anharmonic oscillators, the magnetic bottle and orbit-orbit or spin-orbit resonance in the Solar system. Details are explained in the appendices.

The majority of the articles in this book is intersting to mathematicians and theoretical physicists alike. The book can however not (and is not intended to) serve as an introduction. It can be recommended to graduate students and researchers interested to enter deeper into the rapidly growing fields of Dynamical Systems.

The reader of the book would definitely benefit from an index.

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G. LUSZTIG: **Introduction to Quantum Groups** (Progress in Mathematics: Vol. 10). Boston et al: Birkhäuser Verlag 1993; 2nd rev. ed., 346 pp.

What are quantum groups? The topic is rather popular since several years, but it seems that there does not yet exist a generally accepted definition. Note that objects called "quantum groups" are *not* groups and some of them should be quite useless in quantum physics; so the concept may sound a little odd. Those objects called "quantum groups" are usually Hopf algebras, in this way they are similar (not to groups, but at least) to group algebras, and the first examples were constructed in order to deal with the quantum Yang-Baxter equation. There is a general feeling that symmetries in quantum physics may be related not to group actions, but to the operations by associative algebras or better Hopf algebras – this may give some explanation to the otherwise strange label. The first general account is Drinfeld's famous address to the ICM in Berkeley in 1986, and there is an increasing number of applications in various parts of mathematics (and maybe physics). Thus it is very worthwhile to have an exposition by one of the leading experts. There is a special class of algebras where anyone agrees that they should belong to whatever one may call "quantum groups", namely the Drinfeld-Jimbo quantizations of the Kac-Moody algebras, and these are the quantum groups considered in Lusztig's book. Questions about possible applications inside mathematics or physics are not touched in the book: there are no knots or 3-manifolds, the Yang-Baxter equation is not mentioned, no quantum field theory is constructed. For a general account concerning the different lines of development, the reader may be referred to *A Guide to Quantum Groups* by V. Chari and A. Pressley, which just now has been published by Cambridge University Press; in particular, it contains a list of references of 71 pages.

Let us recall that the Kac-Moody Lie algebras were introduced by V. Kac and R. V. Moody in order to consider Lie algebras which are given in a way similar to the Serre presentation of a finite-dimensional semisimple complex Lie algebra. In particular, a Kac-Moody Lie algebra $\mathbf{g}$ has a triangular decomposition $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ (for a finite-dimensional semisimple complex Lie algebra $\mathbf{g}$, we obtain such a decomposition by fixing a Cartan subalgebra $\mathbf{h}$, and choosing some positive part of the root system: $\mathbf{n}_+$ is the direct sum of the weight spaces for the positive roots, $\mathbf{n}_-$ is that for the negative roots). The universal enveloping algebra $U(\mathbf{g})$ of a Kac-Moody Lie algebra $\mathbf{g}$ has a corresponding triangular decomposition $U(\mathbf{g}) = U(\mathbf{n}_-) \otimes U(\mathbf{h}) \otimes U(\mathbf{n}_+)$. Now, we are going to "quantize" the algebra $U(\mathbf{g})$, by introducing an additional parameter v in the Serre presentation of $U(\mathbf{g})$. Lusztig has always stressed that instead of working over the complex numbers it will be useful to work over the integers; thus, as base field he takes the rational function field $\mathbb{Q}(v)$ in one variable v over the rational numbers. This is the setting of the book: we are dealing with an associative $\mathbb{Q}(v)$-algebra $\mathbf{U}$ which is defined by a presentation similar to the usual Serre presentation of $\mathbf{g}$ or of $U(\mathbf{g})$, but involving in addition the parameter v. This algebra $\mathbf{U}$ again has a triangular decomposition $\mathbf{U} = \mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+$, it is a Hopf

algebra, but usually neither commutative nor cocommutative. The algebra $\mathbf{U}$ occurs at a prominent place in Lusztig's book, and the title refers to it; however the main considerations of the book are dealing with only one part of this "quantum group", namely with $\mathbf{U}^+$. *The book should be considered as a detailed study of the algebra* $\mathbf{U}^+$, and of algebras derived from it, such as $\mathbf{U}$.

Something rather curious should be mentioned: As we have said, it is the algebra $\mathbf{U}^+$ which is the main object studied in the book, but this algebra is exhibited in the book under various names, such as $\mathbf{f}$, $\tilde{\mathbf{f}}$, or $\mathbf{k}$, of course also as $\mathbf{U}^+$, as well as $\tilde{\mathbf{U}}^+$, and at least half of the book is devoted to showing that these differently defined algebras are isomorphic (an isomorphism from $\mathbf{f}$ to $\tilde{\mathbf{f}}$ is given in Theorem 33.1.3, to $\mathbf{k}$ in Theorem 13.2.11, and to $\mathbf{U}^+$ in Corollary 3.2.6; an isomorphism between $\mathbf{U}^+$ and $\tilde{\mathbf{U}}^+$ can be found in Section 15.1.2). The reader will observe that there is a big advantage of using disjoint notations for algebras which only later turn out to be isomorphic: as soon as such a symbol occurs, one knows the setting one is dealing with; the basic operations may have completely different interpretations in the various settings. Why should it be of interest to deal with several realizations of one particular algebra? First of all, the mere existence of these realizations already indicates the importance of this algebra; also, if an algebra arises in diverse contexts, then it may serve as a link connecting these subjects. Second, specific properties may be easy to observe in one realization, but not in some other: thus these realizations also are working tools. Even if there is a large number of realizations of $\mathbf{U}^+$ in the book, it seems that further such results will be of interest in the future; already now there are similar realizations using methods from combinatorics (a recent preprint of J. A. Green shows that $\mathbf{U}^+$ is isomorphic to the so-called generic composition algebra) or from differential geometry (P. B. Kronheimer, H. Nakajima).

We should note that the algebra $\mathbf{U}^+$ usually is *not* a Hopf algebra, not even a bialgebra in the usual sense. But it is very similar to a bialgebra or a Hopf algebra: it has both an associative multiplication with unit and a coassociative comultiplication with counit, however these two structures interrelate in an unusual way: for a bialgebra Λ one requires that the comultiplication $\delta : \Lambda \to \Lambda \otimes \Lambda$ is an algebra homomorphism, where $\Lambda \otimes \Lambda$ is made into an algebra by using componentwise multiplication

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2,$$

whereas the comultiplication $\delta : \mathbf{U}^+ \to \mathbf{U}^+ \otimes \mathbf{U}^+$ will be an algebra homomorphism only in case we use a (just slightly) different multiplication on the tensor product $\mathbf{U}^+ \otimes \mathbf{U}^+$. The new multiplication on $\mathbf{U}^+$ differs from the usual one by a factor of the form $v^{|a_2|\cdot|b_1|}$; here, a symmetric bilinear form $\cdot$ appears, the so-called Cartan datum. As Lusztig points out, the Cartan datum is the essential ingredient for the whole theory, thus he uses it as the starting point of his investigation.

To begin with, Lusztig defines the algebra $\mathbf{f}$ on the first pages of the book in a very straightforward way, and he takes $\mathbf{f}$ as his main reference algebra. The basic data which have to be given, are a (symmetrizable) generalized Cartan matrix, say $A = (a_{ij})_{i,j=1}^n$ and a specific symmetrization, say positive integers d_i (thus $d_i a_{ij} = d_j a_{ji}$); Lusztig calls the symmetric matrix $(d_i a_{ij})_{i,j=1}^n$, or better the corresponding symmetric bilinear form on $\boldsymbol{Z}^n$ a *Cartan datum* and he denotes this bilinear form by $\cdot$. The main relevance of the bilinear form is the following: The algebra $\mathbf{f}$ is (in a natural way) $\boldsymbol{Z}^n$ graded and we may use the bilinear form $\cdot$ in order to define a non-trivial multiplication on the tensor product $\mathbf{f} \otimes \mathbf{f}$ by

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = v^{|a_2|\cdot|b_1|} a_1 b_1 \otimes a_2 b_2;$$

here we assume that a_2 and b_1 are homogeneous elements of degree $|a_2|$ and $|b_1|$, respectively. Using this multiplication on $\mathbf{f} \otimes \mathbf{f}$, the comultiplication $\mathbf{f} \to \mathbf{f} \otimes \mathbf{f}$ is an algebra homomorphism.

It later will be of interest that $\mathbf{f}$ also carries a (non-degenerate) inner product denoted by $(-,-)$. With respect to the inner product, the comultiplication is adjoint to the multiplication; thus multiplication and comultiplication determine each other. Lusztig's introduction of $\mathbf{f}$ focuses attention on this inner product: he starts with the free $\mathbb{Q}(v)$-algebra ${}'\mathbf{f}$ in n generators $\theta_1, \ldots, \theta_n$ and constructs on ${}'\mathbf{f}$ a symmetric bilinear form depending on the given Cartan datum. The radical $\mathcal{I}$ of the form $(-,-)$ is easily shown to be both an ideal and a coideal of ${}'\mathbf{f}$, and $\mathbf{f}$ is defined as ${}'\mathbf{f}/\mathcal{I}$. Lusztig can show without difficulties that $\mathbf{f}$ satisfies the quantum Serre relations (Proposition 1.4.3 on page 11), but he postpones the proof that $\mathbf{f}$ actually is defined by these relations (see Theorem 33.1.3 on page 260; Lusztig denotes by $\tilde{\mathbf{f}}$ the algebra defined by the quantum Serre relations for the positive part, thus Theorem 33.1.3 asserts that the canonical map $\tilde{\mathbf{f}} \to \mathbf{f}$ is an isomorphism). The proof uses results from the representation theory of (unquantized) Kac-Moody Lie algebras and he outlines the needed references to the book of Kac. These pages 258 - 260 are independent of the previous pages, thus the interested reader may jump from page 13 directly to that part of Lusztig's book. The identification of $\mathbf{f}$ and $\tilde{\mathbf{f}}$ will be essential for the experienced reader who knows already the algebra $\tilde{\mathbf{f}}$ and who wants to learn more about it – it should be of less interest to the neophyte, so Lusztig shifts his attention first to topics he considers to be more fundamental.

Besides the $\mathbb{Q}(v)$-algebra $\mathbf{f}$, Lusztig also introduces an integral form ${}_{\mathcal{A}}\mathbf{f}$; it is an $\mathcal{A}$-lattice inside $\mathbf{f}$ where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ is the ring of Laurent polynomials with integer coefficients. This allows to specialize v, as it is done in Part V of the book.

The main considerations of the book center around the so-called canonical basis of $\mathbf{f}$ (or $\mathbf{U}^+$) and its positivity properties. It was a remarkable observation of Lusztig and also Kashiwara that the algebra $\mathbf{U}^+$ has a basis with very desirable properties, and that such a basis is unique, so it is called the *canonical basis*. The specialization $v = 1$ yields a corresponding canonical basis for $U(\mathbf{n}^+)$ and $U(\mathbf{g})$ itself; even in the case of a finite-dimensional semisimple complex Lie algebra $\mathbf{g}$, the existence of a canonical basis of $U(\mathbf{g})$ was not known before. One property of the canonical basis deserves special attention: Given an integrable highest weight module V, the canonical surjection $\mathbf{U}^+ \to \mathbf{V}$ maps a certain subset of the canonical basis of $\mathbf{U}^+$ bijectively onto a basis of V, whereas the remaining elements are mapped to zero (Theorem 14.4.11); in this way, we see that integrable highest weight modules are endowed with special bases.

The easiest way of presenting the canonical basis of $\mathbf{f}$ (at least up to sign) is given in Theorem 14.2.3: We have mentioned above the inner product on $\mathbf{f}$. This inner product is normalized in Part (a) of Proposition 1.2.3 in a way which may look strange at first sight, but it turns out that with respect to this normalization, the elements of the canonical basis can be described up to sign rather easily: an element $x \in {}_{\mathcal{A}}\mathbf{f}$ or its negative belongs to the canonical basis if and only if, first, x is invariant under the automorphism of $\mathbf{f}$ which sends any generator θ_i to itself and v to v^{-1}, and second, the inner product (x, x) belongs to $1 + v^{-1}\mathbb{Z}[[v^{-1}]]$.

Positivity results are known only in the case of a generalized Cartan matrix which is symmetric (and not just symmetrizable): If we write down the multiplication table with respect to the canonical basis, it turns out that all the coefficients are Laurent polynomials in the variable v with *non-negative* coefficients; similarly, also the coefficients of the comultiplication and the inner product are Laurent polynomials with non-negative coefficients (Theorem 14.4.13). The reason for the positivity of the coefficients is the fact that these numbers are the dimensions of certain vector spaces.

There are several different approaches concerning the existence of the canonical basis, but at present only Lusztig's approach using perverse sheaves yields the positivity assertions. Lusztig gives an outline of perverse sheaf theory in Chapter 8. He writes: "The theory of perverse sheaves ... will be reviewed but not explained. Readers who are not comfortable with the theory of perverse sheaves are advised to skip Chapters 8–13, and accept the theorems in

Chapter 14 without proof (the statement of those theorems do not involve perverse sheaves, only their proofs do)."

In order to explain the use of perverse sheaves, let us start with a symmetric generalized Cartan matrix of size n. As usual, one may consider a corresponding graph. Choosing an arbitrary orientation for the edges of the graph, we deal with what is called a quiver. Let $\nu = (\nu)_{i=1}^n$ be an n-tuple of non-negative integers, choose complex vector spaces $\mathbf{V}_i$ of dimension ν_i, and consider $\mathbf{V} = \oplus \mathbf{V}_i$. By definition, the quiver variety is the set of all representations of the quiver using these vector spaces $\mathbf{V}_i$; it is an affine space, and there is an obvious action of the group $G = \prod \mathrm{Gl}(\mathbf{V}_i)$ on $\mathbf{E}_\mathbf{V}$. The perverse sheaves of interest are certain semisimple G-equivariant perverse sheaves on $\mathbf{E}_\mathbf{V}$. The main target is to define the algebra $\mathbf{k}$. In order to obtain $\mathbf{k}$, one starts with the direct sum of some Grothendieck groups of perverse sheaves on $\mathbf{E}_\mathbf{V}$, one for each possible dimension vector ν, and defines a product and a coproduct using a kind of induction and restriction, respectively.

Lusztig's book is divided into six parts. Part I starts with the definition of $\mathbf{f}$ and then introduces the Drinfeld-Jimbo algebra $\mathbf{U} = \mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+$, with $\mathbf{U}^+$ (and also $\mathbf{U}^-$) isomorphic to $\mathbf{f}$; in order to define $\mathbf{U}$ one needs in addition to a Cartan datum a corresponding root datum, this makes the consideration of $\mathbf{U}$ less comfortable. Further, integrable $\mathbf{U}$-modules as well as highest weight $\mathbf{U}$-modules are considered. The use of the quantum Casimir operator allows to show that non-zero integrable highest weight modules are simple (Lemma 6.2.1), thus one obtains results concerning complete irreducibility.

One of the reasons for the introduction of the quantum groups $\mathbf{U}$ was the need for constructing so-called R-matrices; these are quadratic matrices which satisfy a certain cubic equation, the quantum Yang-Baxter equation. Universal R-matrices are obtained in the following way: recall that $\mathbf{U}$ has a comultiplication which usually is non-commutative. Now, one can show that the comultiplication and its opposite are related to each other, and the universal R-matrix for $\mathbf{U}$ appears as an intertwiner. Instead of presenting the R-matrices, Lusztig prefers to exhibit so-called quasi-R-matrices: they differ from the usual R-matrices only on the diagonal.

Part II comprizes the Chapters 8 – 14 mentioned above: here, the perverse sheaf approach is outlined, the algebra $\mathbf{k}$ is constructed and it is shown that $\mathbf{k}$ is isomorphic to $\mathbf{f}$.

We have mentioned above that Theorem 14.2.3 describes the canonical basis up to sign. The distiction between the elements of the canonical basis and their negatives is stated in Theorem 14.4.3. For a symmetric generalized Cartan matrix, this separation follows directly from the identification of $\mathbf{k}$ with $\mathbf{f}$, here the simple perverse sheaves correspond to the elements of the canonical basis. In order to transfer results from symmetric to non-symmetric, but symmetrizable generalized Cartan matrices, Lusztig outlines a procedure to attach to a non-symmetric situation a pair consisting of a symmetrizable generalized Cartan matrix and an automorphism of the underlying graph (Proposition 14.1.2); for the affine cases, he presents these pairs case-by-case in Section 14.1.5. Such kind of reduction seems to be folklore, but there may be no other complete reference.

Part III uses Kashiwara's operators. They were introduced by M. Kashiwara in order to construct the canonical basis from scratch; in Lusztig's treatment, they are needed in order to complete (in Section 19.2.3.) the proof of Theorem 14.2.3 in the non-symmetric case. In addition, Lusztig also considers the canonical basis at ∞, these "crystal" bases where the main impetus for the work of Kashiwara.

Part IV consists of the Chapters 23 – 30, they are devoted again to the representation theory of the Drinfeld-Jimbo algebras. Lusztig replaces in $\mathbf{U}$ the Cartan part $\mathbf{U}^0$ by the direct sum of infinitely many one-dimensional algebras, one for each element of the weight lattice. The algebra obtained in this way is denoted by $\dot{\mathbf{U}}$, it is an algebra without 1, but with sufficiently many idempotents. Note that the (unital) $\dot{\mathbf{U}}$-modules are just those $\mathbf{U}$-modules which have a weight space decomposition.

Part V deals with specializations of v, or, more generally, with base ring changes: suppose that R is a commutative ring with 1, and that $\mathbf{v}$ is some fixed invertible element of R. We may consider R as an $\mathcal{A}$-algebra via the ring homomorphism $\mathcal{A} \to R$ which sends v to $\mathbf{v}$; the tensor product of R and ${}_{\mathcal{A}}\mathbf{f}$ over $\mathcal{A}$ will be denoted by ${}_{R}\mathbf{f}$. Nothing surprising will happen in case $\mathbf{v}$ has infinite multiplicative order in R. Otherwise, one should distinguish the cases whether $\mathbf{v}$ is, or is not, equal to ± 1. The case $\mathbf{v} = 1$ furnishes the relationship between the Drinfeld-Jimbo algebras and the Kac-Moody Lie algebras. This is the place where the isomorphism between $\mathbf{f}$ and $\tilde{\mathbf{f}}$ is shown.

Chapters 34 - 36 investigate the case when $\mathbf{v}$ is a proper root of unit. As one may expect, the structure theory in case $\mathbf{v}$ is a proper root of unity is more involved. The main difference stems from the fact that the elements $\theta_i \otimes 1$ no longer will generate ${}_{R}\mathbf{f}$, even if R is a field. As Lusztig has shown in previous papers, starting with a generalized Cartan matrix of finite type, the elements $\theta_i \otimes 1$ generate a finite-dimensional Hopf algebra. The book now presents corresponding results in the general case.

In the final Part VI (Chapters 37 - 42), braid group operations on $\mathbf{U}$ are considered. Given a generalized Cartan matrix of size n, there is a corresponding braid group B, say with generators $s_1, \ldots, s_n$; adding the relations $s_i^2 = 1$, we obtain the appropriate Weyl group W. Note that W operates on the root system. Even for a finite-dimensional semisimple complex Lie algebra $\mathbf{g}$, this action cannot always be lifted to an action of W on $\mathbf{g}$; but it can be lifted to an action of B on $\mathbf{g}$. In general, there are several such operations of B on $\mathbf{U}$, and many papers investigating the Drinfeld-Jimbo algebras have dealt with these braid group operations.

It is well-known that there are two kinds of irreducible generalized Cartan matrices, which behave rather special: those of finite, and those of affine type (the finite type ones are those which give rise to the finite-dimensional semisimple Lie algebras). There is a special study of the finite type cases at the end of Parts III and VI. But the reader should be aware that Lusztig did not incorporate into his book his more detailed investigations dealing with the affine type, see *Affine quivers and canonical bases* (Publ. Math., IHES, n° 76 (1992)). Also his examples of "tight" monomials (monomials in the generators θ_i which belong to the canonical basis) are missing. In this way, Lusztig's book is really an "Introduction": it presents the general frame work which is necessary for further more specialized studies.

The considerations presented in the book, the results as well as the methods, are purely algebraic, even power series are used only reluctantly. Of course, the central parts rely on the use of perverse sheaf theory, but again, this may be considered as a quite algebraic part of analysis or topology. On the other hand, algebraists have to be careful when reading the book, since some standard conventions used in ring and module theory are not followed. For example, ${}_{\mathcal{A}}\mathbf{f}$ does not mean that $\mathbf{f}$ is considered as an $\mathcal{A}$-module (which also could be done), but it denotes a certain $\mathcal{A}$-lattice inside $\mathbf{f}$.

Lusztig's book is very well written and seems to be flawless, the only misprint the referee found was one missing (but inessential) bracket. Obviously, this will be the standard reference book for the material presented and anyone interested in the Drinfeld-Jimbo algebras will have to study it very carefully.

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