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The braid group action on the set of exceptional sequences of a hereditary artin algebra

CLAUS MICHAEL RINGEL

ABSTRACT. Let A be a hereditary artin algebra with $s = s(A)$ simple modules. The indecomposable A-modules without self-extensions are of great importance, they may be called exceptional modules. Certain sequences $(X_1, \dots, X_s)$ consisting of exceptional modules will be called complete exceptional sequences. Crawley-Boevey has pointed out that the braid group on $s-1$ generators acts naturally on the set of complete exceptional sequences. In case A is finite-dimensional over an algebraically closed field, he has shown that this action is transitive, using a recent result by Schofield. We are going to present a direct proof which is valid for arbitrary hereditary artin algebras. It follows that the endomorphism rings of exceptional modules are just those rings which occur as endomorphism rings of the simple modules. Also, we will exhibit the relationship between complete exceptional sequences and tilting modules.

1. Exceptional modules

Let A be a hereditary artin algebra with $s = s(A)$ simple modules. We recall that an artin algebra is called *hereditary*, provided that its global dimension is at most 1, thus provided that we have $\mathrm{Ext}^2(X, Y) = 0$ for all A-modules X, Y. Note that the center of a hereditary artin algebra is semisimple. An artin algebra A is said to be *connected* provided that the center of A is a field, say k, and then A is actually a finite dimensional k-algebra. We usually will assume that A is connected. The modules we consider will always be finite length modules.

An A-module M is called *exceptional* provided that M is indecomposable and $\mathrm{Ext}^1(M, M) = 0$. The exceptional modules are of great importance for the representation theory of A. (These modules also have been called indecomposable partial tilting modules, Schur modules, or open bricks; the variety of such names indicates

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that these modules have been considered in various circumstances and that several mathematicians have felt that they deserve a name. Since there does not yet exist a unified terminology for dealing with exceptional modules, we will provide a dictionary at the end of the note. The use of the word 'exceptional' is parallel to the analogous terminology introduced for vector bundles by Rudakov, see [Ru].)

We denote by $S_1, \dots, S_s$ a complete set of simple A-modules (one from each isomorphism class); clearly these modules are exceptional, and our aim is to outline an inductive procedure for obtaining all exceptional A-modules starting from the simple A-modules.

Note that the endomorphism ring of an exceptional A-module is a division ring and one may ask which division rings can arise as endomorphism rings of exceptional modules. We will see that the only ones which arise in this way are the algebras $\operatorname{End}(S_i)$.

Attached to the algebra A is a corresponding Kac-Moody algebra, and thus a root system. Namely, we define a generalized Cartan matrix $\Delta(A) = (\Delta_{ij})_{ij}$ as follows: Given two simple modules S_i, S_j, we have $\operatorname{Ext}^1(S_i, S_j) = 0$ or $\operatorname{Ext}^1(S_j, S_i) = 0$. Let us assume that $i \neq j$ and $\operatorname{Ext}^1(S_j, S_i) = 0$. We may consider $\operatorname{Ext}^1(S_i, S_j)$ as a vector space over the division ring $\operatorname{End}(S_i)$ or over the division ring $\operatorname{End}(S_j)$, and we define

$$\Delta_{ij} = -\dim_{\operatorname{End}(S_i)} \operatorname{Ext}^1(S_i, S_j),$$
$$\Delta_{ji} = -\dim_{\operatorname{End}(S_j)} \operatorname{Ext}^1(S_i, S_j).$$

If we denote the k-dimension of $\operatorname{End}(S_i)$ by d_i, then we have

$$d_i \Delta_{ij} = d_j \Delta_{ji}.$$

This shows that $\Delta(A)$ is a symmetrizable generalized Cartan matrix in the sense of [K3].

Given any $n \times n$-matrix Δ which is a generalized Cartan matrix, we may consider the corresponding root system. Here, we are only interested in the real roots; in the case of a symmetrizable generalized Cartan matrix they may be defined as follows: We consider the n-dimensional real space $\mathbf{R}^n$ with basis $\mathbf{e}_1, \dots \mathbf{e}_n$, and we define a symmetric bilinear form given by $(\mathbf{e}_i, \mathbf{e}_j) = d_i \Delta_{ij}$. For any vector $\mathbf{x} \in \mathbf{R}^n$, with $(\mathbf{x}, \mathbf{x}) \neq 0$, let $r_{\mathbf{x}}$ be the reflection relative to $\mathbf{x}$ with respect to this bilinear form; thus, for $\mathbf{y} \in \mathbf{R}^n$,

$$r_{\mathbf{x}}(\mathbf{y}) = \mathbf{y} - \frac{(\mathbf{y}, \mathbf{x})}{(\mathbf{x}, \mathbf{x})}\mathbf{x}.$$

For $1 \le i \le n$, let $r_i = r_{\mathbf{e}_i}$. Note that any r_i maps the subgroup $\mathbf{Z}^n$ of $\mathbf{R}^n$ into itself. The group generated by these reflections r_i is called the *Weyl group*. By definition, the *real roots* are those elements of $\mathbf{Z}^n$ which belong to the orbits of the base vectors $\mathbf{e}_i$ under the Weyl group. The canonical generators $\mathbf{e}_i$ of $\mathbf{Z}^n$ are called the *simple roots*. For any real root $\mathbf{x}$, the reflection $r_{\mathbf{x}}$ is defined and belongs to W, thus it maps the set of real roots into itself.

We may identify $\mathbb{Z}^n$ with the Grothendieck group $K_0(A)$ of all A-modules modulo exact sequences; the element of $K_0(A)$ attached to the A-module M will be called its *dimension vector* and denoted by $\mathbf{dim}\, M$; we identify $\mathbb{Z}^n$ and $K_0(A)$ so that we have $\mathbf{dim}\, S_i = \mathbf{e}_i$. We will show that the dimension vector $\mathbf{dim}\, X$ of any exceptional module X is a real root for $\Delta(A)$.

The results mentioned above are known (and some are trivial) in the case when k is an algebraically closed field; so our main interest lies in the case of an arbitrary base field k. There are some remarks by Kac [K1,K2] concerning this general case, but no proofs seem to be available. The inductive construction of the exceptional modules has been shown, for k algebraically closed, by Schofield [S]. The operation of the braid group for obtaining all exceptional modules from the simple ones was, again for k algebraically closed, introduced by Crawley-Boevey [CB]. The direct algorithm we present here may be new even in the case of an algebraically closed base field.

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2. Preprojective and preinjective modules

We have noted above that the simple A-modules are exceptional modules. Some other exceptional modules are always known. In case A is representation finite (this means: there are only finitely many isomorphism classes of indecomposable modules), then all indecomposable A-modules are exceptional; otherwise, there are countable families of indecomposable modules which are exceptional, namely the preprojective ones and the preinjective ones. Let us recall the corresponding constructions. For $1 \le i \le s$, we denote by P_i the projective cover of S_i, by Q_i the injective envelope of S_i. Then $P_1, \dots, P_s$ is a complete set of indecomposable projective modules, $Q_1, \dots, Q_s$ is a complete set of indecomposable injective modules, and all these modules P_i, Q_i are exceptional.

We denote by $\tau = DTr$ the Auslander-Reiten translation; given any indecomposable non-projective module M, the module τM again is indecomposable, it is the left hand term of an almost split sequence ending in M. Similarly, we denote by $\tau^- = TrD$ its partial inverse: given any indecomposable non-injective module M, the module $\tau^- M$ is indecomposable, it is the right hand term of an almost split sequence starting in M, see [AR]. If M is exceptional, but not projective, then τM is exceptional again; similarly, if M is exceptional, and not injective, then $\tau^- M$ is exceptional. (This follows from the fact that the restriction of τ to the full subcategory of modules without non-zero projektive direct summands is an equivalence onto the full subcategory of modules without non-zero injective direct summands, its inverse being given by τ^-.)

The indecomposable modules of the form $\tau^{-t} P_i$ are said to be *preprojective,* those of the form $\tau^t Q_i$ are said to be *preinjective.* All these modules are exceptional. We note the following: If A is representation finite, then all the indecomposable A-modules are both preprojective as well as preinjective. If A is not representation finite (and connected), then for any $t \in \mathbb{N}_0$ and any $1 \le i \le s$, the modules $\tau^{-t} P_i$

and $\tau^t Q_i$ are indecomposable modules which are pairwise non-isomorphic, and there are additional indecomposable modules, called the *regular* ones.

In case $s(A) = 2$, the exceptional modules are just the preprojective and the preinjective ones; this case will be considered in detail later, since the algorithm for constructing all exceptional modules will be based on dealing with full exact subcategories which are equivalent to the module categories of hereditary artin algebras B with $s(B) = 2$. In case A is representation infinite and $s(A) > 2$ (and A is connected), then there do exist exceptional modules which are neither preprojective nor preinjective, see [R4].

3. The algorithm for obtaining all exceptional modules

For any hereditary artin algebra A, a pair (X, Y) of exceptional A-modules is called an *exceptional pair* provided that we have

$$\operatorname{Hom}(Y, X) = 0, \quad \operatorname{Ext}^1(Y, X) = 0.$$

An exceptional pair (X, Y) will be called *orthogonal*, provided that also

$$\operatorname{Hom}(X, Y) = 0$$

is satisfied. If (X, Y) is an orthogonal exceptional pair, we may consider the category $\mathcal{C}(X, Y)$ of all A-modules M which have a filtration with factors isomorphic to X and Y. This is an exact abelian subcategory with two simple objects, namely X and Y, it is equivalent to the category of all B-modules of a hereditary artin algebra B with $s(B) = 2$ (see, for example [R1], section 1). Since $\mathcal{C}(X, Y)$ is equivalent to the module category for a hereditary artin algebra B, we may consider those objects in $\mathcal{C}(X, Y)$ which correspond under an equivalence to preprojective, or preinjective B-modules, and we will call them preprojective or preinjective objects in $\mathcal{C}(X, Y)$, respectively. Since $\mathcal{C}(X, Y)$ is closed under extensions inside the category of all A-modules, we see that the preprojective and the preinjective objects in $\mathcal{C}(X, Y)$ are exceptional A-modules.

For any natural number n, we define a class $\mathcal{E}_n = \mathcal{E}_n(A)$ of indecomposable modules of length n inductively as follows: Let $\mathcal{E}_1$ be the simple A-modules. Let us asusme now that for some $n > 1$ the classes $\mathcal{E}_1, \ldots, \mathcal{E}_{n-1}$ already have been defined. Let $\mathcal{E}_n$ be the class of indecomposable A-modules M of length n with the following property: there is an orthogonal exceptional pair (X, Y) with both modules X, Y in $\bigcup_{i=1}^{n-1} \mathcal{E}_i$ such that M is preprojective or preinjective in $\mathcal{C}(X, Y)$. Finally, let $\mathcal{E} = \bigcup_{i \geq 1} \mathcal{E}_i$. We obtain in this way a class of exceptional A-modules and we claim that all exceptional A-modules belong to this class:

THEOREM 1. *The class $\mathcal{E}$ is the class of all exceptional A-modules.*

The proof will be given below using the braid group operation on the set of the socalled exceptional sequences.

COROLLARY 1. *If X is an exceptional A-module, then* $\operatorname{End}(X)$ *is isomorphic to* $\operatorname{End}(S)$ *for some simple A-module S.*

Proof: If (X,Y) is an orthogonal pair of exceptional A-modules, and M is preprojective or preinjective in $\mathcal{C} = \mathcal{C}(X,Y)$, then $\mathrm{End}(M)$ is isomorphic to $\mathrm{End}(X)$ or to $\mathrm{End}(Y)$. For example, assume that M is preprojective in $\mathcal{C}$. Observe that Y is simple projective in $\mathcal{C}$. The projective cover Z of X in $\mathcal{C}$ is an indecomposable module with endomorphism ring isomorphic to $\mathrm{End}(X)$. We denote by $\tau_{\mathcal{C}}$ the Auslander-Reiten translation in $\mathcal{C}$. Since M is preprojective in $\mathcal{C}$, it is either of the form $\tau_{\mathcal{C}}^{-t}Y$ for some $t \in \mathbb{N}_0$, and then its endomorphism ring is isomorphic to $\mathrm{End}(Y)$, or else it is of the form $\tau_{\mathcal{C}}^{-t}Z$ for some $t \in \mathbb{N}_0$, and then its endomorphism ring is isomorphic to $\mathrm{End}(Z)$, and therefore isomorphic to $\mathrm{End}(X)$. In a similar way one deals with the case when M is preinjective.

COROLLARY 2. *If X is an exceptional A-module, then* $\mathbf{dim}\,X$ *is a real root for* $\Delta(A)$.

Proof: It is well-known that the bilinear form $\langle -,- \rangle$ on $\mathbb{Z}^n = K_0(A)$ has the following homological interpretation: Given A-modules M, N, let

$$\langle \mathbf{dim}\,M, \mathbf{dim}\,N \rangle = \dim_k \mathrm{Hom}(M,N) - \dim_k \mathrm{Ext}^1(M,N).$$

Then, according to [R1], Lemma 2.2,

$$(\mathbf{dim}\,M, \mathbf{dim}\,N) = \langle \mathbf{dim}\,M, \mathbf{dim}\,N \rangle + \langle \mathbf{dim}\,N, \mathbf{dim}\,M \rangle.$$

Assume now that (X,Y) is an orthogonal pair of exceptional modules, and assume that we know already that $\mathbf{dim}\,X$ and $\mathbf{dim}\,Y$ are real roots. Let W_{XY} be the subgroup of W generated by $r_{\mathbf{dim}\,X}$ and $r_{\mathbf{dim}\,Y}$. Then the dimension vectors of the preprojective and the preinjective objects in $\mathcal{C}$ belong to the orbits of $\mathbf{dim}\,X$ and $\mathbf{dim}\,Y$ under the operation of W_{XY}, see [R1], 3.2. It follows that the dimension vectors of the preprojective and the preinjective objects in $\mathcal{C}$ are real roots for $\Delta(A)$.

4. The exceptional pairs in the case $s = 2$.

Let us now assume that $s = s(A) = 2$, and let P, Q be the simple modules; we may assume that $\mathrm{Ext}^1(P,Q) = 0$.

LEMMA. *If M is an exceptional module, there are exceptional modules M^-, M^+, unique up to isomorphism, such that (M^-, M) and (M, M^+) are exceptional pairs.*

Proof: An exceptional module M is an indecomposable A-module which is either preprojective or preinjective.

In case A is representation finite, let $P_1, \dots, P_m$ be the indecomposable modules, with

$$\mathrm{Hom}(P_i, P_{i+1}) \neq 0, \quad \text{for all} \quad 1 \le i < m.$$

In particular, we have $P_1 = P, P_m = Q$. Let $P_i^+ = P_{i+1}$ for $1 \le i < m$, and $P_i^- = P_{i-1}$ for $1 < i \le m$. Let $P_1^- = P_m$, and $P_m^+ = P_1$.

(Note that in this case $m = 2$, 3, 4 or 6; the corresponding Cartan matrices $\Delta(A)$ are labelled $\mathsf{A}_1 \times \mathsf{A}_1, \mathsf{A}_2, \mathsf{B}_2$ and G_2, respectively - these are just the Cartan matrices arising for the finite-dimensional semisimple Lie algebras of rank 2.)

In case A is representation infinite, there are countable many indecomposable preprojective modules and countably many indecomposable preinjective modules, and we may label them in the following way (see again [R1]): let $\{P_i \mid i \in \mathbb{N}_1\}$ be the indecomposable preprojective modules, and $\{Q_i \mid i \in \mathbb{N}_1\}$ the indecomposable preinjective modules with

$$\operatorname{Hom}(P_i, P_{i+1}) \neq 0, \quad \text{and} \quad \operatorname{Hom}(Q_{i+1}, Q_i) \neq 0 \quad \text{for all} \quad i \geq 1.$$

It follows that $P_1 = P$, $Q_1 = Q$. In this case, $P_i^+ = P_{i+1}$, and $Q_i^- = Q_{i+1}^-$, for all $i \geq 1$; similarly, $P_i^- = P_{i-1}, Q_i^+ = Q_{i-1}$ for all $i \geq 2$, finally, let $P_1^- = Q_1, Q_1^+ = P_1$.

Note that all the exceptional pairs (X, Y) different from (Q, P) satisfy

$$\operatorname{Hom}(X, Y) \neq 0.$$

5. Exceptional sequences and the braid group

A sequence $\mathcal{X} = (X_1, \ldots, X_n)$ is called *exceptional,* provided that any pair (X_i, X_j) with $i < j$ is exceptional. Actually, we only are interested in isomorphism classes of modules, not in the modules themselves: thus, an exceptional sequence will be considered as a sequence of isomorphism classes. An exceptional sequence $\mathcal{X} = (X_1, \ldots, X_n)$ with $n = s(A)$ is said to be *complete.*

Recall that the braid group B_n in $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ is the free group with these generators and the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $1 \leq i < n-1$, and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $j \geq i+2$.

Given an exceptional sequence $(X_1, \ldots, X_n)$, we denote by $\mathcal{C}(X_1, \ldots, X_n)$ the closure of the full subcategory with objects $X_1, \ldots, X_n$ under kernels, images, and extensions. Of course, this is an exact abelian subcategory (in case (X, Y) is an orthogonal exceptional pair, this subcategory $\mathcal{C}(X, Y)$ coincides with the full subcategory of all modules with a filtration with factors isomorphic to X and Y).

The main ingredients for the definition of the operation of B_n on the set of complete exceptional sequences are the following three observations due to Crawley-Boevey:

PROPOSITION 1. *Let $(X_1, \ldots, X_n)$ be an exceptional sequence. $\mathcal{C}(X_1, \ldots, X_n)$ is equivalent to the category of all B-modules for some hereditary artin algebra B with $s(B) = n$.*

PROPOSITION 2. *Let $(X_1, \ldots, X_n)$ be an exceptional sequence of A-modules. If (Y, Y') is an exceptional pair in $\mathcal{C}(X_i, X_{i+1})$, then*

$$(X_1, \ldots, X_{i-1}, Y, Y', X_{i+2}, \ldots, X_n)$$

is an exceptional sequence of A-modules.

PROPOSITION 3. *Let $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_s)$ be an exceptional sequence, where $1 \leq i \leq s$. Then there exists a unique module X_i such that $(X_1, \ldots, X_s)$ is a complete exceptional sequence.*

For a proof of these assertions, we refer to [CB]; his assumption concerning the base field is not used in this part of the paper. We should mention that these results of Crawley-Boevey were motivated by the work mainly of Gorodentsev, Rudakov, and Bondal, but also others, dealing with exceptional sequences of vector bundles (see the papers [GR], [G], [B] and the collection [Ru]); in these papers, a corresponding action of the braid group had been introduced. The natural setting of such braid group actions seems to be in the context of triangulated categories (say the corresponding derived categories). Actually, as Crawley-Boevey has pointed out, one may use the arguments of [G], section 3.3, in order to obtain also the braid group operation on the set of exceptional sequences of a hereditary artin algebra. The proof presented in [CB] is rather straight-forward, it only uses properties of perpendicular categories.

Let (X,Y) be an exceptional pair. Recall that $\mathcal{C}(X,Y)$ is equivalent to the category of all B-modules, where B is a hereditary artin algebra with $s(B)=2$. Thus, we may consider inside $\mathcal{C}(X,Y)$ the exceptional module Y^+; our notation will be $r(X,Y)=Y^+$, this module is the unique object in $\mathcal{C}(X,Y)$ such that $(Y,r(X,Y))$ is again an exceptional sequence. Similarly, let $l(X,Y)=X^-$ be the unique object in $\mathcal{C}(X,Y)$ such that $(l(X,Y),X)$ is an exceptional pair.

Given an exceptional sequence $\mathcal{X}=(X_1,\dots,X_n)$ of A-modules and $1\le i<n$, we define

$$\sigma_i(\mathcal{X})=(X_1,\dots,X_{i-1},X_{i+1},r(X_i,X_{i+1}),X_{i+2},\dots,X_n),$$

and

$$\sigma_i^{-1}(\mathcal{X})=(X_1,\dots,X_{i-1},l(X_i,X_{i+1}),X_i,X_{i+2},\dots,X_n).$$

In this way, we obtain an action of the braid group in $n-1$ generators $\sigma_1,\dots,\sigma_{n-1}$ on the set of exceptional sequences: If $i+2\le j$, then both $\sigma_i\sigma_j(\mathcal{X})$ and $\sigma_j\sigma_i(\mathcal{X})$ will be equal to

$$(\dots,X_{i+1},r(X_i,X_{i+1}),\dots,X_{j+1},r(X_j,X_{j+1}),\dots)$$

(the positions marked are those with index $i,i+1$ and $j,j+1$). On the other hand, let

$$Y=r\big(r(X_i,X_{i+1}),X_{i+2}\big)\quad\text{and}\quad Y'=r\big(r(X_i,X_{i+2}),r(X_{i+1},X_{i+2})\big).$$

Then

$$\sigma_i\sigma_{i+1}\sigma_i(\mathcal{X})=(\dots,X_{i+2},r(X_{i+1},X_{i+2}),Y,\dots)$$

whereas

$$\sigma_{i+1}\sigma_i\sigma_{i+1}(\mathcal{X})=(\dots,X_{i+2},r(X_{i+1},X_{i+2}),Y',\dots)$$

(only the positions $i,i+1,i+2$ are labelled).

Since $\big(X_{i+2},r(X_{i+1},X_{i+2}),Y\big)$ and $\big(X_{i+2},r(X_{i+1},X_{i+2}),Y'\big)$ are complete exceptional sequences in $\mathcal{C}(X_i,X_{i+1},X_{i+2})$, it follows from Proposition 3 that $Y=Y'$.

6. The reduction theorem

Let $\mathcal{X} = (X_1, \dots, X_n)$ be an exceptional sequence, let $1 \leq i < n$. We say that σ_i is a *transposition* for $\mathcal{X}$, provided that for $\mathcal{Y} = \sigma_i \mathcal{X}$, we have $Y_{i+1} = X_i$ (so that $\mathcal{Y}$ is obtained from $\mathcal{X}$ by just transposing X_i and X_{i+1}). The following assertion is easy to verify.

LEMMA. *Let $\mathcal{X} = (X_1, \dots, X_n)$ be an exceptional sequence, let $1 \leq i < n$. Then σ_i is a transposition for $\mathcal{X}$ if and only if* $\operatorname{Hom}(X_i, X_{i+1}) = 0$, *and* $\operatorname{Ext}^1(X_i, X_{i+1}) = 0$.

LEMMA. *Let $\mathcal{X} = (X_1, \dots, X_n)$ be an exceptional sequence, let $1 \leq i < n$. Then there exists $t \in \mathbb{Z}$ such that $\mathcal{Y} = \sigma_i^t \mathcal{X}$ satisfies* $\operatorname{Hom}(Y_i, Y_{i+1}) = 0$.

PROOF. We apply the considerations of section 4 to $\mathcal{C}(X_i, X_{i+1})$.

We say that σ_i^t is a *proper reduction* for $\mathcal{X}$, provided that $\mathcal{Y} = \sigma_i^t \mathcal{X}$ satisfies $\operatorname{Hom}(Y_i, Y_{i+1}) = 0$, whereas $\operatorname{Hom}(X_i, X_{i+1}) \neq 0$.

An exceptional sequence $\mathcal{X} = (X_1, \dots, X_n)$ is said to be *orthogonal*, provided that we have $\operatorname{Hom}(X_i, X_j) = 0$ for all $i \neq j$.

THEOREM 2. *Any exceptional sequence can be shifted by the braid group action to an orthogonal sequence using only transpositions and proper reductions.*

In order to give the proof, we need some preparations. For any A-module M, let $|M|$ be its length. For a sequence $\mathcal{X} = (X_1, \dots, X_n)$ of A-modules, let $\|\mathcal{X}\| = (|X_{\pi(1)}|, \dots, |X_{\pi(n)}|)$, where π is a permutation of $1, \dots, n$ such that $|X_{\pi(1)}| \geq \cdots \geq |X_{\pi(n)}|$. For sequences $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in $\mathbb{N}_0$, we write $x \leq y$ provided that $x_i \leq y_i$, for alle $1 \leq i \leq n$, and $x < y$, provided that $x \leq y$ and $x \neq y$.

LEMMA. *If σ_i is a transposition for $\mathcal{X}$, then $\|\sigma_i \mathcal{X}\| = \|\mathcal{X}\|$. Is σ_i^t is a proper reduction for $\mathcal{X}$, then $\|\sigma_i^t \mathcal{X}\| < \|\mathcal{X}\|$.*

PROOF. Let $\sigma_i \mathcal{X} = (Y_1, \dots, Y_n)$. Then $Y_j = X_j$ for $j \notin \{i, i+1\}$, whereas Y_i, Y_{i+1} are the simple objects in the category $\mathcal{C}(X_i, X_{i+1})$. Since we assume that σ_i^t is a proper reduction, at least one of the modules X_i, X_{i+1} cannot be simple in $\mathcal{C}(X_i, X_{i+1})$. However, if M is an indecomposable non-simple object in $\mathcal{C}(X_i, X_{i+1})$, then M has a filtration with factors of the form X_i and X_{i+1}, and both types of factors appear at least once.

REMARK. If σ_i^t is a proper reduction for $\mathcal{X}$, we have $\|\sigma_i^t \mathcal{X}\| < \|\mathcal{X}\|$, but we cannot demand to have $\|\sigma_i^t \mathcal{X}\| \leq \|\sigma_i^{t-1} \mathcal{X}\| \leq \cdots \leq \|\mathcal{X}\|$. For example, consider a hereditary algebra with two simple modules P, Q, where P is projective, Q injective, such that the injective envelope of P is of length 2, and the projective cover of Q is of length $c+1$, with $c \geq 3$. Let $\mathcal{X} = \sigma_1^2(Q, P)$. Then $\|\mathcal{X}\| = c$, whereas $\|\sigma_1 \mathcal{X}\| = c^2 - c - 1 > c$, and $\|\sigma_1^{-1} \mathcal{X}\| = c + 1 > c$. Of course, σ_1^t with $t = -2$ is a proper reduction for $\mathcal{X}$, and for $c \geq 4$, this is the only possible choice for t.

PROOF OF THEOREM 2. Let $\mathcal{X} = (X_1, \dots, X_n)$ be an exceptional sequence, and assume that $\mathcal{X}$ is not orthogonal. Choose $a < b$ such that $\operatorname{Hom}(X_a, X_b) \neq 0$, but $\operatorname{Hom}(X_i, X_j) = 0$ for the remaining $a \leq i < j \leq b$. Let φ: $X_a \to X_b$ be a non-zero morphism. According to [HR], we know that φ has to be a monomorphism or an epimorphism.

Consider first the case where φ is a monomorphism. This monomorphism induces epimorphisms

$$\mathrm{Ext}^1(X_b, X_i) \to \mathrm{Ext}^1(X_a, X_i)$$

for all i. The first Ext-group is zero for $b \geq i$, thus the second Ext-group also vanishes for these i. We see that both $\mathrm{Hom}(X_a, X_i) = 0$, and $\mathrm{Ext}^1(X_a, X_i) = 0$ for $a < i < b$, thus

$$\sigma_{b-2} \cdots \sigma_{a+1}\sigma_a \mathcal{X} = (X_1, \dots, X_{a-1}, X_{a+1}, \dots, X_{b-1}, X_a, X_b, \dots, X_n),$$

and always we just use transpositions. Now we apply some power of σ_{b-1}, say σ_{b-1}^t, in order to replace (X_a, X_b) by (Q, P) with $\mathrm{Hom}(Q, P) = 0$, thus we use a proper reduction.

The case when φ is an epimorphism, is treated similarly: The map φ induces epimorphisms

$$\mathrm{Ext}^1(X_i, X_a) \to \mathrm{Ext}^1(X_i, X_b).$$

The first Ext-group is zero for $i \geq a$, thus also the second Ext-group vanishes for these i. We see that both $\mathrm{Hom}(X_i, X_b) = 0$, and $\mathrm{Ext}^1(X_i, X_b) = 0$ for all $a < i < b$, thus

$$\sigma_{a+1} \cdots \sigma_{b-1} \mathcal{X} = (X_1, \dots, X_{a-1}, X_a, X_b, X_{a+1}, \dots, X_{b-1}, X_{b+1}, \dots, X_n),$$

always using just transpositions. We now apply some power σ_a^t of σ_a in order to replace (X_a, X_b) by (Q, P) with $\mathrm{Hom}(Q, P) = 0$; thus we use a proper reduction.

Since a proper reduction always decreases $\|\mathcal{X}\|$, this process has to stop after a finite number of steps, and we obtain some orthogonal exceptional sequence.

7. The orthogonal complete exceptional sequences

THEOREM 3. *The orthogonal complete exceptional sequences are just those exceptional sequences which consist of the simple modules.*

PROOF. Let $\mathcal{X}$ be an exceptional sequence. Let $\mathcal{C}(\mathcal{X})$ be be the smallest subcategory containing $\mathcal{X}$ and being closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. If $\mathcal{X}$ is complete, then $\mathcal{C}(\mathcal{X}) = A$-mod, as Crawley-Boevey [CB] has shown (again without using any assumption on the base field).

Since $X_1, \dots, X_s$ are orthogonal modules with division rings as endomorphism rings, $\mathcal{C}(\mathcal{X})$ is just the set of A-modules which have a filtration by modules of the form X_i, see [R1] (the process of simplification). This shows that any simple A-module S_j has a filtration by modules of the form X_i, thus for any S_j, there is some $\pi(j)$ with $S_j = X_{\pi(j)}$. Since the length of the sequence $\mathcal{X}$ is $s = s(A)$, it follows that any X_i is simple.

COROLLARY. *Any complete exceptional sequence can be shifted by the braid group action to an exceptional sequence consisting only of simple modules by using only transpositions and proper reductions.*

LEMMA. *The complete exceptional sequences which consist of simple modules can be obtained from each other by the braid group action using only transpositions.*

This can be verified without difficulties.

COROLLARY. *The braid group acts transitively on the set of complete exceptional sequences.*

As mentioned in the introduction, this assertion, for A finite-dimensional over some algebraically closed field, is the main result of the paper [CB] by Crawley-Boevey. His proof relies on investigations by Schofield [S] dealing with semi-invariants of quivers, and this is the only part of the paper which uses the assumption on the base field. Actually, the decisive Lemma 7 in [CB] may also be shown directly, using considerations similar to the ones above.

8. Endomorphism rings of exceptional modules

Given an artinian ring B, let $J(B)$ be its radical, thus $J(B)$ is the maximal nilpotent ideal of B.

THEOREM 4. *Let $\mathcal{X} = (X_1, \dots, X_s)$ be a complete exceptional sequence, let $B = B(\mathcal{X})$ be the endomorphism ring of $\bigoplus_i X_i$. Then $B/J(B)$ is Morita equivalent to $A/J(A)$.*

PROOF. Let $D(\mathcal{X}) = B/J(B)$. Let D_i be the endomorphism ring of X_i, and note that D_i is a division ring, thus $D = \prod_i D_i$. We claim that $D(\mathcal{X})$ and $D(\sigma\mathcal{X})$ are isomorphic, for any braid group element σ. Of course, it is sufficient to consider a generator σ_i, and we may assume that $s = 2$. But in this case the assertion is obvious.

9. Tilting sequences

An exceptional sequence $\mathcal{X} = (X_1, \dots, X_n)$ is said to be *strongly exceptional*, provided that we have $\operatorname{Ext}^1(X_i, X_j) = 0$ for all i, j. A strongly exceptional sequence which is complete may be called a *tilting sequence.*

We say that σ_i is a *p-extension* for $\mathcal{X}$, provided that $\operatorname{Hom}(X_i, X_{i+1}) = 0$ (the letter p shall indicate that p-extensions are the usual procedure to construct indecomposable projective modules starting with simple modules; also in the general case considered here, we use the p-extensions in order to construct relative projective objects inside the subcategory of modules having a filtration with factors of the form X_i.)

THEOREM 5. *Any exceptional sequence can be shifted by the braid group action to a strongly exceptional sequence using only transpositions and p-extensions.*

PROOF. Let $\mathcal{X} = (X_1, \dots, X_n)$ be an exceptional sequence, and suppose it is not strongly exceptional. Choose $a < b$ such that $\operatorname{Ext}^1(X_a, X_b) \neq 0$, and such that $b-a$ is minimal. We choose t with $a \leq t \leq b$ maximal such that $\operatorname{Hom}(X_a, X_t) \neq 0$. Since $\operatorname{Hom}(X_a, X_b) = 0$, we even have $a \leq t < b$. Let $\varphi\colon X_a \to X_t$ be a non-zero map.

We claim that φ is an epimorphism. Since $\mathrm{Ext}^1(X_t, X_a) = 0$, φ is a monomorphism or an epimorphism. Let us assume that $a < t$ and that φ is a monomorphism; the map φ induces a surjective map

$$\mathrm{Ext}^1(X_t, X_b) \to \mathrm{Ext}^1(X_a, X_b),$$

but the latter group is non-zero, whereas the first one is zero, by the minimality assumption on $b-a$. Thus, we obtain a contradiction. As a consequence, we see that $\mathrm{Hom}(X_t, X_i) = 0$ for $t < i \leq b$. For, a non-zero map ψ: $X_t \to X_i$ can be composed with φ and will give a non-zero map $X_a \to X_i$, contrary to the maximality of t.

For $t > a$, we have both $\mathrm{Hom}(X_t, X_i) = 0$, and $\mathrm{Ext}^1(X_t, X_i) = 0$, for all $t < i \leq b$; thus the consecutive application of first σ_t, then σ_{t+1}, and so on, finally σ_{b-1} yields just transpositions, and reduces $b - a$ by 1. The assertion follows by induction.

Thus, it remains to consider the case $t = a$. Since $\mathrm{Hom}(X_a, X_i) = 0$, and $\mathrm{Ext}^1(X_a, X_i) = 0$, for all $a < i < b$, the application of first σ_a, then σ_{a+1}, and so on, finally σ_{b-2} yields transpositions, and we obtain in this way an exceptional sequence $\mathcal{Y}$ with $\mathrm{Hom}(Y_{b-1}, Y_b) = 0$, and $\mathrm{Ext}^1(Y_{b-1}, Y_b) \neq 0$ (here, $Y_{b-1} = X_a$). Obviously, σ_{b-1} is a p-extension for $\mathcal{Y}$.

In order to see that the process stops, let us introduce a set $E(\mathcal{X})$ as follows: For any pair (u, v) with $1 \leq u < v \leq n$, let $E(\mathcal{X}; u, v)$ be the factor group of $\mathrm{Ext}^1(X_u, X_v)$ modulo the subgroup generated by the images of the induced maps $\mathrm{Ext}^1(X_u, \zeta)$ where ζ: $X_j \to X_v$ is a map with $j < v$. By definition, a pair (u, v) belongs to $E(\mathcal{X})$ if and only if there exists a sequence $u = u_0 < u_1 < \ldots u_m = v$ with $m \geq 1$ such that $E(\mathcal{X}; u_{i-1}, u_i) \neq 0$ for all $1 \leq i \leq m$. If σ_i is a transposition for $\mathcal{X}$, then the sets $E(\mathcal{X})$ and $E(\sigma_i\mathcal{X})$ clearly will have the same number of elements.

Thus, let σ_i be a p-extension for $\mathcal{X}$; and let $\mathcal{Y} = \sigma_i\mathcal{X}$. We claim that in this case the number of elements of $E(\mathcal{Y})$ decreases by (at least) one. Let $u < i$, and $i+1 < v$. We note the following: We have $E(\mathcal{Y}; u, v) = E(\mathcal{X}; u, v)$. Since $Y_i = X_{i+1}$, it is easy to see that $E(\mathcal{Y}; u, i) = E(\mathcal{X}; u, i+1)$ and $E(\mathcal{Y}; i, v) = E(\mathcal{X}; i+1, v)$. There is an epimorphism $Y_{i+1} \to X_i$ (with kernel a direct sum of copies of Y_i); it induces an isomorphism between $E(\mathcal{Y}; u, i+1)$ and $E(\mathcal{X}; u, i)$. Finally, assume that $E(\mathcal{Y}; i+1, v) \neq 0$. Then we have $E(\mathcal{X}; i, v) \neq 0$ or $E(\mathcal{X}; i+1, v) \neq 0$. In both cases, it follows that the pair (i, v) belongs to $E(\mathcal{X})$. Altogether we see that (u, v) belongs to $E(\mathcal{Y})$ if and only if it belongs to $E(\mathcal{X})$; that (u, i) or (i, v) belongs to $E(\mathcal{Y})$, if and only if $(u, i+1)$, or $(i+1, v)$ belongs to $E(\mathcal{X})$, respectively, and that $(u, i+1)$ or $(i+1, v)$ belongs to $E(\mathcal{Y})$, if and only if (u, i) or (i, v) belongs to $E(\mathcal{X})$, respectively. Of course, $(i, i+1)$ belongs to $E(\mathcal{X})$, but not to $E(\mathcal{Y})$. This completes the proof.

Given an exceptional sequence $\mathcal{X}$, we denote by $\mathcal{F}(\mathcal{X})$ the set of A-modules which have a filtration with factors of the form X_i.

THEOREM 6. *Let $\mathcal{X}$ be an exceptional sequence. Assume that $\mathcal{T}$ is a strongly exceptional sequence such that $\mathcal{X}$ can be shifted by the braid group action to $\mathcal{T}$ using only transpositions and p-extensions. Then the modules T_i are just those indecomposable modules M in $\mathcal{F}(\mathcal{X})$ which satisfy* $\mathrm{Ext}^1(M, X_i) = 0$ *for all i.*

PROOF. Let $\mathcal{X} = (X_1, \ldots, X_n)$. Let $\mathcal{P}(\mathcal{X})$ be the set of A-modules M in $\mathcal{F}(\mathcal{X})$ which satisfy $\mathrm{Ext}^1(M, X_i) = 0$ for all i. If $\mathcal{Y}$ is obtained from $\mathcal{X}$ by a transposition or a p-extension, then on the one hand, all the modules Y_i belong to $\mathcal{F}(\mathcal{X})$, thus

$\mathcal{F}(\mathcal{Y}) \subseteq \mathcal{F}(\mathcal{X})$ (and actually $\mathcal{F}(\mathcal{Y}) \subset \mathcal{F}(\mathcal{X})$ in the case of a p-extension), whereas, on the other hand, we have $\mathcal{P}(\mathcal{X}) \subseteq \mathcal{F}(\mathcal{Y})$, thus $\mathcal{P}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{Y})$. The latter implies that $\mathcal{P}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{T})$. Of course, $\mathcal{P}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$ is just the additive subcategory $\operatorname{add}\mathcal{T}$ generated by the modules T_i. It is known (see [DR], Theorem 2) that $\mathcal{P}(\mathcal{X})$ has precisely n isomorphism classes of indecomposable objects, thus $\mathcal{P}(\mathcal{X}) = \operatorname{add}\mathcal{T}$.

THEOREM 7. *Let $\mathcal{T} = (T_1, \ldots, T_n)$ be a strongly exceptional sequence. The number of exceptional sequences which can be shifted by the braid group action to $\mathcal{T}$ using only transpositions and p-extensions is at most $n!$.*

PROOF. Let $\mathcal{X}$ be an exceptional sequence which can be shifted by the braid group action to $\mathcal{T}$ using only transpositions and p-extensions. Consider T_1. There is a surjective map $T_1 \to X_i$ for some (uniquely defined) i such that the kernel, as well as all the other modules $T_2, \ldots T_n$ have filtrations with factors X_j where $j \neq i$. Let $\mathcal{X}'$ be obtained from $\mathcal{X}$ by deleting X_i. Clearly $\mathcal{T}' = (T_2, \ldots, T_n)$ is itself strongly exceptional, and $\mathcal{X}'$ can be shifted by the braid group action to $\mathcal{T}'$ using only transpositions and p-extensions. By induction, we know that there are at most $(n-1)!$ possibilities for $\mathcal{X}'$. Since X_i is uniquely determined by $\mathcal{X}'$ and the index i (see [CB], Lemma 2), there are at most n possibilities for X_i, when $\mathcal{X}'$ is fixed. This completes the proof.

10. Perpendicular categories

We have mentioned above that Crawley-Boevey's definition of the braid group operation relies on properties of perpendicular categories. Perpendicular categories have been studied by Geigle-Lenzing [GL] and Schofield [S]; they are defined as follows: Given a collection $\mathcal{C}$ of A-modules, then $\mathcal{C}^{\perp}$ is the full subcategory of all A-modules Y which satisfy both

$$\operatorname{Hom}(C, Y) = 0 = \operatorname{Ext}^1(C, Y) \quad \text{for all} \quad C \in \mathcal{C};$$

similarly, ${}^{\perp}\mathcal{C}$ is the full subcategory of all A-modules X which satisfy both

$$\operatorname{Hom}(X, C) = 0 = \operatorname{Ext}^1(X, C) \quad \text{for all} \quad C \in \mathcal{C}.$$

Since A is hereditary, both subcategories $\mathcal{C}^{\perp}$ and ${}^{\perp}\mathcal{C}$ are exact (this means: they are abelian, and the inclusion functors are exact).

If X is an exceptional A-module, then it is known that $\mathcal{C}^{\perp}$ is equivalent to the category of all B-modules, similarly, ${}^{\perp}\mathcal{C}$ is equivalent to the category of all B'-modules; where B, B' are artin algebras with $s(B) = s(B') = s(A) - 1$, see [S]. Note that the procedures above yield an effective way of computing B and B', as soon as a complete exceptional sequence containing X is given:

Indeed, let $(X_1, \ldots, X_s)$ be a complete exceptional sequence with $X = X_i$ for some i. We may shift X to the end of the sequence, thus, without loss of generality, we may assume that $X = X_s$. But in this case, we know from [CB] that $X^{\perp}$ is just $\mathcal{C}(X_1, \ldots, X_{s-1})$. Now, using permutations and proper reductions to the sequence $(X_1, \ldots, X_{s-1})$, we may transform it to an orthogonal sequence $(Y_1, \ldots, Y_{s-1})$. But then the modules $Y_1, \ldots, Y_{s-1}$ are just the simple objects of $X^{\perp}$. Finally, using transpositions and p-extensions, we may transform the sequence $(Y_1, \dots, Y_{s-1})$ to a sequence $(Z_1, \dots, Z_{s-1})$ such that we have $\mathrm{Ext}^1(Z_i, Y_j) = 0$ for all $1 \le i, j \le s-1$. The latter means that the modules Z_i are the indecomposable projective modules in $X^{\perp}$; and the endomorphism ring of $\bigoplus_{i=1}^{s-1} Z_i$ is the artin algebra B we are looking for.

Of course, in case we deal with algebras over an algebraically closed field k, we may already stop when we have obtained the orthogonal exceptional sequence $(Y_1, \dots, Y_{s-1})$, since the dimension of the various k-spaces $\mathrm{Ext}^1(Y_i, Y_j)$ yields the quiver of B, and B is just the corresponding path algebra.

Similarly, we also can construct the artin algebra B'.

Dictionary

The terminology *exceptional* and *strongly exceptional* was introduced by Rudakov and his school [Ru] in the analogous situation of vector bundles.

Exceptional modules have also been called *stones* by Kerner [Ke] and *Schur modules* by Unger [U]. Modules with endomorphism ring a division ring have been named *bricks* in [R3].

A sequence $(X_1, \dots, X_n)$ is exceptional if and only if, first, the set of modules $X_1, \dots, X_n$ is *standardizable* in the sense of [DR] and, second, the order of the modules refines the intrinsic partial ordering of this standardizable set.

A pair of modules M, N with $\mathrm{Hom}(M, N) = 0$, $\mathrm{Hom}(N, M) = 0$ has been called *orthogonal* in [R1]. A set of modules consisting of pairwise orthogonal modules having divison rings as endomorphism rings has been called *discrete* by Gabriel and de la Peña [GP]. The p-extensions have been used in a decisive way in [R1,R2].

A sequence $(X_1, \dots, X_n)$ is strongly exceptional if and only if, first, the direct sum of the modules $X_1, \dots, X_n$ is a multiplicity–free *partial tilting module* as defined by Happel-Unger [HU], and, second, the order of the modules refines the partial ordering given by the existence of non-zero maps. Of course, the tilting sequences correspond in the same way to (multiplicity–free) *tilting modules*.

References

[AR] Auslander,M.; Reiten, I.: Representation theory of artin algebras III. Communications in Algebra 3 (1975), 239-294.

[B] Bondal, A.I.: Representations of associative algebras and coherent sheaves. Izv. Akad. Nauk SSSR Ser Mat. 53 (1989), 25-44.

[CB] Crawley-Boevey, W.: Exceptional sequences of representations of quivers. Proceedings of ICRA VI, Carleton-Ottawa Math. LNS 14 (1992)

[DR] Dlab, V., Ringel, C.M.: The module theoretical approach to quasi-hereditary algebras. In: Representations of Algebras and Related Topics. London Math. Soc. Lecture Note Series 168. Cambridge University Press (1992), 200–224

[GP] Gabriel, P., de la Peña, J.: Quotients of representation-finite algebras. Communications in Algebra 15 (1987), 279-307.

[GL] Geigle, W., Lenzing, H.: Perpendicular categories with applications to representations and sheaves. J.Algebra 144 (1991), 273-343

[G] Gorodentsev, A.L.: Exceptional bundles on surfaces with a moving anticanonical class. Izv. Akad. Nauk SSSR Ser Mat. 52 (1988), 740-757.

[GR] Gorodentsev, A.L., Rudakov, A.N.: Exceptional vector bundles on projective space. Duke Math. J. 54 (1987), 115-130.

[HR] Happel, D., Ringel, C.M.: Tilted algebras. Trans. Amer. Math. Soc. 274 (1982), 399–443.

[HU] Happel, D., Unger, L.: Almost complete tilting modules. Proc. Amer. Math. Soc. 106 (1989), 603-610.

[K1] Kac, V.G.: Infinite root systems, representations of graphs and invariant theory. Invent. Math. 56 (1980), 57-92.

[K2] Kac, V.G.: Infinite root systems, representations of graphs and invariant theory II. J. Algebra 77 (1982), 141-162.

[K3] Kac, V.G.: Infinite dimensional Lie algebras. Second Edition. Cambridge University Press (1985).

[Ke] Kerner, O.: Elementary stones. Communications in Algebra. (To appear)

[R1] Ringel, C.M.: Representations of K–species and bimodules. J. Algebra 41 (1976), 269–302.

[R2] Ringel, C.M.: Reflection functors for hereditary algebras. J. London Math. Soc. 21 (1980), 465-479.

[R3] Ringel, C.M.: Bricks in hereditary length categories. Resultate der Mathematik 6 (1983), 64-70

[R4] Ringel, C.M.: The regular components of the Auslander-Reiten quiver of a tilted algebra. Chinese Ann. Math. B 9 (1988), 1-18.

[Ru] Rudakov, A.N.: Helices and Vector Bundles. London Math. Soc. LNS 148.

[S] Schofield, A.: Semi–invariants of quivers. J. London Math. Soc. 43 (1991), 383-395.

[U] Unger, L.: Schur modules over wild, finite dimensional path algebras with three non isomorphic simple modules. J. Pure Appl. Algebra. 64 (1990), 205-222.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT, D-33 613 BIELEFELD

E-mail address: `ringel@mathematik.uni-bielefeld.de`