The above identity however reduces the computation of $c_{r,p}$ to that of a trigonometric sum.

The following lemma gives a circumstance under which the eigenvalues of a cyclic matrix are outside a neighbourhood of zero.

LEMMA 7. *Let X be a cyclic matrix given by (6). If all the A_j, $1 \leq j \leq p$, are invertible, then the eigenvalues of X satisfy the inequality*

$$|\lambda| \geq \left[\prod_{j=1}^{p} \|A_j^{-1}\|^{-1}\right]^{1/p}. \tag{35}$$

PROOF. Let $\lambda_1(T), \ldots, \lambda_r(T)$ denote the eigenvalues of an $r \times r$ matrix T arranged in decreasing order of modulus. Then

$$\lambda_r(A_1A_2\cdots A_p) = \frac{1}{\lambda_1((A_1A_2\cdots A_p)^{-1})} \geq \frac{1}{\|A_1^{-1}\|\,\|A_2^{-1}\|\cdots\|A_p^{-1}\|}. \tag{36}$$

Since X^p is block diagonal whose diagonal blocks are $A_{\sigma(1)}A_{\sigma(2)}\cdots A_{\sigma(p)}$, where σ varies over cyclic permutations of $\{1, 2, \ldots, p\}$, the eigenvalues of X^p are the eigenvalues of $A_1A_2\cdots A_p$, each repeated p times. The inequality (35), therefore, is a consequence of (36). ∎

4. **A class of Hamiltonian matrices.** Let I_r denote the identity matrix of size r and let

$$J = \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix}.$$

A matrix A of order $n = 2r$ is called *Hamiltonian* if

$$JA = (JA)^*. \tag{37}$$

Since

$$J^{-1} = -J = J^*, \tag{38}$$

A is Hamiltonian iff

$$JAJ^{-1} = -A^*. \tag{39}$$

If a matrix of order $2r$ is partitioned into blocks of order r as

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

then it follows from (39) that A is Hamiltonian iff $A_1^* = -A_4, A_3^* = A_3, A_2^* = A_2$. Thus, Hamiltonian matrices can be characterized as those for which

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & -A_1^* \end{bmatrix}, \quad A_2 = A_2^*, A_3 = A_3^*. \tag{40}$$

We will denote the class of these matrices by $\mathcal{M}(r)$. It follows from (39) that λ is an eigenvalue of A iff so is $-\bar{\lambda}$. Thus the spectrum of a Hamiltonian matrix is symmetric about the imaginary axis.

Let $\mathcal{M}_+(r)$ denote the subclass of $\mathcal{M}(r)$ consisting of those matrices A in whose block decomposition (4) the matrices A_2 and A_3 are both positive definite. For such a matrix define

$$\rho_A = \min\big(\lambda_{\min}(A_2), \lambda_{\min}(A_3)\big), \tag{41}$$

where $\lambda_{\min}$ denotes the smallest eigenvalue.

PROPOSITION 8. *If $A \in \mathcal{M}_+(r)$ then all its eigenvalues are outside the strip* $\{z : |\operatorname{Re} z| \le \rho_A\}$.

PROOF. Let $\lambda = a + ib$ be an eigenvalue of A with $a \ge 0$. Let $\omega = (u, v)$ be the corresponding normalized eigenvector, where $u, v \in \mathbb{C}^r$. Then

$$A_1 u + A_2 v = (a + ib)u, \tag{42}$$

$$A_3 u - A_1^* v = (a + ib)v. \tag{43}$$

Take the inner product of (42) with v, of (43) with u and then the complex conjugate of the second resulting equation. This gives the pair of equations

$$\langle A_1 u, v\rangle + \langle A_2 v, v\rangle = (a+ib)\langle u, v\rangle,$$
$$\langle A_3 u, u\rangle - \langle A_1 u, v\rangle = (a-ib)\langle u, v\rangle,$$

which, in turn, lead to the equation

$$\langle A_2 v, v\rangle + \langle A_3 u, u\rangle = 2a\langle u, v\rangle. \tag{44}$$

Since ω has norm one we have

$$2\langle u, v\rangle \le 2\|u\|\,\|v\| \le \|u\|^2 + \|v\|^2 = \|w\|^2 = 1.$$

So, (44) implies

$$a \ge \langle A_2 v, v\rangle + \langle A_3 u, u\rangle. \tag{45}$$

The righthand side of (45) is a positive number lying in the numerical range of the matrix $\begin{bmatrix} A_2 & 0 \\ 0 & A_3 \end{bmatrix}$, and hence is larger than the minimum eigenvalue of this matrix. So $a \ge \rho_A$. Since the spectrum of A is symmetric about the imaginary axis the proposition follows. ∎

Notice that the class $\mathcal{M}_+(r)$ is closed under convex combinations. Further, if $A, B \in \mathcal{M}_+(r)$ and $0 \le t \le 1$, then

$$\rho_{(1-t)A+tB} \ge \min(\rho_A, \rho_B). \tag{46}$$

This is a consequence of the concavity of the function $\lambda_{\min}$ on the class of positive definite matrices.

Our next result is an improved spectral variation bound for matrices in the class $\mathcal{M}_+(r)$. The improvement is of the same kind as in Section 3. The bound involves the power $1/r$ of $\|A - B\|$ rather than $1/n$.

THEOREM 9. *Let A, B be matrices of size $n = 2r$, and let $A, B \in \mathcal{M}_+(r)$. Then*

$$v(A,B) \le 2^{1-1/r}\min(\rho_A, \rho_B)^{-1}(\|A\| + \|B\|)^{2-1/r}\|A - B\|^{1/r}, \tag{47}$$
$$v(A,B) \le 2^{2-1/r}(\rho_A + \rho_B)^{-1}(2M)^{2-1/r}\|A - B\|^{1/r}, \tag{48}$$

where, $M = \max(\|A\|, \|B\|)$.

PROOF. We shall follow the argument in [5]. Let $A(t) = (1-t)A + tB$, $0 \le t \le 1$. Each matrix in this family is in $\mathcal{M}_+(r)$ and hence has no purely imaginary eigenvalue. As t varies from 0 to 1 the eigenvalues of $A(t)$ trace out $2r$ curves, r of which lie in the right half plane and the other r are their reflections in the imaginary axis. An upper bound for the diameters of these curves will be an upper bound for $v(A, B)$.

Let Γ be one such curve in the right half plane. Let $\lambda \in \Gamma$. Let $\rho = \min(\rho_A, \rho_B)$. By (46) $\rho_{A(t)} \ge \rho$ and hence, by Proposition 8, $\operatorname{Re}\lambda \ge \rho$.

To prove (47) we may assume, without loss of generality, that $\|A\| \leq \|B\|$. If A has eigenvalues λ_i, we have

$$
\begin{aligned}
|\det(A-\lambda I)| &= \prod_{i=1}^{2r} |\lambda - \lambda_i| \\
&= \prod_{\operatorname{Re}\lambda_i<0} |\lambda - \lambda_i| \cdot \prod_{\operatorname{Re}\lambda_i>0} |\lambda - \lambda_i| \qquad (49)\\
&\geq (2\rho)^r \prod_{\operatorname{Re}\lambda_i>0} |\lambda - \lambda_i|,
\end{aligned}
$$

because $\operatorname{Re}\lambda \geq \rho$ and $\operatorname{Re}\lambda_i \leq \rho$ for all those λ_i which are in the left half plane.

Now if a, b are any two points on Γ and λ_i, $i = 1, \ldots, r$ are any r points in the plane then there exists a point λ on Γ between a and b such that

$$\prod_{i=1}^{r} |\lambda - \lambda_i| \geq \frac{|a-b|^r}{2^{2r-1}}.$$

See [5, Lemma 1]. So, from (49) we obtain that between any two points a, b on Γ there exists a point λ on Γ such that

$$|\det(A-\lambda I)| \geq \frac{\rho^r |a-b|^r}{2^{r-1}}. \tag{50}$$

On the other hand, we have from [8]

$$|\det(A-\lambda I)| \leq \|A-B\|(\|A\|+\|B\|)^{2r-1}. \tag{51}$$

It follows from (50) and (51) that $|a-b|$ is bounded by the righthandside of (47). As remarked earlier, this proves (47).

To prove (48) assume, without loss of generality, that $\rho_A \geq \rho_B$. (This fixes A and B and we will now not be able to assume $\|A\| \leq \|B\|$ as done in proving (47)). Instead of (49) we can write, with the same notations,

$$|\det(A-\lambda I)| \geq (\rho_A+\rho_B)^r \prod_{\operatorname{Re}\lambda_i>0} |\lambda - \lambda_i|, \tag{52}$$

because $\operatorname{Re}\lambda \geq \rho_B$ and $\operatorname{Re}\lambda_i \leq \rho_A$ for all λ_i in the left half plane. Instead of (50) and (51) we now have

$$\det(A-\lambda I) \geq \frac{(\rho_A+\rho_B)^r |a-b|^r}{2^{2r-1}}, \tag{53}$$

and

$$\det(A-\lambda I) \leq \|A-B\|(2M)^{2r-1}. \tag{54}$$

The inequalities (53) and (54) lead to (48) as before. ∎

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